Attractors for unimodal quasiperiodically forced maps

Lluís Alsedà

in collaboration with M. Misiurewicz

Departament de Matemàtiques
Universitat Autònoma de Barcelona

http://www.mat.uab.cat/~alseda

Outline

Introduction — the setting

In the last two decades a lot of work has been devoted to find and study *Strange Non-chaotic Attractors (SNA)*.

Many of these attractors are found and studied for non autonomous quasiperiodically forced dynamical systems of the type:

$$(1) \qquad \begin{cases} \theta_{n+1} &= R(\theta_n) = \theta_n + \omega \pmod 1, \\ x_{n+1} &= \psi(\theta_n, x_n) \end{cases}$$

where $x \in \mathbb{R},\ \theta \in \mathbb{S}^1 = \mathbb{R}/\mathbb{Z}$ and $\omega \in \mathbb{R} \setminus \mathbb{Q}$.

Similar models are studied also in higher dimensions and for systems that are both discrete and continuous.

Other important studies are developed in the framework of cocycles and spectral theory.

Skew products

Systems of the form

$$\begin{cases} \theta_{n+1} &= \varphi(\theta), \\ x_{n+1} &= \psi(\theta_n, x_n) \end{cases}$$

of which System (1) is a particular case are called *skew products*. The map φ is called the *base map* and each $\{\theta\} \times \mathbb{R}$ is called the *fibre based at θ*. Fixing θ the map

$$\psi(\theta, \cdot)\colon \{\theta\} \times \mathbb{R} \longrightarrow \{\varphi(\theta)\} \times \mathbb{R}$$

is a continuous function from the fibre based at θ into the fibre based at $\varphi(\theta)$.

The origin of the name

The term *Strange Non-chaotic attractor (SNA)* was introduced and coined in

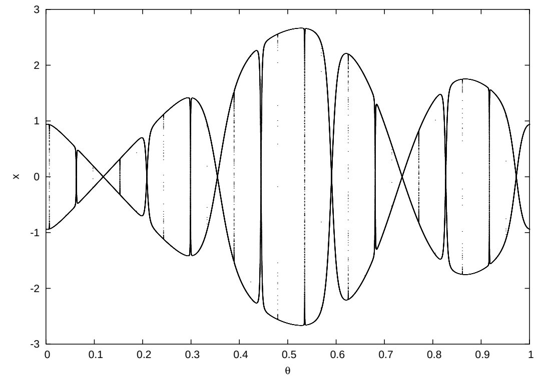
[GOPY] C. Grebogi, E. Ott, S. Pelikan, and J. A. Yorke.
Strange attractors that are not chaotic.
Phys. D, 13(1-2):261–268, 1984.

After this paper the study of these objects became rapidly popular and a number a papers studying different related models appeared.

The [GOPY] model

$$(2) \qquad \begin{cases} \theta_{n+1} &= \theta_n + \omega \pmod 1, \\ x_{n+1} &= 2\sigma \tanh(x_n)\cos(2\pi\theta_n) \end{cases}$$

where $x \in \mathbb{R}, \theta \in \mathbb{S}^1$, $\omega = \frac{\sqrt{5}+1}{2}$ and $\sigma > 1$.



The authors called the attractor of the system an *SNA* since:

- the orbit of the point (θ, x) for almost every $\theta \in \mathbb{S}^1$ and every $x > 0$ *converges* to the SNA (*attractor*).

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- the orbit of the point (θ, x) for almost every $\theta \in \mathbb{S}^1$ and every $x > 0$ *converges* to the SNA (*attractor*).
- it is *strange* because *it is not piecewise differentiable*: The SNA cuts the line $x = 0$ (and then it does so at the orbit of a point which is dense in $x = 0$) and it is different from zero in a set whose projection to $\mathbb{S}^1$ is dense.

Remark

The line $x = 0$ is invariant because $x_{n+1} = \sigma \tanh(x_n)\cos(2\pi\theta_n)$. Moreover this invariant line turns to be a repellor.

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- it is *non-chaotic* because *the Lyapunov exponents are non positive* (computed numerically).

Early results

As pointed out by R. Johnson, constructions of flows containing SNA's can be found in

- **[M1]** V.M. Millionščikov.
  Proof of the existence of irregular systems of linear differential equations with almost periodic coefficients.
  Differ. Uravn., 4 (3): 391–396, 1968.
- **[M2]** V.M. Millionščikov.
  Proof of the existence of irregular systems of linear differential equations with quasi periodic coefficients.
  Differ. Uravn., 5 (11): 1979–1983, 1969.
- **[V]** R.E. Vinograd.
  A problem suggested by N.P. Erugin.
  Differ. Uravn.,11 (4): 632–638, 1975.

Notice that these results were obtained much before than the notion and term SNA was coined.

Remarks

- The notion of SNA is neither unified nor precisely formulated. For instance there problems on whether it has to be imposed that the attracting is closed or not.
- The existence of SNA, often, is not proved rigorously. Some authors just give very rough/rude numerical evidences of their existence that easily can turn out to be wrong.
- The theoretical tools to study these objects and derive these consequences, are often used in a wrong way (Lyapounov exponents).

On the positive side there are some works where the existence of an SNA is rigorously proved. For instance

- **[BO]** Z. I. Bezhaeva and V. I. Oseledets.
  On an example of a “strange nonchaotic attractor”.
  Funktsional. Anal. i Prilozhen., 30(4):1–9, 95, 1996.
- **[Kel]** G. Keller.
  A note on strange nonchaotic attractors.
  Fund. Math., 151(2):139–148, 1996.
- **[H]** A. Haro.
  On strange attractors in a class of pinched skew products
  to appear.

Keller model is an abstract version of the [GOPY] example.

The Keller model

It is a skew product of the form (1) where the function in the second component has separated variables:

$$(3)\qquad \begin{cases} \theta_{n+1} &= R(\theta_n) = \theta_n + \omega \pmod 1, \\ x_{n+1} &= f(x_n) g(\theta_n) \end{cases}$$

where $x \in \mathbb{R}^+, \theta \in \mathbb{S}^1$, $\omega \in \mathbb{R} \setminus \mathbb{Q}$ and

1. $f\colon [0, \infty) \longrightarrow [0, \infty)$ is $\mathcal{C}^1$, bounded, strictly increasing, strictly concave and verifies $f(0) = 0$ (to fix ideas take $f(x) = \tanh(x)$ as in the [GOPY] model). Thus, $x = 0$ will be invariant.
2. $g\colon \mathbb{S}^1 \longrightarrow [0, \infty)$ is bounded and continuous (to fix ideas take $g(\theta) = 2\sigma|\cos(2\pi\theta)|$ with $\sigma > 0$ in a similar way to the [GOPY] model – except for the absolute value).

Pinching

There are big differences between the cases when g takes the value 0 at some point: the *pinched* case and the case when g is strictly positive.

Remark

In the pinched case any T–invariant set has to be 0 on a point and hence on a dense set because the circle $x \equiv 0$ is invariant and the θ-projection of every invariant object must be invariant under R.

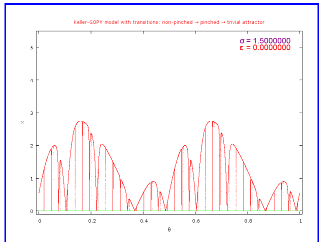
A particular example

$$(4)\qquad \begin{cases} \theta_{n+1} &= \theta_n + \omega \pmod 1, \\ x_{n+1} &= 2\sigma \tanh(x_n)(\varepsilon + |\cos(2\pi\theta_n)|) \end{cases}$$

where $x \in \mathbb{R}, \theta \in \mathbb{S}^1$, $\omega = \frac{\sqrt{5}+1}{2}$. $\sigma > 0$ and $\varepsilon \geq 0$.

Remark

The attractor of the above system (if it exists) will be pinched if and only if $\varepsilon = 0$.



Movie: A family of Keller Attractors

The following theorem due to Keller [Kel] makes the above informal ideas rigorous. Before stating it we need to introduce the constant σ:

Since the line $x = 0$ is invariant, by using Birkhoff Ergodic Theorem, it turns out that

$$\sigma := f'(0) \exp\left(\int_{\mathbb{S}^1} \log g(\theta) d\theta \right) < \infty.$$

is the vertical Lyapunov exponent on the circle $x = 0$.

Keller Theorem

There exists an upper semicontinuous map $\phi\colon \mathbb{S}^1 \longrightarrow [0,\infty)$ whose graph is invariant under the Model (2). Moreover,

1. The Lebesgue measure on the circle, lifted to the graph of ϕ is a Sinai-Ruelle-Bowen measure,
2. if $\sigma \leq 1$ then $\phi \equiv 0$,
3. if $\sigma > 1$ then $\phi(\theta) > 0$ for almost every θ,
4. if $\sigma > 1$ and $g(\theta_0) = 0$ for some θ_0 then the set $\{\theta\colon \phi(\theta) > 0\}$ is meager and ϕ is almost everywhere discontinuous,
5. if $\sigma > 1$ and $g > 0$ then ϕ is positive and continuous; if g is $\mathcal{C}^1$ then so is ϕ,
6. if $\sigma \neq 1$ then $|x_n - \phi(\theta_n)| \to 0$ exponentially fast for almost every θ and every $x > 0$.

Basic ideas about Keller Theorem — invariant functions

A crucial fact is that $f(x)$ is strictly concave.

The Model (2) can be written as $(\theta_{n+1}, x_{n+1}) = F(\theta_n, x_n)$ where $F(\theta, x) = (R(\theta), f(x)g(\theta))$.

Let $\mathcal{P}$ be the space of all functions (not necessarily continuous) from $\mathbb{S}^1$ to $\mathbb{R}$ (or, later, $[0,1]$).

If we look for a functional version of the system (or the iterates of F) in $\mathcal{P}$ then we have to define the *transfer operator* $T\colon \mathcal{P} \longrightarrow \mathcal{P}$ as:

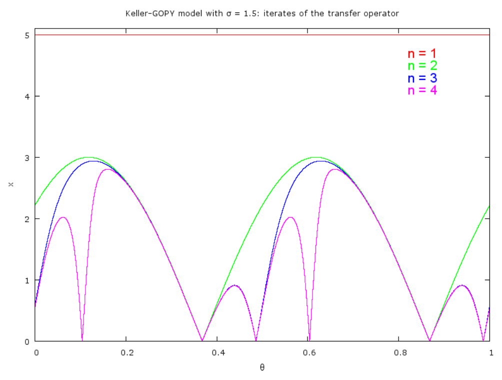
$$(T\psi)(\theta) = f(\psi(R^{-1}(\theta))) \cdot g(R^{-1}(\theta))$$

(the graph of $T\psi$ is the image under F of the graph of ψ).

Observe that ϕ is invariant if and only if $T\phi = \phi$.

Basic ideas about Keller Theorem — invariant functions

To obtain ϕ, Keller takes a sufficiently large constant function u, applies to it the iterates of the transfer operator T and takes the limit (which is the infimum). *This works because the map f is monotone*.



Movie: The iterates of the transfer operator

Statement of the problem

We are interested in extending Keller Theorem to the case when f is not monotone to the fibres. We will stay in the simplest case when f is non-monotone (that is, when f is unimodal) and the most interesting case (for us): the pinched one.

Thus, our assumptions will be:

- $f\colon [0,1] \longrightarrow [0,1]$ is a concave unimodal map with $f(0) = f(1) = 0$ and $f(c) = 1$.
- $g\colon \mathbb{S}^1 \longrightarrow [0,1]$ is a continuous function which takes the value 0 at some point.

The non-monotone model

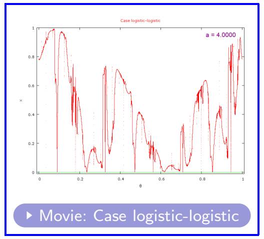
$$\begin{cases} \theta_{n+1} &= \theta_n + \omega \pmod 1, \\ x_{n+1} &= af(x)\theta(1-\theta) \end{cases}$$

where $x \in \mathbb{R}, \theta \in \mathbb{S}^1$, $\omega = \mathbb{R} \setminus \mathbb{Q}$ and f is a concave unimodal map with $f(0) = f(1) = 0$ and $f(c) = 1$ (standard examples are $f(x) = 4x(1-x)$ and $f(x) = 1 - |2x - 1|$).

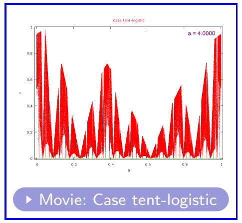
Observe that since $g(\theta) = a\theta(1-\theta)$ we are in the pinched case.

Initial examples

We consider two basic situations. One with strict concavity and another one with non-strict concavity. In any case we take the golden mean as a rotation number: $\omega = \frac{\sqrt{5}+1}{2}$.



$f(x) = 4x(1-x)$
$a \in [1.8, 4]$



$f(x) = 1 - |2x - 1|$
$a \in [3.68, 4]$

Preliminaries

To overcome the non-monotonicity problem we define a *semitransfer operator S* by:

$$(S\psi)(\theta) = \widetilde{f}(\psi(R^{-1}(\theta))) \cdot g(R^{-1}(\theta))$$

where $\widetilde{f}(x) = f(\min\{x, c\}) = \max\{f(x), 1\}$. Clearly, the semitransfer operator is the transfer operator of the map obtained from F by replacing f by the monotone map $\widetilde{f}$.

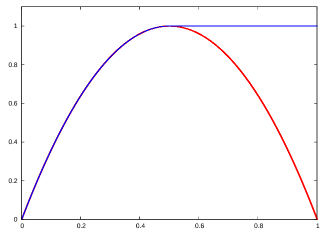


Figure: The map f in red and the map $\widetilde{f}$ in blue.

An invariant graph for the semitransfer operator

From the definition of the semitransfer operator we have

$$F\left(R^{-1}(\theta), \min\{\psi(R^{-1}(\theta)), c\}\right) = (\theta, (S\psi)(\theta)),$$

so the image under F of the graph of the minimum of ψ and c is the graph of $S\psi$.

The sequence $(S^n 1)_{n=0}^{\infty}$ is non-increasing. So, it converges pointwise to

$$\xi^+(\theta) := \inf\{(S^n 1)(\theta) : n \in \mathbb{N}\} \in \mathcal{P}.$$

The function ξ^+ is upper semicontinuous and either zero almost everywhere or positive almost everywhere (although zero on a dense set).

Upper bounds

Set $X^+ = \{(\theta, x) \colon x \leq \xi^+(\theta)\}$.

Proposition

The set X^+ is invariant for F and the ω-limit set of every point of $\mathbb{S}^1 \times [0,1]$ is contained in X^+. Moreover, if $F^n(\theta, x) = (\theta_n, x_n)$ then $x_n \leq (S^n 1)(\theta_n)$.

Remark

Since $f(x) \leq 1$ and $g(\theta) = a\theta(1-\theta)$, it follows that $S1 \leq a/4$. Thus, X^+ is below the circle $x = a/4$.

The *core*

Set

$$\beta = f(\sup \xi^+),$$
$$Y = \{(\theta, x) : \beta\xi^+(\theta) \leq x \leq \xi^+(\theta)\}, \text{ and}$$
$$X = \bigcap_{n=0}^{\infty} F^n(Y).$$

- Y is invariant for F.
- Thus, $X \subset Y$ and X is also invariant for F.
- Since the intersection of Y with every fibre is a closed interval or a point, the same is true for X.

All interesting dynamics of F takes place in the set X which basically plays a role of the *core* of the unimodal map (that is the interval $[f^2(c), f(c)]$).

Vertical Lyapunov exponent at $x = 0$

Since we do not assume that f is smooth, we cannot speak about vertical Lyapunov exponents almost everywhere on the cylinder.

However, since f is concave, there exists a one-sided derivative $f'_+(0)$ of f at 0. Therefore we can consider the vertical Lyapunov exponent at $x = 0$, or more precisely, on $\mathbb{S}^1 \times \{0\}$. It can be defined by (see for instance [Kel]):

$$\Lambda = \log f'_+(0) + \int_{\mathbb{S}^1} \log g(\theta)\, d\theta.$$

Here we assume that $f'_+(0)$ is finite, but admit the possibility of $\int_{\mathbb{S}^1} \log g(\theta)\, d\theta = -\infty$.

The sign of Λ strongly influences the dynamics

Theorem (Negative Lyapunov exponent Theorem)

If $\Lambda < 0$ then $\xi^+ \equiv 0$, and for every $\theta \in \mathbb{S}^1$ and every $x \in [0,1]$ the trajectory of (θ, x) converges exponentially fast to $\mathbb{S}^1 \times \{0\}$.

Theorem

If $\xi^+ = 0$ a.e., then $\Lambda \leq 0$.

Corollary

If $\Lambda > 0$ then ξ^+ is positive a.e.

Existence of an invariant curve–I

If ξ^+ is zero a.e., then the sequence $(T^n\psi))_{n=0}^{\infty}$ converges a.e. to zero for every function $\psi \in \mathcal{P}$.

Assume now that $\Lambda > 0$ and hence ξ^+ is positive a.e. To study this case we:

- additionally assume that the map f is strictly concave,
- moreover we set

$$b := \sup\left\{x \in (c,1]\colon -f'_-(x) < \frac{f(x)}{x}\right\}.$$

Note that $b < 1$. In particular, $f(b) > 0$.

Existence of an invariant curve–II

Theorem (Invariant Curve Theorem)

Assume that $0 < \operatorname{ess\,sup} \xi^+ < b$ and let $\beta' = f(\operatorname{ess\,sup} \xi^+)$. Then there exists a function $\zeta \in \mathcal{P}$ such that

(a) *$0 \le \zeta \le \xi^+$ and $\zeta \ge \beta'\xi^+$ almost everywhere;*

(b) *$T\zeta = \zeta$;*

(c) *if $\psi \in \mathcal{P}$ and $\varepsilon\xi^+ \le \psi \le \xi^+$ for some $\varepsilon > 0$ then $T^n\psi$ converges to ζ almost everywhere as n tends to infinity;*

(d) *ζ is a measurable function;*

(e) *ζ is positive almost everywhere.*

Exponentially fast convergence

Theorem (Exponentially fast convergence Theorem)

Assume that $0 < \operatorname{ess\,sup} \xi^+ < b$. Then for almost every $\theta \in \mathbb{S}^1$ and all $x \in (0,1)$ the trajectory of (θ, x) either converges exponentially fast to the graph of ζ or falls into $\mathbb{S}^1 \times \{1\}$ and then stays in $\mathbb{S}^1 \times \{0\}$. In particular, for almost every $\theta \in \mathbb{S}^1$ and all but countable number of $x \in (0,1)$ the trajectory of (θ, x) converges exponentially fast to the graph of ζ.

Invariant measures

Theorem

If $\Lambda < 0$ then F is uniquely ergodic.

Theorem

Assume that f is strictly concave and $0 < \operatorname{ess\,sup} \xi^+ < b$. Then F has only two invariant ergodic probability measures, namely m_0 and m_ζ. In particular, the topological entropy of F is 0. The measure m_ζ is the (unique) Sinai-Ruelle-Bowen measure for F.

Examples: Looking for examples which cannot be reduced to the monotone case

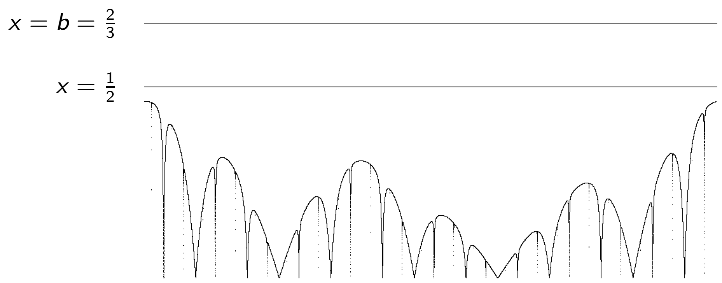
logistic-logistic

The first family of examples consists in taking

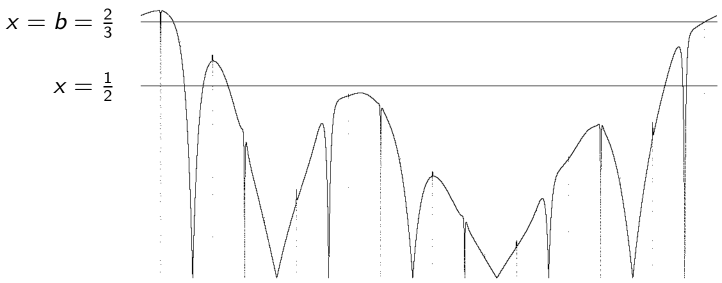
$$f(x) = 4x(1-x).$$

The integral of $\log g$ is equal to $\log a - 2$, and $f'(0) = 4$. Thus, the vertical Lyapunov exponent is $\Lambda = \log(4a/e^2)$. So the interesting case, $\Lambda > 0$, occurs when $a > e^2/4 \approx 1.84726402473266$. However, even for slightly larger values of a (definitely for $a < 2$, because then $S1 < 1/2$) we have $\sup \xi^+ < 1/2$, so this is basically Keller's case. On the other hand, if a is too large, then $\operatorname{ess\,sup} \xi^+ > b = 2/3$ and our main theorems do not apply. Numerical estimates (which do not distinguish between $\sup \xi^+$ and $\operatorname{ess\,sup} \xi^+$) suggest an interval of "good" values of a, in particular $a = 2.6$ (see the next figures).

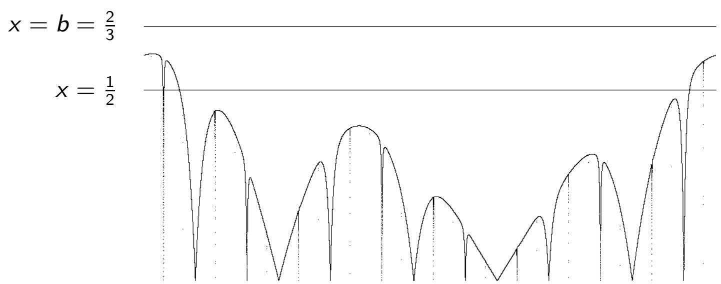
Logistic map forced by a logistic map, $a = 2.3$. The attractor seems to be below the level $x = 1/2$.



Logistic map forced by a logistic map, $a = 2.9$. The attractor seems to stick above the level $x = b = 2/3$.



Logistic map forced by a logistic map, $a = 2.6$. The attractor seems to be between the levels $x = 1/2$ and $x = b = 2/3$.



An example that cannot be reduced to the monotone case

We have to find an f which satisfies $f(1/2) = 1$ and $1/2 < \operatorname{ess\,sup} \xi^+ < b$.

We do it with the help of the following

Lemma

Let g be a logistic map, $g(\theta) = a\theta(1-\theta)$, with $a > e^2/2$, and let f be such that its turning point c is $1/2$. Then $\operatorname{ess\,sup} \xi^+ > 1/2$.

Then we can take $f(x) = 1 - (2x-1)^{2n}$ with sufficiently large n (computations show that $n \geq 13$ is sufficient) to assure that $e^2/8 < b < 1$. Then we only need to choose a value of a such that $e^2/8 < a/4 < b$. This implies that $e^2/2 < a$ and, since $S1 \leq a/4 < b$, we get $1/2 < \operatorname{ess\,sup} \xi^+ < b$ as we wanted.

Thus, the *Invariant Curve Theorem* applies but the study of this system cannot be reduced to the monotone case.

tent-logistic

The next example shows that if we do not assume that f is strictly concave, the situation can be completely different. Namely, let us take again $g(\theta) = a\theta(1-\theta)$ with $a > e^2/2$, but as f we take the tent map,

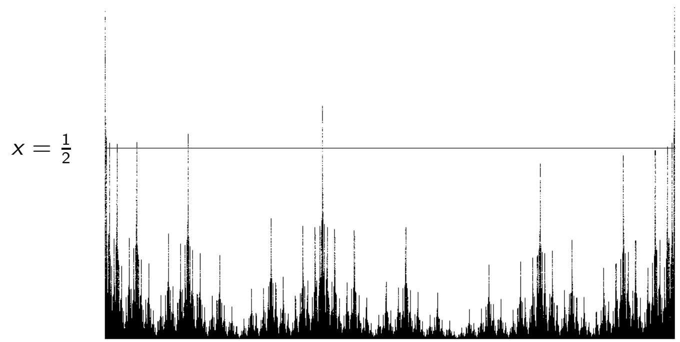
$$f(x) = 1 - |2x-1|.$$

Then the vertical Lyapunov exponent is the same everywhere in $\mathbb{S}^1 \times [0,1]$, and is positive. Thus, by the results of Buzzi, there exists an invariant probability measure for F, absolutely continuous with respect to the Lebesgue measure. This implies that, in this case, the attractor is not a curve. It consists in some region filled by transitive orbits.

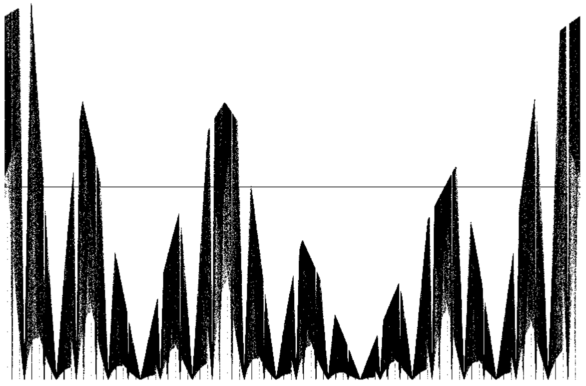
Remark

For the above family, the previous lemma tells us that for all $a > e^2/2$ we have $\operatorname{ess\,sup} \xi^+ > 1/2$, while for all $a < e^2/2$ we have $\Lambda < 0$, and thus by the *Negative Lyapunov exponent Theorem*, $\xi^+ \equiv 0$. This is in a sharp contrast to the family from the first example, where computer experiments suggest continuous dependence of $\operatorname{ess\,sup} \xi^+$ on a.

Tent map forced by a logistic map, $a = 3.696$, slightly more than $e^2/2$. The attractor sticks above the level $x = 1/2$.



Tent map forced by a logistic map, $a = 4$.



Questions

Q 1. Are the *Invariant Curve Theorem* and the *Exponentially fast convergence Theorem* true without the assumption that the essential supremum of ξ^+ is smaller than b?

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Q 2. When both f and g are logistic maps and g depends on the parameter: $g(\theta) = a\theta(1-\theta)$ the computer experiments suggest some kind of continuous dependence of the attractor on the parameter. Is this dependence really continuous? If yes, in what sense (what topology)? If no, is at least the supremum (or the essential supremum) of ξ^+ depending continuously on the parameter? Of course the same question can be asked for other similar families. As we noted at the end of the preceding section, the situation may be different if f is not strictly concave.

Q 3. Are the supremum and the essential supremum of ξ^+ always equal? If not, what natural assumptions imply this?

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Q 4. An attracting invariant graph is an analogue of an attracting fixed point for an interval map. However, for interval maps we see often periodic attracting points of periods $n > 1$. Can in our model an attracting periodic graph occur? To be more specific, we are asking about a possibility of an attracting invariant set that has n points in almost every fibre and is in some sense irreducible.