On the set of periods of sigma maps of degree 1

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Outline

Introduction

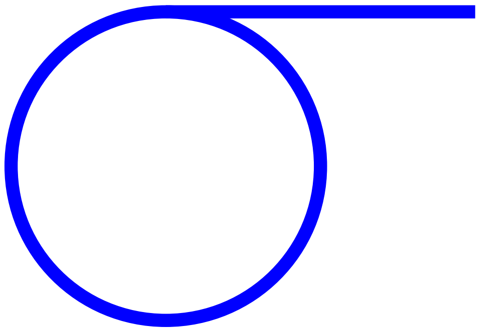
We aim at studying the structure of the set of periods in dimension one, following the path started by the well known Sharkovskii Theorem.

The case of spaces contractible to a point (trees), starting with the interval and stars is the easiest one. The case of graphs with circuits, starting with the circle, is far from being understood. It helps to assume that the branching points are fixed.

As we will see this is an unfinished long project with participation of many people.

Introduction

This talk aims at studying the the simplest case after the circle $\mathbb{S}^1$. That is, the continuous self maps of the space



Background and Motivation
The simplest case: the interval. The Sharkovskii Theorem

We start by introducing

The Sharkovskii Ordering $_{\text{Sh}}\geq$:

$3 \;_{\text{Sh}}> 5 \;_{\text{Sh}}> 7 \;_{\text{Sh}}> \cdots \;_{\text{Sh}}> 2\cdot 3 \;_{\text{Sh}}> 2\cdot 5 \;_{\text{Sh}}> 2\cdot 7 \;_{\text{Sh}}> \cdots \;_{\text{Sh}}>$
$4\cdot 3 \;_{\text{Sh}}> 4\cdot 5 \;_{\text{Sh}}> 4\cdot 7 \;_{\text{Sh}}> \cdots \;_{\text{Sh}}> \cdots \;_{\text{Sh}}>$
$2^n\cdot 3 \;_{\text{Sh}}> 2^n\cdot 5 \;_{\text{Sh}}> 2^n\cdot 7 \;_{\text{Sh}}> \cdots \;_{\text{Sh}}> 2^\infty \;_{\text{Sh}}> \cdots \;_{\text{Sh}}>$
$2^n \;_{\text{Sh}}> \cdots \;_{\text{Sh}}> 16 \;_{\text{Sh}}> 8 \;_{\text{Sh}}> 4 \;_{\text{Sh}}> 2 \;_{\text{Sh}}> 1.$

is defined on the set $\mathbb{N}_{\text{Sh}} = \mathbb{N} \cup \{2^\infty\}$
(we have to include the symbol 2^∞ to assure the existence of supremum for certain sets).

In the ordering $_{\text{Sh}}>$ the least element is 1 and the largest is 3.

The supremum of the set $\{1, 2, 4, \ldots, 2^n, \ldots\}$ is 2^∞.

Initial segments for the Sharkovskii Ordering

For $s \in \mathbb{N}_{\text{Sh}}$, $\mathrm{S}_{\text{sh}}(s)$ denotes the set $\{k \in \mathbb{N} : s \;_{\text{Sh}}\geq k\}$.

Examples of sets of the form $\mathrm{S}_{\text{sh}}(s)$ are:

- $\mathrm{S}_{\text{sh}}(2^\infty) = \{1, 2, 4, \ldots, 2^n, \ldots\}$,
- $\mathrm{S}_{\text{sh}}(3) = \mathbb{N}$,
- $\mathrm{S}_{\text{sh}}(6)$ is the set of all positive even numbers union $\{1\}$, and
- $\mathrm{S}_{\text{sh}}(16) = \{1, 2, 4, 8, 16\}$.

Remark

$\mathrm{S}_{\text{sh}}(s)$ is finite if and only if $s \in \mathrm{S}_{\text{sh}}(2^\infty)$.

Sharkovskii's Theorem

Theorem (Sharkovskii)

For each continuous map g from a closed interval of the real line into itself, there exists $s \in \mathbb{N}_{\text{Sh}}$ such that $\mathrm{Per}(g) = \mathrm{S}_{\text{sh}}(s)$.
Conversely, for each $s \in \mathbb{N}_{\text{Sh}}$ there exists a continuous map g_s from a closed interval of the real line into itself such that $\mathrm{Per}(g_s) = \mathrm{S}_{\text{sh}}(s)$.

$\mathrm{Per}(g)$ denotes the set of (least) periods of all periodic points of g.

The set of periods for star maps — Notation

A *(topological) graph* is a connected Hausdorff space G, which is a finite union of subspaces G_i, each of them homeomorphic to a closed, non-degenerate interval of the real line and $G_i \cap G_j$ is finite for all $i \neq j$. Every graph is compact.

The points from a graph which do not have a neighbourhood homeomorphic to an open interval are called *vertices*. The set of vertices of a graph G is denoted by $\mathrm{V}(G)$ and is clearly finite (or empty — when G is homeomorphic to to the circle).

The closure of any connected component of $G \setminus \mathrm{V}(G)$ is called an *edge of G*. Clearly, a graph has finitely many edges and each of them is homeomorphic to a closed interval or to the circle.

The set of periods for star maps — Notation

A *tree* is a graph which is uniquely arcwise connected.

Let G be a graph, let $z \in G$ and let U be an open neighbourhood (in G) of z such that $\mathrm{Cl}(U)$ is a tree. The number of connected components of $U \setminus \{z\}$ is called *the valence of z* and is denoted by $\mathrm{Val}(z)$. This definition is independent of the choice of U and $\mathrm{Val}(z) \neq 2$ if and only if $z \in V(G)$. A vertex of valence 1 is called an *endpoint of G* whereas a point of valence larger than 2 is called a *branching point of G*.

Let $n \in \mathbb{N} \setminus \{1\}$. An *$n$-star* is a tree with n endpoints and at most one branching point. Note that a 2-star is homeomorphic to an interval (an thus it has no branching point) while an n-star with $n \geq 3$ has a unique branching point b with $\mathrm{Val}(b) = n$. X_n will denote an n-star and $\mathcal{X}_n$ the class of all continuous maps from X_n into X_n.

Baldwin partial orderings. Example: the structure of ${}_4{\geq}$

For each integer $t \geq 2$ we denote:

$$\mathbb{N}_t = (\mathbb{N} \cup \{t \cdot 2^\infty\}) \setminus \{2, 3, \dots, t-1\} \text{ and}$$
$$\mathbb{N}_t^\vee = \{mt \colon m \in \mathbb{N}\} \cup \{1, t \cdot 2^\infty\}.$$

Then, the ordering ${}_t{\geq}$ is defined in $\mathbb{N}_t$ as follows: for $k, m \in \mathbb{N}_t$ we have $m \,{}_t{\geq}\, k$ if one of the following holds:

(i) $k = 1$ or $k = m$,

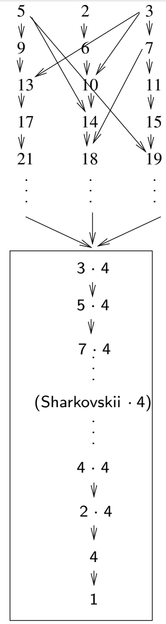
(ii) $k, m \in \mathbb{N}_t^\vee \setminus \{1\}$ and $m/t \,{}_{\mathrm{Sh}}{>}\, k/t$ (here we use the arithmetic rule: $t \cdot 2^\infty / t = 2^\infty$),

(iii) $k \in \mathbb{N}_t^\vee$ and $m \notin \mathbb{N}_t^\vee$,

(iv) $k, m \notin \mathbb{N}_t^\vee$ and $k = im + jt$ with $i, j \in \mathbb{N}$.

Remark

By identifying $2 \cdot 2^\infty$ with 2^∞ we have ${}_2{\geq} = {}_{\mathrm{Sh}}{\geq}$.



$\mathbb{N}_t^\vee$

Baldwin partial orderings: Initial segments

A set $S \subset \mathbb{N}_t \cap \mathbb{N}$ is an *initial segment of the ordering ${}_t{\geq}$* if for every $m \in S$ we have $\{k \in \mathbb{N} : m \,{}_t{\geq}\, k\} \subset S$ (that is, S is closed under predecessors).

Also we set

$$\mathcal{S}_t(s) := \{n \in \mathbb{N} : n \leq_t s\},$$

which is a particular case of an initial segment. Indeed, any initial segment of the $\leq_t$ ordering can be expressed as the union of at most $t-1$ sets of the form $\mathcal{S}_t(s_i)$ because the set $\mathbb{N}_t$ splits in at most $t-1$ branches by the ordering $\leq_t$.

Baldwin's Theorem

Theorem (Baldwin)

Let $f \in \mathcal{X}_n$. Then, $\mathrm{Per}(f)$ is a finite union of initial segments of the orderings ${}_t{\geq}$ with $2 \leq t \leq n$. Conversely, given a set A that is a finite union of initial segments of the orderings ${}_t{\geq}$ with $2 \leq t \leq n$, there exists a map $f \in \mathcal{X}_n$ such that $f(b) = b$ and $\mathrm{Per}(f) = A$.

Stewart Baldwin.
An extension of Šarkovskiĭ's theorem to the n-od.
Ergod. Th. & Dynam. Sys. **11(2)** (1991), 249–271.

Remark

The case $n = 2$ in the above theorem is, indeed, the Sharkovsky's Theorem for interval maps. Moreover, since every tail of ${}_t{\geq}$ contains $1 \in \mathrm{Per}(f)$, then the order ${}_t{\geq}$ does not contribute to $\mathrm{Per}(f)$ if the tail with respect to ${}_t{\geq}$ in the above lemma is reduced to $\{1\}$.

The set of periods for tree maps

The full characterization is known but complicate to state and out of the scope of this talk. It was obtained in the following papers

Ll. Alsedà, D. Juher and P. Mumbrú
[1] *Sets of periods for piecewise monotone tree maps.*
Int. J. of Bifurcation and Chaos **13** (2003), 311–341.

[2] *Periodic behavior on trees.*
Ergodic Theory Dynam. Systems **25(5)** (2005), 1373–1400.

[3] *On the preservation of combinatorial types for maps on trees.*
Annales de l'Institut Fourier **55(7)** (2005) 2375–2398.

[4] *Minimal dynamics for tree maps.*
Discrete and Contin. Dyn. Sys. Ser. A **20(3)** (2008) 511–541.

with the help of the general theory of patterns for trees and graphs

Ll. Alsedà, J. Guaschi, J. Los, F. Mañosas and P. Mumbrú.
Canonical representatives for patterns of tree maps.
Topology **36** (1997), 1123-1153.

Ll. Alsedà, F. Gautero, J. Guaschi, J. Los, F. Mañosas and P. Mumbrú.
Patterns and minimal dynamics for graph maps.
Proc. London Math. Soc. **91(2)** (2005), 9414–442.

The circle case: rotation theory

For interval maps the knowledge of a cycle gives us combinatorial information sufficient to determine the minimal set of periods of cycles of any map having the cycle. However, for circle maps this is not the case, as the following example shows.

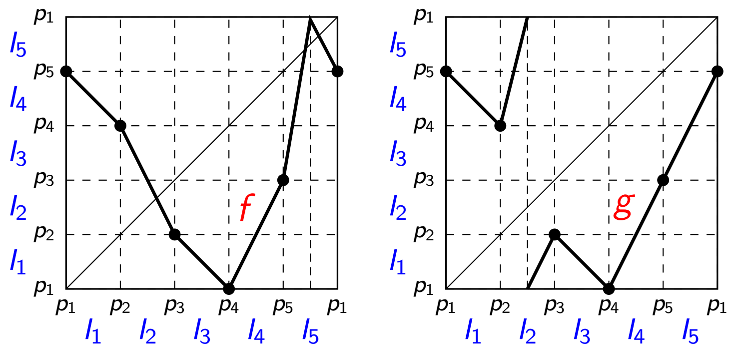


Figure: The graphs of two circle maps drawn on the 2-dimensional torus.

The circle case: rotation theory

We regard the circle $\mathbb{S}^1$ as the set $\{z \in \mathbb{C} \colon |z| = 1\}$, and the natural projection $e \colon \mathbb{R} \longrightarrow \mathbb{S}^1$ is defined by $e(x) = \exp(2\pi i x)$. This map is continuous, surjective and it is a homomorphism from the additive group of $\mathbb{R}$ to the multiplicative group of $\mathbb{S}^1$ (i.e. we have $e(x_1 + x_2) = e(x_1) \cdot e(x_2)$). The kernel of this homomorphism is the group $\mathbb{Z}$ of the integer numbers.

The circle case: rotation theory

Proposition

Any continuous map $f \colon \mathbb{S}^1 \longrightarrow \mathbb{S}^1$ has a continuous lifting $F \colon \mathbb{R} \longrightarrow \mathbb{R}$, which is unique up to translation by an integer and such that the diagram

$$\begin{tikzcd} \mathbb{R} \arrow[r, "F"] \arrow[d, "e"'] & \mathbb{R} \arrow[d, "e"] \\ \mathbb{S}^1 \arrow[r, "f"'] & \mathbb{S}^1 \end{tikzcd}$$

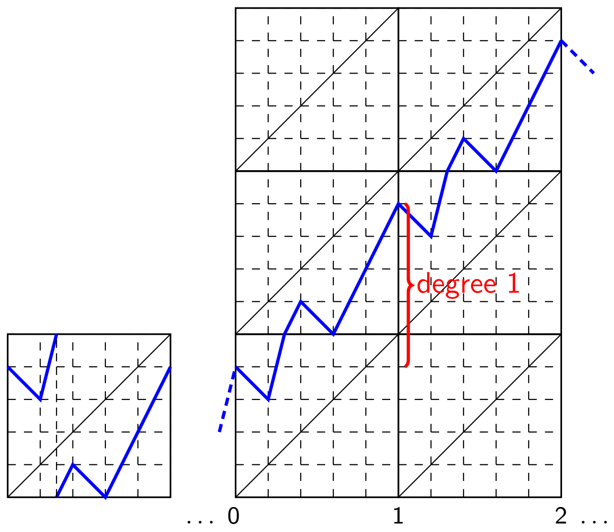
commutes.

The circle case: rotation theory

Since, for each $n \in \mathbb{Z}$, $e(n) = e(n+1) = 1$, we have $e(F(n+1)) = f(e(n+1)) = f(e(n)) = e(F(n))$. Therefore $F(n+1) - F(n)$ is an integer. This integer is called the *degree of f* and is denoted by $\deg f$. Any other lifting $\widetilde{F}$ of f in the interval

Roughly speaking, the degree of a circle map f is the minimum number of times that the image of $\mathbb{S}^1$ by f covers completely $\mathbb{S}^1$ counterclockwise if $\deg f$ is positive, and clockwise if $\deg f$ is negative.

Example a map on the torus (left) and a lifting of it (right)



The circle case: rotation theory

Proposition

Let f, f' be continuous circle maps. Then

(a) $\deg(f \circ f') = \deg f \cdot \deg f'$,

(b) $\deg(f^n) = (\deg f)^n$.

Proposition

Let f be a circle map of degree d and let F be a lifting of f. Then the following statements hold.

(a) *If F' is another lifting of f, then $F = F' + k$ for some integer k.*

(b) *If $k \in \mathbb{Z}$ then $F + k$ is also a lifting of f.*

(c) *$F^n(x + k) = F^n(x) + kd^n$ for all $x \in \mathbb{R}$, $k \in \mathbb{Z}$ and $n \geq 0$.*

(d) *$(F + k)^n(x) = F^n(x) + k(1 + d + d^2 + \cdots + d^{n-1})$ for all $x \in \mathbb{R}$, $k \in \mathbb{Z}$ and $n \geq 0$.*

Liftings of degree one

These are continuous maps $F\colon T \longrightarrow T$ such that $F(x+1) = F(x) + 1$ for every $x \in \mathbb{R}$. The class of these maps will be denoted by $\mathfrak{L}_1$.

Lemma (Behaviour of maps from $\mathfrak{L}_1$ under iteration)

For $n \in \mathbb{N}$, $k \in \mathbb{Z}$ and $x \in \mathbb{R}$:

(a) $F^n \in \mathfrak{L}_1$,

(b) $F^n(x + k) = F^n(x) + k$,

(c) $(F + k)^n(x) = F^n(x) + kn$.

Lifted periods for maps $F \in \mathfrak{L}_1$.

A point $x \in \mathbb{R}$ is periodic $\pmod 1$ if there exists $n \in \mathbb{N}$ such that $F^n(x) \in x + \mathbb{Z}$. The period of x is the least integer n with this property.

That is, $F^n(x) \in x + \mathbb{Z}$ and $F^i(x) \notin x + \mathbb{Z}$ for all $1 \leq i \leq n-1$.

Observation

x is periodic $\pmod 1$ for F if and only if $e(x)$ is periodic for f. Moreover, the F-period $\pmod 1$ of x and the f-period of $e(x)$ coincide.

Lifted Orbits for maps $F \in \mathfrak{L}_1$.

The set

$$\operatorname{Orb}_1(x, F) = \{F^n(x) + m : n \geq 0 \text{ and } m \in \mathbb{Z}\},$$

is called the orbit $\pmod 1$ of x.

Observation

$\operatorname{Orb}_1(x, F) = e^{-1}(\{f^n(e(x)) : n \geq 0\}) = e^{-1}(\operatorname{Orb}(e(x), f))$.

When x is periodic $\pmod 1$ then $\operatorname{Orb}_1(x, F)$ is also called periodic $\pmod 1$. In this case it is not difficult to see that $\operatorname{Card}(\operatorname{Orb}_1(x, F) \cap T_n)$ coincides with the period of x for all $n \in \mathbb{Z}$.

Rotation numbers

For $F \in \mathfrak{L}_1$ and $x \in \mathbb{R}$ we define

$$\overline{\rho}_F(x) = \limsup_{m\to\infty} \frac{F^m(x) - x}{m} \quad \text{and} \quad \underline{\rho}_F(x) = \liminf_{m\to\infty} \frac{F^m(x) - x}{m}.$$

When $\overline{\rho}_F(x) = \underline{\rho}_F(x)$ we write only $\rho_F(x)$ or $\rho(x)$.

The number $\rho_F(x)$ (if it exists) is called the *rotation number of x with respect to F*.

Basic properties of rotation numbers

Lemma (Properties of rotation numbers with respect to the chosen lifting)

Let $F \in \mathfrak{L}_1$, $x \in \mathbb{R}$, $k \in \mathbb{Z}$ and $n \in \mathbb{N}$.

(a) $\overline{\rho}_F(x + k) = \overline{\rho}_F(x)$.

(b) $\overline{\rho}_{(F+k)}(x) = \overline{\rho}_F(x) + k$.

(c) $\overline{\rho}_{F^n}(x) = n\overline{\rho}_F(x)$.

The same statements hold with ρ and $\underline{\rho}_F$ instead of $\overline{\rho}_F$.

Lemma

If $F \in \mathfrak{L}_1$ is non-decreasing then $\rho = \rho(x)$ exists for every $x \in \mathbb{R}$ and is independent on x.

Rotation numbers and periodic points

Definition

Let $F \in \mathfrak{L}_1$. An orbit $\pmod 1$ $P \subset \mathbb{R}$ of F will be called twist if $F\big|_P$ is strictly increasing.

(i) Two points in the same orbit $\pmod 1$ have the same rotation number.

(ii) If $F^q(x) = x + p$ with $q \in \mathbb{N}$ and $p \in \mathbb{Z}$, then $\rho_F(x) = p/q$. Therefore all periodic $\pmod 1$ points have rational rotation numbers.

(iii) Let x be a periodic $\pmod 1$ point of period q and $p \in \mathbb{Z}$ such that $F^q(x) = x + p$. If $\operatorname{Orb}_1(x, F)$ is a twist orbit, then $(p, q) = 1$.

Rotation Set: synthesises the rotation numbers information

Definition

For $F \in \mathfrak{L}_1$ we define:

$$\operatorname{Rot}^+(F) = \{\overline{\rho}_F(x) : x \in \mathbb{R}\},$$
$$\operatorname{Rot}^-(F) = \{\underline{\rho}_F(x) : x \in \mathbb{R}\},$$
$$\operatorname{Rot}(F) = \{\rho F(x) : x \in \mathbb{R} \text{ and } \rho F(x) \text{ exists}\}.$$

Theorem (Ito)

All these sets coincide. They are a closed interval of the real line whose endpoints depend continuously on the map (with respect to the topology of the uniform convergence in the class of continuous liftings of degree one).

[Ito] Ryuichi Ito.
Rotation sets are closed.
Math. Proc. Cambridge Philos. Soc. **89(1)** (1981), 107–111.

Rotation numbers and twist orbits

Theorem

Let $F \in \mathfrak{L}_1$. Then the following statements hold.

1. *For every $a \in \operatorname{Rot}(F)$ there exists a twist lifted orbit of F with rotation number a and disjoint from $\operatorname{Const}(F)$.*
2. *For every $a \in \mathbb{Q} \cap \operatorname{Rot}(F)$ there exists a twist lifted cycle of F with rotation number a and disjoint from $\operatorname{Const}(F)$.*

The set of periods. Notation

For $c \leq d$ we set

$$M(c, d) := \{n \in \mathbb{N} \colon c < k/n < d \text{ for some integer } k\}.$$

Notice that we do not assume here that k and n are coprime. Obviously, $M(c, d) = \emptyset$ if and only if $c = d$.

Given $\rho \in \mathbb{R}$ and $W \subset \mathbb{N}$ we set

$$\Lambda(\rho, W) = \begin{cases} \emptyset & \text{if } \rho \notin \mathbb{Q}, \\ \{nq : q \in W\} & \text{if } \rho = k/n \text{ with } k \text{ and } n \text{ coprime.} \end{cases}$$

The set of periods

Theorem (Misiurewicz)

Let $F \in \mathfrak{L}_1$, and let $\operatorname{Rot}(F) = [c, d]$. Then there exist numbers $s_c, s_d \in \mathbb{N}_{\mathrm{Sh}}$ such that $\operatorname{Per}(F) = \Lambda(c, \mathrm{S_{sh}}(s_c)) \cup M(c, d) \cup \Lambda(d, \mathrm{S_{sh}}(s_d))$. Conversely, for any given $c, d \in \mathbb{R}$ with $c \le d$ and $s_c, s_d \in \mathbb{N}_{\mathrm{Sh}}$, there exists a map $F \in \mathfrak{L}_1$ such that $\operatorname{Rot}(F) = [c, d]$ and $\operatorname{Per}(F) = \Lambda(c, \mathrm{S_{sh}}(s_c)) \cup M(c, d) \cup \Lambda(d, \mathrm{S_{sh}}(s_d))$.

M. Misiurewicz.
Periodic points of maps of degree one of a circle.
Ergod. Th. & Dynam. Sys. **2** (1982), 221–227.

Remark

From the (endpoints of the) rotation interval one also gets lower bounds of the topological entropy of the map under consideration (A-Llibre-Mañosas-Misiurewicz).

The set of periods for σ-maps

A full characterisation of the sets of periods for continuous self maps of the graph σ having the branching fixed is given in

M. Carme Leseduarte and Jaume Llibre.
On the set of periods for σ maps.
Trans. Amer. Math. Soc. **347(12)** (1995), 4899–4942.

Our goal is to extend this result to the general case. The most natural approach is to follow the strategy used in the circle case which consists in dividing the problem according to the degree of the map. The cases considered for the circle are:

- degree different from $\{-1, 0, 1\}$,
- degree -1,
- degree 0, and
- degree 1.

The set of periods for σ-maps

A characterisation of the set of periods of the class of continuous maps from the space σ to itself with degree different from $\{-1, 0, 1\}$ by using Nielsen Numbers can be found in

Alba Málaga.
Dinámica de grafos de un ciclo para funciones de grado diferente de uno (Spanish).
Master thesis, Universidad National de Ingeniería, Peru, 2011.
Available at
http://cybertesis.uni.edu.pe/bitstream/uni/277/1/malaga_sa.pdf.

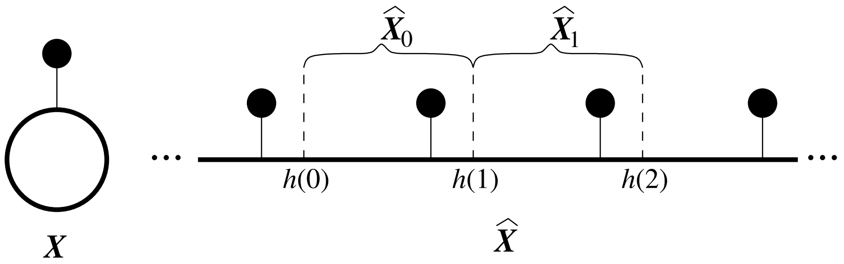
The set of periods for σ-maps

In this talk we aim at studying the set of periods of continuous σ-maps of degree 1. Following again the strategy of the circle case, we shall work at the lifting level and we shall use rotation theory. This theory for graphs with a single circuit was developed in

Lluís Alsedà and Sylvie Ruette.
Rotation sets for graph maps of degree 1.
Ann. Inst. Fourier (Grenoble) **58(4)** (2008), 1233–1294.

and shall be reviewed below.

Lifted spaces: A simple definition



Lifted spaces: A simple definition

A *lifted space* T is a connected closed subset of $\mathbb{C}$ containing $\mathbb{R}$ such that

(i) For every $z \in \mathbb{C}$, $z \in T$ is equivalent to $z + \mathbb{Z} \in T$,
(ii) the closure of each connected component of $T \setminus \mathbb{R}$ is a compact set that intersects $\mathbb{R}$ at a single point, and
(iii) the number of connected components C of $T \setminus \mathbb{R}$ such that $\overline{C} \cap [0,1] \neq \emptyset$ is finite.

The class of all lifted spaces will be denoted by $\mathbf{T}$.

Maps on lifted spaces

Following the circle case, on a lifted space T we will only consider *liftings* of continuous maps of degree one.

These are continuous maps $F \colon T \longrightarrow T$ such that $F(x+1) = F(x) + 1$ for every $x \in T \subset \mathbb{C}$.

The class of these maps will be denoted by $\mathfrak{L}_1$.

Recalling the notion of a lifting.

When $T \in \mathbf{T}$ is obtained by unwinding a loop W contained in a topological space X there exists a continuous map $\pi \colon T \longrightarrow X$, called the *standard projection from T to X*, such that $\pi([0,1]) = W$ and $\pi(x+1) = \pi(x)$ for all $x \in T$.

Then, given $f \colon X \longrightarrow X$ continuous, there exists a (non-unique) continuous map $F \colon T \longrightarrow T$ such that $f \circ \pi = \pi \circ F$.

Each of these maps will be called a *lifting of f*.

Observe that $f \circ \pi = \pi \circ F$ implies that $F(1) - F(0) \in \mathbb{Z}$. This number is called the *degree of f* and denoted by $deg(f)$.

Retraction

Given $T \in \mathbf{T}$ there is a natural retraction $r\colon T \longrightarrow \mathbb{R}$. When $x \in \mathbb{R}$, then clearly $r(x) = x$. When $x \notin \mathbb{R}$, by definition, there exists a connected component C of $T \setminus \mathbb{R}$ such that $x \in C$ and $\operatorname{Clos}(C)$ intersects $\mathbb{R}$ at a single point z. Then $r(x)$ is defined to be, precisely, the point z. In particular, r is constant on $\operatorname{Clos}(C)$.

A point $x \in \mathbb{R}$ such that $r^{-1}(x) \neq \{x\}$ will be called a *branching point of T*. The set of all branching points of T will be denoted by $\mathrm{B}(T)$. It is a subset of $\mathbb{R}$ by definition.

The map $r\colon T \longrightarrow \mathbb{R}$ is continuous and verifies $r(x+1) = r(x) + 1$ for all $x \in T$.

Rotation numbers
Definition

Let $F \in \mathfrak{L}_1$ and $x \in T$. We define the *rotation number of x* as

$$\rho_F(x) := \lim_{n\to+\infty} \frac{r \circ F^n(x) - r(x)}{n}$$

if the limit exists.

We also define the following *rotation sets of F*:

$$\operatorname{Rot}(F) = \{\rho_F(x) : x \in T\},$$
$$\operatorname{Rot}_{\mathbb{R}}(F) = \{\rho_F(x) : x \in \mathbb{R}\}.$$

Remark

Recall that the composition of maps of degree one has degree one. Thus, $r \circ F^n$ has degree one for every n and we can compute rotation numbers.

Rotation numbers

For every $x \in T$, $k \in \mathbb{Z}$ and $n \in \mathbb{N}$, it follows, as in the circle case, that

- $\rho_F(x + k) = \rho_F(x)$,
- $\rho_{(F+k)}(x) = \rho_F(x) + k$ and
- $\rho_{F^n}(x) = n\rho_F(x)$.

The second property implies that, if F, G are two liftings of the same continuous map from σ into itself, then their rotation sets differ from an integer ($\exists k \in \mathbb{Z}$ such that $G = F + k$, and hence $\operatorname{Rot}(G) = \operatorname{Rot}(F) + k$).

Rotation numbers

In what follows we will consider a special subclass of $\mathbf{T}$. Namely, the subclass of all $T \in \mathbf{T}$ such that

$$r^{-1}([0,1]) = \{x \in T : 0 \le r(x) \le 1\}$$

is a finite graph. This class is denoted by $\mathbf{T}^\circ$.

For instance the initial example of lifted space does not belong to $\mathbf{T}^\circ$.

The class $\mathbf{T}^\circ$ has better properties than the general one.

Rotation numbers

Unfortunately, the set $\text{Rot}(F)$*, even when* $T \in \mathbf{T}^\circ$*, need not be connected*, as it happens in the circle case. However, the set $\text{Rot}_{\mathbb{R}}(F)$, which is a subset of $\text{Rot}(F)$, has better properties:

Theorem

For every $T \in \mathbf{T}^\circ$, $F \in \mathfrak{L}_1$, $\text{Rot}_{\mathbb{R}}(F)$ *is a non empty compact interval. Moreover, if* $\alpha \in \text{Rot}_{\mathbb{R}}(F)$*, then:*

1. *There exists a point* $x \in \mathbb{R}$ *such that* $\rho_F(x) = \alpha$ *and* $F^n(x) \in \mathbb{R}$ *for infinitely many* n.
2. *If* $p/q \in \text{Rot}_{\mathbb{R}}(F)$*, then there exists a periodic* $\pmod 1$ *point* $x \in T$ *with* $\rho_F(x) = p/q$.

Remark

If $\min \text{Rot}_{\mathbb{R}}(F) = p/q$ there may not exist a periodic point $x \in \mathbb{R}$ with $\rho(x) = p/q$.

In certain cases the standard rotation set behaves "correctly"

Theorem

If $\overline{\bigcup_{n\geq 0} F^n(\mathbb{R})} = T$ *(including the case when* F *is transitive), then*

$$\text{Rot}_{\mathbb{R}}(F) = \text{Rot}(F)$$

The rotation set

The previous comments tell us that the study of the dynamics of the maps from $\mathfrak{L}_1$ has to be decomposed into two parts:

- $\widehat{T} := \bigcup_{n\geq 0} F^{-n}(\mathbb{R})$; studied with $\text{Rot}_{\mathbb{R}}(F)$.
- and $T \setminus \widehat{T}$ that can be studied with retractions and "tree like" techniques.

Thus, the rotation theory must concentrate on $\text{Rot}_{\mathbb{R}}(F)$ *and its relationship with the set of periods. The dynamics "living" in the other part can be studied with "non rotational" techniques.*

Relation between the rotation set and the set of periods

$\text{Per}(\alpha, F)$ denotes the set of all $n \in \mathbb{N}$ for which $\exists x \in T$ such that x is periodic $\pmod 1$ of period n and $\rho_F(x) = \alpha$.

Theorem

- $\text{Per}(\alpha, F) = \emptyset$ *if and only if* $\alpha \notin \text{Rot}(F) \cap \mathbb{Q}$.
- *Assume that* $p/q \in \text{Int}(\text{Rot}_{\mathbb{R}}(F))$*. Then* $\text{Per}(p/q, F)$ *contains* nq *for all great enough integers* n.
- *If* $\text{Rot}_{\mathbb{R}}(F)$ *is not reduced to a single point, then the set of periods of periodic* $\pmod 1$ *points of* f *contains all but finitely many integers.*

Remark

The theorem does not say that $\text{Per}(p/q, F)$ is equal to $\{n \in \mathbb{N} : n \geq N\}$ for some integer N. There are counterexamples of this statement.

The set of periods for σ maps

The universal covering of the space σ is

$$S = \mathbb{R} \cup B,$$

where

$$B := \{z \in \mathbb{C} : \Re(z) \in \mathbb{Z} \text{ and } \Im(z) \in [0,1]\}$$

and $\Re(z)$ and $\Im(z)$ denote respectively the real and imaginary part of a complex number z.

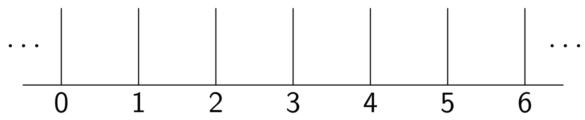


Figure: The space S, universal covering of σ.

Observe that $S = S + \mathbb{Z} = \{x + k : x \in S \text{ and } k \in \mathbb{Z}\}$. Moreover, the real part function defines a retraction from S to $\mathbb{R}$. That is, $\Re(x) = x$ for every $x \in \mathbb{R}$ and, when $x \in S \setminus \mathbb{R}$, then $\Re(x)$ gives the integer in the base of the segment where x lies.

In what follows, $\mathcal{L}_1(S)$ will denote the class of continuous maps F from S into itself of degree 1, that is, $F(x+1) = F(x) + 1$ for all $x \in S$. Also, the set of (true — not $\pmod 1$) periods of all periodic points of F will be denoted by $\operatorname{Per}^\circ(F)$.

For every $m \in \mathbb{Z}$, we set

$$B_m := \{z \in S : \Re(z) = m \text{ and } \Im(z) \in [0,1]\} = S \cap \Re^{-1}(m), \text{ and}$$
$$\mathring{B}_m := B_m \setminus \{m\}.$$

Each of the sets B_m is called *a branch of S*.

Clearly, $B = \cup_{m\in\mathbb{Z}} B_m$, $B_m \cap \mathbb{R} = \{m\}$ and $\mathring{B}_m \cap \mathbb{R} = \emptyset$.

A conjecture on the set of periods for degree one

The maps $F \in \mathcal{L}_1(S)$ such that $F(\mathbb{R}) \subset \mathbb{R}$ and $F(B_m) = F(m)$ for every $m \in \mathbb{Z}$ can be identified with the class of liftings of continuous circle maps of degree 1.

Therefore any possible set of periods of a continuous circle map of degree 1 is a set of periods of a map in $\mathcal{L}_1(S)$.

A conjecture on the set of periods for degree one

Consider the 3-star $Y_0 := B_0 \cup [-1/3, 1/3] \subset S$ and consider the class of maps $F \in \mathcal{L}_1(S)$ such that

- $F(Y_0) \subset Y_0$,
- $F(x) \in Y_0 \cup [1/3, x)$ for every $x \in [1/3, 1/2)$ and
- $F(x) \in (Y_0 + 1) \cup (x, 2/3]$ for every $x \in (1/2, 2/3]$ (in particular $F(1/2) = 1/2$).

This implies that $\operatorname{Per}(F) = \operatorname{Per}^\circ(F\big|_{Y_0})$ and thus, every possible set of periods of a map from a 3-star into itself can be a set of periods of a map from $\mathcal{L}_1(S)$. Clearly, this includes the sets of periods of interval maps.

Moreover, this phenomenon could occur for rotation numbers different from 0. That is, there may exist a map from $\mathcal{X}_3$ with set of periods $A \subset \mathbb{N}$, $p \in \mathbb{Z}$, $q \in \mathbb{N}$ and $\widetilde{S} \subset S$ such that

$$\operatorname{Per}^\circ((F^q - p)\big|_{\widetilde{S}}) = A \quad \text{and} \quad \operatorname{Per}(p/q, F) = q \cdot \operatorname{Per}^\circ((F^q - p)\big|_{\widetilde{S}}).$$

A conjecture on the set of periods for degree one

From the above comments, and Misiurewicz's and Baldwin's Theorems we get the following natural

Conjecture

Let $F \in \mathcal{L}_1(S)$ be with $\operatorname{Rot}_{\mathbb{R}}(F) = [c, d]$. Then there exist sets $E_c, E_d \subset \mathbb{N}$ which are finite unions of of tails of the orderings $\leq_2$ and $\leq_3$ such that $\operatorname{Per}(F) = \Lambda(c, E_c) \cup M(c, d) \cup \Lambda(d, E_d)$.
Conversely, given $c, d \in \mathbb{R}$ with $c \leq d$, and non empty sets $E_c, E_d \subset \mathbb{N}$ which are finite union of of tails of the orderings $\leq_2$ and $\leq_3$, there exists a map $F \in \mathcal{L}_1(S)$ such that $\operatorname{Rot}_{\mathbb{R}}(F) = [c, d]$ and $\operatorname{Per}(F) = \Lambda(c, E_c) \cup M(c, d) \cup \Lambda(d, E_d)$.

As we shall see, some facts seem to indicate that this conjecture is not entirely true (though they do not disprove it). However, we shall use this conjecture as a guideline: on the one hand, we shall prove that it is partly true; on the other hand, we shall stress some difficulties.

On the second (converse) (converse) statement of the conjecture

This statement holds in two particular cases:

Corollary (of Misiurewicz's Theorem)

Given $c, d \in \mathbb{R}$ with $c \leq d$ and $s_c, s_d \in \mathbb{N}_{\mathrm{Sh}}$, there exists a map $F \in \mathcal{L}_1(S)$ such that $\operatorname{Rot}_{\mathbb{R}}(F) = [c, d]$ and
$\operatorname{Per}(F) = \Lambda(c, \mathrm{S}_{\mathrm{sh}}(s_c)) \cup M(c, d) \cup \Lambda(d, \mathrm{S}_{\mathrm{sh}}(s_d))$.

When both c and d are irrational, this corollary implies the second statement of the Conjecture.

On the second (converse) statement of the conjecture

It remains to consider the cases when c and/or d are in $\mathbb{Q}$ and when the order $\leq_3$ is needed (or equivalently when one refers to the set of periods of any 3-star map). The next theorem deals with the case when c (or d) is equal to 0 (or, equivalently, to an integer) and $\leq_3$ is needed only for this endpoint.

Theorem

Let $d \neq 0$ be a real number, $s_d \in \mathbb{N}_{\mathrm{Sh}}$ and $f \in \mathcal{X}_3$. Then there exists a map $F \in \mathcal{L}_1(S)$ such that $\operatorname{Rot}_{\mathbb{R}}(F) = \operatorname{Rot}(F)$ is the closed interval with endpoints 0 and d (i.e., $[c, d]$ or $[d, c]$),
$\operatorname{Per}(0, F) = \operatorname{Per}^{\circ}(f)$ *and*
$\operatorname{Per}(F) = \operatorname{Per}^{\circ}(f) \cup M(0, d) \cup \Lambda(d, \mathrm{S}_{\mathrm{sh}}(s_d))$.

On the second (converse) statement of the conjecture

A natural strategy to prove the second statement of the Conjecture in the general case (i.e. when no endpoint of the rotation interval is an integer) is to construct examples of maps $F \in \mathcal{L}_1(S)$ with a *block structure* over maps $f \in \mathcal{X}_3$ in such a way that p/q is an endpoint of the rotation interval $\operatorname{Rot}_{\mathbb{R}}(F)$ and $\operatorname{Per}(p/q, F) = q \cdot \operatorname{Per}^{\circ}(f)$. The next result shows that this is not possible. Hence, if the second statement of the Conjecture holds, the examples must be built by using some more complicated behaviour of the points of the orbit in $\mathbb{R}$ and on the branches than a block structure.

Bad news for the second statement of the conjecture

Let $F \in \mathcal{L}_1(S)$ and let P be a periodic orbit (mod 1) of F with period nq and rotation number p/q. For every $x \in P$ and $i = 0, 1, \ldots, q-1$, we set

$$P_i(x) := \{F^i(x), G(F^i(x)), G^2(F^i(x)), \ldots, G^{n-1}(F^i(x))\},$$

where $G := F^q - p$. It follows that every $P_i(x)$ is a (true) periodic orbit of G of period n.

Theorem

Let $F \in \mathcal{L}_1(S)$ and let P be a periodic orbit (mod 1) of F with period nq and rotation number p/q. Assume that there exists $x \in P$ such that $\langle P_0(x)\rangle$ is homeomorphic to a 3-star and $\langle P_1(x)\rangle \subset [n, n+1] \subset \mathbb{R}$ for some $n \in \mathbb{Z}$. Assume also that $P_0(x)$ is a periodic orbit of $G := F^q - p$, $F^i(m) \in \langle P_i(x)\rangle$ for $i = 0, 1, \ldots, q-1$ and $G(m) = m$, where $m \in \mathbb{Z} \cap \langle P_0(x)\rangle$ denotes the branching point of $\langle P_0(x)\rangle$. Then $\operatorname{Per}(p/q, F) = q \cdot \mathbb{N}$.

On the first (direct) statement of the conjecture

There are two completely different types of orbits (mod 1) according to the way that they force the existence of other periods:

- the periodic (mod 1) orbits contained in B (viewed at σ level, this means that these periodic orbits do not intersect the circuit of σ), or
- the "rotational orbits" that visit the ground $\mathbb{R}$ of our space S.

We start by studying the periods forced by the periodic (mod 1) orbits contained in B.

Orbits living in the branches

Definition

Let F be a continuous map from S to itself of degree $d \in \mathbb{Z}$ and let P be a periodic (mod 1) orbit of F. We say that P *lives in the branches* when $P \subset B$. Observe that, since P is a (mod 1) orbit, for every $m \in \mathbb{Z}$, $B_m \cap P = (B_0 \cap P) + m$.

The following result extends the results of Leseduarte-Llibre (which deal with σ maps fixing the branching point) to all σ maps.

Theorem

Let F be a continuous map from S to itself of degree $d \in \mathbb{Z}$ and let P be a periodic (mod 1) orbit of F of period p that lives in the branches. Then $\operatorname{Per}(F) \supset \mathrm{S_{sh}}(p)$. Moreover, for every $d \in \mathbb{Z}$ and every $p \in \mathbb{N}_{\mathrm{Sh}}$, there exists a continuous map F_p of S of degree d such that $\operatorname{Per}(F_p) = \mathrm{S_{sh}}(p)$.

Large orbits

Definition

Let F be a continuous map from S to itself of degree $d \in \mathbb{Z}$ and let Q be a (true) periodic orbit of F. We say that Q is a *large orbit* if $\operatorname{diam}(\Re(Q)) \geq 1$, where $\operatorname{diam}(\cdot)$ denotes the diameter of a set.

Observe that a periodic orbit Q living in the branches is large if and only if Q intersects two different branches.

The next result for large orbits living in the branches and degree one is much stronger than the previous one.

Theorem

Let $F \in \mathcal{L}_1(S)$ and let Q be a large orbit of F such that Q lives in the branches. Then $\operatorname{Per}(F) = \mathbb{N}$.

More on large orbits

Large orbits contained in $\mathbb{R}$ work as in the circle case by using $\Re \circ F$. More precisely, if $F \in \mathcal{L}_1(S)$ has a large orbit contained in $\mathbb{R}$, then so does the map $\Re \circ F$. Thus, by the Theorem 2.2 of

Lluís Alsedà and Sylvie Ruette.
Periodic orbits of large diameter for circle maps.
Proc. Amer. Math. Soc. **138(9)** (2010), 3211–3217.

there exists $n \in \mathbb{N}$ such that

$$\left[-\tfrac{1}{n}, \tfrac{1}{n}\right] \subset \mathrm{Rot}(\Re \circ F).$$

It can be shown that, if $0 \in \mathrm{Int}\,\mathrm{Rot}(\Re \circ F)$, then F has a positive horseshoe and $\mathrm{Per}(0, F) = \mathbb{N}$. Consequently, $\mathrm{Per}(F) \supset \mathrm{Per}(0, F) = \mathbb{N}$.

The set of periods of maps from $\mathcal{L}_1(S)$ having a large orbit that intersects both $\mathbb{R}$ and the branches remains unknown. There is an example showing that the existence of a large orbit does not ensure that $\mathrm{Per}(F) = \mathbb{N}$.

Integers in the interior of the rotation set

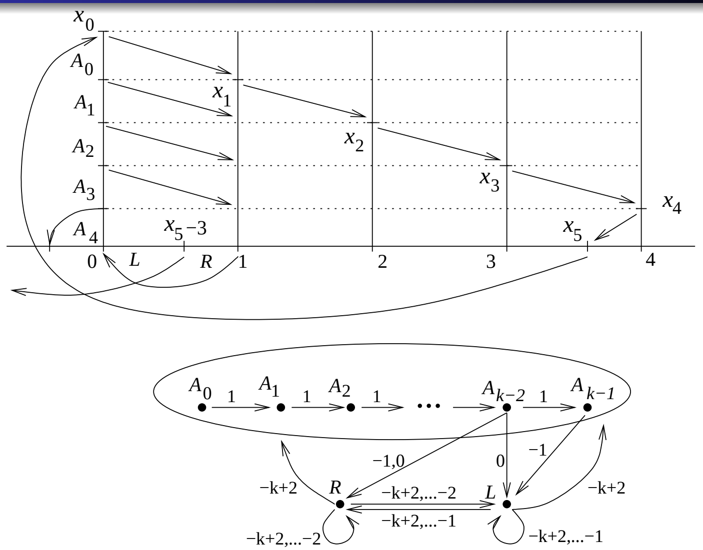
Theorem

Let $F \in \mathcal{L}_1(S)$. If $\mathrm{Int}(\mathrm{Rot}_{\mathbb{R}}(F)) \cap \mathbb{Z} \neq \emptyset$, then $\mathrm{Per}(F)$ is equal to, either $\mathbb{N}$, or $\mathbb{N} \setminus \{1\}$, or $\mathbb{N} \setminus \{2\}$. Moreover, there exist maps $F_0, F_1, F_2 \in \mathcal{L}_1(S)$ with $0 \in \mathrm{Int}(\mathrm{Rot}_{\mathbb{R}}(F_i))$ for $i = 0, 1, 2$ such that $\mathrm{Per}(F_0) = \mathbb{N}$, $\mathrm{Per}(F_1) = \mathbb{N} \setminus \{1\}$ and $\mathrm{Per}(F_2) = \mathbb{N} \setminus \{2\}$.

This theorem is in contrast with the circle case: from one side this results is much more difficult to prove. From another side, in the circle, an integer in the interior of the rotation interval always implies periodic points of all periods while, here, *there exists a map such that* $0 \in \mathrm{Int}(\mathrm{Rot}_{\mathbb{R}}(F))$ *and* $\mathrm{Per}(0, F) = \{k \in \mathbb{N} \mid k \geq n\}$.

The dynamics of certain orbits has complicate underlying structure

The original model



The dynamics of certain orbits has complicate underlying structure

The monotone model: a 5-star

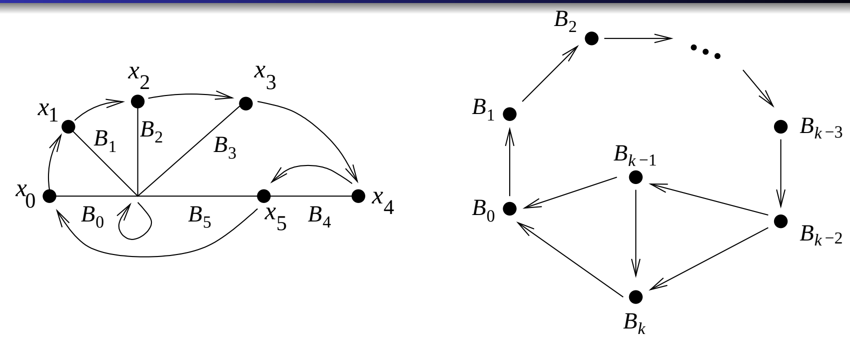
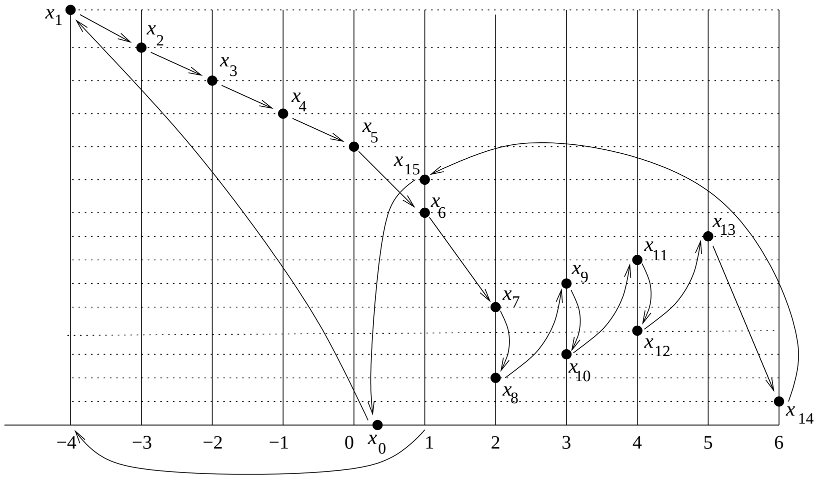


Figure: On the left: the linear model of $[T_P, P, F_P|_P]$, the map being affine on each of the intervals $B_0, \ldots, B_k$. On the right: its Markov graph.

The dynamics of certain orbits has complicate underlying structure

The original model



The dynamics of certain orbits has complicate underlying structure

The monotone model: a bistar