ON PLANAR WEBS WITH INFINITESIMAL AUTOMORPHISMS

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Abstract. We investigate the space of abelian relations of planar webs admitting infinitesimal automorphisms. As an application we construct $4k-14$ new algebraic families of global exceptional k-webs on the projective plane, for each $k \geq 5$.

1. Introduction and statement of the results

Throughout this paper we will work in the holomorphic category.

1.1. **Planar Webs.** A germ of regular k-web $\mathcal{W} = \mathcal{F}_1 \boxtimes \cdots \boxtimes \mathcal{F}_k$ on $(\mathbb{C}^2, 0)$ is a collection of k germs of smooth foliations $\mathcal{F}_i$ subjected to the condition that any two of these foliations have distinct tangent spaces at the origin.

One of the most intriguing invariants of a web is its *space of abelian relations* $\mathcal{A}(\mathcal{W})$. If the foliations $\mathcal{F}_i$ are induced by 1-forms ω_i then by definition

$$\mathcal{A}(\mathcal{W}) = \left\{ \big(\eta_i\big)_{i=1}^k \in (\Omega^1(\mathbb{C}^2,0))^k \;\Big|\; \forall i \ d\eta_i = 0\,,\ \eta_i \wedge \omega_i = 0 \ \text{ and } \sum_{i=1}^k \eta_i = 0 \right\}.$$

The dimension of $\mathcal{A}(\mathcal{W})$ is commonly called the *rank* of $\mathcal{W}$ and noted by $\mathrm{rk}(\mathcal{W})$. It is a theorem of Bol that $\mathcal{A}(\mathcal{W})$ is a finite-dimensional $\mathbb{C}$-vector space and moreover

$$\mathrm{rk}(\mathcal{W}) \leq \frac{1}{2}\,(k-1)(k-2)\,. \tag{1}$$

An interesting chapter of the theory of webs concerns the characterization of webs of *maximal rank*, *i.e* webs for which (1) is in fact an equality. It follows from Abel's Addition Theorem that all the webs $\mathcal{W}_C$ obtained from reduced plane curves C by projective duality are of maximal rank (*cf.* §4.1 for details). The webs analytically equivalent to some $\mathcal{W}_C$ are the so called *algebrizable webs.*

It can be traced back to Lie a remarkable result that says that all 4-webs of maximal rank are in fact algebrizable. In the early 1930's Blaschke claimed to have extended Lie's result to 5-webs of maximal rank. Not much latter Bol came up with a counter-example: a 5-web of maximal rank that is not algebrizable.

The non-algebrizable webs of maximal rank are nowadays called *exceptional webs.* For a long time Bol's web remained as the only example of exceptional planar web in the literature. The following quote illustrates quite well this fact.

> *(...) we cannot refrain from mentioning what we consider to be the fundamental problem on the subject, which is to determine the maximum rank non-linearizable webs. The strong conditions must imply that there are not many. It may not be unreasonable to compare the situation with the exceptional simple Lie groups.*
>
> Chern and Griffiths in [5].

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A comprehensive account of the current state of the art concerning the exceptional webs is available at [11, Introduction §3.2.1], [14] and [12, §1.4]. Here we will just mention that before this work no exceptional k-web with $k \geq 10$ appeared in the literature.

At first glance, the list of known exceptional webs up today does not reveal common features among them. Although at a second look one sees that many of them (but not all, not even the majority) have one property in common: infinitesimal automorphisms.

1.2. **Infinitesimal Automorphisms.** In [2], É. Cartan proves that *a 3-web which admits an 2-dimensional continuous group of transformations is hexagonal.* It is then an exercise to deduce that a k-web ($k > 3$) which admits 2 linearly independent infinitesimal automorphisms is parallelizable and in particular algebrizable.

Cartan's result naturally leads to the following question:

> *What can be said about webs which admit one infinitesimal automorphism?*

In fact, Cartan answers this question for 3-webs. In *loc. cit.* he establishes that such a web is equivalent to those induced by the 1-forms $dx, dy, dy - u(x + y)dx$, where u is a germ of holomorphic function.

It is very surprising that this story stops here... To our knowledge, there is no other study concerning webs with infinitesimal automorphisms, although they are particularly interesting. Indeed, on the one hand their study is considerably simplified by the presence of an infinitesimal automorphism, but on the other hand, these webs can be very interesting from a geometrical point of view: we will show they are connected to the theory of exceptional webs.

1.3. **Variation of the Rank.** Let $\mathcal{W}$ be a regular web in $(\mathbb{C}^2, 0)$ which admits an infinitesimal automorphism X, *i.e.* X is a germ of vector field whose local flow preserves the foliations of $\mathcal{W}$. As we will see in §2 the Lie derivative $L_X = i_X d + d i_X$ with respect to X induces a linear operator on $\mathcal{A}(\mathcal{W})$. Most of our results will follow from an analysis of such operator.

In §3.1 we use this operator to give a simple description of the abelian relations of $\mathcal{W}$ and from this we will deduce in §3.2 what we consider our main result:

Theorem 1. *Let $\mathcal{W}$ be a k–web which admits a transverse infinitesimal automorphism X. Then*

$$\mathrm{rk}(\mathcal{W} \boxtimes \mathcal{F}_X) = \mathrm{rk}(\mathcal{W}) + (k-1)\,.$$

In particular, $\mathcal{W}$ is of maximal rank if and only if $\mathcal{W} \boxtimes \mathcal{F}_X$ is of maximal rank.

We will derive from Theorem 1 the existence of new families of exceptional webs.

1.4. **New Families of Exceptional Webs.** If we start with a reduced plane curve C invariant under an algebraic $\mathbb{C}^*$-action on $\mathbb{P}^2$ then we obtain a dual algebraic $\mathbb{C}^*$-action on $\check{\mathbb{P}}^2$, letting invariant the algebraic web $\mathcal{W}_C$ (*cf.* §4.1 for details). Combining this construction with Theorem 1 we deduce our second main result.

Theorem 2. *For every $k \geq 5$ there exist a family of dimension at least $\lfloor k/2 \rfloor - 1$ of pairwise non-equivalent exceptional global k-webs on $\mathbb{P}^2$.*

In fact, for each $k \geq 5$, we obtain $4k - 15$ other families of smaller dimension.

Theorem 2 is in sharp contrast with the recent algebrization theorem of Trepreau, completing previous works of Bol and Chern-Griffiths, which says that a maximal rank codimension one k-web is algebrizable when the ambient has dimension at least three and k is sufficiently large. For a precise statement the reader should consult [15].

The classification of the exceptional 5-webs of the type $\mathcal{W} \boxtimes \mathcal{F}_X$ where X is an infinitesimal automorphism of $\mathcal{W}$ follows easily from Theorem 2 (*cf.* Corollary 4.1).

2. Generalities on webs with infinitesimal automorphisms

Let $\mathcal{F}$ be a regular foliation on $(\mathbb{C}^2, 0)$ induced by a (germ of) 1-form ω. We say that a (germ of) vector field X is an infinitesimal automorphism of $\mathcal{F}$ if the foliation $\mathcal{F}$ is preserved by the local flow of X. In algebraic terms: $L_X\omega \wedge \omega = 0$.

When the infinitesimal automorphism X is transverse to $\mathcal{F}$, *i.e* when $\omega(X) \neq 0$, then a simple computation (*cf.* [9, Corollary 2]) shows that the 1-form

$$\eta = \frac{\omega}{i_X\omega}$$

is closed and satisfies $L_X\eta = 0$. By definition, the integral

$$u(z) = \int_0^z \eta$$

is the *canonical first integral* of $\mathcal{F}$ (with respect to X). Clearly, we have $u(0) = 0$ and $L_X(u) = 1$.

Keeping in mind that the local flow of X sends leaves into leaves we can geometrically interpret the first integral $u(z)$ as the time that such local flow takes to transport the leaf through 0 to the leaf through z.

Now let $\mathcal{W}$ be a germ of regular k-web on $(\mathbb{C}^2, 0)$ induced by the (germs of) 1-forms $\omega_1, \ldots, \omega_k$ and let X be an infinitesimal automorphism of $\mathcal{W}$. Here, of course, we mean that X is an infinitesimal automorphism for all the foliations in $\mathcal{W}$.

By hypothesis, we have $L_X\,\omega_i \wedge \omega_i = 0$ for $i = 1, \ldots, k$. Then because the Lie derivative L_X is linear and commutes with d, it induces a linear map

$$\begin{array}{rrcl} (2) \qquad & L_X : \mathcal{A}(\mathcal{W}) & \longrightarrow & \mathcal{A}(\mathcal{W}) \\ & (\eta_1, \ldots, \eta_k) & \longmapsto & (L_X\eta_1, \ldots, L_X\eta_k)\,. \end{array}$$

This map is central in this paper: all our results come from an analysis of the L_X-invariant subspaces of $\mathcal{A}(\mathcal{W})$.

3. Abelian relations of webs with infinitesimal automorphisms

3.1. **Description of $\mathcal{A}(\mathcal{W})$ in presence of an infinitesimal automorphism.** In this section, $\mathcal{W} = \mathcal{F}_1 \boxtimes \cdots \boxtimes \mathcal{F}_k$ denotes a k-web in $(\mathbb{C}^2, 0)$ which admits an infinitesimal automorphism X, regular and transverse to the foliations $\mathcal{F}_i$ in a neighborhood of the origin.

Let $i \in \{1, \ldots, k\}$ be fixed. We note $\mathcal{A}^i(\mathcal{W})$ the vector subspace of $\Omega^1(\mathbb{C}^2, 0)$ spanned by the i-th components α_i of abelian relations $(\alpha_1, \ldots, \alpha_k) \in \mathcal{A}(\mathcal{W})$. If $u_i = \int \eta_i$ denotes the canonical first integral of $\mathcal{F}_i$ with respect to X, then for $\alpha_i \in \mathcal{A}^i(\mathcal{W})$, there exists a holomorphic germ $f_i \in \mathbb{C}\{t\}$ such that $\alpha_i = f_i(u_i)\,du_i$.

Assume now that $\mathcal{A}^i(\mathcal{W})$ is not trivial and let $\left\{\alpha_i^\nu = f_\nu(u_i)\,du_i \,|\, \nu = 1, \ldots, n_i\right\}$ be a basis. Since $L_X : \mathcal{A}^i(\mathcal{W}) \to \mathcal{A}^i(\mathcal{W})$ is a linear map, there exist complex constants $c_{\nu\mu}$ such that, for $\nu = 1, \ldots, n_i$ we have

$$L_X(\alpha_i^\nu) = \sum_{\mu=1}^{n_i} c_{\nu\mu}\, \alpha_i^\mu\ . \tag{3}$$

But $L_X(\alpha_i^\nu) = L_X\big(f_\nu(u_i)\, du_i\big) = X\big(f_\nu(u_i)\big) du_i + f_\nu(u_i)\, L_X\big(du_i\big) = f'_\nu(u_i)\, du_i$ for any ν, so relations (3) are equivalent to the scalar ones

$$f'_\nu = \sum_{\mu=1}^{n_i} c_{\nu\mu}\, f_\mu\,, \qquad \nu = 1, \ldots, n_i\,. \tag{4}$$

Now let $\lambda_1, \ldots, \lambda_\tau \in \mathbb{C}$ be the eigenvalues of the map L_X acting on $\mathcal{A}(\mathcal{W})$ corresponding to maximal eigenspaces with corresponding dimensions $\sigma_1, \ldots, \sigma_\tau$. The differential equations (4) give us the following description of $\mathcal{A}(\mathcal{W})$:

Proposition 3.1. *The abelian relations of $\mathcal{W}$ are of the form*

$$P_1(u_1)\, e^{\lambda_i\, u_1}\, du_1 + \cdots + P_k(u_k)\, e^{\lambda_i\, u_k}\, du_k = 0$$

where $P_1, \ldots, P_k$ are polynomials of degree less or equal to $\sigma_i - 1$.

We will now explain how we can use Proposition 3.1 to effectively determine $\mathcal{A}(\mathcal{W})$. The key point is to determine the possible non-zero eigenvalues of the map (2). Once this is done we can easily determine the abelian relations by simple linear algebra.

We claim that 0 is an eigenvalue of (2) if, and only if, for every germ of vector field Y the Wronskian

$$\det \begin{pmatrix} u_1 & \cdots & u_k \\ Y(u_1) & \cdots & Y(u_k) \\ \vdots & \ddots & \vdots \\ Y^{k-1}(u_1) & \cdots & Y^{k-1}(u_k) \end{pmatrix} \tag{5}$$

is identically zero. In fact, if this is the case then we have two possibilities: the functions $u_1, \ldots, u_k$ are $\mathbb{C}$-linearly dependent or all the leaves of Y are cutted out by some element of the linear system generated by $u_1, \ldots, u_k$, *cf.* [8, theorem 4]. In particular if Y is a vector field of the form $Y = \mu x \frac{\partial}{\partial x} + y \frac{\partial}{\partial y}$, with $\mu \in \mathbb{C} \setminus \mathbb{Q}$, then the leaves of Y accumulate at 0 and are not cutted out by any regular holomorphic function. Therefore the vanishing of (5) implies the existence of an abelian relation of the form

$$\sum c_i u_i = 0,$$

where the c_i's are complex constants.

To determine the possible complex numbers λ which are eigenvalues of the map (2) first notice that the corresponding eigenvectors can be readen as a functional equation of the form $c_1\, e^{\lambda\, u_1} + \cdots + c_k\, e^{\lambda\, u_k} = 0$, where, as before, the c_i's are complex constants: just take the interior product of the displayed equation in Proposition 3.1 with X. In the same spirit of what we have just made for the zero eigenvalue case consider the holomorphic function given by

$$\det \begin{pmatrix} \exp(\lambda u_1) & \cdots & \exp(\lambda u_k) \\ Y(\exp(\lambda u_1)) & \cdots & Y(\exp(\lambda u_k)) \\ \vdots & \ddots & \vdots \\ Y^{k-1}(\exp(\lambda u_1)) & \cdots & Y^{k-1}(\exp(\lambda u_k)) \end{pmatrix} \tag{6}$$

for an arbitrary germ of vector field Y.

Notice that (6) is of the form $\exp(\lambda(u_1 + \cdots + u_k))\lambda^{k-1}P_Y(\lambda)$, where P_Y is a polynomial in λ, of degree at most $\frac{(k-1)(k-2)}{2}$, with germs of holomorphic functions as coefficients. The common constant roots of these polynomials, when Y varies, are exactly the eigenvalues of the map (2).

Let us now exemplify these ideas. In practice we do not have to consider all the vector fields Y but just a suitably chosen one.

Example 3.1. *The k-web $\mathcal{W}$ generated by the functions $f_i(x, y) = y + x^i$, $i = 1, \dots, k$, has no abelian relations.*

Proof. Clearly the vector field $X = \frac{\partial}{\partial y}$ is an infinitesimal automorphism of $\mathcal{W}$ and $X(df_i) = 1$, $i = 1, \dots, k$. It follows that $u_i = f_i$ are the canonical first integrals of $\mathcal{W}$. On the other hand, if we consider the vector field $Y = \frac{\partial}{\partial x}$ we can easily see that that $P_Y(\lambda)|_{x=y=0} = (-1)^{k-1}\prod_{n=1}^{k-1} n!$. Consequently, the only candidate for a eigenvalue of the map (2) is $\lambda = 0$. But clearly the functions f_i are linearly independent over $\mathbb{C}$. □

Let us see how to use this approach to recover the abelian relations of one of the exceptional webs found by the third author in [10]

Example 3.2. *The 5-web $\mathcal{W}$ induced by the functions $x, y, x+y, x-y, x^2+y^2$ has rank 6.*

Proof. Clearly the radial vector field $R = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}$ is an infinitesimal automorphism of $\mathcal{W}$. The canonical first integrals are $u_1 = \log x$, $u_2 = \log y$, $u_3 = \log(x+y)$, $u_4 = \log(x-y)$ and $u_5 = \frac{1}{2}\log(x^2+y^2)$. If we take $Y = x\frac{\partial}{\partial x} - y\frac{\partial}{\partial y}$ then one can easily check that the polynomial P_Y is a complex multiple of

$$x^7y^7\lambda(\lambda-1)^2(\lambda-2)^2(\lambda-4)(\lambda-6).$$

According to Proposition 3.1, we have only to look for abelian relations of the form $\sum_{i=1}^5 P_{i\lambda}(\log f_i) f_i^\lambda \frac{df_i}{f_i} = 0$, for $\lambda = 0, 1, 2, 4, 6$, where $P_{i\lambda}$ are polynomials and $f_i = \exp(u_i)$. We have thus reduced our search to a simple problem of linear algebra, i.e. to find linear dependences on finite dimensional vector spaces indexed by $\lambda \in \{0, 1, 2, 4, 6\}$. It turns out that for $\lambda = 1$ we obtain two linearly independent abelian relations

$$f_1 + f_2 - f_3 = 0, \qquad f_1 - f_2 - f_4 = 0.$$

For $\lambda = 2$ one finds two more independent abelian relations

$$f_1^2 + f_2^2 - f_5^2 = 0, \qquad 2f_1^2 + 2f_2^2 - f_3^2 - f_4^2 = 0.$$

Finally, for $\lambda = 4$ and $\lambda = 6$, respectively, there are two more independent abelian relations:

$$5f_1^4 + 5f_2^4 + f_3^4 + f_4^4 - 6f_5^4 = 0, \qquad 8f_1^6 + 8f_2^6 + f_3^6 + f_4^6 - 10f_5^6 = 0.$$

Thus, we have found 6 independent abelian relations. □

3.2. **Proof of Theorem 1.** With Proposition 3.1 at hand we are able to prove our main result.

Let $\mathcal{W} = \mathcal{F}_1 \boxtimes \cdots \boxtimes \mathcal{F}_k$ and for $i = 1 \dots k$, set $\eta_i = du_i$ as the differential of the canonical first integral of $\mathcal{F}_i$ relatively to X. We note x a first integral of the foliation $\mathcal{F}_X$, normalized such that $x(0) = 0$.

When j varies from 2 to k, we have

$$i_X(\eta_1 - \eta_j) = 0 \qquad \text{and} \qquad L_X(\eta_1 - \eta_j) = 0\,.$$

Consequently there exists $g_j \in \mathbb{C}\{x\}$ such that

$$du_1 - du_j - g_j(x)\,dx = 0\,. \tag{7}$$

Clearly these are abelian relations for the web $\mathcal{W} \boxtimes \mathcal{F}_X$. They span a $(k-1)$-dimensional vector subspace $\mathcal{V}$ of the maximal eigenspace of L_X associated to the eigenvalue zero, noted $\mathcal{A}_0(\mathcal{W} \boxtimes \mathcal{F}_X)$.

Observe that $\mathcal{V}$ fits in the following exact sequence (i is the natural inclusion):

$$0 \to \mathcal{V} \xrightarrow{\ i\ } \mathcal{A}_0(\mathcal{W} \boxtimes \mathcal{F}_X) \xrightarrow{L_X} \mathcal{A}_0(\mathcal{W})\,. \tag{8}$$

Indeed, the kernel $K := \ker\{L_X : \mathcal{A}_0(\mathcal{W} \boxtimes \mathcal{F}_X) \to \mathcal{A}_0(\mathcal{W})\}$ is generated by abelian relations of the form $\sum_{i=1}^k c_i du_i + g(x)\,dx = 0$, where $c_i \in \mathbb{C}$ and $g \in \mathbb{C}\{x\}$. Since $i_X du_i = 1$ for each i, it follows that the constants c_i satisfy $\sum_{i=1}^k c_i = 0$. It implies that the abelian relations in the kernel of L_X can be written as linear combinations of abelian relations of the form (7). Therefore

$$K = \mathcal{V} \tag{9}$$

and consequently $\ker L_X \subset \operatorname{Im} i$. The exactness of (8) follows easily.

From general principles we deduce that the sequence

$$0 \to \frac{\mathcal{V}}{\mathcal{A}_0(\mathcal{W}) \cap \mathcal{V}} \xrightarrow{\ i\ } \frac{\mathcal{A}_0(\mathcal{W} \boxtimes \mathcal{F}_X)}{\mathcal{A}_0(\mathcal{W})} \xrightarrow{L_X} \frac{\mathcal{A}_0(\mathcal{W})}{L_X \mathcal{A}_0(\mathcal{W})}\,,$$

is also exact. Thus to prove the Theorem it suffices to verify the following assertions:

(a) $\mathcal{V}$ is isomorphic to

$$\frac{\mathcal{V}}{\mathcal{A}_0(\mathcal{W}) \cap \mathcal{V}} \oplus \frac{\mathcal{A}_0(\mathcal{W})}{L_X \mathcal{A}_0(\mathcal{W})}\,;$$

(b) the morphism $L_X : \mathcal{A}_0(\mathcal{W} \boxtimes \mathcal{F}_X) \to \mathcal{A}_0(\mathcal{W})$ is surjective;

(c) the vector spaces

$$\frac{\mathcal{A}_0(\mathcal{W} \boxtimes \mathcal{F}_X)}{\mathcal{A}_0(\mathcal{W})} \quad \text{and} \quad \frac{\mathcal{A}(\mathcal{W} \boxtimes \mathcal{F}_X)}{\mathcal{A}(\mathcal{W})}$$

are isomorphic.

To verify assertion (a), notice that the nilpotence of L_X on $\mathcal{A}_0(\mathcal{W})$ implies that $\frac{\mathcal{A}_0(\mathcal{W})}{L_X \mathcal{A}_0(\mathcal{W})}$ is isomorphic to $\mathcal{A}_0(\mathcal{W}) \cap K$. Combined with (9), it implies assertion (a).

To prove assertion (b), it suffices to construct a map $\Phi : \mathcal{A}_0(\mathcal{W}) \to \mathcal{A}_0(\mathcal{W} \boxtimes \mathcal{F}_X)$ such that $L_X \circ \Phi = \mathrm{Id}$. Proposition 3.1 implies that $\mathcal{A}_0(\mathcal{W})$ is spanned by abelian relations of the form $\sum_{i=1}^k c_i u_i^r du_i = 0$, where c_i are complex numbers and r is a non-negative integer. For such an abelian relation, since

$$\sum_{i=1}^k c_i u_i^r du_i = \frac{1}{r+1}\, L_X\Big(\sum_{i=1}^k c_i u_i^{r+1} du_i\Big) = 0\,,$$

there exists an unique $g \in \mathbb{C}\{x\}$ satisfying $\sum_{i=1}^k c_i u_i^{r+1} du_i + g(x)\,dx = 0$. If we set

$$\Phi\Big(\sum_{i=1}^k c_i u_i^r du_i\Big) = \frac{1}{r+1}\,\Big(\sum_{i=1}^k c_i u_i^{r+1} du_i + g(x)\,dx\Big)$$

then $L_X \circ \Phi = \mathrm{Id}$ on $\mathcal{A}_0(\mathcal{W})$ and assertion (b) follows.

To prove assertion (c) we first notice that

$$\mathcal{A}(\mathcal{W} \boxtimes \mathcal{F}_X) = \mathcal{A}_0(\mathcal{W} \boxtimes \mathcal{F}_X) \oplus \mathcal{A}_*(\mathcal{W} \boxtimes \mathcal{F}_X)$$

where $\mathcal{A}_*(\mathcal{W} \boxtimes \mathcal{F}_X)$ denotes the sum of eigenspaces corresponding to non-zero eigenvalues. Of course $\mathcal{A}_*(\mathcal{W} \boxtimes \mathcal{F}_X)$ is invariant and moreover we have the equality

$$L_X\big(\mathcal{A}_*(\mathcal{W} \boxtimes \mathcal{F}_X)\big) = \mathcal{A}_*(\mathcal{W} \boxtimes \mathcal{F}_X)\,.$$

But L_X *kills* the $\mathcal{F}_X$-components of abelian relations. In particular, it implies

$$L_X\big(\mathcal{A}_*(\mathcal{W} \boxtimes \mathcal{F}_X)\big) \subset \mathcal{A}_*(\mathcal{W}).$$

This is sufficient to show that $\mathcal{A}_*(\mathcal{W} \boxtimes \mathcal{F}_X) = \mathcal{A}_*(\mathcal{W})$ and deduce assertion (c) and, consequently that

$$\mathrm{rk}(\mathcal{W} \boxtimes \mathcal{F}_X) = \mathrm{rk}(\mathcal{W}) + (k-1)\,.$$

Because $k(k-1)/2 = (k-1)(k-2)/2 + (k-1)$, the above inequality implies immediately the last assertion of Theorem 1. □

4. New Families of exceptional webs

4.1. **Algebrizable Webs with Infinitesimal Automorphisms.** Let $C \subset \mathbb{P}^2$ be a degree k reduced curve. If $U \subset \check{\mathbb{P}}^2$ is a simply-connected open set not intersecting $\check{C}$ and if $\gamma_1, \ldots, \gamma_k : U \to C$ are the holomorphic maps defined by the intersections of lines in U with C then Abel's Theorem implies that

$$\mathrm{Tr}(\omega) = \sum_{i=1}^{k} \gamma_i^* \omega = 0$$

for every $\omega \in H^0(C, \omega_C)$, where ω_C denotes the dualizing sheaf of C.

The maps γ_i define the k-web $\mathcal{W}_C$ on U and the trace formula above associates an abelian relation of $\mathcal{W}_C$ to each $\omega \in H^0(C, \omega_C)$. Since $h^0(C, \omega_C) = (k-1)(k-2)/2$, the web $\mathcal{W}_C$ is of maximal rank.

Suppose now that C is invariant by a $\mathbb{C}^*$-action $\varphi : \mathbb{C}^* \times \mathbb{P}^2 \to \mathbb{P}^2$. Notice that φ induces a dual action $\check{\varphi} : \mathbb{C}^* \times \check{\mathbb{P}}^2 \to \check{\mathbb{P}}^2$ satisfying $\varphi_t \circ \gamma_i = \gamma_i \circ \check{\varphi}_t$ for $i = 1, \ldots, k$. Consequently the web $\mathcal{W}_C$ admits an infinitesimal automorphism.

In a suitable projective coordinate system $[x : y : z]$, a plane curve C invariant by a $\mathbb{C}^*$-action is cut out by an equation of the form

$$x^{\epsilon_1} \cdot y^{\epsilon_2} \cdot z^{\epsilon_3} \cdot \prod_{i=1}^{k} (x^a + \lambda_i y^b z^{a-b}) \tag{10}$$

where $\epsilon_1, \epsilon_2, \epsilon_3 \in \{0, 1\}$, $k, a, b \in \mathbb{N}$ are such that $k \geq 1$, $a \geq 2$, $1 \leq b \leq a/2$, $\gcd(a, b) = 1$ and the λ_i are distinct non zero complex numbers (*cf.* [1, §1] for instance). Notice that here the $\mathbb{C}^*$-action in question is

$$\begin{array}{cccl} \varphi & : & \mathbb{C}^* \times \mathbb{P}^2 & \to \quad \mathbb{P}^2 \\ & & (t, [x : y : z]) & \mapsto \quad [t^{b(a-b)} x : t^{a(a-b)} y : t^{ab} z]\,. \end{array} \tag{11}$$

Moreover once we fix $\epsilon_1, \epsilon_2, \epsilon_3, k, a, b$ we can always choose $\lambda_1 = 1$ and in this case the set of $k-1$ complex numbers $\{\lambda_2, \ldots, \lambda_k\}$ projectively characterizes the curve C. In particular one promptly sees that there exists a $(d-1)$-dimensional family of degree $2d$ (or $2d+1$) reduced plane curves all projectively distinct and invariant by the same $\mathbb{C}^*$-action: for a given $2d + \delta$ with $\delta \in \{0, 1\}$ set $a = 2$, $b = 1$, $\epsilon_1 = \delta$ and $\epsilon_2 = \epsilon_3 = 0$.

A moment of reflection shows that the number of discrete parameters giving distinct families of degree d curves of the form (10) is

$$\underbrace{\left\lfloor \frac{d}{2} \right\rfloor}_{\epsilon_1=\epsilon_2=\epsilon_3=0} + \underbrace{3\left\lfloor \frac{d-1}{2} \right\rfloor}_{\epsilon_i=\epsilon_j=0,\, \epsilon_k=1} + \underbrace{3\left\lfloor \frac{d-2}{2} \right\rfloor}_{\epsilon_i=\epsilon_j=1,\, \epsilon_k=0} + \underbrace{\left\lfloor \frac{d-3}{2} \right\rfloor}_{\epsilon_1=\epsilon_2=\epsilon_3=1} - 2 = 4d-10\,.$$

Notice that the -2 appears on left hand side because the curves $\{y=0\}$ and $\{z=0\}$ are indistinguishable when $a=2$.

4.2. **Proof of Theorem 2.** If C is a reduced curve of the form (10) then $\mathcal{W}_C$ is invariant by an algebraic $\mathbb{C}^*$-action $\check{\varphi}$. We will note by X the infinitesimal generator of $\check{\varphi}$ and by $\mathcal{F}_X$ the corresponding foliation. From the discussion on the last paragraph, Theorem 2 follows at once from the stronger:

Theorem 4.1. *If* $\deg C \geq 4$ *then* $\mathcal{W}_C \boxtimes \mathcal{F}_X$ *is exceptional. Moreover if* C' *is another curve invariant by* φ *then* $\mathcal{W}_C \boxtimes \mathcal{F}_X$ *is analytically equivalent to* $\mathcal{W}_{C'} \boxtimes \mathcal{F}_X$ *if and only if the curve* C *is projectively equivalent to* C'.

Proof. Since $\mathcal{W}_C$ has maximal rank it follows from Theorem 1 that $\mathcal{W}_C \boxtimes \mathcal{F}_X$ is also of maximal rank. Suppose that its localization at a point $p \in \mathbb{P}^2$ is algebrizable and let $\psi : (\mathbb{P}^2, p) \to (\mathbb{C}^2, 0)$ be a holomorphic algebrization. Since both $\mathcal{W}_C$ and $\psi_*(\mathcal{W}_C)$ are linear webs of maximal rank it follows from a result of Nakai [7] that ψ is the localization of an automorphism of $\mathbb{P}^2$. But the generic leaf of $\mathcal{F}_X$ is not contained in any line of $\mathbb{P}^2$ and consequently $\psi_*(\mathcal{W} \boxtimes \mathcal{F}_X)$ is not linear. This concludes the proof of the theorem. □

Remark 4.1. We do not know if the families above are *irreducible* in the sense that the generic element does not admit a deformation as an exceptional web that is not contained in the family. Due to the presence of automorphism one could imagine that they are indeed degenerations of some other exceptional webs.

4.3. **A characterization result.** Combining Theorem 1 with Lie's Theorem we can easily prove the

Corollary 4.1. *Let* $\mathcal{W}$ *be a 4-web that admits a transverse infinitesimal automorphism* Y. *If* $\mathcal{W} \boxtimes \mathcal{F}_Y$ *is exceptional then it is analytically equivalent to an exceptional 5-web* $\mathcal{W}_C \boxtimes \mathcal{F}_X$ *described in Theorem 4.1.*

Proof. It follows from Theorem 1 that $\mathcal{W}$ is of maximal rank. Lie's Theorem implies that $\mathcal{W}$ is analytically equivalent to $\mathcal{W}_C$ for some reduced plane quartic C. Since the local flow of Y preserves $\mathcal{W}$ there exists a (germ) of vector field X whose local flow preserves $\mathcal{W}_C$. Using again Nakai's result we deduce that the germs of automorphisms on the local flow of X are indeed projective automorphisms. This is sufficient to conclude that X is a global vector field preserving $\mathcal{W}_C$. □

Remark 4.2. Theorem 4.1 does not give all the exceptional webs admitting an infinitesimal automorphism. As we have seen in the Example 3.2, the web $\mathcal{W}$ induced by the functions $x, y, x+y, x-y, x^2+y^2$ is exceptional and it admits the radial vector field $R = x\,\partial/\partial_x + y\,\partial/\partial_y$ as a transverse infinitesimal automorphism. Theorem 1 implies that the 6-web $\mathcal{W} \boxtimes \mathcal{F}_R$ is also exceptional. This result was previously obtained by determining an explicit basis of the space of abelian relations, see [11, p. 253].

5. Problems

5.1. **A conjecture about the nature of the abelian relations.** It is clear from Proposition 3.1 that for webs $\mathcal{W}$ admitting infinitesimal automorphisms there exists a Liouvillian extension of the field of definition of $\mathcal{W}$ containing all its abelian relations. We believe that a similar statement should hold for arbitrary webs $\mathcal{W}$.

Conjecture 5.1. *The abelian relations of a web $\mathcal{W}$ are defined on a Liouvillian extension of the definition field of $\mathcal{W}$.*

Our belief is supported by the recent works of Hénaut [6] and Ripoll [13] on abelian relations and of Casale [3] on non-linear differential Galois Theory.

When $\mathcal{W}$ is of maximal rank the main result of [6] shows that there exists a Picard-Vessiot extension of the field of definition of $\mathcal{W}$ containing all the abelian relations. In the general case, one should be able to deduce a similar result from the above mentioned work of Ripoll.

On the other hand, and at least over polydiscs, [3, Theorem 6.4] implies that the foliations with first integrals on Picard-Vessiot extension are transversely projective. Since the first integrals in question are components of abelian relations they are of finite determinacy and hopefully this should imply that they are indeed Liouvillian.

5.2. **Restricted Chern's Problem.** With the techniques now available, the classification of all exceptional 5-webs ("*Chern's problem*") seems completely out of reach. So we propose the

Problem 5.1. *Classify exceptional 5-webs admitting infinitesimal automorphisms.*

Notice that this restricted version is not completely hopeless. The linear map L_X can be "integrated" giving birth to a holomorphic action on $\mathbb{P}(\mathcal{A}(\mathcal{W}))$. The Poincaré-Blaschke curves will be orbits of this action and the dual action will induce an automorphism of the associated Blaschke surface. This seems valuable extra data that may lead to a solution of the restricted Chern's problem.

For a definition of the above mentioned concepts see [11, Chapter 8].

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