Transitivity without (relative) specification in dendrites

Vladimír Špitalský

Matej Bel University, Banská Bystrica, Slovakia

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Transitivity on trees

Theorem (Blokh 1987)

If X is a tree and $f : X \to X$ is transitive, then

- *f has the relative specification property*

Transitivity on trees

Theorem (Blokh 1987)

If X is a tree and $f : X \to X$ is transitive, then

- *f has the relative specification property*

$f : X \to X$ is transitive if

- $\forall U, V$ – nonempty open $\exists n \in \mathbb{N} : \quad f^n(U) \cap V \neq \emptyset$

Transitivity on trees

Theorem (Blokh 1987)

If X is a tree and $f : X \to X$ is transitive, then

- *f has the relative specification property*

$f : X \to X$ has the specification property if [Bowen 1971]

- $\forall \varepsilon > 0 \quad \exists m \quad \forall k \geq 2 \quad \forall x_1, \dots, x_k \in X$
  $\forall a_1 \leq b_1 < \dots < a_k \leq b_k$ with $a_i - b_{i-1} \geq m$ $(i = 2, \dots, k)$
  and $\forall p \geq m + b_k - a_1$,
  there is a point $x \in X$ with $f^p(x) = x$ and

$$d(f^n(x), f^n(x_i)) \leq \varepsilon \qquad \text{for} \quad a_i \leq n \leq b_i,\ 1 \leq i \leq k.$$

Transitivity on trees

Theorem (Blokh 1987)

If X is a tree and $f : X \to X$ is transitive, then

- *f has the relative specification property*

$f : X \to X$ has the relative property $\mathcal{P}$ if [Banks 1997]

- there exist regular closed sets $D_0, \dots, D_{m-1}$ covering X such that, for every $0 \leq i < j < m$, $D_i \cap D_j$ is nowhere dense,

$$f(D_i) \subseteq D_{(i+1) \bmod m}$$

  and

$$f^m|_{D_i} : D_i \to D_i \text{ has the property } \mathcal{P}.$$

Transitivity on trees

Theorem (Blokh 1987)

If X is a tree and $f : X \to X$ is transitive, then

- *f has the relative specification property*

Consequently, every transitive tree map

- is relatively mixing
- has positive entropy
- has dense periodic points
- ...

Transitivity on dendrites

1. Transitivity on trees
2. Transitivity on dendrites
3. Proof of the main result

Dendrites

Dendrite

- a locally connected metric continuum which contains no circle

A point x of a dendrite X is

- end point if $X \setminus \{x\}$ is connected
- cut point if $X \setminus \{x\}$ is not connected
  - branch point if $X \setminus \{x\}$ has at least 3 components

$E(X)$ and $B(X)$

- the sets of all end points and branch points

Tree

- a dendrite with finitely many end points

Dendrites

An arc $A = [a, b]$ in a dendrite X is called free if

- $A \setminus \{a, b\}$ is open in X

For a dendrite X the following are equivalent

- X does not contain a free arc
- branch points of X are dense in X
- end points of X are dense (i.e. residual) in X

Transitivity on dendrites: Positive results

Theorem (Alsedà-Kolyada-Llibre-Snoha 1999; Kwietniak 2011; Harańczyk-Kwietniak-Oprocha 2011; Dirbák-Snoha-Š. 2012)

If X is a dendrite containing a free arc and $f : X \to X$ is transitive, then

- *f is relatively mixing*
- *f has positive entropy*
- *f has dense periodic points*

Transitivity on dendrites: Positive results

Theorem (Alsedà-Kolyada-Llibre-Snoha 1999; Kwietniak 2011; Harańczyk-Kwietniak-Oprocha 2011; Dirbák-Snoha-Š. 2012)

If X is a dendrite containing a free arc and $f : X \to X$ is transitive, then

- *f is relatively mixing*
- *f has positive entropy*
- *f has dense periodic points*

Theorem (Acosta-Hernández-Naghmouchi-Oprocha 2013)

If X is a dendrite and $f : X \to X$ has a transitive cut point, then

- *f is relatively weakly mixing*
- *f has dense periodic points*

Transitivity on dendrites: Negative results

Theorem (Hoehn-Mouron 2013)

There is a dendrite X (with dense $B(X)$) admitting a map $f : X \to X$ which is

- *weakly mixing but*
- *not mixing*

Moreover, [Acosta-Hernández-Naghmouchi-Oprocha 2013]

- *f is proximal, and thus*
- *it has a unique periodic (= fixed) point*

Transitivity on dendrites: Negative results

Theorem (Š.)

There is a dendrite X (with dense $B(X)$) admitting a map $f : X \to X$ such that

- *f is transitive*
- *f has infinite decomposition ideal (that is, f is not relatively totally transitive)*
- *f has a unique periodic (= fixed) point*

Transitivity on dendrites: The main theorem

$C(X)$

- the space of all subcontinua (= subdendrites) of X equipped with the Hausdorff metric

$N_f(U, V)$

- the return time set $\{n \in \mathbb{N} : f^n(U) \cap V \neq \emptyset\}$

Transitivity on dendrites: The main theorem

Theorem (Š.)

Let $\sigma : \Sigma \to \Sigma$ be a subshift. Then there are a dendrite X (with dense $B(X)$) and maps $f = f_\sigma : X \to X$ and $D : \Sigma \to C(X)$ s.t.

- *$f \circ D = D \circ \sigma$; i.e. $f(D(\gamma)) = D(\sigma(\gamma))$ for every $\gamma \in \Sigma$*
- *for every cylinders $[\alpha], [\beta]$ in Σ and every non-empty open sets $U \subseteq D[\alpha]$, $V \subseteq D[\beta]$ in X there is $n_0 \in \mathbb{N}$ such that, for every $n \geq n_0$,*

$$n \in N_\sigma([\alpha], [\beta]) \iff n \in N_f(U, V)$$

  consequently, f is transitive (totally transitive, weakly mixing, mixing) if and only if σ is
- *if σ is aperiodic then f has a unique periodic (= fixed) point*

Transitivity on dendrites: The main theorem

Corollary

There is a dendrite X and maps $f, g, h : X \to X$ such that

- f *is transitive and has infinite decomposition ideal*
- g *is weakly mixing but not mixing*
- h *is mixing but has not dense periodic points*

3. Proof of the main result

Structure of the proof

Theorem

For a subshift $\sigma : \Sigma \to \Sigma$ there is a dendrite X, a continuous map $f : X \to X$ and a map $D : \Sigma \to C(X)$ such that

- $f \circ D = D \circ \sigma$
- $N_f(U, V) \approx N_\sigma([\alpha], [\beta])$ *for every ...*
- *if σ is aperiodic then $\mathrm{Per}(f) = \mathrm{Fix}(f)$ is a singleton*

Main steps of the proof.

1. construct the dendrite X
2. define $D : \Sigma \to C(X)$
3. construct the map $f : X \to X$
4. prove the properties of f

Step 1: The dendrite X

The dendrite X: the universal dendrite of order 3

- branch points are dense
- every branch point has order 3

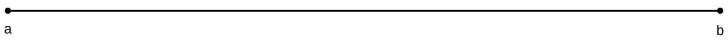
We can write

$$X = \bigcup_{m=0}^{\infty} X_m \ \cup \ X_\infty$$

- $X_0 = [a, b]$ is a segment
- $X_1 = X_0 \cup \bigcup_{r \in Q} [a_r, b_r]$
- $X_2 = X_1 \cup \bigcup_{rs \in Q^2} [a_{rs}, b_{rs}]$
- ...
- $X_\infty = \{b_{\mathbf{r}} : \mathbf{r} \in Q^\infty\}$ totally disconnected dense G_δ set

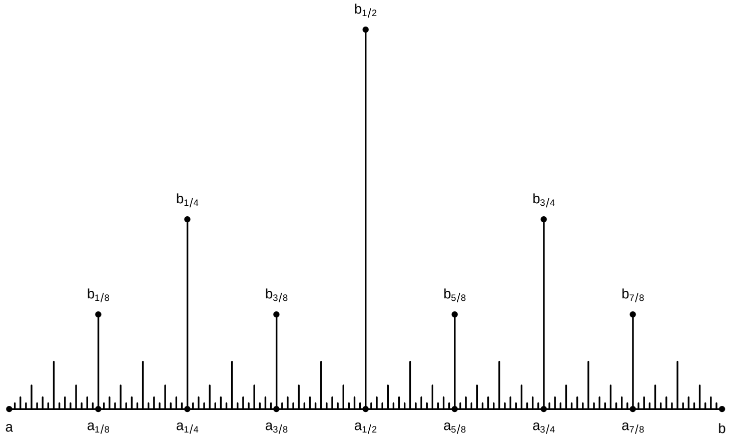
where Q is the set of all dyadic rationals in $(0, 1)$

Step 1: The dendrite X



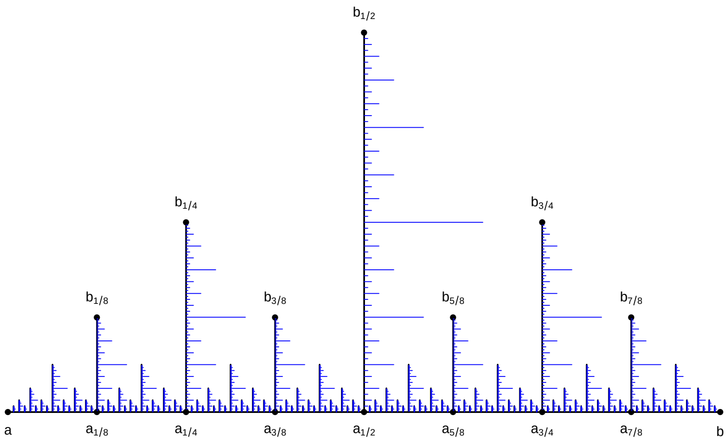
The set X_0

Step 1: The dendrite X



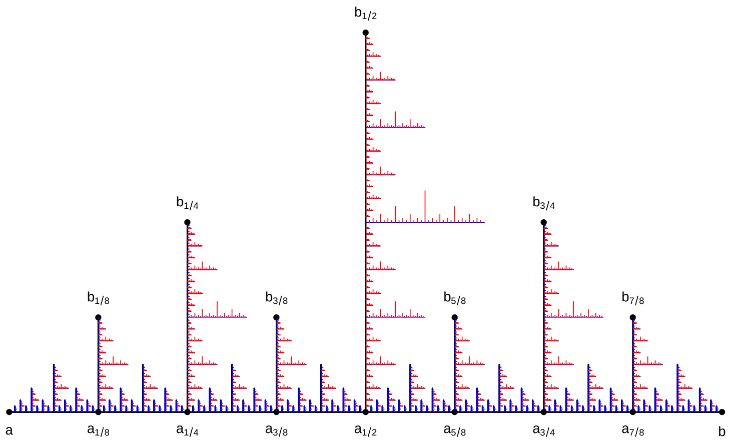
The set X_1

Step 1: The dendrite X



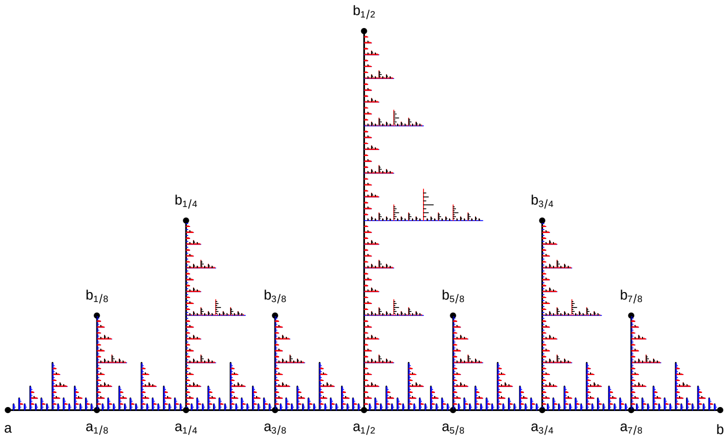
The set X_2

Step 1: The dendrite X



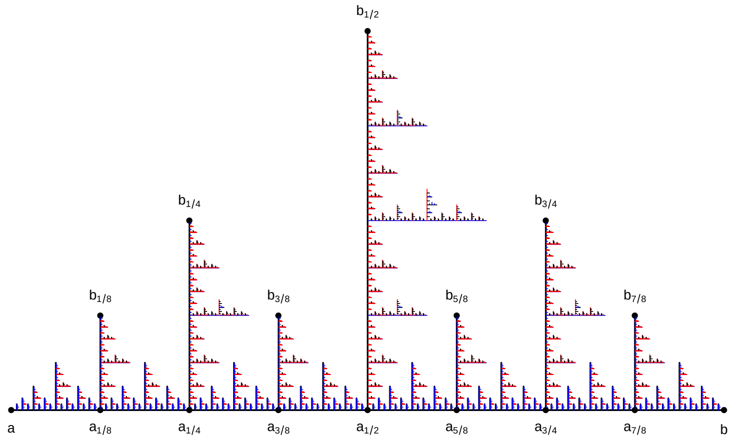
The set X_3

Step 1: The dendrite X



The set X_4

Step 1: The dendrite X



The set X_5

Step 1: The dendrite X

We can write

$$X = \text{closure}\left(\bigcup_{m=0}^{\infty} X_m\right) = \bigcup_{m=0}^{\infty} X_m \ \cup \ X_\infty$$

- $X_0 = [a, b]$ is a segment
- $X_1 = X_0 \cup \bigcup_{r \in Q} [a_r, b_r]$
- $X_2 = X_1 \cup \bigcup_{rs \in Q^2} [a_{rs}, b_{rs}]$
- ...
- $X_\infty = \{b_{\mathbf{r}} : \ \mathbf{r} \in Q^\infty\}$ totally disconnected dense G_δ set

where Q is the set of all dyadic rationals in $(0, 1)$

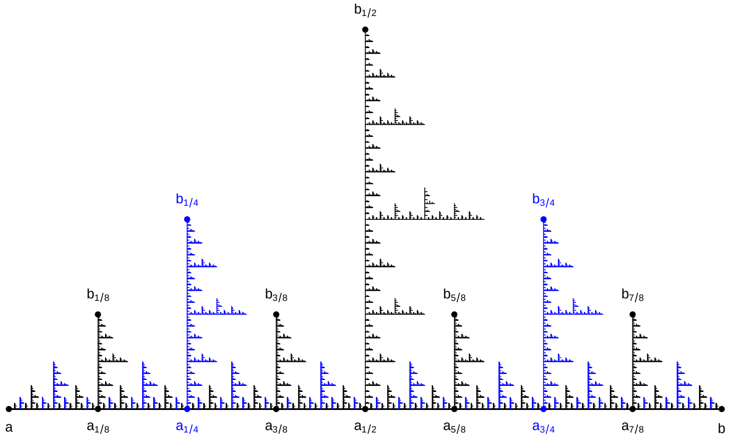
Step 2: The subdendrites $D(\gamma)$

Assume that $\Sigma = \{0, 1\}^{\mathbb{N}}$ and $\sigma : \Sigma \to \Sigma$ is the full shift

D_0, D_1

- D_0, D_1 are regular closed subdendrites of X
- $D_0 \cup D_1 = X$
- $D_0 \cap D_1 = X_0$ is nowhere dense in X

Step 2: The subdendrites $D(\gamma)$



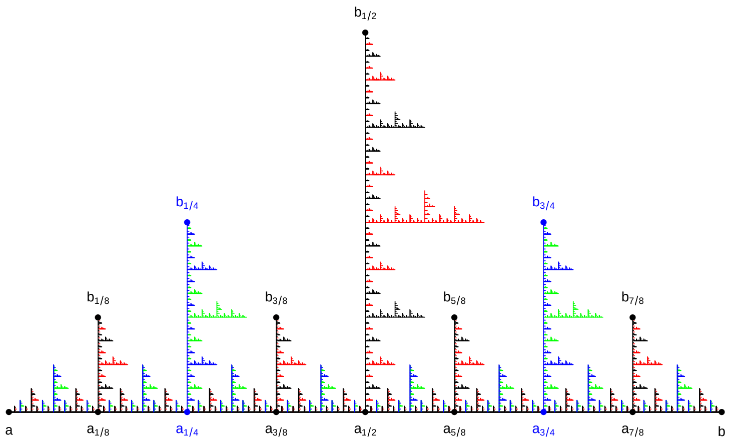
D_0, D_1

Step 2: The subdendrites $D(\gamma)$

$D_{00}, D_{01}, D_{10}, D_{11}$

- $D_{00}, \ldots, D_{11}$ are regular closed subdendrites of X
- $D_0 = D_{00} \cup D_{01}$, $D_1 = D_{10} \cup D_{11}$
- $\bigcup_{i_0 i_1 \neq j_0 j_1} (D_{i_0 i_1} \cap D_{j_0 j_1}) = X_1$ is nowhere dense in X

Step 2: The subdendrites $D(\gamma)$



$D_{00}, D_{01}, D_{10}, D_{11}$

Step 2: The subdendrites $D(\gamma)$

For $\gamma = \gamma_0 \gamma_1 \cdots \in \Sigma$:

- $D(\gamma) = D_{\gamma_0} \cap D_{\gamma_0 \gamma_1} \cap D_{\gamma_0 \gamma_1 \gamma_2} \cap \ldots$

Step 3: The map $f : X \to X$

$f_0 : X_0 \to X_0$ $\qquad X_0 = [a, b]$

- a surjective map
- "agrees with" the shift σ
- $f_0(x) < x$ for $x \in (a, b)$
- $\lim_n f_0^n(x) = a$ for every $x \neq b$

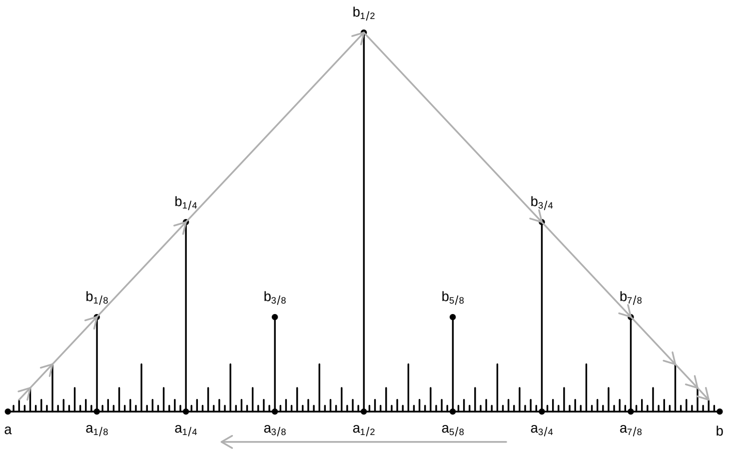
Step 3: The map $f : X \to X$

$f_1 : X_1 \to X_1$ $\qquad X_1 = X_0 \cup \bigcup_{r \in Q} [a_r, b_r]$

- $f_1(a_r) = f_0(a_r)$
- maps every end point b_r onto an end point $b_{\varrho(r)}$ in such a way that

$$\lim_{n \to \infty} f_1^n(b_r) = b \qquad \text{for every } r \in Q$$

Step 3: The map $f : X \to X$



The map $f_1 : X_1 \to X_1$

Step 3: The map $f : X \to X$

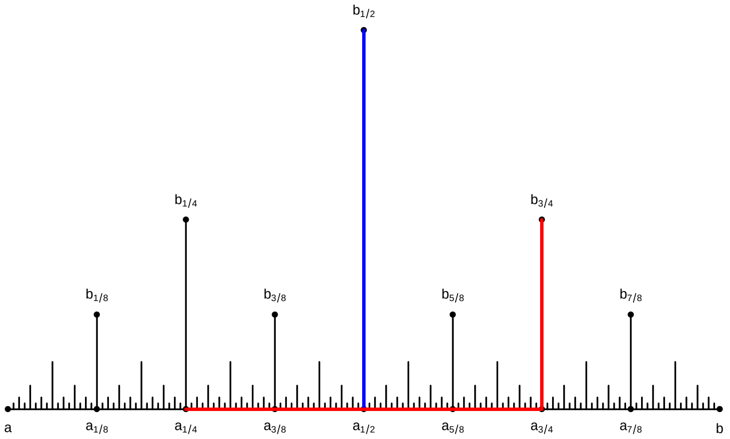
$f_1 : X_1 \to X_1$ $\qquad X_1 = X_0 \cup \bigcup_{r \in Q} [a_r, b_r]$

- continuous surjective extension of f_0
- "agrees with" the shift σ
- maps every $[a_r, b_r]$ onto

$$[f_0(a_r), b_{\varrho(r)}] = [f_0(a_r), a_{\varrho(r)}] \cup [a_{\varrho(r)}, b_{\varrho(r)}]$$

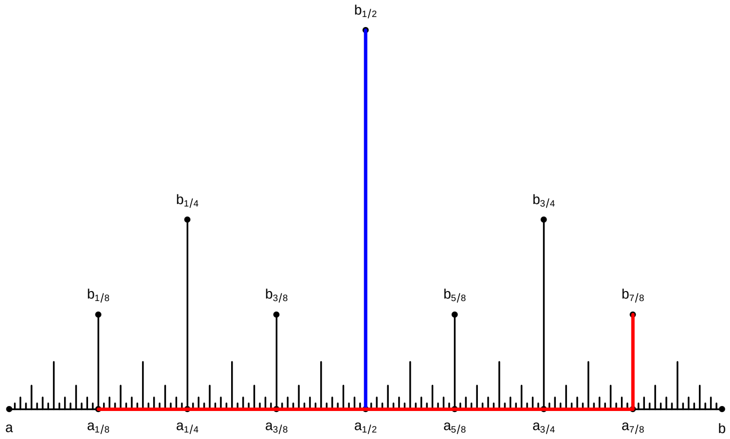
- $\lim_n f_1^n(x) = a$ for every $x \neq b, b_r$ $\quad (r \in Q)$

Step 3: The map $f : X \to X$



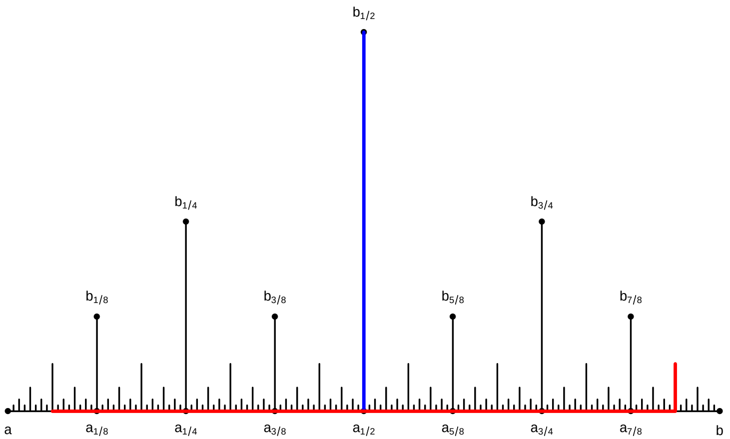
The image of $[a_{1/2}, b_{1/2}]$

Step 3: The map $f : X \to X$



The second image of $[a_{1/2}, b_{1/2}]$

Step 3: The map $f : X \to X$



The third image of $[a_{1/2}, b_{1/2}]$

Step 3: The map $f : X \to X$

$f_m : X_m \to X_m \ (m \geq 2)$ $\qquad X_m = X_{m-1} \cup \bigcup_{\mathbf{r} \in Q^m} [a_\mathbf{r}, b_\mathbf{r}]$

- continuous surjective extension of f_{m-1}
- "agrees with" the shift σ
- maps every $[a_\mathbf{r}, b_\mathbf{r}]$ onto $[a_{\varrho(\mathbf{r})}, b_{\varrho(\mathbf{r})}]$
  - $\varrho : Q^m \to (Q^m \cup Q^{m-1})$ is such that every $\mathbf{r} \in Q^m$ eventually falls into Q^{m-1}

- $\lim_n f_m^n(b_\mathbf{r}) = b$ for every $\mathbf{r} \in Q^m$
- $\lim_n f_m^n(x) = a$ for every $x \neq b, b_\mathbf{r}$ $\quad (\mathbf{r} \in Q^1 \cup \cdots \cup Q^m)$

Step 3: The map $f : X \to X$

$f : X \to X$ $\qquad X = \bigcup_m X_m \cup X_\infty,\ X_\infty = \{b_{\mathbf{r}} :\ \mathbf{r} \in Q^\infty\}$

$$f(x) = \begin{cases} f_m(x) & \text{if } x \in X_m,\ m \geq 0 \\ b_{\varrho(\mathbf{r})} & \text{if } x = b_{\mathbf{r}},\ \mathbf{r} \in Q^\infty \end{cases}$$

- $\varrho : Q^\infty \to Q^\infty$ is determined by $\varrho|_{Q^m}$ $(m \geq 1)$
- X_∞ is an f-invariant (not closed) set

Step 4: Properties of $f : X \to X$

f is a continuous surjection

- every X_m is a closed invariant set with "trivial" dynamics
- X_∞ is an invariant set with "shift-like" dynamics

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f "agrees" with the shift σ

- f maps $D(\gamma)$ onto $D(\sigma(\gamma))$

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f is a continuous surjection

- every X_m is a closed invariant set with "trivial" dynamics
- X_∞ is an invariant set with "shift-like" dynamics

f "agrees" with the shift σ

- f maps $D(\gamma)$ onto $D(\sigma(\gamma))$

f has the "same" return time sets as σ

- ...

Step 4: Properties of $f : X \to X$

Subshifts $\tilde{\sigma} : \tilde{\Sigma} \to \tilde{\Sigma}$ $\quad (\tilde{\Sigma} \subseteq \Sigma)$

- correspond to subsystems

$$\tilde{f} = f|_{D(\tilde{\Sigma})} : D(\tilde{\Sigma}) \to D(\tilde{\Sigma})$$

Thanks for your attention!