Rotation sets for graph maps of degree 1

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Outline

Motivation

A key point in the study of the dynamics of the continuous self maps of the circle (of degree one) is the rotation theory. From the rotation interval one can obtain, for instance,

- The set of periods (which consists –essentially– on the set of all denominators of all rationals –not necessarily written in irreducible form– in the interior of the rotation interval).

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- Lower bounds of the topological entropy.
- Lower bounds of the number of periodic points of each period.

Aim

We want to use this strategy to study the dynamics of the continuous self maps of graphs with a unique circuit.

To this end we need to develop a rotation theory for continuous maps homotopic to the identity on graphs with a unique circuit; and study its relation with the dynamics of these maps.

Note

As a reward we will obtain a theory which is valid for a certain class of compact metric spaces with a circuit.

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We will better show this idea with a couple of examples.

Example 1

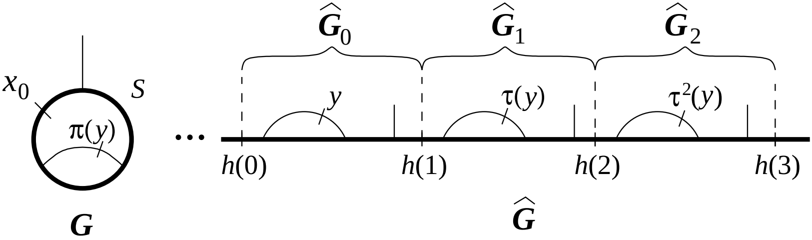


Figure: The graph G, on the left, is unwound with respect to the bold loop S, on the right. This unwinding is made up of infinitely many subspaces $\widehat{G}_n$ that are all homeomorphic by a translation τ. Moreover, there is a continuous projection $\pi\colon \widehat{G} \longrightarrow G$ such that $\pi\big|_{\mathrm{Int}(\widehat{G}_n)}$ is a homeomorphism onto $G \setminus \{x_0\}$ for each n, and $\pi(\tau(y)) = \pi(y)$ for all $y \in \widehat{G}$. The set $\pi^{-1}(S)$ is homeomorphic to the real line.

Example 1 continued

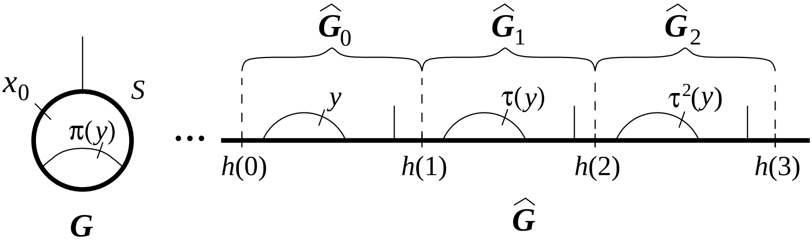


Figure: If we suppose that the loop S has length 1 and that x_0 is the origin, then it is natural to consider a homeomorphism $h\colon \mathbb{R} \longrightarrow \pi^{-1}(S)$ such that $\pi^{-1}(x_0) = h(\mathbb{Z})$. In this setting, $\tau(h(x)) = h(x+1)$ for all $x \in \mathbb{R}$.

Example 2

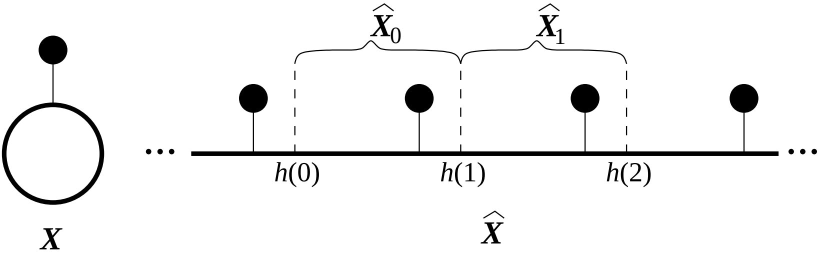


Figure: The unwinding of a connected compact topological space with a unique circuit. In this example $\widehat{X}$ can be retracted to $h(\mathbb{R})$ because the closure of any connected component of $\widehat{X} \setminus h(\mathbb{R})$ meets $h(\mathbb{R})$ at a single point.

Remarks

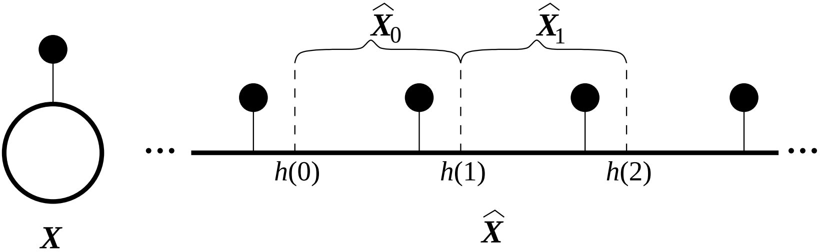
- Example 1: Since G has more than one loop, $\widehat{G}$ is not the universal covering of G and it cannot be retracted to $\pi^{-1}(S) = h(\mathbb{R})$. Therefore, $\widehat{G}$ in this example will not be considered as a lifted space.

Remarks

- Example 1: Since G has more than one loop, $\widehat{G}$ is not the universal covering of G and it cannot be retracted to $\pi^{-1}(S) = h(\mathbb{R})$. Therefore, $\widehat{G}$ in this example will not be considered as a lifted space.
- Example 2: The unwinding of a graph with a single loop always can be retracted to $\pi^{-1}(S) = h(\mathbb{R})$.

Lifted spaces: A simple definition

To have in mind: Example 2.



Lifted spaces: A simple definition

A *lifted space* T is a connected closed subset of $\mathbb{C}$ containing $\mathbb{R}$ such that

(i) For every $z \in \mathbb{C}$, $z \in T$ is equivalent to $z + \mathbb{Z} \in T$,

(ii) the closure of each connected component of $T \setminus \mathbb{R}$ is a compact set that intersects $\mathbb{R}$ at a single point, and

(iii) the number of connected components C of $T \setminus \mathbb{R}$ such that $\overline{C} \cap [0, 1] \neq \emptyset$ is finite.

The class of all lifted spaces will be denoted by $\mathbf{T}$.

Lifted spaces: branching points and retraction

Given $T \in \mathbf{T}$ there is a natural retraction $r\colon T \longrightarrow \mathbb{R}$:

$$r(x) = \begin{cases} x & \text{when } x \in \mathbb{R}, \\ z & \text{when } x \notin \mathbb{R}, \end{cases}$$

where z is the unique point in $\overline{C} \cap \mathbb{R}$ and C is the connected component of $T \setminus \mathbb{R}$ containing x.

Note

r is constant on $\overline{C}$.

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Definition

A point $x \in \mathbb{R}$ such that $r^{-1}(x) \neq \{x\}$ will be called a branching point of T. $\mathrm{B}(T) \subset \mathbb{R}$ denotes the set of all branching points of T.

Maps on lifted spaces

On a lifted space T we will only consider *liftings* of continuous maps of degree one.

These are continuous maps $F\colon T \longrightarrow T$ such that $F(x+1) = F(x)+1$ for every $x \in T \subset \mathbb{C}$.

The class of these maps will be denoted by $\mathcal{C}_1(T)$.

Recalling the notion of a lifting.

When $T \in \mathbf{T}$ is obtained by unwinding a loop S contained in a topological space X there exists a continuous map $\pi\colon T \longrightarrow X$, called the *standard projection from T to X*, such that $\pi([0,1]) = S$ and $\pi(x+1) = \pi(x)$ for all $x \in T$.

Then, given $f\colon X \longrightarrow X$ continuous, there exists a (non-unique) continuous map $F\colon T \longrightarrow T$ such that $f \circ \pi = \pi \circ F$.

Each of these maps will be called a *lifting of f*.

Observe that $f \circ \pi = \pi \circ F$ implies that $F(1) - F(0) \in \mathbb{Z}$. This number is called the *degree of f* and denoted by $deg(f)$.

Remark (Properties of liftings)

- F is a lifting of f if and only if $F(x+1) = F(x) + deg(f)$ for every $x \in \mathbb{R}$.
- F' is a lifting of f if and only if $F = F' + k$ for some $k \in \mathbb{Z}$.

Lemma (Behaviour of maps from $\mathcal{C}_1(T)$ under iteration)

For $n \in \mathbb{N}$, $k \in \mathbb{Z}$ and $x \in T$:

(a) $F^n \in \mathcal{C}_1(T)$,

(b) $F^n(x+k) = F^n(x) + k$,

(c) $(F+k)^n(x) = F^n(x) + kn$.

Observation: Why degree one?

Rotation theory only makes sense for maps homotopic to the identity (degree one). The dynamics of the other maps have to be studied with other techniques (Nielsen Numbers, reduction to the tree case, ...).

Rotation numbers

We define

$$\underline{\rho}_F(x) := \liminf_{n\to+\infty} \frac{r \circ F^n(x) - r(x)}{n}$$
$$\overline{\rho}_F(x) := \limsup_{n\to+\infty} \frac{r \circ F^n(x) - r(x)}{n}.$$

When $\underline{\rho}_F(x) = \overline{\rho}_F(x)$ then this number is denoted by $\rho_F(x)$ and called the *rotation number of x*. The numbers $\underline{\rho}_F(x)$ and $\overline{\rho}_F(x)$ are called the *lower rotation number of x* and *upper rotation number of x*, respectively.

Historical remarks

- This definition extends straightforwardly the usual definition of the circle to this setting by using the retraction and the fact that we have a unitary translation on $\mathbb{R}$.
- Initially, the rotation number was defined by Poincaré for homeomorphisms of the circle. Is is a number and is independent on the point.
- Later on, Poincaré's definition was extended to the non-invertible case by Newhouse, Palis and Takens by using $\limsup$ instead of $\lim$. It is not independent on the point. Ito showed that the set of all rotation numbers it is a closed interval of the real line.

[Po] H. Poincaré, *Sur les courbes définies par les équations différentielles*, Oeuvres completes, vol. 1, 137–158, Gauthier-Villars, Paris, 1952.

[It3] R. Ito, *Note on rotation set*, Proc. Amer. Math. Soc. **89** (1983), 730–732.

[NPT] S. Newhouse, J. Palis and F. Takens, *Bifurcations and stability of families of diffeomorphisms*, Inst. Hautes Études Sci. Publ. Math. **57** (1983), 5–71.

Basic properties of rotation numbers

The definition of rotation number given before has properties analogous to the corresponding definition for circle maps.

Lemma (Properties of rotation numbers with respect to the chosen lifting)

Let $T \in \mathbf{T}$, $F \in \mathcal{C}_1(T)$, $x \in T$, $k \in \mathbb{Z}$ and $n \in \mathbb{N}$.

(a) $\overline{\rho}_F(x+k) = \overline{\rho}_F(x)$.

(b) $\overline{\rho}_{(F+k)}(x) = \overline{\rho}_F(x) + k$.

(c) $\overline{\rho}_{F^n}(x) = n\overline{\rho}_F(x)$.

The same statements hold with ρ and $\underline{\rho}$ instead of $\overline{\rho}$.

Next we want to study the relation between rotation numbers and periodic points.

Lifted periods for maps $F \in \mathcal{C}_1(T)$.

A point $x \in T$ is periodic $\pmod 1$ if there exists $n \in \mathbb{N}$ such that $F^n(x) \in x + \mathbb{Z}$. The period of x is the least integer n with this property.

That is, $F^n(x) \in x + \mathbb{Z}$ and $F^i(x) \notin x + \mathbb{Z}$ for all $1 \leq i \leq n-1$.

Observation

x is periodic $\pmod 1$ for F if and only if $\pi(x)$ is periodic for f. Moreover, the F-period $\pmod 1$ of x and the f-period of $\pi(x)$ coincide.

Lifted Orbits for maps $F \in \mathcal{C}_1(T)$.

The set

$$\mathrm{Orb}_1(x, F) = \{F^n(x) + m : n \geq 0 \text{ and } m \in \mathbb{Z}\},$$

is called the orbit (mod 1) of x.

Observation

$\mathrm{Orb}_1(x, F) = \pi^{-1}(\{f^n(\pi(x)) : n \geq 0\}) = \pi^{-1}(\mathrm{Orb}(\pi(x), f))$.

When x is periodic (mod 1) then $\mathrm{Orb}_1(x, F)$ is also called periodic (mod 1). In this case it is not difficult to see that $\mathrm{Card}(\mathrm{Orb}_1(x, F) \cap T_n)$ coincides with the period of x for all $n \in \mathbb{Z}$.

Definition

Let $T \in \mathbf{T}$ and $F \in \mathcal{C}_1(T)$. An orbit (mod 1) $P \subset \mathbb{R}$ of F will be called twist if $F\big|_P$ is strictly increasing.

Rotation numbers and periodic points

(i) Two points in the same orbit $\pmod 1$ have the same rotation number.

(ii) If $F^q(x) = x + p$ with $q \in \mathbb{N}$ and $p \in \mathbb{Z}$, then $\rho_F(x) = p/q$. Therefore all periodic $\pmod 1$ points have rational rotation numbers.

(iii) Let x be a periodic $\pmod 1$ point of period q and $p \in \mathbb{Z}$ such that $F^q(x) = x + p$. If $\mathrm{Orb}_1(x, F)$ is a twist orbit, then $(p, q) = 1$.

The Rotation Set

It is an important object that synthesises all the information about rotation numbers is the *rotation set* (i.e., the set of all rotation numbers).

Definition

For $T \in \mathbf{T}$ and $F \in \mathcal{C}_1(T)$ we define:

$$\mathrm{Rot}^+(F) = \{\overline{\rho}_F(x) : x \in T\},$$
$$\mathrm{Rot}^-(F) = \{\underline{\rho}_F(x) : x \in T\},$$
$$\mathrm{Rot}(F) = \{\rho_F(x) : x \in T \text{ and } \rho_F(x) \text{ exists}\}; \text{ and finally}$$
$$\mathrm{Rot}_{\mathbb{R}}(F) = \{\rho_F(x) : x \in \mathbb{R} \text{ and } \rho_F(x) \text{ exists}\}.$$

The Rotation Set: Comments

- For continuous degree one circle maps, all these sets are known to coincide. They are a closed interval of the real line ([Ito]) whose endpoints depend continuously on the map (with respect to the topology of the uniform convergence in the class of continuous liftings of degree one).

The Rotation Set: Comments

- For continuous degree one circle maps, all these sets are known to coincide. They are a closed interval of the real line ([Ito]) whose endpoints depend continuously on the map (with respect to the topology of the uniform convergence in the class of continuous liftings of degree one).
- For lifted spaces the rotation set $\mathrm{Rot}(F)$ may not be connected and endpoints do not depend continuously on the map. In general we do not know if it is closed.

Key example

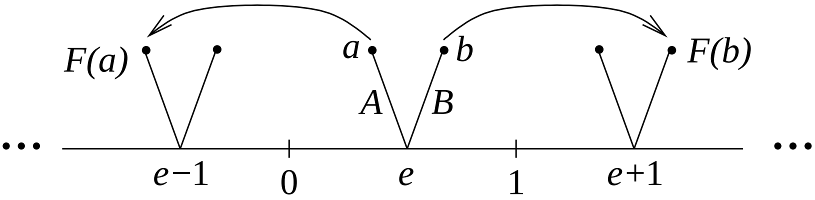

Figure: $F|_{\mathbb{R}} = \mathrm{Id}$, $F(A) = [a-1, e]$ and $F|_A$ is affine, $F(B) = [e, b+1]$ and $F|_B$ is affine.

Properties of the rotation set

- $\mathrm{Rot}(F) = \{-1, 0, 1\}$ is not connected.
- $\mathrm{Rot}_{\mathbb{R}}(F) = \{0\} \neq \mathrm{Rot}(F)$.

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- The same hold for $x \in B$, $x \neq b$.
- Hence $\mathrm{Rot}(F) = \{-1, 0, 1\}$.
- Moreover $\mathrm{Rot}(F) \neq \mathrm{Rot}_{\mathbb{R}}(F)$ although the set

$$\bigcup_{n \geq 0} F^{-n}(\mathbb{R}) = T \setminus (\{a, b\} + \mathbb{Z})$$

is dense in T.

Negative remark

There exist maps F such that $\mathrm{Rot}(F)$ has

- n connected components for any n arbitrarily large (even when there is a single branch).
- connected components outside $\mathrm{Rot}_{\mathbb{R}}(F)$ which are non degenerate intervals (e.g., $F|_{\mathbb{R}} = \mathrm{Id}$ and $F(A) \supset (A+1) \cup (A+2)$ — a “horseshoe”).

Generally, when the dynamics of parts of the branches has no relation with the dynamics of $\mathbb{R}$, disconnectedness of the rotation set is likely to occur.

How to overcome this problem?

The previous example suggests that the study of the dynamics of such maps has to be decomposed into two parts:

- $\widehat{T} := \bigcup_{n \geq 0} F^{-n}(\mathbb{R})$; studied with $\mathrm{Rot}_{\mathbb{R}}(F)$.
- and $T \setminus \widehat{T}$ studied with retractions and "tree like" techniques.

Conclusion

To develop a rotation theory we must concentrate on $\mathrm{Rot}_{\mathbb{R}}(F)$ and its relationship with the general $\mathrm{Rot}(F)$. The dynamics "living" in the other part can be studied with "non rotational" techniques.

Main tasks

In the above setting two main tasks arise:

(I) Study the properties of the rotation set and its relation with the set of periods $\pmod 1$. As we will see, it turns out that this theory gives a lot of information on the dynamics but, at the same time, this information is not satisfactory. The situation is clearly worse than the circle case.

(II) Try to find a subclass of $\mathcal{C}_1(T)$ for which the rotation theory works well as in the circle case. These are called *combed maps*.

Basic results on $\mathrm{Rot}_{\mathbb{R}}(F)$

Given $F \in \mathcal{C}_1(T)$ and $n \in \mathbb{N}$ we consider

$$r \circ F^n\big|_{\mathbb{R}} : \mathbb{R} \longrightarrow \mathbb{R}.$$

The map $r \circ F^n$ is a lifting of a circle map of degree 1. Thus the results on rotation sets for circle maps apply.

Theorem

- *Assume that* $\mathrm{Orb}_1(x, F^n) \subset \mathbb{R}$. *Then,* $\rho_{r \circ F^n}(x) = \rho_{F^n}(x) = n\rho_F(x)$.

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- *Conversely, for each* $\alpha \in \mathrm{Rot}(r \circ F^n)$ *there exists* $x \in \mathbb{R}$ *such that* $\alpha = \rho_{r\circ F^n}(x) = \rho_{F^n}(x) = n\rho_F(x)$, $\mathrm{Orb}_1(x, F^n) = \mathrm{Orb}_1(x, r \circ F^n) \subset \mathbb{R}$ *and* $\mathrm{Orb}_1(x, F^n)$ *is twist.*

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- *Moreover, if* $\alpha \in \mathbb{Q}$ *then* x *can be chosen to be periodic* (mod 1) *of* F. *In particular, for each* $n \in \mathbb{N}$,

$$\frac{1}{n}\,\mathrm{Rot}(r \circ F^n) \subset \mathrm{Rot}_{\mathbb{R}}(F)$$

Corollary

Let $F \in \mathcal{C}_1(T)$ and let $n \in \mathbb{N}$. Then,

$$\mathrm{Rot}(r \circ F) \subset \tfrac{1}{n}\,\mathrm{Rot}(r \circ F^n).$$

Consequently, the set

$$\bigcup_{n \geq 1} \tfrac{1}{n}\,\mathrm{Rot}(r \circ F^n)$$

is a nonempty interval contained in $\mathrm{Rot}_{\mathbb{R}}(F)$.

Remark

In general, the interval $\bigcup_{n\geq 1} \frac{1}{n}\,\mathrm{Rot}(r \circ F^n)$ need not be closed.

Characterisation of $\mathrm{Rot}_{\mathbb{R}}(F)$

Theorem

- $\mathrm{Rot}_{\mathbb{R}}(F)$ *is a non empty compact interval and*

$$\mathrm{Rot}_{\mathbb{R}}(F) = \mathrm{Rot}^{+}_{\mathbb{R}}(F) = \mathrm{Rot}^{-}_{\mathbb{R}}(F) = \overline{\bigcup_{n \geq 1} \frac{1}{n} \mathrm{Rot}(r \circ F^n)}.$$

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- *If* $\alpha \in \mathrm{Rot}_{\mathbb{R}}(F)$, *then there exists a point* $x \in \mathbb{R}$ *such that* $\rho_F(x) = \alpha$ *and* $F^n(x) \in \mathbb{R}$ *for infinitely many* n.

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- *If* $p/q \in \mathrm{Int}(\mathrm{Rot}_{\mathbb{R}}(F))$, *then there exists a periodic point* $x \in \mathbb{R}$ *with* $\rho_F(x) = p/q$.

Remark

$\mathrm{Rot}_{\mathbb{R}}(F)$ is a subset of $\mathrm{Rot}(F)$. Clearly, if $\bigcup_{n\in\mathbb{Z}} F^n(\mathbb{R}) = T$, then

$$\mathrm{Rot}_{\mathbb{R}}(F) = \mathrm{Rot}(F) = \mathrm{Rot}^+(F) = \mathrm{Rot}^-(F).$$

Relation between the rotation set and the set of periods

$\mathrm{Per}(\alpha, F)$ denotes the set of all $n \in \mathbb{N}$ for which $\exists x \in T$ such that x is periodic $\pmod 1$ of period n and $\rho_F(x) = \alpha$.

Theorem

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- *Assume that* $p/q \in \mathrm{Int}(\mathrm{Rot}_{\mathbb{R}}(F))$. *Then* $\mathrm{Per}(p/q, F)$ *contains* nq *for all great enough integers* n.
- *If* $\mathrm{Rot}_{\mathbb{R}}(F)$ *is not reduced to a single point, then the set of periods of periodic* $\pmod 1$ *points of* f *contains all but finitely many integers.*

Remark

The theorem does not say that $\mathrm{Per}(p/q, F)$ is equal to $\{n \in \mathbb{N} : n \geq N\}$ for some integer N. There are counterexamples of this statement.

Additional results for lifted spaces of graphs

Assumption: $\{x \in T : 0 \leq r(x) \leq 1\}$ is now a finite graph.

Theorem

If $\overline{\bigcup_{n\geq 0} F^n(\mathbb{R})} = T$ *(including the case when F is transitive), then*

$$\mathrm{Rot}_{\mathbb{R}}(F) = \mathrm{Rot}(F) = \mathrm{Rot}^+(F) = \mathrm{Rot}^-(F).$$

Theorem

If $\min \mathrm{Rot}_{\mathbb{R}}(F) = p/q$ *(resp.* $\max \mathrm{Rot}_{\mathbb{R}}(F) = p/q$*), then there exists a periodic point* $x \in T$ *such that* $\rho_F(x) = p/q$.

Remark

If $\min \mathrm{Rot}_{\mathbb{R}}(F) = p/q$ there may not exist a periodic point $x \in \mathbb{R}$ with $\rho(x) = p/q$.

A bad example

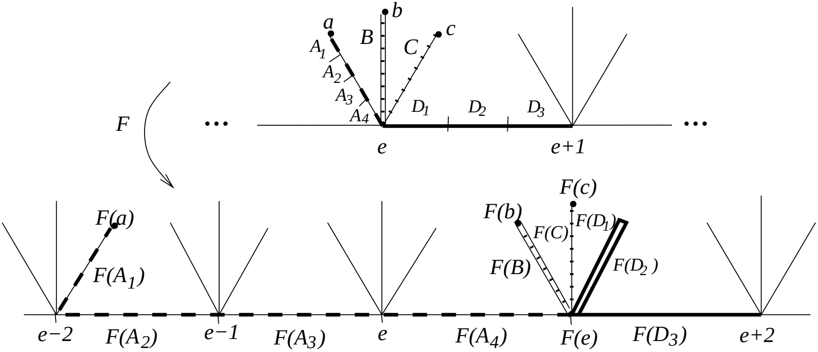

Figure: F is a linear Markov, topologically mixing map

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- The set of periods is:

		$\mathrm{Per}(p/q, F)$
$p = 1$	$q \equiv 0 \mod 3$	$\{nq : n \geq 3\}$
	$q \equiv 1 \mod 3$	$q\mathbb{N}$
	$q \equiv 2 \mod 3$	$\{nq : n \geq 2\}$
$p = 2$	$q \equiv 0 \mod 3$	$\{nq : n \geq 2\}$
	$q \equiv 1, 2 \mod 3$	$q\mathbb{N}$
$p \geq 3$		$q\mathbb{N}$

Table: Values of $\mathrm{Per}(p/q, F)$ for $p/q \in (0, 1)$ and p, q coprime.

Combed maps

In the previous part we have developed a rotation theory for lifted spaces and has studied the differences with the nice rotation theory for continuous degree one circle maps (and also for *old heavy maps*).

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In this part we will take the somewhat converse approach. we will look for a "natural" class of maps on lifted spaces (the *combed maps*) which has a rotation theory analogous to the continuous degree one circle maps. Of course, this class of maps will be rather restrictive.

For the class of combed maps we will also go further in developing the rotation theory in the spirit of the continuous case. In particular we will be able to define and use in this more general setting the *upper* and *lower* maps and also the one-parameter family of *water functions*.

An example of a combed map to fix ideas

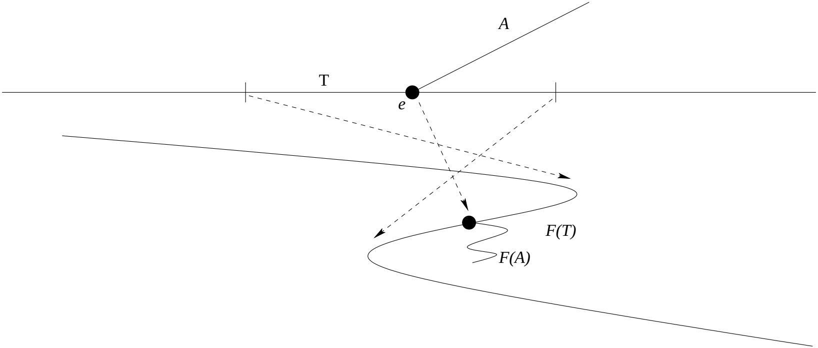

Figure: The image of the branch A gets "hidden" inside $F(\mathbb{R})$. An observer looking at $F(T)$ from above or below does not distinguish this map from a "pure circle map".

Combed maps – The definition

Recall that r denotes the retraction from T to $\mathbb{R} \subset T$.

Definition

- F is *left-combed at $x \in \mathbb{R}$* if
$$r \circ F(\{y \in \mathbb{R} : y \leq x\} \supset r \circ F(r^{-1}(x)).$$
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Remark

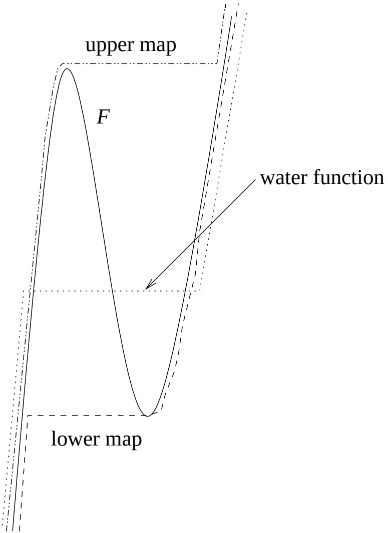
If x is not a branching point of T, then $r^{-1}(x) = \{x\}$. Therefore, F is combed at x.

Upper, lower and water maps

To study the rotation set and set of periods for combed maps, as in the circle case we need to introduce thethe *upper* and *lower* maps and the *water functions*.

See the next figure to illustrate them:

Upper, lower and water maps



Upper, lower maps: formal definition

- For each $x, y \in T$, the relation $r(x) \leq r(y)$ defines a linear pre-ordering on T (denoted by $x \preccurlyeq y$).

Definition

A map F such that $F(x) \preccurlyeq F(y)$ whenever $x \preccurlyeq y$ will be called non-decreasing.

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Remarks

- The map $r\colon T \longrightarrow \mathbb{R}$ is non-decreasing.
- If F is non-decreasing and $r(x) = r(y)$, then $r(F(x)) = r(F(y))$.

Upper, lower maps: formal definition

Now we are ready to define the

- *upper map* $F_u \colon \mathbb{R} \longrightarrow \mathbb{R}$ by

$$\begin{aligned} F_u(x) &:= \sup\{r(F(y)) : y \preccurlyeq x\} \\ &:= \max\{r(F(y)) : x - 1 \preccurlyeq y \preccurlyeq x\}; \text{ and} \end{aligned}$$

- *lower map* $F_l \colon \mathbb{R} \longrightarrow \mathbb{R}$ by

$$\begin{aligned} F_l(x) &:= \inf\{r(F(y)) : y \succcurlyeq x\} \\ &:= \min\{r(F(y)) : x + 1 \succcurlyeq y \succcurlyeq x\}. \end{aligned}$$

Properties of upper and lower maps

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- The map F_u is continuous from the right whereas F_l is continuous from the left.
- The map F_u (respectively F_l) is continuous at $x \in \mathbb{R}$ if and only if it is left-combed (respectively right-combed) at x. In particular, F_u and F_l are continuous at any point which is not a branching point.

Properties of upper and lower maps II

- The maps $F \mapsto r \circ F$, $F \mapsto F_l$ and $F \mapsto F_u$ are Lipschitz continuous with constant 1 (with the topology of the uniform convergence in $\mathcal{C}_1(T)$).

Proposition

The map F_u is continuous if and only if F is left-combed at all $x \in \mathbb{R}$ whereas F_l is continuous if and only if F is right-combed at all $x \in \mathbb{R}$.

Rotation numbers and upper and lower maps

The fact that the maps F_l and F_u are non-decreasing implies ([Rodes and Thompson]) that $\rho_{F_l}(x)$ and $\rho_{F_u}(x)$ exist for each $x \in \mathbb{R}$ and are independent of the choice of the point x.

These two numbers will be denoted by $\rho(F_l)$ and $\rho(F_u)$ respectively.

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Corollary

- $\rho(F_l) \leq \rho(F_u)$, *and*
- $\mathrm{Rot}^+(F) \subset [\rho(F_l), \rho(F_u)]$, $\mathrm{Rot}^-(F) \subset [\rho(F_l), \rho(F_u)]$ *and* $\mathrm{Rot}(F) \subset [\rho(F_l), \rho(F_u)]$.

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In view to his corollary a natural question arises: do the equalities in the above inclusions hold? To study this (and other issues) we need the:

Water functions

Definition

Let us define a one-parameter family of maps from $\mathbb{R}$ to itself:

$$F_\mu = \left(\min(r \circ F\big|_{\mathbb{R}}, F_l + \mu)\right)_u,$$

where $0 \le \mu \le \mu_1 = \sup_{x \in \mathbb{R}} r(F(x)) - F_l(x)$.

See the figure on water functions.

Water functions

Proposition (Properties of water functions)

If F is combed at each $x \in \mathbb{R}$, then the maps F_μ are non-decreasing continuous liftings of degree one circle maps that satisfy:

(a) $F_0 = F_l$ *and* $F_{\mu_1} = F_u$.

(b) *If* $0 \leq \lambda \leq \mu \leq \mu_1$, *then* $F_\lambda \leq F_\mu$.

(c) $\mathrm{Const}(r \circ F) \subset \mathrm{Const}(F_\mu)$ *for each* μ.

(d) *Each* F_μ *coincides with* $r \circ F$ *outside* $\mathrm{Const}(F_\mu)$.

(e) *The function* $\mu \mapsto F_\mu$ *is Lipschitz continuous with constant 1.*

where $\mathrm{Const}(F)$ denotes the set of points $x \in T$ such that F is constant in a neighbourhood of x.

The rotation set of combed maps

The next result extends o combed maps the corresponding one for continuous degree one circle maps.

Theorem

For each map $F \in \mathcal{C}_1(T)$ which is combed at each $x \in \mathbb{R}$ the following statements hold

(a) $\mathrm{Rot}(r \circ F) = \mathrm{Rot}_{\mathbb{R}}(F) = \mathrm{Rot}(F) = \mathrm{Rot}^{+}(F) = \mathrm{Rot}^{-}(F) = [\rho(F_l), \rho(F_u)]$.

(b) *For every $\alpha \in \mathrm{Rot}(F)$, there exists a twist orbit $\pmod 1$ of F contained in $\mathbb{R}$, disjoint from $\mathrm{Const}(F\big|_{\mathbb{R}})$ and having rotation number α.*

(c) *For every $\alpha \in \mathbb{Q} \cap \mathrm{Rot}(F)$, the orbit $\pmod 1$ given by (b) can be taken periodic $\pmod 1$.*

(d) *The endpoints of the rotation interval, $\rho(F_l)$ and $\rho(F_u)$ depend continuously on F.*

The set of periods of combed maps

Given two real numbers $a \leq b$ we denote by $M(a, b)$ the set $\{n \in \mathbb{N} : a < k/n < b \text{ for some integer } k\}$. Clearly $M(a, b) = \emptyset$ whenever $a = b$ and, if $a \neq b$, $M(a, b) \supset \{n \in \mathbb{N} : n > \frac{1}{b-a}\}$.

Also, $\mathrm{Per}(a, F)$ denotes the set of periods $\pmod 1$ of F with rotation number a.

Theorem

If $F \in \mathcal{C}_1(T)$ is combed and $\mathrm{Rot}(F) = [a, b]$, then the following statements hold:

1. *If p, q are coprime and $p/q \in (a, b)$, then $\mathrm{Per}(p/q, F) = q\mathbb{N}$.*
2. $\mathrm{Per}(F) = \mathrm{Per}(a, F) \cup M(a, b) \cup \mathrm{Per}(b, F)$.

Remark

This result is the analogue to this setting of the same result for continuous degree one circle maps. However, contrary to the case of circle maps, the characterisation of the sets $\mathrm{Per}(a, F)$ and $\mathrm{Per}(b, F)$ (where a and b are the endpoints of $\mathrm{Rot}(F)$) is not possible without completely knowing the lifted space T.

Conclusions

- The combed maps are the analogues of the continuous degree one circle maps in the setting of lifted spaces, according to their dynamical properties.

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- The combed maps are the analogues of the continuous degree one circle maps in the setting of lifted spaces, according to their dynamical properties.
- An open problem is whether there exists a more general class displaying the same features. This could be a more general class of continuous maps of degree one in lifted spaces or a class of discontinuous ones analogous to the *old heavy* degree one circle maps.