Hofbauer Towers and Inverse Limit Spaces

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Motivation

Our goal is to better understand the topological structure of inverse limit spaces.

We use combinatoric tools, including Hofbauer towers, to study the collection of endpoints of the inverse limit space (I, f) where f is a unimodal map with $\lim_{k\to\infty} Q(k) = \infty$.

Unimodal Maps

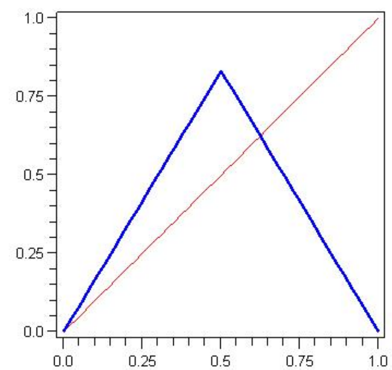
A unimodal map is a continuous map $f : [0,1] \to [0,1]$ for which there exists a point $c \in (0,1)$ such that $f|_{[0,c)}$ is strictly increasing and $f|_{(c,1]}$ is strictly decreasing.

The point c is called the turning point and we set c_i to be the ith iterate of c; i.e., $c_i = f^i(c)$.

Symmetric Tent Maps

The symmetric tent map $T_a : [0,1] \to [0,1]$ with $a \in [0,2]$ is given by

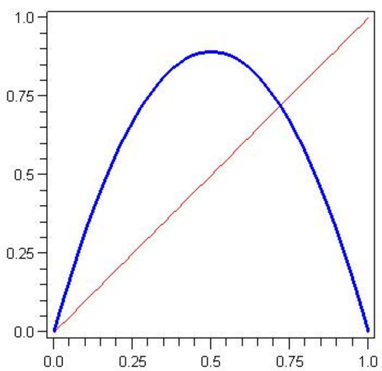
$$T_a(x) = \begin{cases} ax & \text{if } x \le \frac{1}{2}, \\ a(1-x) & \text{if } x \ge \frac{1}{2}. \end{cases}$$



Logistic Maps

The logistic map $g_a : [0,1] \to [0,1]$ with $a \in [0,4]$ is defined by

$$g_a(x) = ax(1-x).$$



Kneading Sequences

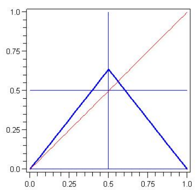
For a unimodal map f and a point $x \in [0,1]$, the itinerary of x under f is given by $I(x) = I_0 I_1 I_2 \cdots$, where

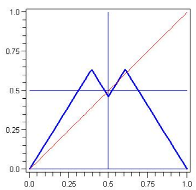
$$I_j = \begin{cases} 0 & \text{if } f^j(x) < c, \\ * & \text{if } f^j(x) = c, \\ 1 & \text{if } f^j(x) > c. \end{cases}$$

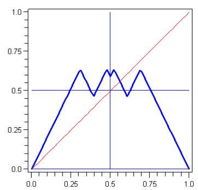
The kneading sequence of a map f, denoted $\mathcal{K}(f)$, is the sequence $I(c_1) = e_1 e_2 e_3 \cdots$.

Cutting Times and Kneading Maps

An iterate n is called a cutting time if the image of the central branch of f^n contains c. The cutting times are denoted $S_0, S_1, S_2, \ldots$, where $S_0 = 1$ and $S_1 = 2$.







Cutting Times and Kneading Maps

An integer function $Q : \mathbb{N} \to \mathbb{N} \cup \{0\}$, called the kneading map, may be defined by $S_k - S_{k-1} = S_{Q(k)}$.

The kneading sequence, kneading map, and cutting times each completely determine the combinatorics of the map f.

Hofbauer Towers

Given a unimodal map f, the associated *Hofbauer tower* is the disjoint union of intervals $\{D_n\}_{n\geq 1}$ where $D_1 = [0, c_1]$ and, for $n \geq 1$,

$$D_{n+1} = \begin{cases} f(D_n) & \text{if } c \notin D_n, \\ [c_{n+1}, c_1] & \text{if } c \in D_n. \end{cases}$$

Hofbauer Towers

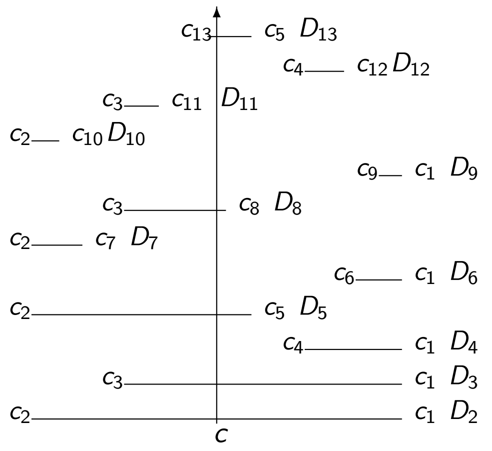


Figure: Hofbauer tower for Fibonacci combinatorics

Inverse Limit Spaces

Here a continuum is a compact connected metrizable space. Given a continuum I and a continuous map $f : I \to I$, the associated inverse limit space (I, f) is defined by

$$(I, f) = \{x = (x_0, x_1, \ldots) \mid x_n \in I \text{ and } f(x_{n+1}) = x_n \text{ for all } n \in \mathbb{N}\}$$

and has metric

$$d(x, y) = \sum_{i=0}^{\infty} \frac{|x_i - y_i|}{2^i}.$$

Ingram's Conjecture

Inverse limit spaces are difficult to classify.

Ingram's Conjecture, dating to the early 1990s, states that the inverse limit spaces (I, f) and (I, g) are not topologically homeomorphic when f and g are distinct symmetric tent maps.

There have been many partial results over the past two decades, and most recently Barge, Bruin, and Štimac establish Ingram's Conjecture.

Endpoints and $\mathcal{E}$

In our case, a point $x \in (I, f)$ is an endpoint of (I, f) provided for every pair A and B of subcontinua of (I, f) with $x \in A \cap B$, either $A \subset B$ or $B \subset A$.

Given a unimodal map f, define
$\mathcal{E}_f := \{(x_0, x_1, \dots) \in (I, f) \mid x_i \in \omega(c, f) \text{ for all } i \in \mathbb{N}\}$

Lemma (2010, Alvin and Brucks, Fund. Math.)

Let f be a unimodal map with $\mathcal{K}(f) \neq 10^\infty$ and suppose $x = (x_0, x_1, \cdots) \in (I, f) \setminus \mathcal{E}$. Then x is not an endpoint of (I, f).

Backward Itineraries

The backward itinerary of a point $x \in (I, f)$ is defined coordinate-wise by $\mathcal{I}_j(x)$, where $\mathcal{I}_j(x) = 1$ if $x_j > c$, $\mathcal{I}_j(x) = 0$ if $x_j < c$, and $\mathcal{I}_j(x) = *$ if $x_j = c$.

Backward Itineraries

For each $x \in (I, f)$ such that $x_i \neq c$ for all $i > 0$, set

$$\tau_R(x) = \sup\{n \geq 1 \mid \mathcal{I}_{n-1}(x)\mathcal{I}_{n-2}(x)\cdots\mathcal{I}_1(x) = e_1 e_2 \cdots e_{n-1} \text{ and } \#\{1 \leq i \leq n-1 \mid e_i = 1\} \text{ is even }\}, \text{ and}$$

$$\tau_L(x) = \sup\{n \geq 1 \mid \mathcal{I}_{n-1}(x)\mathcal{I}_{n-2}(x)\cdots\mathcal{I}_1(x) = e_1 e_2 \cdots e_{n-1} \text{ and } \#\{1 \leq i \leq n-1 \mid e_i = 1\} \text{ is odd }\}.$$

Known Results About Endpoints

Bruin provides a characterization with both a combinatoric and analytic component when f is unimodal and the turning point is not periodic.

Proposition (1999, Bruin, *Topology Appl.*)

Let f be a unimodal map and $x \in (I, f)$ be such that $x_i \neq c$ for all $i \geq 0$. Then x is an endpoint of (I, f) if and only if $\tau_R(x) = \infty$ and $x_0 = \sup \pi_0(\Gamma(x))$ (or $\tau_L(x) = \infty$ and $x_0 = \inf \pi_0(\Gamma(x))$).

The Adding Machine Map

Let $\alpha = \langle q_1, q_2, \ldots \rangle$ be a sequence of integers where each $q_i \geq 2$. Denote by Δ_α the set of all sequences $(a_1, a_2, \ldots)$ such that $0 \leq a_i \leq q_i - 1$ for each i.

The map $f_\alpha : \Delta_\alpha \to \Delta_\alpha$, defined by

$$f_\alpha((x_1, x_2, \ldots)) = (x_1, x_2, x_3, \ldots) + (1, 0, 0, \ldots),$$

is called the α-adic adding machine map.

Relating Endpoints and Renormalization

Theorem (2010, Alvin and Brucks, Fund. Math.)

Let f be an infinitely renormalizable logistic map. Then $\mathcal{E}$ is precisely the collection of endpoints of (I, f).

In this case $\lim\limits_{k\to\infty} Q(k) = \infty$.

Kneading Maps, Adding Machines, and Endpoints

Theorem (2011, Alvin and Brucks, Topology Appl.)

Let $f \in \mathcal{A}$ be such that $\lim_{k\to\infty} Q(k) = \infty$. Then $\mathcal{E}$ is precisely the collection of endpoints of (I, f).

Further, if $f \in \mathcal{A}$ and $\lim_{k\to\infty} Q(k) \neq \infty$, then it may be that $\mathcal{E}$ is exactly the collection of endpoints of (I, f), or it may be that $\mathcal{E}$ properly contains the collection of endpoints of (I, f).

Kneading Maps and Endpoints

Is it possible that every unimodal map f with $\lim\limits_{k\to\infty} Q(k) = \infty$ is such that $\mathcal{E}$ is the collection of endpoints of (I, f)?

Recall that if $f|_{\omega(c)}$ is topologically conjugate to an adding machine, then $f|_{\omega(c)}$ is one-to-one.

Kneading Maps and Endpoints

Theorem (Alvin, Proc. AMS, to appear)

Let f be a unimodal map such that $\lim_{k\to\infty} Q(k) = \infty$ and $f|_{\omega(c)}$ is one-to-one. Then $\mathcal{E}$ is precisely the collection of endpoints of (I, f).

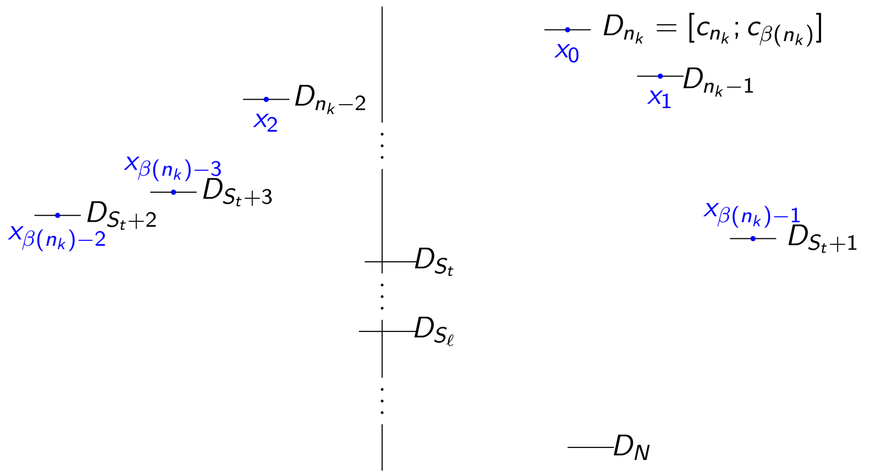
Proof of Main Result

Let $x = (x_0, x_1, x_2, \dots) \in \mathcal{E}$ be such that $x_i \neq c$ for all $i \geq 0$. Recall that $x_0 \in \omega(c)$.

We can find an increasing sequence of D_{n_k} such that $x_0 \in D_{n_k}$ for all $k \in \mathbb{N}$.

As $Q(k) \to \infty$ and $f|_{\omega(c)}$ is one-to-one, there exists some level D_N of the Hofbauer tower where if $x_0 \in D_n$ for some $n \geq N$, then the unique preimage $x_1 \in \omega(c)$ lies in D_{n-1}.
WLOG take $\{n_k\}$ such that $n_1 > S_l > N$.

Proof of Main Result



Proof of Main Result

Hence $\mathcal{I}_{\beta(n_k)-1}(x) \cdots \mathcal{I}_1(x) = e_1 e_2 \cdots e_{\beta(n_k)-1}$.

Note that $\beta(n_k) \to \infty$.

$\tau_R(x) = \infty$ or $\tau_L(x) = \infty$.

In both cases we show x must be an endpoint of (I, f), using Bruin's characterization.

Summary of Results and Open Questions

Is it the case that for all unimodal maps f with $\lim_{k\to\infty} Q(k) = \infty$ the collection $\mathcal{E}$ is precisely the collection of endpoints for (I, f)?

Summary of Results and open questions

How will this better understanding of the collection of endpoints help us to understand the topological structure of the inverse limit space?

Can we use the behavior of the endpoints to distinguish between two inverse limit spaces?

Thank you for your attention.