Entropy and Switching Systems

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OUTLINE

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1. Introduction

Given

- two dissipative, continuous maps $f_{\pm 1} : \mathbb{R}^d \longrightarrow \mathbb{R}^d$,
- a *switching* or *control sequence*

$$\mathbf{s} = (..., s_{-n}, ..., s_0, ...s_n, ...) \in \{-1, +1\}^{\mathbb{Z}},$$

the corresponding (discrete) *time-switched system* is defined as

$$x_{n+1} = f_{s_n}(x_n).$$

Time-switched systems (or switching systems) are an instance of *non-autonomous dynamical systems*.

1. Introduction

Remark. $\mathcal{S} = \{-1, +1\}^{\mathbb{Z}}$ endowed with

$$\mathrm{dist}_{\mathcal{S}}(\mathbf{s}, \mathbf{s}') = \sum_{n \in \mathbb{Z}} 2^{-|n|} \left| s_n - s'_n \right|,$$

is a compact metric space.

Set

$$\boxed{\text{Complexity(control)} := h_{top}(\sigma)}$$

where

$$\sigma : (\cdots, s_n, s_{n+1}, \cdots) \mapsto (\cdots, s_{n+1}, s_{n+2}, \cdots).$$

is the shift on $\mathcal{S}$.

1. Introduction

Let $\tilde{\Sigma}$ be the shift on the "entire solutions" of the switched dynamics. Set

$$\boxed{\text{Complexity(switched dynamics)} \;:= h_{top}(\tilde{\Sigma}).}$$

Result. Under some provisos,

$$\text{Complexity(control)} \leq \text{Complexity(switched dynamics)}$$

Corollary. (*Complexity increase via switching*) If

$$\text{Complexity(control)} > h_{top}(f_+), h_{top}(f_-)$$

then

$$\text{Complexity(switched dynamics)} > h(f_+), h(f_-).$$

1. Introduction

In general, the emergence of different properties to those of the constituent maps via switching is called *Parrondo's paradox*.

- Original version[1]: Switching two loosing games can produce a winning game.
- Dynamical version[2]: Periodic switching of chaotic maps can produce order.
- A possible topological version: Switching of noncomplex dynamics can produce a complex dynamics.

[1] J.M.R. Parrondo, G.P. Harner, D. Abbott, Phys. Rev. Lett. 85 (2000).

[2] J. Almeida, D. Peralta-Salas, M. Romera, Physica D 200 (2006).

2. Switching systems

Switching systems can be studied by means of *cocycle maps*, which are continuous maps

$$\varphi : \mathbb{N}_0 \times \{-1,+1\}^{\mathbb{Z}} \times \mathbb{R}^d \to \mathbb{R}^d$$

with

$$\begin{aligned} \varphi(0,\mathbf{s},x_0) &= x_0 \\ \varphi(n,\mathbf{s},x_0) &= f_{s_{n-1}} \circ \cdots \circ f_{s_1} \circ f_{s_0}(x_0), \qquad n \geq 1. \end{aligned}$$

Then (*cocycle property*)

$$\varphi(n+k,\mathbf{s},x_0) = \varphi(n,\sigma^k\mathbf{s},\varphi(k,\mathbf{s},x_0)),\ \forall n,k \geq 0.$$

Def.[3] (σ,φ) is a **skew product flow** on $\{-1,+1\}^{\mathbb{Z}} \times \mathbb{R}^d$

[3] P.E. Kloeden, M. Rasmussen, *Nonautonomous Dynamical Systems*, AMS, 2010.

2. Switching systems

Def. An *entire solution* of (σ, φ) is a map $\chi : \mathcal{S} \to \mathbb{R}^d$ such that

$$\chi(\sigma^n \mathbf{s}) = \varphi(n, \mathbf{s}, \chi(\mathbf{s})) \text{ for all } n \geq 0.$$

More generally,

$$\chi(\sigma^n \mathbf{s}) = \varphi(n - k, \sigma^k \mathbf{s}, \chi(\sigma^k \mathbf{s})),$$

for all $\mathbf{s} \in \mathcal{S}$ and $n, k \in \mathbb{Z}$ with $k \leq n$.

Interpretation. $\chi(\mathbf{s})$ is the point of the *orbit*

$$\{\chi(\sigma^n \mathbf{s}) : n \in \mathbb{Z}\}$$

at time $n = 0$.

2. Switching systems

Def. The space $\mathcal{K}$ of compact subsets of $\mathbb{R}^d$ is a complete metric space with the *Hausdorff metric*

$$\mathrm{dist}_H(A,B) := \max\{\rho(A,B), \rho(B,A)\}$$

where $\rho(A,B)$ is the Hausdorff semi-distance defined by

$$\rho(A,B) := \max_{a\in A} \mathrm{dist}(a,B), \qquad \mathrm{dist}(a,B) := \min_{b\in B} |a-b| .$$

Def. A *pullback attractor* is a family of nonempty compact subsets,

$$\mathfrak{A} = \{A(\mathbf{s}), \mathbf{s} \in \mathcal{S}\} \subset \mathcal{K},$$

which

(i) is φ-invariant, i.e.,

$$\varphi(n, \mathbf{s}, A(\mathbf{s})) = A(\sigma^n \mathbf{s}), \qquad n \geq 0,$$

(ii) pullback attracts, i.e.

$$\mathrm{dist}_H\left(\varphi(n, \sigma^{-n}\mathbf{s}, D), A(\mathbf{s})\right) \to 0 \qquad \text{for } n \to \infty$$

for every nonempty bounded subset $D \subset \mathbb{R}^d$.

The $A(\mathbf{s})$ are called the *component sets* of the attractor $\mathfrak{A}$.

Remarks.

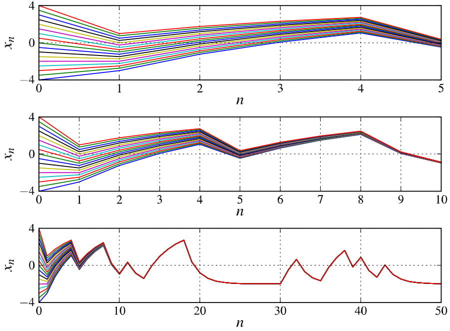
- The component sets $A(\mathbf{s})$ consist of entire solutions bounded in the past.
- Pullback attractors exist under more general conditions than forward attractors.

3. Simple case: 1D affine constituent maps

Constituent maps: $f_{\pm 1} : \mathbb{R} \to \mathbb{R}$,

$$f_{\pm 1}(x) = \theta_{\pm} x \pm 1 \ \ (0 < \theta_{+}, \theta_{-} < 1, \theta_{+} \neq \theta_{-}).$$

Remark: $h_{top}(f_{\pm 1}) = 0$.



3. Simple case: 1D affine constituent maps

- The *component sets* of the attractor $\mathfrak{A} = \{A(\mathbf{s}) : \mathbf{s} \in \mathcal{S}\}$ are singletons:

$$A(\mathbf{s}) = \{\chi(\mathbf{s})\} \text{ with } \chi(\mathbf{s}) \in \left[\frac{-1}{1-\theta_-}, \frac{1}{1-\theta_+}\right],$$

where $\chi(\mathbf{s})$ are the *entire solutions* of the skew product (σ, φ).

- Thus, Hausdorff distance = Hausdorff semidistance = Euclidean distance:

$$\mathrm{dist}_H\left(\chi(\mathbf{s}), \chi(\mathbf{s}^*)\right) = \rho\left(\chi(\mathbf{s}), \chi(\mathbf{s}^*)\right) = \left|\chi(\mathbf{s}) - \chi(\mathbf{s}^*)\right|.$$

It follows that the mapping

$$\begin{array}{ccc} \mathcal{S} & \to & \mathfrak{A} = \left[\frac{-1}{1-\theta_-}, \frac{1}{1-\theta_+}\right] \\ \mathbf{s} & \mapsto & \chi(\mathbf{s}) \end{array}$$

is continuous.

3. Simple case: 1D affine constituent maps

Proposition. Define

$$\begin{array}{rccc} \Phi : & \mathcal{S} & \to & \mathfrak{A}^{\mathbb{Z}} \\ & \mathbf{s} & \mapsto & (\chi(\sigma^n \mathbf{s}))_{n\in\mathbb{Z}} \end{array}$$

(a) Then Φ is 1-to-1 and bicontinuous.

(b) If Σ is the shift on $\mathfrak{A}^{\mathbb{Z}}$, then

$$\begin{array}{ccc} \mathcal{S} & \stackrel{\sigma}{\to} & \mathcal{S} \\ \Phi \downarrow & & \downarrow \Phi \\ \mathfrak{A}^{\mathbb{Z}} & \stackrel{\Sigma}{\to} & \mathfrak{A}^{\mathbb{Z}} \end{array}$$

commutes.

Here

$$\mathrm{dist}\left((\chi(\sigma^n \mathbf{s}))_{n\in\mathbb{Z}}, (\chi(\sigma^n \mathbf{s}^*))_{n\in\mathbb{Z}}\right) := \sum_{n\in\mathbb{Z}} \frac{|\chi(\sigma^n \mathbf{s}) - \chi(\sigma^n \mathbf{s}^*)|}{2^{|n|}}$$

3. Simple case: 1D affine constituent maps

Therefore

$$h_{top}(\Sigma|_{\Phi(\mathcal{S})}) = h_{top}(\sigma) := \text{ Complexity(control)}.$$

Call

$$\text{Complexity(switched dynamics)} := h_{top}(\Sigma|_{\Phi(\mathcal{S})}).$$

Thus:

$$\boxed{\text{Complexity(switched dynamics)} = \text{Complexity(control)}.}$$

Corollary. *Sufficient condition for entropy increase via switching*: If

$$h_{top}(\sigma) > 0$$

then

$$\text{Complexity(switched dynamics)} > 0 = h_{top}(f_{\pm}).$$

4. General case

General assumptions for switched dynamics on $\mathbb{R}^d$, $d \geq 1$:

- The constituent mappings have attractors.
- The switched dynamics has a pullback attractor

$$\mathfrak{A} = \{A(\mathbf{s}) : \mathbf{s} \in \mathcal{S}\}$$

such that $A(\mathbf{s})$ are nonempty, uniformly bounded compact subsets of $\mathbb{R}^d$, i.e., there is a closed ball $\bar{B}_R(0) \subset R^d$, such that

$$A(\mathbf{s}) \subset \bar{B}_R(0), \ \ \forall \mathbf{s} \in \mathcal{S}.$$

Call $\mathcal{K}_R$ the family of nonempty compact subsets of $\mathbb{R}^d$ contained in $\bar{B}_R(0)$.

4. General case

Technical difficulties:

- The component sets $A(\mathbf{s})$ are not singletons in general.
- $\mathrm{dist}_H(A(\mathbf{s}), A(\mathbf{s}^*))$ is not continuous.

Proposition[4]**.** The map $\mathbf{s} \mapsto A(\mathbf{s})$ is upper semi-continuous in $(\mathcal{K}_R, \mathrm{dist}_H)$, i.e.,

$$\rho\left(A(\mathbf{s}), A(\mathbf{s}^*)\right) \to 0 \quad \text{as} \quad \mathrm{dist}_{\mathcal{S}}(\mathbf{s}, \mathbf{s}^*) \to 0,$$

here $\rho(\cdot, \cdot)$ is the Hausdorff semi-distance.

[4] P.E. Kloeden, M. Rasmussen, *Nonautonomous Dynamical Systems*, AMS, 2010.

4. General case

To replicate the approach in the affine case, some additional assumptions seem necessary:

1. **First possibility**. Guarantee that

$$\Phi : \mathbf{s} \mapsto (A(\sigma^n \mathbf{s}))_{n\in\mathbb{Z}}$$

is Borel bimeasurable.

2. **Second possibility**. Guarantee that $\mathbf{s} \to A(\mathbf{s})$ is continuous.

Remarks.

- There are several sufficient conditions for (1). *For example*, (2) implies (1).
- There are several sufficient conditions for (2). *For example*, suppose that

$$\mathrm{dist}_H(\varphi(n, \sigma^{-n}\mathbf{s}, D), A(\mathbf{s})) \to 0$$

uniformly in $\mathbf{s}$ for some nonempty set $D \subset \mathbb{R}^d$.

4. General case

Consider

$$\begin{array}{cccc} \Phi: & \mathcal{S} & \to & \mathcal{K}_R^{\mathbb{Z}} \\ & \mathbf{s} & \mapsto & (A(\sigma^n \mathbf{s}))_{n\in\mathbb{Z}} \end{array}$$

where

$$\mathrm{dist}_{\mathfrak{A}^{\mathbb{Z}}}((A(\sigma^n \mathbf{s}))_{n\in\mathbb{Z}}, (A(\sigma^n \mathbf{s}^*))_{n\in\mathbb{Z}} = \sum_{n\in\mathbb{Z}} \frac{\mathrm{dist}_H \ (A(\sigma^n \mathbf{s}), A(\sigma^n \mathbf{s}^*))}{2^{|n|}}$$

Remark. If $\chi(\mathbf{s})$ is an entire solution and $\chi(\mathbf{s}) \in A(\mathbf{s})$, then

$$(\chi(\sigma^n \mathbf{s}))_{n\in\mathbb{Z}} \in (A(\sigma^n \mathbf{s}))_{n\in\mathbb{Z}}.$$

We call $(A(\sigma^n \mathbf{s}))_{n\in\mathbb{Z}}$ the *lumped trajectory*.

4. General case

Proposition. If one of the assumptions (1) or (2) holds and

$$\begin{array}{rcl}\Phi: \ \mathcal{S} & \rightarrow & \mathcal{K}_R^{\mathbb{Z}} \\ \mathbf{s} & \mapsto & (A(\sigma^n \mathbf{s}))_{n\in\mathbb{Z}}\end{array}$$

is 1-to-1, then Φ a homeomorphism from $\mathcal{S}$ to $\Phi(\mathcal{S})$, and the diagram

$$\begin{array}{ccc}\mathcal{S} & \overset{\sigma}{\rightarrow} & \mathcal{S} \\ \Phi\downarrow & & \downarrow\Phi \\ \mathcal{K}_R^{\mathbb{Z}} & \overset{\Sigma}{\rightarrow} & \mathcal{K}_R^{\mathbb{Z}}\end{array}$$

commutes, where σ is the shift on $\mathcal{S}$ and Σ is the shift on $\mathcal{K}_R^{\mathbb{Z}}$ (the *lumped dynamics*).

- There are several sufficient conditions[5] for the injectivity of Φ.

[5] J.M.A., P.E. Kloeden, A. Giménez, *Entropy* 15 (2013).

4. General case

Hence (as in the 1D affine case)

$$\boxed{h_{top}(\sigma) = h_{top}(\Sigma|_{\Phi(\mathcal{S})}) =: \text{Complexity(lumped dynamics)}.}$$

Consider the shift on the *lumped trajectories*

$$\Sigma : (A(\sigma^n \mathbf{s}))_{n\in\mathbb{Z}} \mapsto (A(\sigma^{n+1}\mathbf{s}))_{n\in\mathbb{Z}}$$

and the shift on the *sharp trajectories*

$$\tilde{\Sigma} : (\chi(\sigma^n \mathbf{s}))_{n\in\mathbb{Z}} \mapsto (\chi(\sigma^{n+1}\mathbf{s}))_{n\in\mathbb{Z}}.$$

Then

$$\boxed{h_{top}(\Sigma|_{\Phi(\mathcal{S})}) \leq h_{top}(\tilde{\Sigma}|_{\Phi(\mathcal{S})}) =: \text{Complexity(switched dynamics)}.}$$

4. General case

In sum:

$$\text{Complexity(control)} = \text{Complexity(lumped dynamics)}$$

and

$$\text{Complexity(lumped dynamics)} \leq \text{Complexity(switched dynamics)}.$$

Thus

$$\boxed{\text{Complexity(control)} \leq \text{Complexity(switched dynamics)}.}$$

Corollary. (*Entropy increase via switching*) If $h_{top}(\sigma) > h_{top}(f_\pm)$, then

$$\text{Complexity(switched dynamics)} \geq h_{top}(f_\pm)$$

5. Conclusion

- We provided a sufficient condition for the topological entropy of a switching system to increase wrt to the topological entropy of its two constituent maps.
- Generalization to more than two constituent maps possible.
- The complexity of non-autonomous systems, as measured by the topological entropy, can be studied via pullback attractors.

References

1. J.M. Amigó, P.E. Kloeden, and A. Giménez, Switching systems and entropy. *J. Diff. Eq. Appl.* 19 (2013) 1872-1888.
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4. S. Kolyada, L. Snoha, Topological entropy of nonautonomous dynamical systems, *Random Comp. Dyn.* 4 (1996) 205-233.