On the return time function along monodromic polycycles

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Abstract. In this paper we study the period function of centers of planar polynomial differential systems. With a convenient compactification of the phase portrait, the boundary of the period annulus of the center has two connected components: the center itself and a polycycle. We are interested in the behaviour of the period function near the polycycle. The desingularization of its critical points gives rise to a new polycycle (monodromic as well) with hyperbolic saddles or saddle-nodes at the vertices. In this paper we compute the first terms in the asymptotic development of the time function around any saddle that may come from this desingularization process. In addition, we use these developments to study the bifurcation diagram of the period function of the dehomogenized Loud's centers. More generally, the tools developed here can be used to study the return time function around a monodromic polycycle. This work is a continuation of the results in [7, 8].

1 Introduction and setting of the problem

The present paper deals with planar polynomial differential systems and our goal is to develop tools in order to study the qualitative properties of the period function of a center. Although this is perhaps the most natural framework of our work, it will be clear later that the results can be applied in more general settings. Recall that a critical point p of a planar differential system is a *center* if it has a punctured neighbourhood that consists entirely of periodic orbits surrounding p. The largest punctured neighbourhood with this property is called the *period annulus* of the center and in what follows it will be denoted by $\mathcal{P}$. The *period function* of the center assigns to each periodic orbit in $\mathcal{P}$ its period. Questions related to the behaviour of the period function have been extensively studied. Let us quote, for instance, the problems of isochronicity (see [4, 5, 6]), monotonicity (see [1, 2, 11]) or bifurcation of critical periods (see [3, 10, 13]).

Compactifying the phase portrait in $\mathbb{RP}^2$, the boundary of $\mathcal{P}$ has two connected components: the center itself and a polycycle. We call them respectively the *inner* and *outer boundary* of the period annulus. In this paper we are interested in the behaviour of the period function near the outer boundary. The vertices of the outer boundary are critical points with a hyperbolic sector inside $\mathcal{P}$ and, in case of unbounded period annuli, some of them are located at infinity. Note in addition that the polycycle may have degenerated critical points and then it is necessary to desingularize them by means of a blow-up process. One obtains in this way a desingularized polycycle with hyperbolic saddles or saddle-nodes at its vertices. This allows to reduce the study of the period function to a local problem, namely the time function associated to the passage around a saddle or a saddle-node. In this paper we consider the time function around any saddle that may come from this blow-up process. Taking local coordinates on the separatrices of such a saddle, the desingularized vector field writes as

$$X(x,y)=\frac{1}{x^m y^n}\big(xP(x,y)\partial_x+yQ(x,y)\partial_y\big)$$

with $m,n\in\mathbb{Z}$. The result that we obtain extends a previous one [7] that treats the case $m=0$ and $n\in\mathbb{N}$, which is useful basically to study only those period annuli such that its outer boundary has all the vertices at infinity and being hyperbolic saddles (in particular, such that no blow-up process is needed). As in that paper and since it will be important for subsequent applications, we suppose that the vector field X depends on a parameter $\mu\in\Lambda\subset\mathbb{R}^k$. Most part of this paper is devoted to compute the first terms in the asymptotic development of the time function (see Figure 2) associated to the passage around the saddle of the family $\{X_\mu,\,\mu\in\Lambda\}$. The development that we obtain is uniform with respect to the parameter and this is important to remark because this property is essential to investigate the bifurcation diagram of the period function in a family of centers.

The above study was motivated by the necessity of such a development for the investigation of the period function of the dehomogenized Loud's centers, namely

$$\begin{cases}\dot{x}=-y+xy,\\ \dot{y}=x+Dx^2+Fy^2.\end{cases}\tag{1}$$

At this point, in order to put our study in context, we must recall the results in [8] and to this end some definitions are needed. The period function of a center is *monotonous increasing* (respectively, *decreasing*) if for any pair of periodic orbits inside $\mathcal{P}$, say γ_1 and γ_2 with $\gamma_1\subset\mathrm{Int}(\gamma_2)$, we have that the period of γ_2 is greater (respectively, smaller) than the one of γ_1. (Here by $\mathrm{Int}(\gamma)$ we mean the bounded connected component of $\mathbb{R}^2\setminus\{\gamma\}$.) It is important to note that the period function is defined on the set of periodic orbits in $\mathcal{P}$. So usually the first step is to parametrize this set, let us say $\{\gamma_s\}_{s\in(0,1)}$, and then one can study the qualitative properties of the period function by means of the map $s\longmapsto$ period of γ_s, which is smooth on $(0,1)$. The *critical periods* are the critical points of this function and its number, character (maximum or minimum) and distribution do not depend on the particular parametrization of the set of periodic orbits used. In case that the differential system depends on a parameter $\mu\in\Lambda$, as it occurs with (1), then the problem is to obtain the *bifurcation diagram* of the period function of the center. That is, to decompose the parameter space as $\Lambda=\cup V_i$ in such a way that if μ_1 and μ_2 belong to the same set V_i then the corresponding period functions are qualitatively the same. (With this we mean that their critical periods are equal in number, character and distribution.) The *bifurcation values* are the boundaries of the sets V_i (roughly speaking, those parameters $\mu_0\in\Lambda$ for which some critical period emerges or disappears as μ tends to μ_0) and there are three different cases to consider:

(a) Bifurcations of critical periods from the inner boundary (i.e., the center).

(b) Bifurcations of critical periods from the interior of the period annulus.

(c) Bifurcations of critical periods from the outer boundary (i.e., the polycycle).

The interested reader is referred to [8] for precise definitions. Chicone and Jacobs [3] described completely the bifurcation of critical periods from the inner boundary for the whole quadratic family. The bifurcations from the outer boundary for the subfamily (1) are studied in [8]. Let us recall the main result in that paper and to this end denote by Γ_U the union of dotted straight lines in Figure 1. Consider also the bold curve Γ_B. (Here the subscripts B and U stand for bifurcation and unspecified respectively.) Note in particular that Γ_B is a Jordan curve. We can consider therefore the bounded and unbounded components of $\mathbb{R}^2\setminus\Gamma_B$, which we denote by $\mathcal{D}_B$ and $\mathcal{I}_B$ (for decreasing and increasing) respectively. With this notation, the main result in [8] is the following:

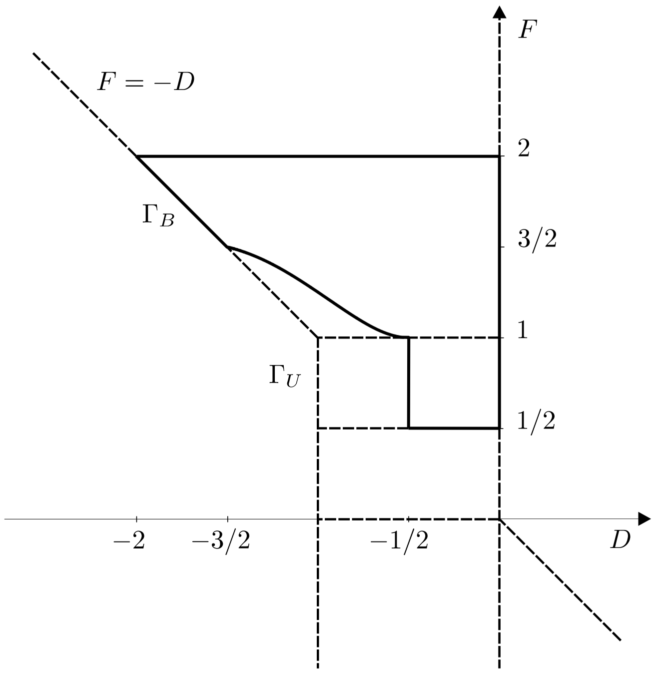

Figure 1: Bifurcation diagram of the period function at the outer boundary

Theorem 1.1 (Mardešić-Marín-Villadelprat). *Denoting $\mu = (D, F)$, let $\{X_\mu,\ \mu \in \mathbb{R}^2\}$ be the family of vector fields in* (1) *and consider the period function of the center at the origin. Then the open set $\mathbb{R}^2 \setminus \{\Gamma_B \cup \Gamma_U\}$ corresponds to regular values of the period function at the outer boundary of the period annulus. In addition,*

(*a*) *If $\mu_0 \in \mathcal{I}_B \setminus \Gamma_U$ then the period function of X_{μ_0} is monotonous increasing near the outer boundary.*

(*b*) *If $\mu_0 \in \mathcal{D}_B \setminus \Gamma_U$ then the period function of X_{μ_0} is monotonous decreasing near the outer boundary.*

Finally, the parameters in Γ_B are bifurcation values of the period function at the outer boundary of the period annulus.

The curve Γ_U corresponds, except for the segment $\left(-1, -\frac{1}{2}\right) \times \{\frac{1}{2}\}$, to parameters for which the corresponding period annulus has degenerate critical points at its outer boundary. The blow-up process of these critical points leads to hyperbolic saddles, but the tools developed in [7] are not general enough to study the associated time functions. We conjectured however that the parameters in Γ_U are not bifurcation values except for the the segment $\{0\} \times \left[0, \frac{1}{2}\right]$. The results obtained in the present paper allow us to show that this is indeed the case for half of the segment. More concretely, we prove the following:

Theorem A. *Denoting $\mu = (D, F)$, let $\{X_\mu,\ \mu \in \mathbb{R}^2\}$ be the family of vector fields in* (1) *and consider the period function of the center at the origin.*

(*a*) *If $\mu_0 \in \{0\} \times \left(0, \frac{1}{4}\right)$ then the period function of X_{μ_0} is monotonous increasing near the outer boundary.*

(*b*) *If $\mu_0 \in \{0\} \times \left(\frac{1}{4}, \frac{1}{2}\right)$ then the period function of X_{μ_0} is monotonous decreasing near the outer boundary.*

Moreover the parameters in $\{0\} \times \left[\frac{1}{4}, \frac{1}{2}\right]$ are bifurcation values of the period function at the outer boundary of the period annulus.

It remains of course to show that the segment $\{0\}\times\left[0,\frac{1}{4}\right]$ consists of bifurcation values as well and, even more difficult, that the rest of the parameters in Γ_U are not. The machinery developed here will be very useful to tackle this second issue because the key point to verify that a parameter is not a bifurcation value (i.e., it is a regular value) is to use developments that are uniform with respect to parameters.

The paper is organized in the following way. Section 2 is devoted to prove Theorem 2.7, which provides the first order development of the time function around a hyperbolic saddle. This result contemplates all the possible cases, in the sense that we consider any saddle that may come from the desingularization of a monodromic polycycle. We introduce moreover the notation and definitions used henceforth. In Section 3 we obtain higher order developments of this time function, but only those cases required for the proof of Theorem A are considered. More precisely, Teorems 3.1 and 3.3 give respectively the second and third order developments. Finally in Section 4 we prove Theorem A.

2 First order development

Let W be an open set of $\mathbb{R}^k$ and consider an analytic family of meromorphic vector fields $\{X_\mu,\, \mu\in W\}$ of the form

$$X_\mu(x,y)=\frac{1}{x^m y^n}\big(xP(x,y;\mu)\partial_x+yQ(x,y;\mu)\partial_y\big) \tag{2}$$

with $m,n\in\mathbb{Z}$. We also assume that P and Q are analytic functions on $V\times W$, where V is an open set of $\mathbb{R}^2$ containing the origin, and that verify $P(x,0;\mu)>0$ and $Q(0,y;\mu)<0$. Note then that, for each $\mu\in W$, $x^m y^n X_\mu(x,y)$ is an analytic vector field on V that has a hyperbolic saddle at the origin with *hyperbolicity ratio* given by

$$\lambda(\mu):=-\frac{Q(0,0;\mu)}{P(0,0;\mu)}>0.$$

The family $\{X_\mu,\, \mu\in W\}$ can be thought as a single vector field Y defined on $V\times W\subset\mathbb{R}^{2+k}$ whose trajectories lie on the submanifolds $\{\mu=\text{const}\}$. Let $\sigma: I\times W\longrightarrow\Sigma_\sigma$ and $\tau: I\times W\longrightarrow\Sigma_\tau$ be two analytic transverse sections to Y defined by

$$\sigma(s;\mu)=\big(\sigma_1(s;\mu),\sigma_2(s;\mu);\mu\big)\ \text{ and }\ \tau(s;\mu)=\big(\tau_1(s;\mu),\tau_2(s;\mu);\mu\big)$$

such that $\sigma_1(0;\mu)=0$ and $\tau_2(0;\mu)=0$. (Here I stands for a small interval of $\mathbb{R}$ containing 0.) We denote the Dulac and time mappings between the transverse sections Σ_σ and Σ_τ by R and T respectively. More precisely (see Figure 2), if $\varphi\big(t,(x_0,y_0);\mu\big)$ is the solution of X_μ passing through (x_0,y_0) at $t=0$, for each $s>0$ we define $R(s;\mu)$ and $T(s;\mu)$ by means of the relation

$$\varphi\big(T(s;\mu),\sigma(s);\mu\big)=\tau\big(R(s;\mu)\big). \tag{3}$$

Definition 2.1 We say that $\{X_\mu,\, \mu\in W\}$ verifies the *family linearization property* (FLP in short) if there exist an open set $U\subset\mathbb{R}^2$ containing the origin and an analytic *local* diffeomorphism $\Phi: U\times W\to V\times W$ of the form $\Phi(x,y;\mu)=\big(x+\text{h.o.t.},y+\text{h.o.t.},\mu\big)$ such that

$$X_\mu=\Phi_*\left(\frac{1}{f(x,y;\mu)}\big(x\partial_x-\lambda(\mu)y\partial_y\big)\right),$$

where f is an analytic function on $U\times W$. □

Remark 2.2 Since, by assumption, the invariant manifolds of the saddle point are located on the axes, from Definition 2.1 it follows easily that

$$\Phi_1(x,y;\mu)=x\psi_1(x,y;\mu)\text{ and }\Phi_2(x,y;\mu)=y\psi_2(x,y;\mu)$$

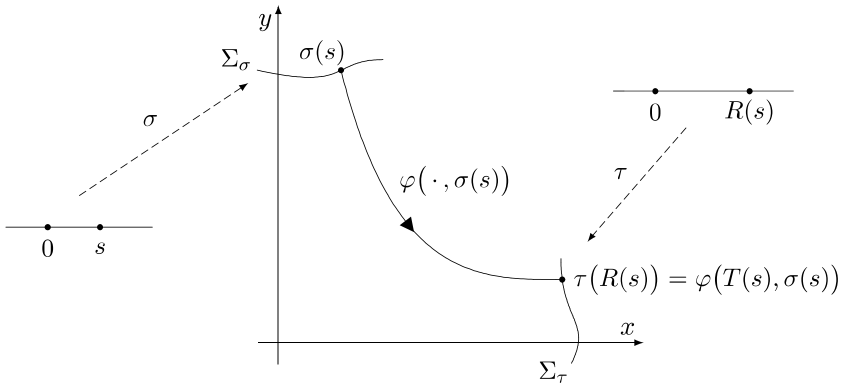

Figure 2: Definition of T in Theorem 2.7.

with $\psi_i(0,0;\mu) \equiv 1$. In addition, $f(x,y;\mu) = x^m y^n g(x,y;\mu)$ where g is an analytic function verifying that $g(0,0;\mu) \neq 0$. $\square$

Remark 2.3 It is easy to show that the family of meromorphic vector fields $\{X_\mu,\, \mu \in W\}$ defined in (2) verifies FLP if it has a Darboux first integral

$$H_\mu(x,y) = f_1(x,y;\mu)^{\beta_1(\mu)} \cdots f_k(x,y;\mu)^{\beta_k(\mu)},$$

where $f_j \in \mathcal{C}^\omega(U \times W)$ for some open set $U \subset \mathbb{R}^2$ containing the origin and $\beta_j \in \mathcal{C}^\omega(W)$. $\square$

Definition 2.4 Let W be any open subset of $\mathbb{R}^k$. We denote by $\mathcal{I}(W)$ the set of germs of analytic functions $h(s;\mu)$ defined on $(0,\varepsilon) \times W$ for some $\varepsilon > 0$ such that

$$\lim_{s\to 0} h(s;\mu) = 0 \text{ and } \lim_{s\to 0} s\,\frac{\partial h(s;\mu)}{\partial s} = 0$$

uniformly (on μ) on every compact subset of W. We denote moreover by $\mathcal{I}_0(W)$ the set of germs of analytic functions $h(s;\mu)$ defined on $(-\varepsilon,\varepsilon) \times W$ for some $\varepsilon > 0$ verifying that $h(0;\mu) \equiv 0$. $\square$

Remark 2.5 It is clear that $\mathcal{I}(W)$ is closed under the addition and product. Moreover, $\mathcal{I}_0(W) \subset \mathcal{I}(W)$. Note finally that if $f \in \mathcal{I}(U) \cap \mathcal{I}(V)$, where U and V are two open subsets of $\mathbb{R}^k$, then $f \in \mathcal{I}(U \cup V)$. $\square$

Definition 2.6 The function defined for $s > 0$ and $\alpha \in \mathbb{R}$ by means of

$$\omega(s;\alpha) = \begin{cases} \frac{s^{\alpha-1}-1}{\alpha-1} & \text{if } \alpha \neq 1, \\ \log s & \text{if } \alpha = 1, \end{cases}$$

is called the *Roussarie-Ecalle compensator* [9]. $\square$

In order to simplify the expressions that appear in the statement of the next result we introduce the functions

$$L(u;\mu) := \exp\left(\int_{\sigma_2(0)}^{u} \left(\frac{P(0,y)}{Q(0,y)} + \frac{1}{\lambda}\right)\frac{dy}{y}\right),$$

$$M(u;\mu) := \exp\left(\int_{0}^{u} \left(\frac{Q(x,0)}{P(x,0)} + \lambda\right)\frac{dx}{x}\right),$$

and the covering of the parameter space W given by the open subsets

$$W_1 := \big\{\mu \in W : m - \lambda(\mu)n < 0\big\},$$
$$W_2 := \big\{\mu \in W : m - \lambda(\mu)n > 0\big\}, \tag{4}$$
$$W_3 := \big\{\mu \in W : -1 < m - \lambda(\mu)n < \lambda(\mu)\big\}.$$

Theorem 2.7 (First order development). *Let $\{X_\mu,\, \mu \in W\}$ be the family of vector fields defined in* (2) *and assume that it verifies the FLP. Let T be the time function associated to the transverse sections Σ_σ and Σ_τ as introduced in* (3)*. Then the following holds:*

(*a*) *If $\mu \in W_1$ then $T(s;\mu) = s^m\big(\Delta_1(\mu) + \mathcal{I}(W_1)\big)$, where*

$$\Delta_1(\mu) = \sigma_1'(0)^m \sigma_2(0)^{\frac{m}{\lambda}} \int_{\sigma_2(0)}^{0} \frac{L(x)^m x^{n-\frac{m}{\lambda}}}{Q(0,x)} \frac{dx}{x}.$$

(*b*) *If $\mu \in W_2$ then $T(s;\mu) = s^{\lambda n}\big(\Delta_2(\mu) + \mathcal{I}(W_2)\big)$, where*

$$\Delta_2(\mu) = \big(\sigma_1'(0)^\lambda \sigma_2(0) L(0)^\lambda\big)^n \int_0^{\tau_1(0)} \frac{M(x)^n x^{m-\lambda n}}{P(x,0)} \frac{dx}{x}.$$

(*c*) *If $\mu \in W_3$ then $T(s;\mu) = s^{\lambda n}\big(\Delta_3(\mu)\omega(s;m+1-\lambda n) + \Delta_4(\mu) + \mathcal{I}(W_3)\big)$, where $\Delta_3(\mu)$ and $\Delta_4(\mu)$ are analytic on W_3. Furthermore, if $m - \lambda(\mu_0)n = 0$ then*

$$\Delta_3(\mu_0) = -\frac{\big(\sigma_1'(0)^\lambda L(0)^\lambda \sigma_2(0)\big)^n}{P(0,0)}.$$

In many situations henceforth we shall study the expansion of the composition or product of two given functions. By applying the next result (see [7]) we shall obtain the corresponding remainder terms.

Lemma 2.8. *Let a, k and r be analytic functions on W and let $f(s;\mu)$ and $g(s;\mu)$ be analytic functions on $(0,\varepsilon) \times W$ for some $\varepsilon > 0$. Assume furthermore that $a(\mu)$ and $r(\mu)$ are positive on W and define $\varphi(s;\mu) := s^{r(\mu)}\big(a(\mu) + f(s;\mu)\big)$.*

(*a*) *If $f \in \mathcal{I}$ and $g \in \mathcal{I}_0$ then $g \circ f \in \mathcal{I}$.*

(*b*) *If $f \in \mathcal{I}$ (respectively $\mathcal{I}_0$) then $s^k \circ \varphi - a^k s^{kr}$ belongs to $s^{kr}\mathcal{I}$ (respectively $s^{kr}\mathcal{I}_0$).*

(*c*) *If $f, g \in \mathcal{I}$ then $(s^k g) \circ \varphi$ belongs to $s^{kr}\mathcal{I}$.*

(*d*) *If $g \in \mathcal{I}_0$ then $g\omega(s;r) \in \mathcal{I}$.*

(*e*) *If $g \in \mathcal{I}_0$ then $\big(s\omega(s;r)\big) \circ \big(s(a+g)\big) = s\big(a^r \omega(s;r) + a\omega(a;r) + \mathcal{I}\big)$.*

In the statement of the above result in [7] it is also required that k is positive. Let us remark however that the proof follows exactly the same way without this assumption. This is not the case of the following result, that will be applied several times in what follows.

Corollary 2.9. *Let k and r be analytic functions on W. If $k(\mu) > 0$ and $k(\mu) + r(\mu) > 1$ then $s^k\omega(s;r) \in \mathcal{I}$.*

Proof. An easy manipulation shows that

$$s^k\omega(s;r) = \tfrac{1}{k}\, s^k\omega\big(s^k; \tfrac{r+k-1}{k}\big) = \tfrac{1}{k}\,\big(s\omega(s;\tfrac{r+k-1}{k})\big) \circ s^k.$$

Therefore $s^k\omega(s;r) = h(s^k)$ with $h(s;\mu) := \frac{1}{k} s\omega\big(s;\frac{r+k-1}{k}\big)$, which belongs to $\mathcal{I}$ by (*d*) in Lemma 2.8 since by assumption it holds $\frac{r+k-1}{k} > 0$. Accordingly, $s^k\omega(s;r) = h \circ \varphi$ with $h \in \mathcal{I}$ and, taking $f(s;\mu) \equiv 0$, $\varphi(s;\mu) := s^k\big(1 + f(s;\mu)\big)$. Since it is obvious that $f \in \mathcal{I}$, by (*c*) in Lemma 2.8 with $k = 0$ we conclude that $s^k\omega(s;r) = h \circ \varphi$ belongs to $\mathcal{I}$. ■

Consider next an analytic family of meromorphic vector fields of the form

$$Y_\mu = \frac{1}{x^m y^n G(x,y;\mu)}\big(x\partial_x - \lambda(\mu)y\partial_y\big),$$

where $m,n \in \mathbb{Z}$, G is an analytic function and $\lambda(\mu)>0$ for all $\mu \in W$. Let $V(s;\mu)$ be the time that spends the solution of Y_μ passing through $(s,1)\in\mathbb{R}^2$ with $s>0$ to reach $\{x=1\}$. It is clear that

$$V(s;\mu) = \int_{\mathcal{C}} x^m y^n G(x,y;\mu)\frac{dx}{x},$$

where $\mathcal{C}_{(s,\mu)} := \{(x,y) : y=(s/x)^{\lambda(\mu)}, s\le x\le 1\}$. The following result provides the first order development of $V(s;\mu)$ at $s=0$ for $\mu\in W$ and the expression of its leading coefficient. In addition, since it will be necessary for the subsequent application, we also give the second order expansion for the case $\mu \in W_1$ (see (4) for the definition). With this aim in view we introduce the covering of W_1 given by the open subsets

$$(5)\qquad \begin{aligned} W_{11} &:= \big\{\mu\in W_1 : \lambda(\mu)n-m>1\big\},\\ W_{12} &:= \big\{\mu\in W_1 : \lambda(\mu)n-m<1\big\},\\ W_{13} &:= \big\{\mu\in W_1 : 2>\lambda(\mu)n-m>1-\lambda(\mu)\big\}. \end{aligned}$$

Proposition 2.10. *With the above definitions, the following holds:*

(a) *If $\mu\in W_1$ then $V(s;\mu)=s^m\big(a_1(\mu)+f_1(s;\mu)\big)$, where $f_1\in\mathcal{I}(W_1)$ and $a_1(\mu)=\int_0^1 u^{\lambda n-m}G(0,u^\lambda)\frac{du}{u}$. Moreover the remainder term is given by*

$$f_1(s;\mu) = \begin{cases} s\big(a_{11}(\mu)+\mathcal{I}(W_{11})\big) & \text{if } \mu\in W_{11},\\ s^{\lambda n-m}\big(a_{12}(\mu)+\mathcal{I}(W_{12})\big) & \text{if } \mu\in W_{12},\\ s\big(a_{13}(\mu)\omega(s;\lambda n-m)+a_{14}(\mu)+\mathcal{I}(W_{13})\big) & \text{if } \mu\in W_{13}.\end{cases}$$

(b) *If $\mu\in W_2$ then $V(s;\mu)=s^{\lambda n}\big(a_2(\mu)+\mathcal{I}(W_2)\big)$, where $a_2(\mu)=\int_0^1 u^{m-\lambda n}G(u,0)\frac{du}{u}$.*

(c) *If $\mu\in W_3$ then $V(s;\mu)=s^{\lambda n}\big(a_3(\mu)\omega(s;m+1-\lambda n)+a_4(\mu)+\mathcal{I}(W_3)\big)$, where a_3 and a_4 are analytic functions on W_3. Moreover, if $m-\lambda(\mu_0)n=0$ then $a_3(\mu_0)=-G(0,0)$.*

Proof. The idea to show this is to take advantage of a similar result proved in [7] that holds for $m=0$ and $n>0$. Let us consider first the case $\mu\in W_1$, i.e., $\lambda(\mu)n-m>0$. To apply the above-mentioned result we write the function as

$$V(s;\mu)=\int_{\mathcal{C}} x^m y^n G(x,y;\mu)\frac{dx}{x} = s^m\int_{\mathcal{C}} y^{n-m/\lambda}G(x,y;\mu)\frac{dx}{x}.$$

(Here we use that $y=(s/x)^\lambda$ on $\mathcal{C}$.) Then by applying Theorem 3.3 in [7] with $\widehat{n}=n-m/\lambda$, which is positive due to $\mu\in W_1$, it follows that

$$V(s;\mu)=\begin{cases} s^m\big(a_1(\mu)+a_{11}(\mu)s+sf_{11}(s;\mu)\big) & \text{if } \mu\in W_{11},\\ s^m\big(a_1(\mu)+a_{12}(\mu)s^{\lambda n-m}+s^{\lambda n-m}f_{12}(s;\mu)\big) & \text{if } \mu\in W_{12},\\ s^m\big(a_1(\mu)+a_{13}(\mu)\,s\omega(s;\lambda n-m)+a_{14}(\mu)s+sf_{13}(s;\mu)\big) & \text{if } \mu\in W_{13},\end{cases}$$

where $f_{1i}\in\mathcal{I}(W_{1i})$ for $i=1,2,3$ and $a_1(\mu)=\int_0^1 u^{\lambda n-m}G(0,u^\lambda)\frac{du}{u}$. The "second order" coefficients also follow from that result. For instance,

$$a_{12}(\mu)=\frac{G(0,0)}{m-\lambda n}+\int_0^1\frac{G(u,0)-G(0,0)}{u^{\lambda n-m}}\frac{du}{u}.$$

We claim that $V(s;\mu)=s^m\big(a_1(\mu)+\mathcal{I}(W_1)\big)$ and note that (a) will follow once we prove this. The fact that s and $s^{\lambda n-m}$ belong to $\mathcal{I}(W_1)$ is obvious and, by (d) in Lemma 2.8, this is also the case of $s\omega(s;\lambda n-m)$. Finally, since $f_{1i}\in\mathcal{I}(W_{1i})$ and $W_1=W_{11}\cup W_{12}\cup W_{13}$, the claim follows from Remark 2.5.

Let us consider next (b), which corresponds to $\mu\in W_2$, i.e., $m-\lambda(\mu)n>0$. It follows by applying (a) to the vector field $\widehat{Y}_\mu:=-\varphi_*\big(Y_\mu\big)$ with $\varphi(x,y)=(y,x)$. Indeed, following the obvious notation, it turns out that $V(s;\mu)=\widehat{V}(s^\lambda;\mu)$ and one can check that $\widehat{G}(x,y)=\frac{1}{\lambda}G(y,x)$, $\big(\widehat{m},\widehat{n}\big)=(n,m)$ and $\widehat{\lambda}=1/\lambda$. Accordingly, since $\widehat{\lambda}\widehat{n}-\widehat{m}=m/\lambda-n$ is positive for $\mu\in W_2$, we can take advantage of (a) to conclude that $\widehat{V}(s;\mu)=s^n\big(\widehat{a}_1(\mu)+f(s;\mu)\big)$ with $f\in\mathcal{I}(W_2)$ and

$$\widehat{a}_1(\mu)=\frac{1}{\lambda}\int_0^1 u^{m/\lambda-n}G\big(u^{1/\lambda},0\big)\frac{du}{u}=\int_0^1 u^{m-\lambda n}G(u,0)\frac{du}{u}.$$

Hence, due to $f(s^\lambda;\mu)\in\mathcal{I}(W_2)$ by (c) in Lemma 2.8, we have that $V(s;\mu)=\widehat{V}(s^\lambda;\mu)=s^{\lambda n}\big(\widehat{a}_1(\mu)+\mathcal{I}(W_2)\big)$.

To prove (c) we take two analytic functions G_1 and G_2 so that $G(x,y)=G(0,0)+xG_1(x,y)+yG_2(x,y)$. This enables us to decompose the function under consideration as $V(s;\mu)=V_0(s;\mu)+V_1(s;\mu)+V_2(s;\mu)$, where $V_2(s;\mu):=\int_{\mathcal{C}}x^my^{n+1}G_2(x,y)\frac{dx}{x}$, $V_1(s;\mu):=\int_{\mathcal{C}}x^{m+1}y^nG_1(x,y)\frac{dx}{x}$ and

$$V_0(s;\mu):=G(0,0)\int_{\mathcal{C}}x^my^n\frac{dx}{x}=G(0,0)s^{\lambda n}\int_s^1 x^{m-\lambda n}\frac{dx}{x}=-G(0,0)s^{\lambda n}\omega(s;m+1-\lambda n). \tag{6}$$

Note that we can apply (b) with $\big(\widehat{m},\widehat{n}\big)=(m+1,n)$ to study V_1 since $m+1-\lambda n>0$ for $\mu\in W_3$. Hence

$$V_1(s;\mu)=s^{\lambda n}\big(b_1(\mu)+\mathcal{I}(W_3)\big). \tag{7}$$

On the other hand, since $m-\lambda(n+1)<0$ for $\mu\in W_3$, we can apply (a) with $\big(\widehat{m},\widehat{n}\big)=(m,n+1)$ to study V_2. We obtain in this way $V_2(s;\mu)=s^m\big(b_2(\mu)+\mathcal{I}(W_3)\big)$. However the first order development of V_2 does not suffices for our purpose because we need to show that

$$V_2(s;\mu)=b_2(\mu)s^m+s^{\lambda n}g(s;\mu)\text{ with }g\in\mathcal{I}(W_3). \tag{8}$$

The expression of g follows by applying the second part of (a) with $\big(\widehat{m},\widehat{n}\big)=(m,n+1)$. Indeed, setting

$$\begin{aligned}W_{31}&=\big\{\mu\in W_3:\lambda(\mu)(n+1)-m>1\big\},\\ W_{32}&=\big\{\mu\in W_3:\lambda(\mu)(n+1)-m<1\big\},\\ W_{33}&=\big\{\mu\in W_3:2>\lambda(\mu)(n+1)-m>1-\lambda(\mu)\big\},\end{aligned}$$

one can verify that it is given by

$$g(s;\mu)=\begin{cases}s^{m+1-\lambda n}\big(b_{21}(\mu)+\mathcal{I}(W_{31})\big)\ \text{ if }\mu\in W_{31},\\ s^\lambda\big(b_{22}(\mu)+\mathcal{I}(W_{32})\big)\ \text{ if }\mu\in W_{32},\\ s^{m+1-\lambda n}\big(b_{23}(\mu)\omega(s;\lambda(n+1)-m)+b_{24}(\mu)+\mathcal{I}(W_{33})\big)\ \text{ if }\mu\in W_{33}.\end{cases}$$

The first row in the above equality shows that $g\in\mathcal{I}(W_{31})$ because $s^{m+1-\lambda n}\in\mathcal{I}(W_3)$ and $W_{31}\subset W_3$. Clearly from the second one it turns out that $g\in\mathcal{I}(W_{32})$. Finally the third one shows that $g\in\mathcal{I}(W_{33})$. To see this it suffices to check that $s^{m+1-\lambda n}\omega\big(s;\lambda(n+1)-m\big)\in\mathcal{I}(W_{33})$, and this follows by applying Corollary 2.9 with $k=m+1-\lambda n$ (which is positive on W_3) and $r=\lambda(n+1)-m$ because $k+r=\lambda+1>1$. Thus, due to $W_3=W_{31}\cup W_{32}\cup W_{33}$, by Remark 2.5 we can assert that $g\in\mathcal{I}(W_3)$. This shows the validity of (8). Now, using that $s^m=s^{\lambda n}\big(1+(m-\lambda n)\omega(s;m+1-\lambda n)\big)$, the expansion in (8) yields to

$$V_2(s;\mu)=s^{\lambda n}\big((m-\lambda n)b_2(\mu)\omega(s;m+1-\lambda n)+b_2(\mu)+\mathcal{I}(W_3)\big).$$

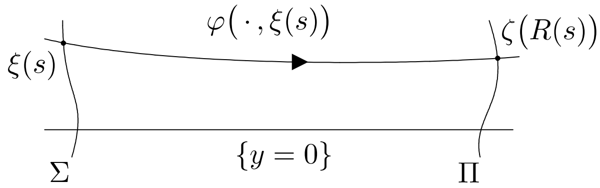

Figure 3: Transverse sections in Lemma 2.12.

Finally, the combination of this with the expansions in (6) and (7) shows that

$$V(s;\mu) = V_0(s;\mu) + V_1(s;\mu) + V_2(s;\mu) = s^{\lambda n}\big(a_3(\mu)\omega(s;m+1-\lambda n) + a_4(\mu) + \mathcal{I}(W_3)\big)$$

with $a_3(\mu) = \big(m-\lambda n\big)b_2(\mu) - G(0,0)$ and $a_4(\mu) = b_1(\mu) + b_2(\mu)$. This concludes the proof of the result. ■

Remark 2.11 With the notation introduced in Proposition 2.10, we have shown in its proof that

$$a_{12}(\mu) = \frac{G(0,0)}{m-\lambda n} + \int_0^1 \frac{G(u,0)-G(0,0)}{u^{\lambda n - m}}\frac{du}{u}.$$

□

Let us consider a family of vector fields of the form

$$X_\mu = \frac{1}{y^n}\big(f(x,y;\mu)\partial_x + yg(x,y;\mu)\partial_y\big), \tag{9}$$

where $n \in \mathbb{Z}$ and $\mu \in W$. The functions $f(x,y;\mu)$ and $g(x,y;\mu)$ are assumed to be analytic on a neighbourhood of $\{y=0\}$ and depending also analytically on the parameter μ. We also consider (see Figure 3) two analytic transverse sections $\xi(\,\cdot\,;\mu)\colon I \longrightarrow \Sigma_\mu$ and $\zeta(\,\cdot\,;\mu)\colon I \longrightarrow \Pi_\mu$ to the integral curve $\{y=0\}$. The next result (see [7]) provides the first nontrivial term of the Poincaré and time mappings between Σ_μ and Π_μ. More concretely, denoting by $\varphi\big(t,(x_0,y_0);\mu\big)$ the solution of (9) with initial condition (x_0,y_0), we define $R(s;\mu)$ and $T(s;\mu)$ by means of $\varphi\big(T(s),\xi(s)\big) = \zeta\big(R(s)\big)$.

Lemma 2.12. *If $f(x,0) \neq 0$ for all $x \in [\xi_1(0),\zeta_1(0)]$, then*

(a) $R(s;\mu) = s\Big(\rho(\mu) + \mathcal{I}_0(W)\Big)$ *with*

$$\rho(\mu) = \frac{\xi_2'(0)}{\zeta_2'(0)}\exp\left(\int_{\xi_1(0)}^{\zeta_1(0)} \frac{g(x,0)}{f(x,0)}\,dx\right).$$

(b) $T(s;\mu) = s^n\Big(\Delta_1(\mu) + \mathcal{I}_0(W)\Big)$ *with*

$$\Delta_1(\mu) = \xi_2'(0)^n \int_{\xi_1(0)}^{\zeta_1(0)} \exp\left(n\int_{\xi_1(0)}^{x} \frac{g(u,0)}{f(u,0)}\,du\right)\frac{dx}{f(x,0)}.$$

Moreover, if $n=0$ then $T(s;\mu) = \Delta_1(\mu) + \Delta_2(\mu)s + s\mathcal{I}_0(W)$ with

$$\Delta_2(\mu) = \frac{\zeta_1'(0)\rho(\mu)}{f\big(\zeta(0)\big)} - \frac{\xi_1'(0)}{f\big(\xi(0)\big)} - \xi_2'(0)\int_{\xi_1(0)}^{\zeta_1(0)} \frac{f_y(x,0)}{f(x,0)^2}\exp\left(\int_{\xi_1(0)}^{x}\frac{g(u,0)}{f(u,0)}\,du\right)dx.$$

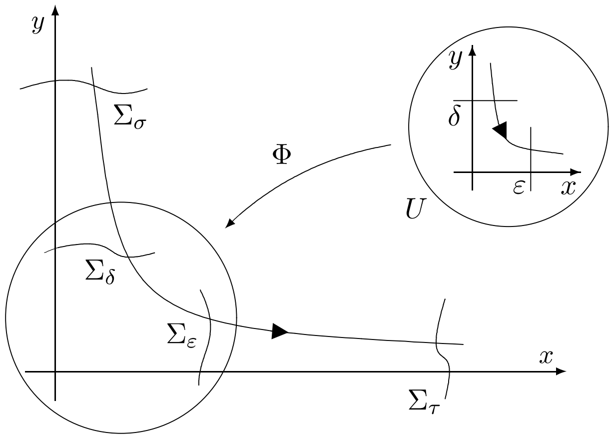

Figure 4: Auxiliary sections in the proof of Theorem 2.7.

Proof of the Theorem 2.7. For the sake of simplicity in the formulas we shall omit the parameter dependence when there is no risk of ambiguity.

Take $\delta > 0$ and $\varepsilon > 0$ small enough so that the points $(0, \delta)$ and $(\varepsilon, 0)$ belong to the linearizing domain U (recall Definition 2.1). Thus, taking advantage of the linearizing local diffeomorphism Φ, we define two auxiliary transverse sections Σ_δ and Σ_ε to X parameterized by $s \longmapsto \Phi(s, \delta)$ and $s \longmapsto \Phi(\varepsilon, s)$ respectively (see Figure 4). Next we consider the Dulac and time mappings between Σ_σ and Σ_δ. To this end we use the parametrization of the corresponding transverse sections. More precisely, if $\varphi\big(t, (x_0, y_0); \mu\big)$ denotes the solution of X_μ passing through (x_0, y_0) at $t = 0$, we define $R_1(s; \mu)$ and $T_1(s; \mu)$ by means of the relation

$$\varphi\big(T_1(s; \mu), \sigma(s)\big) = \Phi\big(R_1(s; \mu), \delta\big).$$

We also consider the mappings between Σ_δ and Σ_ε, say $R_2(s; \mu)$ and $T_2(s; \mu)$, and the ones between Σ_ε and Σ_τ, say $R_3(s; \mu)$ and $T_3(s; \mu)$. Exactly as before, these mappings are defined by means of

$$\varphi\big(T_2(s; \mu), \Phi(s, \delta)\big) = \Phi\big(\varepsilon, R_2(s; \mu)\big) \text{ and } \varphi\big(T_3(s; \mu), \Phi(\varepsilon, s)\big) = \tau\big(R_3(s; \mu)\big).$$

Now according to these definitions we can split up the Dulac and time mappings as

$$R(s; \mu) = R_3\big(R_2(R_1(s))\big) \text{ and } T(s; \mu) = T_1(s) + T_2\big(R_1(s)\big) + T_3\big(R_2(R_1(s))\big).$$

It is to be pointed out that $T_i(s)$ depend on δ and ε but that $T(s)$ as a whole does not. This will be the key point in order to compute its first nontrivial coefficient.

Lemma 2.12 provides us the expansions of the (regular) mappings from Σ_σ to Σ_δ and from Σ_ε to Σ_τ. Indeed, one can show in this way that

$$R_1(s) = s\big(\rho_1 + \mathcal{I}_0(W)\big), \; T_1(s) = s^m\big(a_1 + \mathcal{I}_0(W)\big) \text{ and } T_3(s) = s^n\big(c_1 + \mathcal{I}_0(W)\big). \tag{10}$$

Let us remark here that in order to study R_1 and T_1 by means of Lemma 2.12 it is first necessary to perform the coordinate transformation $(x, y) \longmapsto (y, x)$. Taking this into account, some computations yield to

$$\rho_1(\mu) = \frac{\sigma_1'(0)}{\psi_1(0,\delta)} \exp\left(\int_{\sigma_2(0)}^{\delta\psi_2(0,\delta)} \frac{P(0,u)}{Q(0,u)} \frac{du}{u} \right) = \frac{\sigma_1'(0) L\big(\delta\psi_2(0,\delta)\big)}{\psi_1(0,\delta)} \left(\frac{\sigma_2(0)}{\delta\psi_2(0,\delta)} \right)^{\frac{1}{\lambda}} \tag{11}$$

and

$$a_1(\mu) = \sigma_1'(0)^m \int_{\sigma_2(0)}^{\delta\psi_2(0,\delta)} \exp\left(m \int_{\sigma_2(0)}^{x} \frac{P(0,u)}{Q(0,u)} \frac{du}{u} \right) \frac{x^{n-1}dx}{Q(0,x)} \tag{12}$$
$$= \sigma_1'(0)^m \sigma_2(0)^{\frac{m}{\lambda}} \int_{\sigma_2(0)}^{\delta\psi_2(0,\delta)} \frac{L(x)^m x^{n-\frac{m}{\lambda}}}{Q(0,x)} \frac{dx}{x}.$$

In both equalities above we used that

$$\exp\left(\int_{\sigma_2(0)}^{x} \frac{P(0,u)}{Q(0,u)} \frac{du}{u} \right) = \exp\left(\int_{\sigma_2(0)}^{x} \left(\frac{P(0,u)}{Q(0,u)} + \frac{1}{\lambda} \right) \frac{du}{u} \right) \exp\left(\frac{1}{\lambda} \int_{x}^{\sigma_2(0)} \frac{du}{u} \right) = L(x) \left(\frac{\sigma_2(0)}{x} \right)^{1/\lambda}.$$

Note moreover that $R_2(s) = \delta(s/\varepsilon)^\lambda$ by the FLP. Thus, by (b) in Lemma 2.8, from (10) it follows that

$$R_2\big(R_1(s)\big) = s^\lambda\big(\rho_2 + \mathcal{I}_0(W)\big) \text{ with } \rho_2 = \delta\varepsilon^{-\lambda}\rho_1^\lambda.$$

Therefore, on account of the expansion of T_3 in (10) and by applying Lemma 2.8 again, it turns out that

$$T_3\big(R_2(R_1(s))\big) = s^{\lambda n}\big(\rho_2^n + \mathcal{I}_0(W)\big)\big(c_1 + \mathcal{I}(W)\big) = s^{\lambda n}\big(c_1\rho_2^n + \mathcal{I}(W)\big). \tag{13}$$

It remains to study $T_2(s;\mu)$ and this will be done by means of Proposition 2.10. Since in this result the transverse sections are assumed to be on $\{y = 1\}$ and $\{x = 1\}$, we must compose the linearizing diffeomorphism Φ with $(x,y) \longmapsto (\varepsilon x, \delta y)$. We thus consider $\widetilde{\Phi}(x,y) := \Phi(\varepsilon x, \delta y)$ and then from Remark 2.2 it follows that

$$X_\mu = \widetilde{\Phi}_* \left(\frac{1}{x^m y^n G(x,y)} \big(x\partial_x - \lambda y \partial_y\big) \right) \text{ with } G(x,y) := \varepsilon^m \delta^n g(\varepsilon x, \delta y).$$

(Recall that the existence of g is a consequence of Definition 2.1.) Hence we have that $T_2(s;\mu) = V(s/\varepsilon;\mu)$, where $V(s;\mu)$ is the function considered in Proposition 2.10 taking $G(x,y)$ as above.

At this point we can begin with the proof of (a). So assume that $\mu \in W_1$, i.e., $\lambda(\mu)n - m > 0$. In this case from (a) in Proposition 2.10 it turns out that

$$T_2(s;\mu) = V(s/\varepsilon;\mu) = s^m\big(b_1 + \mathcal{I}(W_1)\big), \text{ where } b_1(\mu) = \delta^n \int_0^1 u^{\lambda n - m} g(0, \delta^\lambda u^\lambda) \frac{du}{u}, \tag{14}$$

and then, taking (10) into account, the application of Lemma 2.8 shows that $T_2\big(R_1(s)\big) = s^m\big(b_1\rho_1^m + \mathcal{I}(W_1)\big)$. Note on the other hand that $s^{\lambda n} = s^m s^{\lambda n - m} \in s^m\mathcal{I}(W_1)$ and so, from (13), we can assert that $T_3\big(R_2(R_1(s))\big)$ belongs to $s^m\mathcal{I}(W_1)$. Therefore, gathering this with the expression of T_1 in (10) yields to

$$T(s) = T_1(s) + T_2\big(R_1(s)\big) + T_3\big(R_2(R_1(s))\big) = s^m\big(\underbrace{a_1 + b_1\rho_1^m}_{\Delta_1} + \mathcal{I}(W_1)\big).$$

This shows the validity of the expansion of the time function in (a). In order to compute Δ_1 explicitly note first that it does not depend on δ or ε. Using the expression of the coefficients in (11), (12) and (14), one can easily verify that

$$\Delta_1 = a_1 + b_1\rho_1^m = \sigma_1'(0)^m \sigma_2(0)^{\frac{m}{\lambda}} \int_{\sigma_2(0)}^{\delta\psi_2(0,\delta)} \frac{L(x)^m x^{n-\frac{m}{\lambda}}}{Q(0,x)} \frac{dx}{x}$$
$$+ \delta^{n-\frac{m}{\lambda}} \left(\frac{\sigma_1'(0) L\big(\delta\psi_2(0,\delta)\big)}{\psi_1(0,\delta)} \right)^m \left(\frac{\sigma_2(0)}{\psi_2(0,\delta)} \right)^{\frac{m}{\lambda}} \int_0^1 u^{\lambda n - m} g(0, \delta^\lambda u^\lambda) \frac{du}{u}.$$

Consequently, since $\lambda(\mu)n - m > 0$ for $\mu \in W_1$ and $\psi_i(0,0) = 1$, we can assert that

$$\Delta_1 = \lim_{\delta \longrightarrow 0} (a_1 + b_1 \rho_1^m) = \sigma_1'(0)^m \sigma_2(0)^{\frac{m}{\lambda}} \int_{\sigma_2(0)}^0 \frac{L(x)^m x^{n-\frac{m}{\lambda}}}{Q(0,x)} \frac{dx}{x}$$

and this concludes the proof of (a).

Let us turn now to prove (b). So assume that $\mu \in W_2$, i.e., $m - \lambda(\mu)n > 0$. Consider $\widehat{X}_\mu := -\varphi_*(X_\mu)$ with $\varphi(x,y) = (y,x)$ and note that then, following the obvious notation, $T(s;\mu) = \widehat{T}\big(R(s;\mu);\mu\big)$. Moreover

$$\widehat{X}_\mu = \frac{1}{x^{\widehat{m}} y^{\widehat{n}}} \big(x\widehat{P}(x,y;\mu)\partial_x + y\widehat{Q}(x,y;\mu)\partial_y \big),$$

where $\widehat{P}(x,y) = -Q(y,x)$, $\widehat{Q}(x,y) = -P(y,x)$, $\big(\widehat{m},\widehat{n}\big) = (n,m)$ and $\widehat{\lambda} = 1/\lambda$. Since $\widehat{\lambda}\widehat{n} - \widehat{m} = m/\lambda - n$ is positive on W_2 we can apply the previous case. Accordingly $\widehat{T}(s;\mu) = s^n\big(\widehat{\Delta}_1(\mu) + \mathcal{I}(W_2)\big)$ with

$$\widehat{\Delta}_1(\mu) = -\tau_2'(0)^n \tau_1(0)^{\lambda n} \int_{\tau_1(0)}^0 \frac{\widehat{L}(x)^n x^{m-\lambda n}}{P(x,0)} \frac{dx}{x} = \frac{\tau_2'(0)^n \tau_1(0)^{\lambda n}}{M\big(\tau_1(0)\big)^n} \int_0^{\tau_1(0)} \frac{M(x)^n x^{m-\lambda n}}{P(x,0)} \frac{dx}{x}. \tag{15}$$

Here we took $\widehat{\sigma}(s) = \big(\tau_2(s), \tau_1(s)\big)$ and $\widehat{\tau}(s) = \big(\sigma_2(s), \sigma_1(s)\big)$ into account and we used that

$$\widehat{L}(x) = \exp\left(\int_{\tau_1(0)}^0 \left(\frac{Q(u,0)}{P(u,0)} + \lambda \right) \frac{du}{u} + \int_0^x \left(\frac{Q(u,0)}{P(u,0)} + \lambda \right) \frac{du}{u} \right) = \frac{M(x)}{M(\tau_1(0))}.$$

On the other hand, since the Dulac map depends only on the foliation, by applying Theorem A in [7] to the vector fields $x^m y^n X_\mu$ we get that $R(s;\mu) = s^\lambda\big(\rho(\mu) + \mathcal{I}(W)\big)$ with

$$\rho(\mu) = \frac{\sigma_1'(0)^\lambda \sigma_2(0)}{\tau_2'(0)\tau_1(0)^\lambda} L(0)^\lambda M\big(\tau_1(0)\big). \tag{16}$$

Consequently by using (b) and (c) in Lemma 2.8 it follows that

$$T(s) = \widehat{T}\big(R(s)\big) = s^{\lambda n}\big(\rho + \mathcal{I}(W)\big)^n \big(\widehat{\Delta}_1 + \mathcal{I}(W_2)\big) = s^{\lambda n}\big(\underbrace{\rho^n \widehat{\Delta}_1}_{\Delta_2} + \mathcal{I}(W_2)\big).$$

Finally, from (15) and (16), an straightforward simplification shows that

$$\Delta_2 = \rho^n \widehat{\Delta}_1 = \big(\sigma_1'(0)^\lambda \sigma_2(0) L(0)^\lambda\big)^n \int_0^{\tau_1(0)} \frac{M(x)^n x^{m-\lambda n}}{P(x,0)} \frac{dx}{x}$$

and this completes the proof of the assertion in (b).

Let us show next (c). Assume therefore that $\mu \in W_3$, i.e., $-1 < m - \lambda(\mu)\, n < \lambda(\mu)$. Note first of all that the substitution $s^m = s^{\lambda n}\big(1 + (m - \lambda n)\omega(s; m + 1 - \lambda n)\big)$ in the expression of T_1 in (10) yields to

$$\begin{aligned} T_1(s) &= s^{\lambda n}\big(1 + (m - \lambda n)\omega(s; m + 1 - \lambda n)\big)\big(a_1 + \mathcal{I}_0(W)\big) \\ &= s^{\lambda n}\big(a_1(m - \lambda n)\omega(s; m + 1 - \lambda n) + a_1 + \mathcal{I}(W_3)\big). \end{aligned}$$

In the second equality above we use that, since $m + 1 - \lambda n > 0$ on W_3, $\omega(s; m + 1 - \lambda n)\mathcal{I}_0(W) \in \mathcal{I}(W_3)$ by (d) in Lemma 2.8. Recall on the other hand that $T_2(s;\mu) = V(s/\varepsilon;\mu)$, where $V(s;\mu)$ is the function studied in Proposition 2.10 taking $G(x,y) = \varepsilon^m \delta^n g(\varepsilon x, \delta y)$. Hence, by applying (c) in Proposition 2.10, $V(s;\mu) = s^{\lambda n}\big(a_3(\mu)\omega(s; m + 1 - \lambda n) + a_4(\mu) + \mathcal{I}(W_3)\big)$, where $a_3(\mu_0) = -G(0,0)$ for those $\mu_0 \in W_3$ such that $\lambda(\mu_0)n - m = 0$. Thus, on account of (10) and applying Lemma 2.8, one can verify that

$$T_2\big(R_1(s)\big) = V\big(s(\varepsilon^{-1}\rho_1 + \mathcal{I}_0(W))\big) = s^{\lambda n}\big(a_3 \varepsilon^{-m} \rho_1^m \omega(s; m + 1 - \lambda n) + \widetilde{a}_4 + \mathcal{I}(W_3)\big).$$

In addition from (13) it follows that $T_3\big(R_2(R_1(s))\big) = s^{\lambda n}\big(c_1\rho_2^n + \mathcal{I}(W)\big)$. The combination of these expansions gives

$$T(s;\mu) = T_1(s) + T_2\big(R_1(s)\big) + T_3\big(R_2(R_1(s))\big) = s^{\lambda n}\big(\Delta_3(\mu)\omega(s; m+1-\lambda n) + \Delta_4(\mu) + \mathcal{I}(W_3)\big),$$

where $\Delta_3(\mu) = a_1(m-\lambda n) + a_3\varepsilon^{-m}\rho_1^m$ and $\Delta_4(\mu) = a_1 + \widetilde{a}_4 + c_1\rho_2^n$. Consider finally some $\mu_0 \in W_3$ such that $m - \lambda(\mu_0)n = 0$. Then, since $a_3(\mu_0) = -G(0,0) = -\varepsilon^m\delta^n g(0,0)$ and taking (11) into account, we obtain

$$\Delta_3(\mu_0) = -g(0,0)\delta^n\rho_1^{\lambda n} = -g(0,0)\frac{\sigma_1'(0)^{\lambda n}L\big(\delta\psi_2(0,\delta)\big)^{\lambda n}}{\psi_1(0,\delta)^{\lambda n}}\frac{\sigma_2(0)^n}{\psi_2(0,\delta)^n},$$

which tends to $-g(0,0)\big(\sigma_1'(0)^\lambda L(0)^\lambda\sigma_2(0)\big)^n$ as $\delta \longrightarrow 0$ due to $\psi_i(0,0) = 1$. Consequently this shows that $\Delta_3(\mu_0) = -g(0,0)\big(\sigma_1'(0)^\lambda L(0)^\lambda\sigma_2(0)\big)^n$ because Δ_3 does not depend on δ. We claim that

$$g(0,0) = \frac{1}{P(0,0)} \tag{17}$$

and note that the result will follow once we prove this. To show the claim note first that, from Remark 2.2,

$$X_\mu\big(\Phi(x,y)\big) = \frac{1}{x^m y^n g(x,y)}\begin{pmatrix} \Phi_{1x} & \Phi_{1y} \\ \Phi_{2x} & \Phi_{2y}\end{pmatrix}\begin{pmatrix} x \\ -\lambda y\end{pmatrix}.$$

Since $\Phi_1(x,y) = x\psi_1(x,y)$ and $\Phi_2(x,y) = y\psi_2(x,y)$, taking the first component of the vectors above one can easily conclude that

$$g(x,y) = \frac{\psi_1^m\psi_2^n}{P\big(\Phi(x,y)\big)}\left(1 + \frac{x\psi_{1x} - \lambda y\psi_{1y}}{\psi_1}\right),$$

which on account of $\psi_i(0,0) = 1$ proves (17). This completes the proof of the result. ∎

We conclude this section with the following result about the Dulac map. For the sake of convenience it refers to the family X_μ in (2) but, since it is clear that this map depends only on the foliation, one may consider $\widetilde{X}_\mu = xP(x,y;\mu)\partial_x + yQ(x,y;\mu)\partial_y$ instead.

Lemma 2.13. *Let $\{X_\mu,\ \mu \in W\}$ be the family of vector fields defined in* (2) *and assume that it verifies the FLP. Let R be the Dulac map from Σ_σ to Σ_τ as introduced in* (3). *Then $R(s;\mu) = s^\lambda\big(\rho_1(\mu) + f(s;\mu)\big)$ with $f \in \mathcal{I}(W)$ and*

$$\rho_1(\mu) = \frac{\sigma_1'(0)^\lambda\sigma_2(0)}{\tau_2'(0)\tau_1(0)^\lambda}L(0)^\lambda M\big(\tau_1(0)\big).$$

Moreover, in case that $\lambda(\mu) < 1$ for all $\mu \in W$, the remainder term is given by $f(s;\mu) = s^\lambda\big(\rho_2(\mu) + \mathcal{I}(W)\big)$ where ρ_2 is an analytic function on W.

Proof. The first part of the result follows by applying Theorem A in [7]. In order to prove the assertion concerning the remainder term we take advantage of the fact that X_μ verifies the FLP and introduce the auxiliary transverse sections as in the proof of Theorem 2.7 (see Figure 4). Accordingly

$$R(s;\mu) = R_3\big(R_2(R_1(s))\big),$$

where R_1 and R_3 are analytic diffeomorphisms by (a) in Lemma 2.12 and $R_2(s) = \delta(s/\varepsilon)^\lambda$. Thus we have that $R_1(s) = s\big(a_1 + \mathcal{I}_0(W)\big)$ and hence, from (b) in Lemma 2.8,

$$R_2\big(R_1(s)\big) = s^\lambda\big(b_1 + \mathcal{I}_0(W)\big) = s^\lambda\big(b_1 + b_2 s + s\mathcal{I}_0(W)\big)$$

for some $b_i \in \mathcal{C}^\omega(W)$. Finally, since $R_3(s) = c_1 s + c_2 s^2 + s^2\mathcal{I}_0(W)$, by Lemma 2.8 once again we obtain that

$$\begin{aligned} R(s) &= c_1 s^\lambda\big(b_1 + b_2 s + s\mathcal{I}_0(W)\big) + c_2 s^{2\lambda}(b_1^2 + \mathcal{I}_0(W)) + s^{2\lambda}\mathcal{I}(W) \\ &= \rho_1 s^\lambda + \rho_2 s^{2\lambda} + s^{2\lambda}\mathcal{I}(W) \end{aligned}$$

where $\rho_1 = b_1c_1$ and $\rho_2 = b_1^2c_2$. In the second equality above we take the hypothesis $\lambda < 1$ into account to conclude that $s^{\lambda+1} = s^{2\lambda}s^{1-\lambda} = s^{2\lambda}\mathcal{I}(W)$. This shows the validity of the result. ∎

3 Higher order developments

In the previous section we obtained the first order development of $T(s;\mu)$ at $s=0$ for any $\mu \in W$ and we computed its leading coefficient. In the present section we study higher order developments but we restrict ourselves to those cases that are strictly necessary for the subsequent application, namely $\mu \in W_1$, see (4).

3.1 Second order developments

Theorem 3.1 (Second order development). *Let $\{X_\mu,\, \mu \in W\}$ be the family of vector fields defined in (2) and assume that it verifies the FLP. Let T be the time function associated to the transverse sections Σ_σ and Σ_τ as introduced in (3). Finally assume that $\mu \in W_1$ with $\lambda(\mu)n - m \neq 1$.*

(a) If $\mu \in W_{11}$ then $T(s;\mu) = s^m\big(\Delta_1(\mu) + \Delta_{11}(\mu)s + s\mathcal{I}(W_{11})\big)$, where Δ_{11} is an analytic function on W_{11}.

(b) If $\mu \in W_{12}$ then $T(s;\mu) = s^m\big(\Delta_1(\mu) + \Delta_{12}(\mu)s^{\lambda n-m} + s^{\lambda n - m}\mathcal{I}(W_{12})\big)$, where

$$\Delta_{12}(\mu) = \big(\sigma_1'(0)^\lambda L(0)^\lambda \sigma_2(0)\big)^n \left\{ \frac{\tau_1(0)^{m-\lambda n}}{P(0,0)(m-\lambda n)} + \int_0^{\tau_1(0)} \frac{1}{u}\left(\frac{M(u)^n}{P(u,0)} - \frac{M(0)^n}{P(0,0)}\right)\frac{du}{u^{\lambda n - m}} \right\}.$$

Proof. By means of the same auxiliary transverse sections used in the proof of Theorem 2.7 we split up the time function as

$$T(s;\mu) = T_1(s) + T_2\big(R_1(s)\big) + T_3\big(R_2(R_1(s))\big).$$

The first and third terms in the above expression were already computed in the proof of Theorem 2.7. Let us take advantage of them for the sake of shortness. Thus, (10) and (13) show respectively that

$$T_1(s) = s^m\big(a_1 + \mathcal{I}_0(W)\big) \text{ and } T_3\big(R_2(R_1(s))\big) = s^{\lambda n}\big(c_1\rho_2^n + \mathcal{I}(W)\big), \text{ where } \rho_2 = \delta\varepsilon^{-\lambda}\rho_1^\lambda.$$

Let us prove (a) first. So assume that $\mu \in W_{11}$, i.e., $\lambda(\mu)n - m > 1$. From (a) in Proposition 2.10 it follows that $T_2(s) = V(s/\varepsilon) = s^m\big(b_1 + b_2 s + s\mathcal{I}(W_{11})\big)$ with $b_i \in \mathcal{C}^\omega(W_{11})$. Hence, since $R_1(s) = s\big(\rho_1 + \mathcal{I}_0(W)\big)$ due to (a) in Lemma 2.12,

$$\begin{aligned} T_2\big(R_1(s)\big) &= s^m\big(\rho_1^m + \mathcal{I}_0(W)\big)\big(b_1 + b_2(\rho_1 + \mathcal{I}_0(W))s + s\mathcal{I}(W_{11})\big) \\ &= s^m\big(\widehat{b}_1 + \widehat{b}_2 s + s\mathcal{I}(W_{11})\big) \end{aligned}$$

for some $\widehat{b}_i \in \mathcal{C}^\omega(W_{11})$. (In the first equality above we used Lemma 2.8 to get the remainder terms.) Note on the other hand that $T_3\big(R_2(R_1(s))\big) = s^{\lambda n}\big(c_1\rho_2^n + \mathcal{I}(W)\big) = s^{m+1}\mathcal{I}(W_{11})$ because $s^{\lambda n} \in s^{m+1}\mathcal{I}(W_{11})$ due to $\lambda n > m+1$. Finally the combination of the three developments gives

$$\begin{aligned} T(s) &= \underbrace{s^m\big(a_1 + a_2 s + s\mathcal{I}_0(W)\big)}_{T_1} + \underbrace{s^m\big(\widehat{b}_1 + \widehat{b}_2 s + s\mathcal{I}(W_{11})\big)}_{T_2} + \underbrace{s^{m+1}\mathcal{I}(W_{11})}_{T_3} \\ &= s^m\big(\Delta_1 + \Delta_{11}s + s\mathcal{I}(W_{11})\big), \end{aligned}$$

where $\Delta_1 = a_1 + \widehat{b}_1$ and $\Delta_{11} = a_2 + \widehat{b}_2$, and this completes the proof of (a).

Let us turn now to the assertion in (b). So assume that $\mu \in W_{12}$, i.e., $0 < \lambda(\mu)n - m < 1$. Setting $G(x,y) := \varepsilon^m\delta^n g(\varepsilon x, \delta y)$, from (a) in Proposition 2.10 it turns out that $T_2(s) = V(s/\varepsilon)$ where

$$V(s) = s^m\big(b_0 + b_2 s^{\lambda n - m} + s^{\lambda n - m}\mathcal{I}(W_{12})\big).$$

In addition, see Remark 2.11,

$$b_2 = \frac{G(0,0)}{m-\lambda n} + \int_0^1 \frac{G(u,0) - G(0,0)}{u^{\lambda n - m}}\frac{du}{u} = \varepsilon^m\delta^n\left(\frac{g(0,0)}{m-\lambda n} + \varepsilon^{\lambda n - m}\int_0^\varepsilon \frac{g(u,0) - g(0,0)}{u^{\lambda n - m}}\frac{du}{u}\right).$$

Recall on the other hand that $R_1(s;\mu) = s\big(\rho_1(\mu) + \mathcal{I}_0(W)\big)$, where ρ_1 is given explicitly in (11). Therefore

$$\begin{aligned}
T_2\big(R_1(s)\big) &= V\Big(s\big(\rho_1\varepsilon^{-1} + \mathcal{I}_0(W)\big)\Big)\\
&= s^m\big(\rho_1^m\varepsilon^{-m} + \mathcal{I}_0(W)\big)\Big(b_0 + \big(\rho_1^{\lambda n-m}\varepsilon^{m-\lambda n} + \mathcal{I}_0(W)\big)b_2 s^{\lambda n-m} + s^{\lambda n-m}\mathcal{I}(W_{12})\Big)\\
&= s^m\big(b_0\rho_1^m\varepsilon^{-m} + b_2\rho_1^{\lambda n}\varepsilon^{-\lambda n}s^{\lambda n-m} + s^{\lambda n-m}\mathcal{I}(W_{12})\big),
\end{aligned}$$

where in the second equality we use Lemma 2.8 and in the third one that $\mathcal{I}_0(W) \subset s^{\lambda n-m}\mathcal{I}(W_{12})$. This inclusion follows from the fact that if $g \in \mathcal{I}_0(W)$ then $g(s) = s\widehat{g}(s)$ with $\widehat{g}$ analytic on $s = 0$, and hence we can write it as $g(s) = s^{\lambda n-m}s^{m+1-\lambda n}\widehat{g}(s) = s^{\lambda n-m}\mathcal{I}(W_{12})$ because $m + 1 - \lambda n > 0$ on W_{12}. Similarly

$$T_3\big(R_2(R_1(s))\big) = s^{\lambda n}\big(c_1\rho_2^n + \mathcal{I}(W)\big) = s^m\big(c_1\rho_2^n s^{\lambda n-m} + s^{\lambda n-m}\mathcal{I}(W)\big),$$

where recall that $\rho_2 = \delta\varepsilon^{-\lambda}\rho_1^{\lambda}$. Now the combination of the three developments yields to

$$T(s) = s^m\Big(\underbrace{a_1 + b_0\rho_1^m\varepsilon^{-m}}_{\Delta_1} + \underbrace{\rho_1^{\lambda n}\varepsilon^{-\lambda n}\big(b_2 + c_1\delta^n\big)}_{\Delta_{12}}s^{\lambda n-m} + s^{\lambda n-m}\mathcal{I}(W_{12})\Big),$$

where we used $\mathcal{I}_0(W) \subset s^{\lambda n-m}\mathcal{I}(W_{12})$ again, and this proves the assertion in (b) concerning the expansion of the time function. Our next goal is to compute Δ_{12} explicitly. To this end note first that, by applying Lemma 2.12 to

$$\frac{1}{y^n}\Big(f(x,y)\partial_x + yg(x,y)\partial_y\Big) \text{ with } f(x,y) = \frac{P(x,y)}{x^{m-1}} \text{ and } g(x,y) = \frac{Q(x,y)}{x^m},$$

we obtain the leading coefficient of $T_3(s) = s^n\big(c_1 + \mathcal{I}_0(W)\big)$, namely

$$c_1 = \psi_2(\varepsilon,0)^n\int_{\varepsilon\psi_1(\varepsilon,0)}^{\tau_1(0)}\exp\left(n\int_{\varepsilon\psi_1(\varepsilon,0)}^{x}\frac{Q(u,0)}{P(u,0)}\frac{du}{u}\right)\frac{x^{m-1}dx}{P(x,0)}.$$

As usual the key point will be the fact that Δ_{12} does not depend on δ or ε. Therefore to obtain a simpler expression we can take limits when both parameters tend to zero. To do this we must first rewrite c_1 in terms of $M(u)$ as follows. With this aim in view observe first that

$$\int_{\varepsilon\psi_1(\varepsilon,0)}^{x}\frac{Q(u,0)}{P(u,0)}\frac{du}{u} = \int_{\varepsilon\psi_1(\varepsilon,0)}^{x}\left(\frac{Q(u,0)}{P(u,0)} + \lambda\right)\frac{du}{u} - \ln\left(\frac{x}{\varepsilon\psi_1(\varepsilon,0)}\right)^{\lambda},$$

and hence

$$\exp\left(n\int_{\varepsilon\psi_1(\varepsilon,0)}^{x}\frac{Q(u,0)}{P(u,0)}\frac{du}{u}\right) = \left(\frac{\varepsilon\psi_1(\varepsilon,0)}{x}\right)^{\lambda n}\frac{M(x)^n}{M\big(\varepsilon\psi_1(\varepsilon,0)\big)^n}.$$

Consequently

$$c_1 = \frac{\psi_2(\varepsilon,0)^n\varepsilon^{\lambda n}\psi_1(\varepsilon,0)^{\lambda n}}{M\big(\varepsilon\psi_1(\varepsilon,0)\big)^n}\int_{\varepsilon\psi_1(\varepsilon,0)}^{\tau_1(0)}\frac{M(u)^n u^{m-\lambda n}}{P(u,0)}\frac{du}{u}.$$

Thus, since $\Delta_{12} = \rho_1^{\lambda n}\varepsilon^{-\lambda n}\big(b_2 + c_1\delta^n\big)$, gathering the expressions of ρ_1 in (11), b_2 and c_1 together, and taking (17) into account, some easy simplifications show that

$$\begin{aligned}
\Delta_{12} = \frac{\sigma_1'(0)^{\lambda n}L\big(\delta\psi_2(0,\delta)\big)^{\lambda n}\sigma_2(0)^n}{\psi_1(0,\delta)^{\lambda n}\psi_2(0,\delta)^n}&\left\{\frac{\varepsilon^{m-\lambda n}}{P(0,0)(m-\lambda n)} + \int_0^{\varepsilon}\frac{g(u,0)-g(0,0)}{u^{\lambda n-m}}\frac{du}{u}\right.\\
&\left.+\frac{\psi_2(\varepsilon,0)^n\psi_1(\varepsilon,0)^{\lambda n}}{M\big(\varepsilon\psi_1(\varepsilon,0)\big)}\int_{\varepsilon\psi_1(\varepsilon,0)}^{\tau_1(0)}\frac{M(u)^n u^{m-\lambda n}}{P(u,0)}\frac{du}{u}\right\}.
\end{aligned}$$

The first factor in the above expression does not depend on ε and it tends to $\big(\sigma_1'(0)^\lambda L(0)^\lambda \sigma_2(0)\big)^n$ as $\delta \longrightarrow 0$ since $\psi_i(0,0)=1$. Similarly, the second one does not depend on δ, but its limit as $\varepsilon \longrightarrow 0$ is more delicate. This factor consists of the addition of three terms, say κ_1, κ_2 and κ_3 respectively. One can easily see that $\kappa_2 \longrightarrow 0$ as $\varepsilon \longrightarrow 0$. Concerning the other two we claim that

$$\lim_{\varepsilon \longrightarrow 0}\big(\kappa_1+\kappa_3\big) = \frac{\tau_1(0)^{m-\lambda n}}{P(0,0)(m-\lambda n)} + \int_0^{\tau_1(0)} \frac{1}{u}\left(\frac{M(u)^n}{P(u,0)} - \frac{M(0)^n}{P(0,0)}\right)\frac{du}{u^{\lambda n - m}}$$

and notice that (b) will follow once we prove this. In order to show the claim we introduce the function

$$N(u) := \begin{cases} \frac{1}{u}\left(\frac{M(u)^n}{P(u,0)} - \frac{M(0)^n}{P(0,0)}\right) & \text{if } u \neq 0, \\ \frac{d}{du}\left(\frac{M(u)^n}{P(u,0)}\right)\Big|_{u=0} & \text{if } u = 0. \end{cases}$$

Then

$$\begin{aligned} \int_{\varepsilon\psi_1(\varepsilon,0)}^{\tau_1(0)} \frac{M(u)^n u^{m-\lambda n}}{P(u,0)}\frac{du}{u} &= \int_{\varepsilon\psi_1(\varepsilon,0)}^{\tau_1(0)} \left(\frac{M(u)^n}{P(u,0)} - \frac{M(0)^n}{P(0,0)}\right)\frac{u^{m-\lambda n}du}{u} + \frac{M(0)^n}{P(0,0)}\int_{\varepsilon\psi_1(\varepsilon,0)}^{\tau_1(0)} u^{m-\lambda n}\frac{du}{u} \\ &= \int_{\varepsilon\psi_1(\varepsilon,0)}^{\tau_1(0)} N(u)\, u^{m-\lambda n} du + \frac{M(0)^n}{P(0,0)}\,\frac{\tau_1(0)^{m-\lambda n} - \big(\varepsilon\psi_1(\varepsilon,0)\big)^{m-\lambda n}}{m-\lambda n}. \end{aligned}$$

Therefore, due to $M(0)=\psi_i(0,0)=1$,

$$\begin{aligned} \kappa_1+\kappa_3 &= \frac{\varepsilon^{m-\lambda n}}{P(0,0)(m-\lambda n)} + \frac{\psi_2(\varepsilon,0)^n \psi_1(\varepsilon,0)^{\lambda n}}{M\big(\varepsilon\psi_1(\varepsilon,0)\big)}\int_{\varepsilon\psi_1(\varepsilon,0)}^{\tau_1(0)} \frac{M(u)^n u^{m-\lambda n}}{P(u,0)}\frac{du}{u} \\ &= \frac{\psi_2(\varepsilon,0)^n \psi_1(\varepsilon,0)^{\lambda n}}{M\big(\varepsilon\psi_1(\varepsilon,0)\big)}\left(\frac{M(0)^n}{P(0,0)}\,\frac{\tau_1(0)^{m-\lambda n}}{m-\lambda n} + \int_{\varepsilon\psi_1(\varepsilon,0)}^{\tau_1(0)} N(u)\, u^{m-\lambda n} du\right) \\ &\quad + \frac{\varepsilon^{m-\lambda n}}{P(0,0)(m-\lambda n)}\underbrace{\left(1 - \frac{\psi_1(\varepsilon,0)^m \psi_2(\varepsilon,0)^n}{M\big(\varepsilon,\psi_1(\varepsilon,0)\big)^n}\right)}_{\varepsilon O(\varepsilon)}, \end{aligned}$$

which tends to

$$\frac{\tau_1(0)^{m-\lambda n}}{P(0,0)(m-\lambda n)} + \int_0^{\tau_1(0)} N(u)\, u^{m-\lambda n} du$$

as $\varepsilon \longrightarrow 0$ because $m+1-\lambda n > 0$ on W_{12} and $N(u)$ is analytic at $u=0$. So the claim is true and (b) follows. This completes the proof of the result. ■

3.2 Third order developments for $\mathbf{m=0}$

The rest of the present section is devoted to study the case $m=0$ and $n \in \mathbb{N}$. Assuming this, our aim is to obtain the third order development of the time function of X_μ for $\mu \in W_{12} = \big\{\mu \in W : \lambda(\mu) < \frac{1}{n}\big\}$, cf. (5). To this end we introduce, following the usual notation,

$$W_{121} := \big\{\mu \in W : \tfrac{1}{n+1} < \lambda(\mu) < \tfrac{1}{n}\big\} \text{ and } W_{122} := \big\{\mu \in W : \lambda(\mu) < \tfrac{1}{n+1}\big\}.$$

Proposition 3.2. *With the notation in Proposition 2.10 and the above assumptions, if $\mu \in W_{122}$ then*

$$V(s;\mu) = a_1(\mu) + a_{12}(\mu)s^{\lambda n} + a_{122}(\mu)s^{\lambda(n+1)} + s^{\lambda(n+1)}\mathcal{I}(W_{122}),$$

where the coefficients are analytic functions on W_{122}.

Proof. Let G_1 be the analytic function verifying that $G(x,y) = G(0,y) + xG_1(x,y)$. Then

$$V(s) = \int_{\mathcal{C}} y^n G(0,y)\frac{dx}{x} + V_1(s) \ \text{ where } \ V_1(s) := \int_{\mathcal{C}} xy^n G_1(x,y)\frac{dx}{x}.$$

Taking $yx^\lambda = s^\lambda$ into account, an straightforward computation shows that

$$\begin{aligned}\int_{\mathcal{C}} y^n G(0,y)\frac{dx}{x} &= \frac{1}{\lambda}\int_{s^\lambda}^1 y^n G(0,y)\frac{dy}{y} \\ &= \underbrace{\frac{1}{\lambda}\int_0^1 y^n G(0,y)\frac{dy}{y}}_{a_1} - \underbrace{\frac{1}{\lambda}\int_0^{s^\lambda} y^n G(0,y)\frac{dy}{y}}_{F(s^\lambda)}.\end{aligned} \tag{18}$$

Note that a_1 is the coefficient in Proposition 2.10 and that $x \longmapsto F(x)$ is an analytic function on $x = 0$ with

$$F(x) = \tfrac{G(0,0)}{\lambda n}\, x^n + \tfrac{G_y(0,0)}{\lambda(n+1)}\, x^{n+1} + \mathrm{o}(x^{n+1}).$$

In order to study V_1 we consider the analytic function G_2 with $G_1(x,y) = G_1(x,0) + yG_2(x,y)$. Then

$$V_1(s) = \int_{\mathcal{C}} xy^n G_1(x,0)\frac{dx}{x} + V_2(s) \ \text{ where } \ V_2(s) := \int_{\mathcal{C}} xy^{n+1} G_2(x,y)\frac{dx}{x}. \tag{19}$$

By applying (a) in Proposition 2.10 with $(\widehat{m},\widehat{n}) = (0,n+1)$ and $\widehat{G}(x,y) = xG_2(x,y)$ it turns out that

$$V_2(s) = \widehat{a}_1 + \widehat{a}_{12}s^{\lambda(n+1)} + s^{\lambda(n+1)}\mathcal{I}(W_{122}) \tag{20}$$

because $\lambda\widehat{n} - \widehat{m} < 1$ on W_{122}. Note that in fact $\widehat{a}_1 = 0$ due to $\widehat{G}(0,y) \equiv 0$. On the other hand,

$$\int_{\mathcal{C}} xy^n G_1(x,0)\frac{dx}{x} = s^{\lambda n}\int_s^1 G_1(x,0)\frac{dx}{x^{\lambda n}} = s^{\lambda n}\Big(\underbrace{\int_0^1 G_1(x,0)\frac{dx}{x^{\lambda n}}}_{c_1} - \underbrace{\int_0^s G_1(x,0)\frac{dx}{x^{\lambda n}}}_{f(s)}\Big)$$

and we claim that $f \in s^\lambda\mathcal{I}(W_{122})$. To show this take a compact subset K of W_{122} and let M be a positive constant such that $|G_1(x,0)| \leqslant M$ for $x \approx 0$. Then, if $\mu \in K$,

$$\left|s^{-\lambda}\int_0^s G_1(x,0)\frac{dx}{x^{\lambda n}}\right| \leqslant M\frac{s^{1-\lambda(n+1)}}{1-\lambda n} \longrightarrow 0 \ \text{ as } \ s \longrightarrow 0$$

uniformly on K since $1 - \lambda(n+1) > 0$. On the other hand

$$\begin{aligned}\left|s\frac{d}{ds}\left(s^{-\lambda}\int_0^s G_1(x,0)\frac{dx}{x^{\lambda n}}\right)\right| &\leqslant \lambda s^{-\lambda}\int_0^s |G_1(x,0)|\frac{dx}{x^{\lambda n}} + s^{1-\lambda(n+1)}|G_1(s,0)| \\ &\leqslant \lambda M\frac{s^{1-\lambda(n+1)}}{1-\lambda n} + s^{1-\lambda(n+1)}|G_1(s,0)| \longrightarrow 0 \ \text{ as } \ s \longrightarrow 0\end{aligned}$$

uniformly on K. Therefore the claim is true and thus, from (19) and (20), it follows that

$$V_1(s) = c_1 s^{\lambda n} + \widehat{a}_{12}s^{\lambda(n+1)} + s^{\lambda(n+1)}\mathcal{I}(W_{122}).$$

Finally, taking also (18) into account,

$$V(s) = a_1 + \left(c_1 - \tfrac{G(0,0)}{\lambda n}\right)s^{\lambda n} + \left(\widehat{a}_{12} - \tfrac{G_y(0,0)}{\lambda(n+1)}\right)s^{\lambda(n+1)} + s^{\lambda(n+1)}\mathcal{I}(W_{122}),$$

and this completes the proof of the result. (Note that the coefficients a_1 and $a_{12} = c_1 - \frac{G(0,0)}{\lambda n}$ are the ones in Proposition 2.10 with $m = 0$.) ∎

Theorem 3.3 (Third order development). *Let $\{X_\mu,\, \mu\in W\}$ be the family of vector fields defined in* (2) *with $m=0$ and $n\in\mathbb{N}$. Assume that it verifies the FLP and let T be the time function associated to the transverse sections Σ_σ and Σ_τ as introduced in* (3)*. Finally assume that $\mu\in W_{12}=\big\{\mu\in W:\lambda(\mu)<\frac{1}{n}\big\}$ with $\lambda(\mu)\neq\frac{1}{n+1}$.*

(*a*) *If $\mu\in W_{121}$ then $T(s;\mu)=\Delta_1(\mu)+\Delta_{12}(\mu)s^{\lambda n}+\Delta_{121}(\mu)s+s\mathcal{I}(W_{121})$, where*

$$\begin{aligned}\Delta_{121}(\mu)=&\,\frac{\sigma_1'(0)\sigma_2(0)^n}{n-1/\lambda}\frac{Q_x(0,0)L(0)}{Q(0,0)^2}-\frac{\sigma_2'(0)\,\sigma_2(0)^{n-1}}{Q(0,\sigma_2(0))}\\ &+\sigma_1'(0)\sigma_2(0)^{1/\lambda}\int_0^{\sigma_2(0)}\left(\frac{Q_x(0,v)L(v)}{Q(0,v)^2}-\frac{Q_x(0,0)L(0)}{Q(0,0)^2}\right)\frac{v^{n-1}dv}{v^{1/\lambda}}.\end{aligned}$$

(*b*) *If $\mu\in W_{122}$ then $T(s;\mu)=\Delta_1(\mu)+\Delta_{12}(\mu)s^{\lambda n}+\Delta_{122}(\mu)s^{\lambda(n+1)}+s^{\lambda(n+1)}\mathcal{I}(W_{122})$, where Δ_{122} is an analytic function on W_{122}.*

Proof. Concerning the assertion in (*a*), we shall not compute the explicit expression of the coefficient Δ_{121}. This follows by means of the same approach as the preceding cases but the computations involved are even longer. Thus, for the sake of shortness, we prefer not to include it here. Moreover we can obtain the development of $T(s;\mu)$ at $s=0$ in a very short way by means of a previous result. Indeed, Theorem A in [7] shows that if $\mu\in U:=\big\{\mu\in W:\frac{1}{n+1}<\lambda(\mu)<\frac{2}{n}\big\}$ then

$$T(s;\mu)=\bar\Delta_0(\mu)+\bar\Delta_3(\mu)s\omega(s;\lambda n)+\bar\Delta_4(\mu)s+s\mathcal{I}(U),$$

where $\bar\Delta_i$ are some analytic functions on U. Note that if $\lambda(\mu)\neq\frac{1}{n}$ then $s\omega(s;\lambda n)=\frac{s^{\lambda n}-s}{\lambda n-1}$. Accordingly, since $W_{121}\subset U$, if $\mu\in W_{121}$ then

$$T(s)=\bar\Delta_0+\bar\Delta_3\tfrac{s^{\lambda n}-s}{\lambda n-1}+\bar\Delta_4 s+s\mathcal{I}(W_{121})=\bar\Delta_0+\tfrac{\bar\Delta_3}{\lambda n-1}\,s^{\lambda n}+\big(\bar\Delta_4-\tfrac{\bar\Delta_3}{\lambda n-1}\big)s+s\mathcal{I}(W_{121}).$$

Setting $\Delta_1=\bar\Delta_0$, $\Delta_{12}=\frac{\bar\Delta_3}{\lambda n-1}$ and $\Delta_{121}=\bar\Delta_4-\frac{\bar\Delta_3}{\lambda n-1}$, which are analytic functions on W_{121}, this clearly shows the validity of the expansion in (*a*).

Finally let us prove (*b*). So assume that $\mu\in W_{122}$ and note that then, in particular, $\lambda(\mu)<1$. Exactly the same way as in the proof of Theorem 2.7, the FLP enables us to split up the time function as

$$T(s;\mu)=T_1(s)+T_2\big(R_1(s)\big)+T_3\big(R_2(R_1(s))\big),$$

where T_2 is the time associated to the passage through the saddle between "normalized" transverse sections. From Proposition 3.2 it follows that

$$T_2(s)=V(s/\varepsilon)=b_0+b_1s^{\lambda n}+b_2s^{\lambda(n+1)}+s^{\lambda(n+1)}\mathcal{I}(W_{122})$$

for some $b_i\in\mathcal{C}^\omega(W_{122})$. Thus, since $R_1(s)=s\big(\rho_1+\mathcal{I}_0(W)\big)$ by (*a*) in Lemma 2.12,

$$\begin{aligned}T_2\big(R_1(s)\big)&=b_0+b_1\big(\rho_1^{\lambda n}+\mathcal{I}_0(W)\big)s^{\lambda n}+b_2\big(\rho_1^{\lambda(n+1)}+\mathcal{I}_0(W)\big)s^{\lambda(n+1)}+s^{\lambda(n+1)}\mathcal{I}(W_{122})\\ &=\widehat{b}_0+\widehat{b}_1s^{\lambda n}+\widehat{b}_2s^{\lambda(n+1)}+s^{\lambda(n+1)}\mathcal{I}(W_{122}).\end{aligned}$$

In the first equality above we use (*c*) in Lemma 2.8 to obtain the remainder term and in the second one the fact that $\mathcal{I}_0(W)\subset s^\lambda\mathcal{I}(W_{122})$ due to $\lambda<1$. On the other hand, by applying (*b*) in Lemma 2.12,

$$T_1(s)=a_1+\mathcal{I}_0(W)\ \text{ and }\ T_3(s)=s^n\big(c_1+c_2s+s\mathcal{I}_0(W)\big).$$

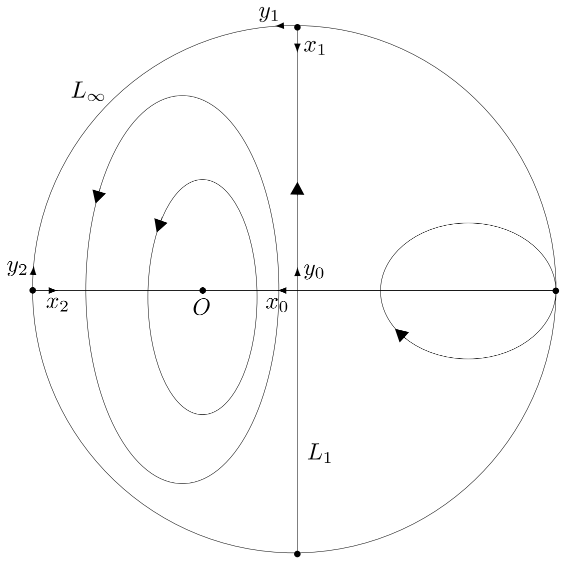

Figure 5: Phase portrait of the restricted Loud family (21).

Note moreover that $R_2\big(R_1(s)\big) = s^\lambda\big(\rho_2 + \mathcal{I}_0(W)\big)$ because $R_2(s) = \delta\big(s/\varepsilon\big)^\lambda$. Then, by Lemma 2.8 and using again that $\mathcal{I}_0(W) \subset s^\lambda \mathcal{I}(W_{122})$,

$$\begin{aligned} T_3\big(R_2(R_1(s))\big) &= c_1 s^{\lambda n}\big(\rho_2^n + \mathcal{I}_0(W)\big) + c_2 s^{\lambda(n+1)}\big(\rho_2^{n+1} + \mathcal{I}_0(W)\big) + s^{\lambda(n+1)}\mathcal{I}(W) \\ &= \widehat{c}_1 s^{\lambda n} + \widehat{c}_2 s^{\lambda(n+1)} + s^{\lambda(n+1)}\mathcal{I}(W_{122}). \end{aligned}$$

Finally, since $\mathcal{I}_0(W) \subset s^{\lambda(n+1)}\mathcal{I}(W_{122})$ due to $\lambda(n+1) < 1$, gathering the three developments together we conclude that

$$T(s) = \underbrace{a_1 + \widehat{b}_0}_{\Delta_1} + \underbrace{\big(\widehat{b}_1 + \widehat{c}_1\big)}_{\Delta_{12}} s^{\lambda n} + \underbrace{\big(\widehat{b}_2 + \widehat{c}_2\big)}_{\Delta_{122}} s^{\lambda(n+1)} + s^{\lambda(n+1)}\mathcal{I}(W_{122})$$

and this completes the proof of (b). ■

4 Proof of Theorem A

This section is devoted to study the period function of the center at the origin of the vector field

$$(21) \qquad X_F(x,y) := y(x-1)\partial_x + (x + Fy^2)\partial_y \text{ with } F \in \big(0, \tfrac{1}{2}\big).$$

Note that this is precisely the subfamily of Loud's centers (1) that Theorem A refers to. Since the period annulus is unbounded, it is first of all necessary to compactify $\mathbb{R}^2$ and to this end we use the real projective plane $\mathbb{RP}^2 = \mathbb{R}^2 \cup L_\infty$. The outer boundary of the period annulus in $\mathbb{RP}^2$ (see Figure 5) is a polycycle made up with the straight line $L_1 = \{x = 1\}$ and a piece of the line at infinity L_∞. Taking $(x_0, y_0) = (1-x, y)$, let us consider the coordinates of $\mathbb{RP}^2$ given by

$$(x_1, y_1) = \left(\tfrac{1}{y_0}, \tfrac{x_0}{y_0}\right) \text{ and } (x_2, y_2) = \left(\tfrac{1}{x_0}, \tfrac{y_0}{x_0}\right).$$

It is easy to check that the expression of the vector field in these coordinates is

$$X_F(x_1, y_1) = \frac{1}{x_1}\big(x_1(-F - x_1^2 + x_1 y_1)\partial_{x_1} + y_1(1 - F - x_1^2 + x_1 y_1)\partial_{y_1}\big)$$

and

$$X_F(x_2, y_2) = \frac{1}{x_2}\big(-x_2y_2\partial_{x_2} + (-x_2 + x_2^2 + (F-1)y_2^2)\partial_{y_2}\big)$$

respectively. Note that $(x_1, y_1) = (0, 0)$ is a hyperbolic saddle of x_1X_F. However $(x_2, y_2) = (0, 0)$ is a degenerate singularity of x_2X_F and so we must perform a blow-up. The blow-up of $\mathbb{RP}^2$ at this singularity has an ambient space S_1 that can be described topologically as the connected sum of two copies of $\mathbb{RP}^2$. We can cover a neighbourhood of the exceptional divisor (that can be identified with $\mathbb{RP}^1$) with two charts coordinated by (t_1, x_2) and (s_1, y_2), where $y_2 = t_1x_2$ and $x_2 = s_1y_2$. Then one can easily verify that the pull-back of X_F in S_1 is given by

$$X_F(t_1, x_2) = \frac{1}{x_2}\big((-1 + x_2 + Ft_1^2x_2)\partial_{t_1} - t_1x_2^2\partial_{x_2}\big)$$

and

$$X_F(s_1, y_2) = \frac{1}{s_1y_2}\big(s_1(s_1 - Fy_2 - s_1^2y_2)\partial_{s_1} + y_2(-s_1 + (F-1)y_2 + s_1^2y_2)\partial_{y_2}\big)$$

respectively. Notice that $x_2X_F(t_1, x_2)$ has not any singularity along the exceptional divisor $x_2 = 0$. In the second chart, $s_1y_2X_F(s_1, y_2)$ still has a degenerate singularity at $(s_1, y_2) = (0, 0)$ and so we must blow-up again. We obtain in this way a new algebraic surface S_2, topologically equivalent to the connected sum of three projective planes, where the singularities of the pull-back of the foliation determined by X_F are all hyperbolic saddles. Indeed, we can cover a neighbourhood of the second exceptional divisor with two new charts coordinated by (s_1, t_2) and (s_2, y_2) so that $y_2 = t_2s_1$ and $s_1 = s_2y_2$. The expression of X_F in these charts is given by

$$X_F(s_1, t_2) = \frac{1}{s_1t_2}\big(s_1(1 - Ft_2 - s_1^2t_2)\partial_{s_1} + t_2(-2 + (2F-1)t_2 + 2s_1^2t_2)\partial_{t_2}\big)$$

and

$$X_F(s_2, y_2) = \frac{1}{s_2y_2}\big(s_2(1 - 2F + 2s_2 - 2s_2^2y_2^2)\partial_{s_2} + y_2(F - 1 - s_2 + s_2^2y_2^2)\partial_{y_2}\big)$$

respectively. At this point we rename the new coordinates in order to unify the notation and we also give their expressions in terms of the original (x, y) coordinates:

$$(u_0, v_0) = (y_0, x_0) = (y, 1-x) \qquad (u_3, v_3) = (s_1, t_2) = \left(\tfrac{1}{y}, \tfrac{y^2}{1-x}\right)$$
$$(u_1, v_1) = (y_1, x_1) = \left(\tfrac{1-x}{y}, \tfrac{1}{y}\right) \qquad (u_4, v_4) = (t_1, x_2) = \left(y, \tfrac{1}{1-x}\right)$$
$$(u_2, v_2) = (s_2, y_2) = \left(\tfrac{1-x}{y^2}, \tfrac{y}{1-x}\right)$$

Moreover, to study the period function associated to the center of (21), we introduce several auxiliary transverse sections (see Figure 6) at the desingularized polycycle, namely $\sigma^i : I \longrightarrow \Sigma_i$ for $i = 0, 1, \ldots, 5$. To make easier the application of the tools developed in the preceding sections, setting

$$X_F(u_i, v_i) = \frac{1}{u_i^{m_i}v_i^{n_i}}\big(u_iP_i(u_i, v_i)\partial_{u_i} + v_iQ_i(u_i, v_i)\partial_{v_i}\big)$$

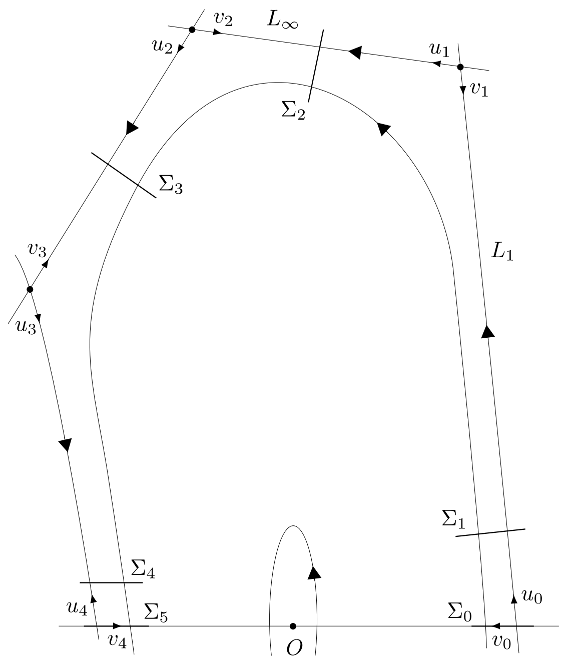

Figure 6: Desingularization of the outer boundary.

for $i = 1, 2, 3$, we summarize the relevant information for the passage through each saddle as follows:

$$
\begin{array}{lll}
P_1(u,v) = 1 - F + uv - v^2 & (m_1, n_1) = (0,1) & \sigma^1(s) = \left(\frac{s}{\eta}, \frac{1}{\eta}\right) \\
Q_1(u,v) = -F + uv - v^2 & \lambda_1 = \frac{F}{1-F} & \tau^1(s) = (1, s)
\end{array}
\tag{22}
$$

$$
\begin{array}{lll}
P_2(u,v) = 1 - 2F + 2u - 2u^2v^2 & (m_2, n_2) = (1,1) & \sigma^2(s) = (s, 1) \\
Q_2(u,v) = F - 1 - u + u^2v^2 & \lambda_2 = \frac{1-F}{1-2F} & \tau^2(s) = (1, s)
\end{array}
\tag{23}
$$

$$
\begin{array}{lll}
P_3(u,v) = 1 - Fv - u^2v & (m_3, n_3) = (1,1) & \sigma^3(s) = (s, 1) \\
Q_3(u,v) = -2 + (2F-1)v + 2u^2v & \lambda_3 = 2 & \tau^3(s) = \left(\frac{1}{\eta}, s\right)
\end{array}
\tag{24}
$$

These expressions, where we took an arbitrary $\eta > 0$, will be used to study the time function from Σ_1 to Σ_2, the one from Σ_2 to Σ_3 and the one from Σ_3 to Σ_4 respectively. Note in particular that $\tau^i = \sigma^{i+1}$. On the other hand, setting

$$X_F(u_i, v_i) = \frac{1}{v_i^{n_i}}\big(f_i(u_i, v_i)\partial_{u_i} + v_i g_i(u_i, v_i)\partial_{v_i}\big)$$

for $i=0$ and $i=4$, we have that

$$(25)\qquad \begin{aligned} &f_0(u,v)=1-v+Fu^2 &&\sigma^0(s)=(0,s) &&n_0=0\\ &g_0(u,v)=u &&\tau^0(s)=(\eta,s) \end{aligned}$$

$$(26)\qquad \begin{aligned} &f_4(u,v)=-1+v+Fu^2v &&\sigma^4(s)=(\eta,s) &&n_4=1\\ &g_4(u,v)=-uv &&\tau^4(s)=(0,s) \end{aligned}$$

These expressions will be used to study the time function associated to the regular passage from Σ_0 to Σ_1 and the one from Σ_4 to Σ_5.

Let us turn now to the study of the period function of the center. Note first that to this end it is enough to consider the time function from Σ_0 to Σ_5. Indeed, this is so because it gives half of the period of each periodic orbit due to the symmetry of X_F with respect to $\{y=0\}$. However, for the sake of convenience, we shall compute this function with respect to the transverse section Σ_1. With this aim in view, let us denote by T_0 the time function for $-X_F$ from Σ_1 to Σ_0. Moreover, for $i=1,2,3,4$, denote the Dulac and time mappings for X_F from Σ_i to Σ_{i+1} by R_i and T_i respectively. According to these definitions, it is clear that the period of the periodic orbit of X_F passing through the point $(1-s,\eta)\in\Sigma_1$ is precisely $2T(s;F)$, where

$$T(s)=T_0(s)+T_1(s)+T_2\big(R_1(s)\big)+T_3\big((R_2\circ R_1)(s)\big)+T_4\big((R_3\circ R_2\circ R_1)(s)\big).$$

As we said before, we shall apply the results of the preceding sections to study the mappings R_i and T_i. To do so we first define $\mathscr{J}:=\big(0,\frac{1}{2}\big)$ and set $\mathscr{J}\setminus\{\frac{1}{3}\}=\mathscr{J}_1\cup\mathscr{J}_2$ with

$$\mathscr{J}_1:=\big(\tfrac{1}{3},\tfrac{1}{2}\big) \text{ and } \mathscr{J}_2:=\big(0,\tfrac{1}{3}\big).$$

Let us consider first the passage from Σ_1 to Σ_2. Thus, taking (22) into account, the direct application of Theorem 3.3 and Lemma 2.13 yield to the following:

Lemma 4.1. *Set $\lambda_1=\frac{F}{1-F}$. If $F\in\mathscr{J}$ then $R_1(s;F)=s^{\lambda_1}\big(\rho_1^1+\rho_2^1s^{\lambda_1}+s^{\lambda_1}\mathcal{I}(\mathscr{J})\big)$ with $\rho_1^1=\Big(\frac{F}{1+\eta^2F}\Big)^{\frac{1}{2(1-F)}}$. Moreover, setting*

$$\Delta_1^1=\tfrac{1}{\sqrt{F}}\arctan\Big(\tfrac{1}{\eta\sqrt{F}}\Big) \text{ and } \Delta_{12}^1=-\tfrac{1}{F}\Big(\tfrac{F}{1+\eta^2F}\Big)^{\frac{1}{2(1-F)}},$$

the following holds:

(a) If $F\in\mathscr{J}_1$ then $T_1(s;F)=\Delta_1^1+\Delta_{12}^1s^{\lambda_1}+\Delta_{121}^1s+s\mathcal{I}(\mathscr{J}_1)$ with

$$\Delta_{121}^1=\big(1+\eta^2F\big)^{-\frac{1}{2F}}\int_0^{1/\eta}(F+x^2)^{\frac{1}{2F}-2}\frac{dx}{x^{\frac{1}{F}-2}}.$$

(b) If $F\in\mathscr{J}_2$ then $T_1(s;F)=\Delta_1^1+\Delta_{12}^1s^{\lambda_1}+\Delta_{122}^1s^{2\lambda_1}+s^{2\lambda_1}\mathcal{I}(\mathscr{J}_2)$.

It is important to mention that the family of vector fields under consideration verifies the FLP because it has a Darboux first integral (see [12] for instance). To study the passage from Σ_2 to Σ_3 we apply Theorem 3.1 and Lemma 2.13. From (23) it easily follows:

Lemma 4.2. *Set $\lambda_2=\frac{F-1}{2F-1}$. If $F\in\mathscr{J}$ then $R_2(s;F)=s^{\lambda_2}\big(\rho_1^2+\mathcal{I}(\mathscr{J})\big)$ with $\rho_1^2=\Big(\frac{2F-1}{2F-3}\Big)^{\frac{1}{4F-2}}$. Moreover, setting $\Delta_1^2=\frac{1}{F}$, the following holds:*

(a) If $F\in\mathscr{J}_1$ then $T_2(s;F)=\Delta_1^2s+\Delta_{11}^2s^2+s^2\mathcal{I}(\mathscr{J}_1)$.

(*b*) *If* $F\in\mathcal{J}_2$ *then* $T_2(s;F)=\Delta_1^2 s+\Delta_{12}^2 s^{\lambda_2}+s^{\lambda_2}\mathcal{I}(\mathcal{J}_2)$ *with*

$$\Delta_{12}^2=-\frac{1}{F}+\frac{1}{1-2F}\int_0^1\left(\left(1+\frac{2x}{1-2F}\right)^{\frac{4F-1}{2-4F}}-1\right)\frac{dx}{x^{\frac{F-1}{2F-1}}}.$$

It remains only the passage from Σ_3 to Σ_4. In this case for our purpose it suffices the first order expansion of the time function. Thus, on account of (24), Theorem 2.7 and Lemma 2.13 show that:

Lemma 4.3. *Set* $\lambda_3=2$. *If* $F\in\mathcal{J}$ *then* $R_3(s;F)=s^{\lambda_3}\big(\rho_1^3+\mathcal{I}(\mathcal{J})\big)$ *and* $T_3(s;F)=s\big(\Delta_1^3+\mathcal{I}(\mathcal{J})\big)$, *where*

$$\Delta_1^3=(3-2F)^{\frac{1}{4F-2}}\int_0^1\Big(2+(1-2F)x\Big)^{\frac{4F-1}{2-4F}}\frac{dx}{\sqrt{x}}.$$

We can now gather all this to obtain the expansion of the period function. This is done in the following result, which refers to the time function T. Recall that $T(s;F)$ is precisely half of the period of the periodic orbit of X_F passing through the point $(1-s,\eta)\in\Sigma_1$.

Proposition 4.4. *Setting* $\Delta_1=\frac{\pi}{2\sqrt{F}}$, *the following holds:*

(*a*) *If* $F\in\mathcal{J}_1$ *then* $T(s;F)=\Delta_1+\Delta_2 s+s\mathcal{I}(\mathcal{J}_1)$ *with*

$$\Delta_2=\frac{1}{2}\sqrt{\frac{\pi}{F}}\,\frac{\Gamma\left(\frac{3F-1}{2F}\right)}{\Gamma\left(\frac{4F-1}{2F}\right)}.$$

(*b*) *If* $F\in\mathcal{J}_2$ *then* $T(s;F)=\Delta_1+\Delta_3 s^{\frac{F}{1-2F}}+s^{\frac{F}{1-2F}}\mathcal{I}(\mathcal{J}_2)$ *with*

$$\Delta_3=\frac{F^{\frac{1}{2-4F}}}{1-2F}\int_0^\infty\left(\left(1+\frac{2x}{1-2F}\right)^{\frac{4F-1}{2-4F}}-1\right)\frac{dx}{x^{\frac{F-1}{2F-1}}}.$$

Proof. By means of the transverse sections introduced after the desingularization of the polycycle we can split up the time function as

$$T(s)=T_0(s)+T_1(s)+T_2\big(R_1(s)\big)+T_3\big(R_2(R_1(s))\big)+T_4\big(\widehat{R}_3(s)\big),$$

where $\widehat{R}_3:=R_3\circ R_2\circ R_1$. Let us point out that T depends only on F but T_i and R_i depend on η as well. This will be the key point in order to simplify the leading coefficient of its expansion.

The mappings T_1, T_2 and T_3 are associated to the passage through saddles, whereas T_0 and T_4 correspond to "regular" passages. The expansion of the latter ones at $s=0$ follows by applying Lemma 2.12. Indeed, taking (25) and (26) into account we obtain

$$T_0(s)=\Delta_1^0+\Delta_2^0 s+s\mathcal{I}_0(\mathcal{J})\ \text{ and }\ T_4(s)=s\big(\Delta_1^4+\mathcal{I}_0(\mathcal{J})\big)\tag{27}$$

respectively. Lemma 2.12 provides also the concrete expression of Δ_1^0, Δ_2^0 and Δ_1^4 but this is not relevant for our purposes. We shall use however that these coefficients tend to zero as $\eta\longrightarrow 0$. (This is so because then Σ_1 collapses to Σ_0 and Σ_4 collapses to Σ_5.) On the other hand, Lemma 4.1, 4.2 and 4.3 provide respectively the developments of R_1, R_2 and R_3. Taking them into account, by applying Lemma 2.8 we obtain

$$R_2\big(R_1(s)\big)=s^{\lambda_1\lambda_2}\big(\rho_1^2(\rho_1^1)^{\lambda_2}+\mathcal{I}(\mathcal{J})\big)\ \text{ and }\ \widehat{R}_3(s)=s^{2\lambda_1\lambda_2}\big(\widehat{\rho}+\mathcal{I}(\mathcal{J})\big)\ \text{ with }\widehat{\rho}=\rho_1^3(\rho_1^2)^2(\rho_1^1)^{2\lambda_2}.\tag{28}$$

By using Lemma 2.8 once again, the first equality above and Lemma 4.3 yield to

$$T_3\big(R_2(R_1(s))\big)=s^{\lambda_1\lambda_2}\big(\Delta_1^3\rho_1^2(\rho_1^1)^{\lambda_2}+\mathcal{I}(\mathcal{J})\big).\tag{29}$$

Let us consider the case $F \in \mathcal{J}_1$ first. Then, due to $R_1(s) = s^{\lambda_1}\big(\rho_1^1+\rho_2^1 s^{\lambda_1}+s^{\lambda_1}\mathcal{I}(\mathcal{J})\big) = s^{\lambda_1}\big(\rho_1^1+\mathcal{I}(\mathcal{J})\big)$, taking (a) in Lemma 4.2 into account we get

$$\begin{aligned} T_2\big(R_1(s)\big) &= \Delta_1^2 s^{\lambda_1}\big(\rho_1^1+\rho_2^1 s^{\lambda_1}+s^{\lambda_1}\mathcal{I}(\mathcal{J})\big) + \Delta_{11}^2 s^{2\lambda_1}\big((\rho_1^1)^2+\mathcal{I}(\mathcal{J})\big) + s^{2\lambda_1}\mathcal{I}(\mathcal{J}_1) \\ &= \Delta_1^2\rho_1^1 s^{\lambda_1} + \big(\Delta_1^2\rho_2^1+\Delta_{11}^2(\rho_1^1)^2\big)s^{2\lambda_1} + s^{2\lambda_1}\mathcal{I}(\mathcal{J}_1) \\ &= \Delta_1^2\rho_1^1 s^{\lambda_1} + s\mathcal{I}(\mathcal{J}_1). \end{aligned}$$

Here we use Lemma 2.8 in the first equality and the fact that $2\lambda_1 > 1$ for $F \in \mathcal{J}_1$ in the third one. On the other hand from (29) it follows that $T_3\big(R_2(R_1(s))\big) = s\mathcal{I}(\mathcal{J}_1)$ because one can easily verify that $\lambda_1\lambda_2 > 1$ for $F \in \mathcal{J}_1$. Therefore, using also the expansion of T_1 given by (a) in Lemma 4.1 and the ones in (27),

$$\begin{aligned} T(s) = &\underbrace{\Delta_1^0+\Delta_2^0 s+s\mathcal{I}_0(\mathcal{J})}_{T_0} + \underbrace{\Delta_1^1+\Delta_{12}^1 s^{\lambda_1}+\Delta_{121}^1 s+s\mathcal{I}(\mathcal{J}_1)}_{T_1} \\ &+ \underbrace{\Delta_1^2\rho_1^1 s^{\lambda_1}+s\mathcal{I}(\mathcal{J}_1)}_{T_2} + \underbrace{s\mathcal{I}(\mathcal{J}_1)}_{T_3} + \underbrace{s\big(\Delta_1^4+\mathcal{I}_0(\mathcal{J})\big)}_{T_4} \\ = &\underbrace{\Delta_1^0+\Delta_1^1}_{\Delta_1} + \big(\Delta_{12}^1+\Delta_1^2\rho_1^1\big)s^{\lambda_1} + \underbrace{\big(\Delta_2^0+\Delta_{121}^1+\Delta_1^4\big)}_{\Delta_2} s + s\mathcal{I}(\mathcal{J}_1). \end{aligned}$$

This shows the validity of the expansion for $F \in \mathcal{J}_1$ because, from Lemma 4.1 and 4.2, one can easily check that $\Delta_{12}^1+\Delta_1^2\rho_1^1 = 0$. Recall on the other hand that $T(s;F)$ does not depend on η and so neither do the coefficients Δ_1 and Δ_2. Thus, since Δ_1^0, Δ_2^0 and Δ_1^4 tend to zero as $\eta \longrightarrow 0$,

$$\Delta_1 = \lim_{\eta\longrightarrow 0^+}(\Delta_1^0+\Delta_1^1) = \lim_{\eta\longrightarrow 0^+}\frac{1}{\sqrt{F}}\arctan\left(\frac{1}{\eta\sqrt{F}}\right) = \frac{\pi}{2\sqrt{F}}$$

and

$$\begin{aligned} \Delta_2 &= \lim_{\eta\longrightarrow 0^+}(\Delta_2^0+\Delta_{121}^1+\Delta_1^4) = \lim_{\eta\longrightarrow 0^+}\big(1+\eta^2 F\big)^{-\frac{1}{2F}}\int_0^{1/\eta}(F+x^2)^{\frac{1}{2F}-2}\frac{dx}{x^{\frac{1}{F}-2}} \\ &= \int_0^{+\infty}(F+x^2)^{\frac{1}{2F}-2}\frac{dx}{x^{\frac{1}{F}-2}} = \frac{1}{2}\sqrt{\frac{\pi}{F}}\,\frac{\Gamma\left(\frac{3F-1}{2F}\right)}{\Gamma\left(\frac{4F-1}{2F}\right)}. \end{aligned}$$

This completes the proof of (a). Let us turn now to the assertion in (b) and so assume that $F \in \mathcal{J}_2$. In this case, by (b) in Lemma 4.2 and using again that $R_1(s) = s^{\lambda_1}\big(\rho_1^1+\rho_2^1 s^{\lambda_1}+s^{\lambda_1}\mathcal{I}(\mathcal{J})\big) = s^{\lambda_1}\big(\rho_1^1+\mathcal{I}(\mathcal{J})\big)$, we obtain

$$\begin{aligned} T_2\big(R_1(s)\big) &= \Delta_1^2 s^{\lambda_1}\big(\rho_1^1+\rho_2^1 s^{\lambda_1}+s^{\lambda_1}\mathcal{I}(\mathcal{J})\big) + \Delta_{12}^2 s^{\lambda_1\lambda_2}\big((\rho_1^1)^{\lambda_2}+\mathcal{I}(\mathcal{J})\big) + s^{\lambda_1\lambda_2}\mathcal{I}(\mathcal{J}_2) \\ &= \Delta_1^2\rho_1^1 s^{\lambda_1} + \Delta_{12}^2(\rho_1^1)^{\lambda_2}s^{\lambda_1\lambda_2} + s^{2\lambda_1}\big(\rho_2^1+\mathcal{I}(\mathcal{J})\big) + s^{\lambda_1\lambda_2}\mathcal{I}(\mathcal{J}_2) \\ &= \Delta_1^2\rho_1^1 s^{\lambda_1} + \Delta_{12}^2(\rho_1^1)^{\lambda_2}s^{\lambda_1\lambda_2} + s^{\lambda_1\lambda_2}\mathcal{I}(\mathcal{J}_2). \end{aligned}$$

Here we used Lemma 2.8 in the first equality and that $s^{2\lambda_1} \in s^{\lambda_1\lambda_2}\mathcal{I}(\mathcal{J}_2)$, due to $\lambda_2 < 2$ for $F \in \mathcal{J}_2$, in the third one. Gathering this with the expansion of T_1 given by (b) in Lemma 4.1 and the ones in (27) and (29) we obtain

$$
\begin{aligned}
T(s) = & \underbrace{\Delta_1^0 + \Delta_2^0 s + s\mathcal{I}_0(\mathcal{J})}_{T_0} + \underbrace{\Delta_1^1 + \Delta_{12}^1 s^{\lambda_1} + s^{2\lambda_1}\big(\Delta_{122}^1 + \mathcal{I}(\mathcal{J}_2)\big)}_{T_1} \\
& + \underbrace{\Delta_1^2\rho_1^1 s^{\lambda_1} + s^{\lambda_1\lambda_2}\big(\Delta_{12}^2(\rho_1^1)^{\lambda_2} + \mathcal{I}(\mathcal{J}_2)\big)}_{T_2} + \underbrace{s^{\lambda_1\lambda_2}\big(\Delta_1^3\rho_1^2(\rho_1^1)^{\lambda_2} + \mathcal{I}(\mathcal{J})\big)}_{T_3} + \underbrace{s\big(\Delta_1^4 + \mathcal{I}_0(\mathcal{J})\big)}_{T_4} \\
= & \underbrace{\Delta_1^0 + \Delta_1^1}_{\Delta_1} + \big(\Delta_{12}^1 + \Delta_1^2\rho_1^1\big)s^{\lambda_1} + \underbrace{(\rho_1^1)^{\lambda_2}\big(\Delta_{12}^2 + \Delta_1^3\rho_1^2\big)}_{\Delta_3} s^{\lambda_1\lambda_2} + s^{\lambda_1\lambda_2}\mathcal{I}(\mathcal{J}_2).
\end{aligned}
$$

In the second equality above we use that s and $s^{2\lambda_1}$ belong to $s^{\lambda_1\lambda_2}\mathcal{I}(\mathcal{J}_2)$ due to the fact that $\lambda_1\lambda_2 < 2\lambda_1 < 1$ for $F \in \mathcal{J}_2$. This proves the validity of the development for $F \in \mathcal{J}_2$ because $\lambda_1\lambda_2 = \frac{F}{1-2F}$ and we showed previously that $\Delta_{12}^1 + \Delta_1^2\rho_1^1 = 0$. Consequently it only remains to compute the coefficient Δ_3. With this aim in view notice first that, from Lemma 4.2 and 4.3,

$$
\begin{aligned}
\Delta_{12}^2 + \Delta_1^3\rho_1^2 = & -\frac{1}{F} + \frac{1}{1-2F}\int_0^1 \left(\left(1 + \frac{2x}{1-2F}\right)^{\frac{4F-1}{2-4F}} - 1\right)\frac{dx}{x^{\frac{F-1}{2F-1}}} \\
& + (1-2F)^{\frac{1}{4F-2}}\int_0^1 \Big(2 + (1-2F)x\Big)^{\frac{4F-1}{2-4F}}\frac{dx}{\sqrt{x}} \\
= & -\frac{1}{F} + \frac{1}{1-2F}\int_0^1 \left(\left(1 + \frac{2x}{1-2F}\right)^{\frac{4F-1}{2-4F}} - 1\right)\frac{dx}{x^{\frac{F-1}{2F-1}}} \\
& + \frac{1}{1-2F}\int_1^{+\infty}\left(1 + \frac{2y}{1-2F}\right)^{\frac{4F-1}{2-4F}}\frac{dy}{y^{\frac{F-1}{2F-1}}} \\
= & \frac{1}{1-2F}\int_0^{+\infty}\left(\left(1 + \frac{2x}{1-2F}\right)^{\frac{4F-1}{2-4F}} - 1\right)\frac{dx}{x^{\frac{F-1}{2F-1}}},
\end{aligned}
$$

where the second equality follows after performing the change $y = 1/x$ in the second integral and the last one by using that $\frac{1}{F} = \frac{1}{1-2F}\int_1^\infty x^{-\frac{F-1}{2F-1}}dx$. Finally, since $(\rho_1^1)^{\lambda_2} \longrightarrow F^{\frac{1}{2-4F}}$ as $\eta \longrightarrow 0$, the result follows. ■

Proof of Theorem A. Let us prove the assertions in (a) and (b) first. More concretely, we have to show that if $F \in (0, \frac{1}{4})$ (respectively, $F \in (\frac{1}{4}, \frac{1}{2})$) then the period function of the center at the origin of (21) is monotonous increasing (respectively, decreasing) near the outer boundary of its period annulus. To this end, since the vector field (21) is symmetric with respect to $\{y = 0\}$, it suffices to study the time function considered in Proposition 4.4. This is so because the period of the periodic orbit of (21) passing through the point $(1-s, \eta) \in \mathbb{R}^2$ is precisely $2T(s; F)$. Note in addition that this periodic orbit *approaches* to the outer boundary of the period annulus as s *decreases* to zero. Taking this into account, we must prove that if $F \in (0, \frac{1}{4})$ (respectively, $F \in (\frac{1}{4}, \frac{1}{2})$) then there exists $\varepsilon > 0$ such that $T'(s; F)$ is negative (respectively, positive) for $s \in (0, \varepsilon)$.

Let us assume first that $F \in \mathcal{J}_2 = (0, \frac{1}{3})$. In this case, setting $\lambda = \frac{F}{1-2F}$, from (b) in Proposition 4.4 it follows that $T(s; F) = \Delta_1 + \Delta_3 s^\lambda + s^\lambda f(s; F)$, where $f \in \mathcal{I}(\mathcal{J}_2)$ and

$$
\Delta_3(F) = \frac{F^{\frac{1}{2-4F}}}{1-2F}\int_0^\infty \left(\left(1 + \frac{2x}{1-2F}\right)^{\frac{4F-1}{2-4F}} - 1\right)\frac{dx}{x^{\frac{F-1}{2F-1}}}.
$$

Accordingly $T'(s; F) = \lambda\Delta_3 s^{\lambda-1} + \lambda s^{\lambda-1}f(s; F) + s^\lambda f'(s; F)$ and then, taking Definition 2.4 into account,

$$
\frac{T'(s; F)}{s^{\lambda-1}} = \lambda\Delta_3 + \lambda f(s; F) + sf'(s; F) \longrightarrow \lambda\Delta_3 \text{ as } s \longrightarrow 0.
$$

Due to $\lambda > 0$, this shows that if $s \approx 0$ then $T'(s; F)$ has the same signum as $\Delta_3(F)$. Since one can easily check that $\Delta_3(F) < 0$ for $F \in (0, \frac{1}{4})$ and $\Delta_3(F) > 0$ for $F \in (\frac{1}{4}, \frac{1}{3})$, this proves the result for $F \in (0, \frac{1}{3}) \setminus \{\frac{1}{4}\}$. (Note that $\Delta_3(\frac{1}{4}) = 0$ because $F = \frac{1}{4}$ corresponds to an isochronous center.)

Let us consider now the case $F \in \mathcal{J}_1 = (\frac{1}{3}, \frac{1}{2})$. Then, by applying (*a*) in Proposition 4.4, we can assert that $T(s; F) = \Delta_1 + \Delta_2 s + s g(s; F)$, where $g \in \mathcal{I}(\mathcal{J}_1)$ and

$$\Delta_2(F) = \frac{1}{2}\sqrt{\frac{\pi}{F}}\,\frac{\Gamma\left(\frac{3F-1}{2F}\right)}{\Gamma\left(\frac{4F-1}{2F}\right)}.$$

Thus $T'(s; F) = \Delta_2 + g(s; F) + s g'(s; F)$ and hence, on account of Definition 2.4, $T'(s; F) \longrightarrow \Delta_2$ as s tends to 0. Since it is clear that $\Delta_2(F) > 0$ for $F > \frac{1}{3}$, this shows that $T'(s; F)$ is positive for $s \approx 0$.

It remains to consider $F = \frac{1}{3}$. This case follows from the results of Zhao [14]. In that paper the author studies the period function of a subfamily of quadratic centers that intersects the one in (21) at $F = \frac{1}{3}$. Taking advantage of his result we can assert that the period function for $F = \frac{1}{3}$ is globally (i.e., in the whole period annulus) monotonous decreasing. This completes the proof of (*a*) and (*b*).

We can now prove the last assertion in the statement. To this end consider the period function of the center at the origin of the differential system (1), which depends on the parameter $\mu := (D, F) \in \mathbb{R}^2$. Recall that system (21) corresponds to $\mu \in \{0\} \times \big(0, \frac{1}{2}\big)$. Fix some $\widehat{\mu} \in \{0\} \times \big[\frac{1}{4}, \frac{1}{2}\big]$, see Figure 1, and let us show that it is a bifurcation value of the period function at the outer boundary. To this end it is enough to verify that any neighbourhood $U \subset \mathbb{R}^2$ of $\widehat{\mu}$ contains two parameters μ_+ and μ_- such that the corresponding period functions have different behaviour near the outer boundary, let us say increasing for μ_+ and decreasing for μ_-. The existence of μ_+ is guaranteed by (*a*) in Theorem 1.1 (see Figure 1), whereas the existence of μ_- follows precisely from (*b*) in the present result, which has already been proved. ∎

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