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On the time function of the Dulac map for families of meromorphic vector fields

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Abstract
Given an analytic family of vector fields in $\mathbb{R}^2$ having a saddle point, we study the asymptotic development of the time function along the union of the two separatrices. We obtain a result (depending uniformly on the parameters) which we apply to investigate the bifurcation of critical periods of quadratic centres.

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1. Introduction and setting of the problem

The aim of this paper is to calculate the first terms in the development of the time function of the passage around a saddle point of a family of meromorphic vector fields. A good scale permitting a uniform development (with respect to the parameter) involves the compensator function of Ecalle–Roussarie. The study was motivated by the necessity of such a result for the investigation of the bifurcation diagram of the period function of quadratic centres. Let $\{X_\mu, \mu \in \Lambda\}$ be the family of quadratic vector fields having a centre at the origin (parametrized by means of the coefficients), the problem is to decompose the parameter space as $\Lambda = \bigcup V_i$ in such a way that if μ_1 and μ_2 belong to the same set V_i then the period function of the centre of X_{μ_1} and the one of X_{μ_2} are qualitatively the same. With this we mean that their critical periods are equal in number, character (minimum or maximum) and distribution.

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Chicone [2] has conjectured that if a quadratic system has a centre with a period function which is not monotonic then, by an affine transformation and a constant rescaling of time, it can be brought to the Loud normal form:

$$\begin{cases} \dot{x} = -y + Bxy, \\ \dot{y} = x + Dx^2 + Fy^2, \end{cases}$$

and that the period function of these centres has at most two critical periods. We remark that if the first part of this conjecture is true then, in order to solve the problem posed above, it is enough to study the bifurcation diagram of the Loud systems. In fact, there is much analytic evidence that the conjecture is true (see [7, 17, 28] for instance).

We are studying precisely the bifurcation diagram of the period function of the Loud systems. Numerically, in the parameter space there are regions with two, one and zero critical periods. The first step in order to solve the problem is to find the boundaries of these regions. Some of these boundaries correspond to the bifurcation of critical periods from the boundary of the period annulus. The bifurcation from the inner boundary (i.e. the centre) was already solved by Chicone and Jacobs [5]. Our goal is to investigate the bifurcation from the outer boundary, which in the Poincaré disc is a polycycle. The polycycle consists of regular trajectories and critical points with a hyperbolic sector, which after the desingularization process give rise to saddles and saddle-nodes. It is necessary therefore to have the asymptotic development of the time function associated to this type of critical points. In addition, since these critical points are eventually located at infinity (i.e. the equator of the Poincaré disc), we must also study those vector fields that are the product of a meromorphic function with an analytic vector field. Many polycycles appearing in the Loud family present two points at infinity. Moreover, by the symmetry in Loud systems, it is enough to study half of the period, which involves only one passage through the corner (see section 5). The main result of this paper, namely theorem A, deals precisely with this situation. Theorem A allows us to compute the first nontrivial coefficient of this development in the case that the critical point is an orbitally linearizable saddle located at infinity. By means of this coefficient it will be possible to determine from which parameters the critical periods bifurcate. In order to determine the exact number of critical periods that bifurcate it will be necessary to compute, at least, the next two coefficients. We shall also need a similar study for resonant saddles and saddle-nodes. Once we have these tools we will be able to determine the bifurcation curves. They will split up the parameter space into several regions, and we will have an approximate idea of the number of critical periods of the period function associated to the parameters of each region.

As has already been noted, the goal of this paper is to prove theorem A. We devote a subsequent paper (see [15]) to treat specifically the problem concerning the bifurcation diagram. Nevertheless, in this paper, we study the period function of a subfamily of quadratic centres to illustrate the application of theorem A. Let us finally point out that the techniques developed to approach this problem will allow us to investigate other situations. Our techniques can be used to study the period function of degenerate centres or, more generally, continuums of periodic orbits.

Questions related to the behaviour of the period function have been extensively studied by a number of authors. Let us quote for instance the problems of isochronicity (see [6, 13, 14]), monotonicity (see [3, 4, 23]) or bifurcation of critical periods (see [5, 20, 25]). Most of the work on plane polynomial vector fields, including the present paper, is related to the questions surrounding Hilbert's 16th problem (see [1,10,11,19,27] and references therein) and its various weakened versions (i.e. problems which ask for the number of occurrences of some property in a system given by polynomials in terms of the degrees of the defining polynomials).

The paper is organized as follows. In section 2 there are the definitions of the notions that we shall use and the statement of theorem A. In section 3 we develop some tools that will be used in the proof of theorem A, which is given in section 4. Finally, in section 5, we apply this result to the problem that was our initial motivation, the bifurcation of critical periods in the quadratic centres. The precise definitions concerning this particular problem are deferred to this section.

2. Definitions and statement of the result

Let W be an open set of $\mathbb{R}^m$ and let $\{\tilde{X}_\mu, \mu \in W\}$ be an analytic family of vector fields defined on some open set V of $\mathbb{R}^2$. Assume that each vector field $\tilde{X}_\mu$ has a hyperbolic saddle p_μ as the unique critical point inside V. In this situation, it is well known that there exist exactly two analytic transverse invariant curves $\mathcal{S}_\mu$ and $\mathcal{T}_\mu$, the stable and unstable manifolds, passing through p_μ (depending also analytically on μ). We consider an analytic family of meromorphic vector fields X_μ proportional to $\tilde{X}_\mu$ with a pole of order $n > 0$ along $\mathcal{T}_\mu$. We can take a coordinate system (u, v, μ) on $V \times W \subset \mathbb{R}^{2+m}$ such that $p_\mu = (0, 0, \mu)$, $\mathcal{S}_\mu = \{(u, v, \mu) : u = 0\}$ and $\mathcal{T}_\mu = \{(u, v, \mu) : v = 0\}$. In these coordinates, the family $\{X_\mu, \mu \in W\}$ can be written as

$$X_\mu(u, v) = \frac{1}{v^n}(uP(u, v; \mu)\partial_u + vQ(u, v; \mu)\partial_v), \tag{1}$$

where P and Q are analytic functions such that $P(u, 0; \mu) \neq 0$ and $Q(0, v; \mu) \neq 0$ for any $(0, v, \mu) \in \mathcal{S}_\mu$ and $(u, 0, \mu) \in \mathcal{T}_\mu$. Moreover, by hypothesis, we have that

$$\lambda(\mu) := -\frac{Q(0, 0; \mu)}{P(0, 0; \mu)} > 0.$$

Throughout the paper, we shall deal with functions that depend on 'spatial' coordinates and on parameters. To distinguish them we use the following convention. A function F defined on $V \times W$ will be written as $F(x, y; \mu)$, meaning that $(x, y) \in V$ and $\mu \in W$.

The family $\{X_\mu, \mu \in W\}$ can be thought of as a single vector field X defined on $V \times W \subset \mathbb{R}^{2+m}$ whose trajectories are contained inside the submanifolds $\{\mu = \text{const.}\}$. Let $\sigma : I \times W \to \Sigma_\sigma$ and $\tau : I \times W \to \Sigma_\tau$ be two analytic transverse sections to X defined by

$$\sigma(s; \mu) = (\sigma_1(s; \mu), \sigma_2(s; \mu); \mu) \text{ and } \tau(s; \mu) = (\tau_1(s; \mu), \tau_2(s; \mu); \mu)$$

such that $\sigma(0; \mu) \in \mathcal{S}_\mu$ and $\tau(0; \mu) \in \mathcal{T}_\mu$. Here, I denotes a small interval of $\mathbb{R}$ containing 0.

We denote the Dulac and time mappings between the transverse sections Σ_σ and Σ_τ by R and T, respectively. More precisely (see figure 1), if $\varphi(t, (u_0, v_0); \mu)$ is the solution of X_μ passing through (u_0, v_0) at $t = 0$, for each $s > 0$ we define $R(s; \mu)$ and $T(s; \mu)$ by means of the relation

$$\varphi(T(s; \mu), \sigma(s); \mu) = \tau(R(s; \mu)). \tag{2}$$

It is well known that $R(s; \mu)$ and $T(s; \mu)$ are analytic on (s, μ) for $s > 0$ small enough. But both functions can fail to be analytic on $s = 0$. For the Dulac map this problem has been extensively treated (see [8, 12] for instance). In relation to the time function, Saavedra proves in [21, 22] that, for a given vector field, $T(s)$ has an asymptotic expansion in $s = 0$ similar to the series of Dulac but with negative powers. Unfortunately, the case of families is not treated there. Our purpose is to give an uniform asymptotic expansion and to compute the first nontrivial term. To this end, we shall require that the family of vector fields $\{X_\mu, \mu \in W\}$ satisfies the following.

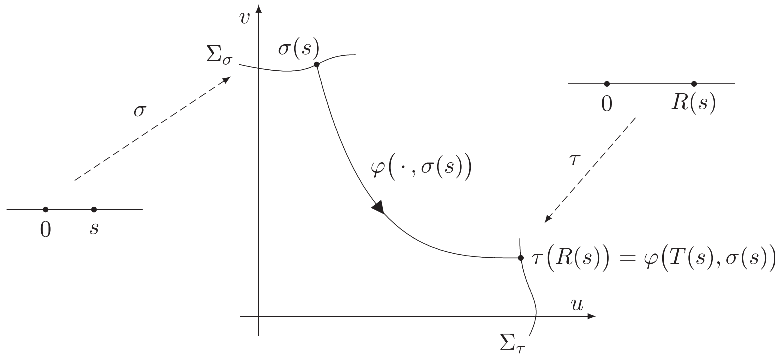

Figure 1. Definition of T and R.

Definition 2.1. *We will say that $\{X_\mu, \mu \in W\}$ verifies the family linearization property (FLP in short) if there exist an open set $U \subset \mathbb{R}^2$ containing the origin and an analytic local diffeomorphism $\Phi : U \times W \to V \times W$ of the form $\Phi(x, y; \mu) = \big(x + h.o.t., y + h.o.t., \mu\big)$ such that*

$$X_\mu = \Phi_* \left(\frac{1}{f(x, y; \mu)} (x\partial_x - \lambda(\mu) y \partial_y) \right),$$

where f is an analytic function on $U \times W$.

Remark 2.2. Since, by assumption, the invariant manifolds of the saddle point are located on the axes, from definition 2.1 it follows easily that

$$\Phi_1(x, y; \mu) = x\psi_1(x, y; \mu) \qquad \text{and} \qquad \Phi_2(x, y; \mu) = y\psi_2(x, y; \mu)$$

with $\psi_i(0, 0; \mu) \equiv 1$. In addition, taking into account that X_μ has a pole of order n on $v = 0$, it turns out that $f(x, y; \mu) = y^n g(x, y; \mu)$ where g is an analytic function with $g(0, 0; \mu) \neq 0$.

Remark 2.3. It is easy to show that the family of meromorphic vector fields $\{X_\mu, \mu \in W\}$ defined in (1) verifies FLP if it has a Darboux first integral

$$H_\mu(x, y) = f_1(x, y; \mu)^{\beta_1(\mu)} \cdots f_k(x, y; \mu)^{\beta_k(\mu)},$$

where f_j and β_j are analytic functions on $V \times W$ and W, respectively. Note that the Loud family has a Darboux first integral if $FB(F - B)(2F - B) \neq 0$ (see [24] for instance), so the FLP is verified in these cases.

In order to control the rest in the asymptotic expansions we need the following definition, which is an adaptation of the one used by Mourtada [16] and Roussarie [19].

Definition 2.4. *Let W be an open subset of $\mathbb{R}^m$. We denote by $\mathcal{I}(W)$ the set of germs of analytic functions $h(s; \mu)$ defined on $(0, \varepsilon) \times W$ for some $\varepsilon > 0$ such that*

$$\lim_{s \to 0} h(s; \mu) = 0 \qquad \textit{and} \qquad \lim_{s \to 0} s \frac{\partial h(s; \mu)}{\partial s} = 0$$

uniformly (on μ) on every compact subset of W.

Definition 2.5. *The function defined for $s > 0$ and $\alpha \in \mathbb{R}$ by means of*

$$\omega(s; \alpha) = \begin{cases} \dfrac{s^{\alpha-1} - 1}{\alpha - 1} & \textit{if } \alpha \neq 1, \\ \log s & \textit{if } \alpha = 1, \end{cases}$$

is called the Roussarie–Ecalle compensator.

Remark 2.6. In [18], it is proved that $s\omega(s;\alpha) \in \mathcal{I}((0,+\infty))$.

In order to simplify the expressions that appear in the statement of the main result, we introduce the functions

$$L(u;\mu) := \exp\left(\int_{\sigma_2(0)}^{u}\left(\frac{P(0,y)}{Q(0,y)}+\frac{1}{\lambda}\right)\frac{\mathrm{d}y}{y}\right),$$
$$M(u;\mu) := \exp\left(\int_{0}^{u}\left(\frac{Q(x,0)}{P(x,0)}+\lambda\right)\frac{\mathrm{d}x}{x}\right)$$

and the covering of the parameter space W given by the open subsets

$$W_1 := \left\{\mu \in W : \lambda > \frac{1}{n}\right\},\ W_2 := \left\{\mu \in W : \lambda < \frac{1}{n}\right\},\ W_3 := \left\{\mu \in W : \frac{1}{n+1} < \lambda < \frac{2}{n}\right\}.$$

Theorem A. *Let $\{X_\mu, \mu \in W\}$ be the family of vector fields defined in* (1) *and assume that it verifies FLP. Let R and T be respectively the Dulac map and the time function associated to the transverse sections Σ_σ and Σ_τ as introduced in* (2). *Denote*

$$\rho(\mu) = \frac{\sigma_1'(0)^\lambda \sigma_2(0)}{\tau_2'(0)\tau_1(0)^\lambda} L(0)^\lambda M(\tau_1(0)) \qquad \textit{and} \qquad \Delta_0(\mu) = \int_{\sigma_2(0)}^{0} \frac{v^{n-1}}{Q(0,v)}\,\mathrm{d}v.$$

Then $R(s;\mu) = \rho(\mu)s^\lambda + s^\lambda f_0(s;\mu)$ with $f_0 \in \mathcal{I}(W)$. In addition, the time function $T(s;\mu)$ verifies the following:

(a) If $\mu \in W_1$ then $T(s;\mu) = \Delta_0(\mu) + \Delta_1(\mu)s + sf_1(s;\mu)$, where $f_1 \in \mathcal{I}(W_1)$ and

$$\Delta_1(\mu) = -\frac{\sigma_2'(0)\,\sigma_2(0)^{n-1}}{Q(0,\sigma_2(0))} + \sigma_1'(0)\sigma_2(0)^{1/\lambda}\int_0^{\sigma_2(0)} \frac{Q_u(0,v)L(v)v^{n-1/\lambda}}{Q(0,v)^2}\frac{\mathrm{d}v}{v}.$$

(b) If $\mu \in W_2$ then $T(s;\mu) = \Delta_0(\mu) + \Delta_2(\mu)s^{\lambda n} + s^{\lambda n} f_2(s;\mu)$, where $f_2 \in \mathcal{I}(W_2)$ and

$$\Delta_2(\mu) = \sigma_1'(0)^{\lambda n}\sigma_2(0)^n L(0)^{\lambda n}\left\{\frac{\tau_1(0)^{-\lambda n}}{nQ(0,0)} + \int_0^{\tau_1(0)}\left(\frac{M(u)^n}{P(u,0)} - \frac{M(0)^n}{P(0,0)}\right)\frac{\mathrm{d}u}{u^{\lambda n+1}}\right\}.$$

(c) If $\mu \in W_3$ then $T(s;\mu) = \Delta_0(\mu) + \Delta_3(\mu)s\omega(s;\lambda n) + \Delta_4(\mu)s + sf_3(s;\mu)$, where $f_3 \in \mathcal{I}(W_3)$ and the functions $\Delta_3(\mu)$ and $\Delta_4(\mu)$ are analytic on W_3. Furthermore, if $\lambda(\mu_0) = 1/n$ then

$$\Delta_3(\mu_0) = -n\sigma_1'(0)\sigma_2(0)^n L(0)\frac{Q_u(0,0)}{P(0,0)^2}.$$

The coefficients ρ and Δ_i in theorem A depend on μ. For instance, to be precise we should write Δ_0 as

$$\Delta_0(\mu) = \int_{\sigma_2(0;\mu)}^{0} \frac{v^{n-1}}{Q(0,v;\mu)}\,\mathrm{d}v.$$

In the statement of theorem A, the parameter dependence on σ, τ, λ, P, Q, L and M is omitted to avoid lengthy formulae. Theorem A will be applied in our investigation of the bifurcations of the critical points of the function $s \longmapsto T(s;\mu)$. More precisely, if $F(s;\mu) := T_s(s;\mu)$, our goal is to study the solutions of the equation $F(s;\mu) = 0$ near $s = 0$ as the parameter μ varies. Therefore, in order to investigate if there are critical points bifurcating from $\mu^\star$, it is necessary to have the asymptotic expansion of $T(s;\mu)$ near $s = 0$ in a neighbourhood $U^\star$ of $\mu^\star$, and that the remainder term of this expansion is uniform on $U^\star$. This is the reason why we define $\mathcal{I}(W_i)$ and we need that the open sets W_i cover W. Note in addition that the expression of Δ_3 is only necessary for those $\mu_0 \in W_3$ such that $\lambda(\mu_0) = 1/n$, otherwise the problem can be studied by applying (a) or (b).

3. Machinery

This section is devoted to develop the tools that will be used in the proof of theorem A. The idea is to decompose the time function in three parts by introducing two auxiliary transverse sections Σ_δ and Σ_ε (see figure 3) inside the linearizing domain U (recall definition 2.1). In this way we must consider two situations: first, the passage around the corner of a vector field which is already orbitally linearized (i.e. proportional to $x\partial_x - \lambda y\partial_y$), and second the passage between two sections transverse to the same invariant manifold of the saddle point. This is done in theorem 3.3 and lemma 3.2, respectively. The combination of these two results provides an expression of the time function that depends also on the linearizing diffeomorphism Φ. Thus the main difficulty in the proof of theorem A, which is done in section 4, will be to manipulate this expression in order that Φ does not appear.

Let us consider a family of vector fields of the form

$$X_\mu = \frac{1}{y^n}(f(x, y; \mu)\partial_x + yg(x, y; \mu)\partial_y), \tag{3}$$

where $n \geqslant 0$ and $\mu \in W$. The functions $f(x, y; \mu)$ and $g(x, y; \mu)$ are assumed to be analytic on a neighbourhood of $\{y = 0\}$ and depending also analytically on the parameter μ. We also consider (see figure 2) two analytic transverse sections $\xi(\cdot; \mu) : I \to \Sigma_\mu$ and $\zeta(\cdot; \mu) : I \to \Pi_\mu$ to the integral curve $\{y = 0\}$. Our goal is to give the first nontrivial term of the time mapping between Σ_μ and Π_μ. Thus, denoting by $\varphi(t, (x_0, y_0); \mu)$ the solution of (3) with initial condition (x_0, y_0), define $R(s; \mu)$ and $T(s; \mu)$ by means of $\varphi(T(s), \xi(s)) = \zeta(R(s))$.

Definition 3.1. *In the sequel $\mathcal{I}_0(W)$ will denote the set of germs of analytic functions $h(s; \mu)$ defined on $(-\varepsilon, \varepsilon) \times W$ for some $\varepsilon > 0$ such that $h(0; \mu) \equiv 0$. Note then that $\mathcal{I}_0(W) \subset \mathcal{I}(W)$.*

Now, with this definition, we prove the following result.

Lemma 3.2. *Assume that $f(x, 0) \neq 0$ for all $x \in [\xi_1(0), \zeta_1(0)]$. Then the functions $R(s; \mu)$ and $T(s; \mu)$ defined above are analytic on $s = 0$ for all $\mu \in W$. In addition,*

(a) $R(s; \mu) = \Delta_1(\mu)s + sh_1(s; \mu)$ with $h_1 \in \mathcal{I}_0(W)$ and

$$\Delta_1(\mu) = \frac{\xi_2'(0)}{\zeta_2'(0)} \exp\left(\int_{\xi_1(0)}^{\zeta_1(0)} \frac{g(x, 0)}{f(x, 0)}\, \mathrm{d}x\right).$$

(b) $T(s; \mu) = \Delta_2(\mu)s^n + s^n h_2(s; \mu)$ with $h_2 \in \mathcal{I}_0(W)$ and

$$\Delta_2(\mu) = \xi_2'(0)^n \int_{\xi_1(0)}^{\zeta_1(0)} \exp\left(n \int_{\xi_1(0)}^{x} \frac{g(u, 0)}{f(u, 0)}\, \mathrm{d}u\right) \frac{\mathrm{d}x}{f(x, 0)}.$$

Moreover, if $n = 0$ then $T(s; \mu) = \Delta_2(\mu) + \Delta_3(\mu)s + sh_3(s; \mu)$ with $h_3 \in \mathcal{I}_0(W)$ and

$$\Delta_3(\mu) = \frac{\zeta_1'(0)\Delta_1(\mu)}{f(\zeta(0))} - \frac{\xi_1'(0)}{f(\xi(0))} - \xi_2'(0) \int_{\xi_1(0)}^{\zeta_1(0)} \frac{f_y(x, 0)}{f(x, 0)^2} \exp\left(\int_{\xi_1(0)}^{x} \frac{g(u, 0)}{f(u, 0)}\, \mathrm{d}u\right) \mathrm{d}x.$$

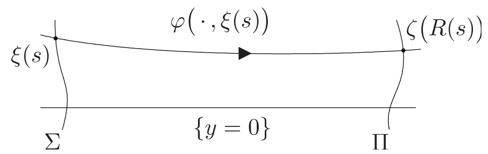

Figure 2. Transverse sections in lemma 3.2.

Proof. Let us denote the solution of $\bar{X}_\mu := (\partial_x + y(g(x,y)/f(x,y))\partial_y)$ with initial condition (x_0, y_0) by $\bar{\varphi}(t, (x_0, y_0); \mu)$. Note that this provides us a parametrization of the integral curves of X_μ. In particular, the integral curve passing through $\xi(s)$ is given by $y = \bar{\varphi}_2(x - \xi_1(s), \xi(s))$. Consequently we have

$$\zeta_2(R(s)) = \bar{\varphi}_2(\zeta_1(R(s)) - \xi_1(s), \xi(s)). \tag{4}$$

By an application of the Implicit Function Theorem, $R(s;\mu)$ is analytic. On the other hand, since $\xi_2(0) = 0$ and $\bar{\varphi}_2(x, (x_0, 0)) \equiv 0$, it turns out that

$$\bar{\varphi}_2(x - \xi_1(s), \xi(s)) = a(x)s + s^2 r(s, x), \tag{5}$$

where $a(x)$ and $r(s,x)$ are analytic functions. Hence, taking (4) into account, we can assert that

$$\Delta_1(\mu) = R'(0) = \frac{a(\zeta_1(0))}{\zeta_2'(0)}. \tag{6}$$

Therefore, in order to prove (a), we must compute $a(\zeta_1(0))$. To this end notice first that

$$a(x) = \frac{\mathrm{d}}{\mathrm{d}s}\bar{\varphi}_2(x - \xi_1(s), \xi(s))\bigg|_{s=0} = \xi_1'(0)\frac{\mathrm{d}\bar{\varphi}_2}{\mathrm{d}x_0}(x - \xi_1(0), \xi(0)) + \xi_2'(0)\frac{\mathrm{d}\bar{\varphi}_2}{\mathrm{d}y_0}(x - \xi_1(0), \xi(0)).$$

Here, we used that $\{y = 0\}$ is an integral curve of $\bar{X}_\mu$. Recall that the derivatives of $\bar{\varphi}(t, \xi(0))$ with respect to the initial conditions satisfy the initial value problem

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} \dfrac{\partial\bar{\varphi}_1}{\partial x_0} & \dfrac{\partial\bar{\varphi}_1}{\partial y_0} \\ \dfrac{\partial\bar{\varphi}_2}{\partial x_0} & \dfrac{\partial\bar{\varphi}_2}{\partial y_0} \end{pmatrix} = (D\bar{X})_{\bar{\varphi}(t,\xi(0))}\begin{pmatrix} \dfrac{\partial\bar{\varphi}_1}{\partial x_0} & \dfrac{\partial\bar{\varphi}_1}{\partial y_0} \\ \dfrac{\partial\bar{\varphi}_2}{\partial x_0} & \dfrac{\partial\bar{\varphi}_2}{\partial y_0} \end{pmatrix} \quad \text{with} \quad \begin{pmatrix} \dfrac{\partial\bar{\varphi}_1}{\partial x_0} & \dfrac{\partial\bar{\varphi}_1}{\partial y_0} \\ \dfrac{\partial\bar{\varphi}_2}{\partial x_0} & \dfrac{\partial\bar{\varphi}_2}{\partial y_0} \end{pmatrix}\Bigg|_{t=0} = Id_2.$$

In this case, since $\bar{\varphi}(t, \xi(0)) = (t + \xi_1(0), 0)$, it turns out that

$$(D\bar{X})_{\bar{\varphi}(t,\xi(0))} = \begin{pmatrix} 0 & 0 \\ 0 & \dfrac{g(t+\xi_1(0), 0)}{f(t+\xi_1(0), 0)} \end{pmatrix}$$

and so one can solve the above linear differential equation. Indeed, the solutions are given by

$$\frac{\partial\bar{\varphi}_1}{\partial x_0} \equiv 1, \qquad \frac{\partial\bar{\varphi}_1}{\partial y_0} \equiv 0, \qquad \frac{\partial\bar{\varphi}_2}{\partial x_0} \equiv 0, \qquad \frac{\partial\bar{\varphi}_2}{\partial y_0}(t, \xi(0)) = \exp\left(\int_{\xi_1(0)}^{t+\xi_1(0)} \frac{g(u,0)}{f(u,0)}\,\mathrm{d}u\right).$$

Consequently, we can assert that

$$a(x) = \xi_2'(0)\exp\left(\int_{\xi_1(0)}^{x} \frac{g(u,0)}{f(u,0)}\,\mathrm{d}u\right),$$

and this, on account of (6), proves (a). Next, in order to prove (b) notice first that $T(s;\mu)$ is analytic because

$$T(s;\mu) = \int_{\xi_1(s)}^{\zeta_1(R(s))} \frac{y^n}{f(x,y)}\bigg|_{y=\bar{\varphi}_2(x-\xi_1(s),\xi(s))} \mathrm{d}x. \tag{7}$$

Thus, from (5) and applying the dominated convergence theorem, it follows that

$$\Delta_2(\mu) = \lim_{s\to 0}\frac{T(s)}{s^n} = \int_{\xi_1(0)}^{\zeta_1(0)} \frac{a(x)^n}{f(x,0)}\,\mathrm{d}x$$

and this shows the first part of the assertion. Finally, in case that $n = 0$, the coefficient $\Delta_3(\mu)$ can be computed from (7) using the previous calculations. Indeed, using that $\xi_2(0) = 0$ and that $\bar{\varphi}_2\big(x, (x_0, 0)\big) \equiv 0$, we obtain

$$\Delta_3(\mu) = T'(0;\mu) = \frac{\zeta_1'(0)R'(0)}{f(\zeta_1(0), 0)} - \frac{\xi_1'(0)}{f(\xi_1(0), 0)} - \int_{\xi_1(0)}^{\zeta_1(0)} \frac{f_y(x,0)a(x)}{f(x,0)^2}\,\mathrm{d}x.$$

This concludes the proof of the result since the assertions concerning the remainder terms follow easily on account of the analyticity of $R(s;\mu)$ and $T(s;\mu)$. $\square$

Consider now a family of vector fields of the form

$$X_\nu = \frac{1}{y^n G(x,y;\mu)}(x\partial_x - \lambda y \partial_y), \text{ where } \nu := (\lambda,\mu) \text{ and } n > 0.$$

We assume that $G(x,y;\mu)$ is a nonvanishing analytic function for each $\mu \in W$ and that the family depends analytically on the parameter. Our goal is to study the time function between the transverse sections $\{y = 1\}$ and $\{x = 1\}$. More concretely, we consider

$$V(s;\nu) := \int_{\mathcal{C}_{(s,\lambda)}} y^n G(x,y;\mu)\frac{\mathrm{d}x}{x},$$

where $\mathcal{C}_{(s,\lambda)} := \{(x,y) : y = (s/x)^\lambda, s \leqslant x \leqslant 1\}$. Following the approach of Saavedra in [21] we prove the theorem.

Theorem 3.3. *With the above definitions, for each $\mu \in W$ let $G_1(x,y;\mu)$ be the analytic function satisfying $G(x,y) = G(0,y) + xG_1(x,y)$ and denote $b_0(\nu) = \int_0^1 u^{\lambda n} G(0,u^{\lambda n})(\mathrm{d}u/u)$.*

(a) If $\lambda \in (1/(n+1), 2/n)$ then $V(s;\nu) = b_0(\nu) + a_1(\nu)s\omega(s;\lambda n) + a_2(\nu)s + sf(s;\nu)$ with $f \in \mathcal{I}((1/(n+1),2/n)\times W)$ and

$$a_1(\nu) = (\lambda n - 1)\left(\int_0^1 \frac{G_1(u,0) - G_1(0,0)}{u^{\lambda n}}\,\mathrm{d}u - \frac{G(0,0)}{\lambda n}\right) - G_x(0,0),$$

$$a_2(\nu) = \int_0^1 \frac{G_1(u,0) - G_1(0,0)}{u^{\lambda n}}\,\mathrm{d}u + \int_0^1 \frac{G_x(0,u^\lambda) - G_x(0,0)}{u^{2-\lambda n}}\,\mathrm{d}u - \frac{G(0,0)}{\lambda n}.$$

(b) If $\lambda \in (1/n,+\infty)$ then $V(s;\nu) = b_0(\nu) + b_1(\nu)s + sf(s;\nu)$, with $f \in \mathcal{I}((1/n,+\infty)\times W)$ and

$$b_1(\nu) = \int_0^1 \frac{G_x(0,u^\lambda)}{u^{2-\lambda n}}\,\mathrm{d}u.$$

(c) If $\lambda \in (0,1/n)$ then $V(s;\nu) = b_0(\nu) + b_2(\nu)s^{\lambda n} + s^{\lambda n} f(s;\nu)$, with $f \in \mathcal{I}((0,1/n)\times W)$ and

$$b_2(\nu) = -\frac{G(0,0)}{\lambda n} + \int_0^1 \frac{G(u,0) - G(0,0)}{u^{\lambda n+1}}\,\mathrm{d}u.$$

Proof. For the sake of simplicity in the exposition, when it is possible we shall omit the parameter dependence in the formulae.

We begin by showing (a) and during its proof we shall use the following convention. We shall say that a function $\psi(s;\nu)$ satisfies the good remainder property if $\psi(s;\nu)/s \in \mathcal{I}((1/(n+1),2/n)\times W)$. Clearly, this is equivalent to requiring that

$$\lim_{s\to 0}\frac{\psi(s;\nu)}{s} = 0 \qquad \text{and} \qquad \lim_{s\to 0}\psi'(s;\nu) = 0$$

uniformly on every compact subset of $(1/(n+1),2/n)\times W$.

Let us assume first that $(\lambda,\mu) \in (1/n,2/n)\times W$. Thus, since $G(x,y) = G(0,y) + xG_1(x,y)$, we have that

$$V(s) = \int_{\mathcal{C}_s} y^n G(x,y)\frac{\mathrm{d}x}{x} = \int_{\mathcal{C}_s} y^n G(0,y)\frac{\mathrm{d}x}{x} + V_1(s),$$

where

$$V_1(s) := \int_{\mathcal{C}_s} x y^n G_1(x, y) \frac{\mathrm{d}x}{x}.$$

Using the formula $y = (s/x)^{\lambda}$, the change of variables $u = s/x$ yields

$$V(s) = \int_s^1 u^{\lambda n} G(0, u^{\lambda}) \frac{\mathrm{d}u}{u} + V_1(s) = \int_0^1 u^{\lambda n} G(0, u^{\lambda}) \frac{\mathrm{d}u}{u} + h_1(s) + V_1(s),$$

where

$$h_1(s) := -\int_0^s u^{\lambda n} G(0, u^{\lambda}) \frac{\mathrm{d}u}{u}.$$

Let g_1 be the analytic function such that $G(0, y) = G(0, 0) + y g_1(y)$. Then

$$h_1(s) = -\frac{G(0,0)}{\lambda n} s^{\lambda n} - \int_0^s u^{\lambda(n+1)} g_1(u^{\lambda}) \frac{\mathrm{d}u}{u}$$

and we claim that the second function in this expression satisfies the good remainder property. To show this consider a compact subset K of $(1/(n+1), 2/n) \times W$. It is clear that there is no loss of generality in assuming that $K = [\kappa_1, \kappa_2] \times S$, where S is a compact subset of W. Then, there exists a positive constant M such that $|g_1(u^{\lambda}; \mu)| \leqslant M$ for all $\nu \in K$ and $u \in [0, 1]$. Consequently, if $s \in (0, 1)$ then

$$\left| \frac{1}{s} \int_0^s u^{\lambda(n+1)} g_1(u^{\lambda}) \frac{\mathrm{d}u}{u} \right| \leqslant \frac{M}{s} \int_0^s u^{\lambda(n+1)} \frac{\mathrm{d}u}{u} = M \frac{s^{\lambda(n+1)-1}}{\lambda(n+1)} \leqslant M \frac{s^{\kappa_1(n+1)-1}}{\kappa_1(n+1)}.$$

Clearly, this uniform upper bound tends to 0 as $s \to 0$ since $\kappa_1 > 1/(n+1)$. On the other hand,

$$\left| \frac{\mathrm{d}}{\mathrm{d}s} \left(\int_0^s u^{\lambda(n+1)} g_1(u^{\lambda}) \frac{\mathrm{d}u}{u} \right) \right| = |s^{\lambda(n+1)-1} g_1(s^{\lambda})| \leqslant M s^{\kappa_1(n+1)-1}$$

and again this uniform upper bound tends to 0 as $s \to 0$.

Consider next the function V_1 and let G_2 denote the analytic function such that $G_1(x, y) = G_1(0, y) + x G_2(x, y)$. Then

$$V_1(s) = \int_{\mathcal{C}_s} x y^n G_1(0, y) \frac{\mathrm{d}x}{x} + V_2(s), \text{ where } V_2(s) := \int_{\mathcal{C}_s} x^2 y^n G_2(x, y) \frac{\mathrm{d}x}{x}.$$

Again after the change $u = s/x$ it turns out that

$$V_1(s) = s \int_0^1 u^{\lambda n - 1} G_1(0, u^{\lambda}) \frac{\mathrm{d}u}{u} + h_2(s) + V_2(s),$$

where

$$h_2(s) := -s \int_0^s u^{\lambda n - 1} G_1(0, u^{\lambda}) \frac{\mathrm{d}u}{u}.$$

Denote now by g_2 the analytic function such that $G_1(0, y) = G_1(0, 0) + y g_2(y)$. Then

$$h_2(s) = \frac{G_1(0,0)}{1 - \lambda n} s^{\lambda n} - s \int_0^s u^{\lambda(n+1)-1} g_2(u^{\lambda}) \frac{\mathrm{d}u}{u}$$

and one can prove exactly as before that the second function in this expression satisfies the good remainder property. We point out that this second function is well defined for all $\lambda > 1/(n+1)$. This is important because to prove the good remainder property we must consider compact subsets of $(1/(n+1), 2/n) \times W$. Note that h_2 on the whole is only well defined when $\lambda > 1/n$. The last step will be to study V_2, and to do so we denote by G_3 the analytic function which satisfies that $G_2(x, y) = G_2(x, 0) + y G_3(x, y)$. Then

$$V_2(s) = \int_{\mathcal{C}_s} x^2 y^n G_2(x, 0) \frac{\mathrm{d}x}{x} + V_3(s), \text{ where } V_3(s) := \int_{\mathcal{C}_s} x^2 y^{n+1} G_3(x, y) \frac{\mathrm{d}x}{x}.$$

Hence, using that $y = (s/x)^\lambda$, we can write

$$V_2(s) = s^{\lambda n}\int_0^1 x^{2-\lambda n} G_2(x,0)\frac{\mathrm{d}x}{x} + h_3(s) + V_3(s),$$

where

$$h_3(s) := -s^{\lambda n}\int_0^s x^{2-\lambda n} G_2(x,0)\frac{\mathrm{d}x}{x}.$$

We remark that here we use the assumption $\lambda < 2/n$. Exactly as before one can check that h_3 satisfies the good remainder property. We claim that this is also the case of V_3. To see this, notice first that

$$V_3(s;\nu) = s^{\lambda(n+1)}\int_s^1 x^{2-\lambda(n+1)} G_3\left(x, \left(\frac{s}{x}\right)^\lambda; \mu\right)\frac{\mathrm{d}x}{x}$$

and consider a compact subset K of $(1/(n+1), 2/n) \times W$. Since

$$\left\{\left(x, \left(\frac{s}{x}\right)^\lambda\right) : s \leqslant x \leqslant 1, 0 < s < 1\right\} \subset [0,1]^2, \tag{8}$$

there exists a positive constant M_1 such that if $s \in (0,1)$ and $\nu = (\lambda, \mu) \in K$ then

$$\left|\frac{V_3(s;\nu)}{s}\right| \leqslant M_1 s^{\lambda(n+1)-1}\int_s^1 x^{1-\lambda(n+1)}\mathrm{d}x = -M_1 s\omega(s; \lambda(n+1)-1).$$

Clearly, see remark 2.6, the last function tends to 0 as $s \to 0$ uniformly on K since $\lambda(n+1) - 1 > 0$ for all $\nu \in K$. On the other hand, a computation shows that

$$V_3'(s;\nu) = \frac{V_3(s;\nu)}{s} - s\,G_3(s,1) + \lambda s^{\lambda(n+2)-1}\int_s^1 x^{1-\lambda(n+2)}(G_3)_y\left(s, \left(\frac{s}{x}\right)^\lambda\right)\mathrm{d}x.$$

It is clear that the claim will follow once we prove that the last term in the above expression tends to 0 as $s \to 0$ uniformly on K. To see this notice that from (8) it follows that there exists $M_2 > 0$ such that if $s \in (0,1)$ and $\nu = (\lambda, \mu) \in K$ then

$$\begin{aligned}\left|s^{\lambda(n+2)-1}\int_s^1 x^{1-\lambda(n+2)}(G_3)_y\left(s, \left(\frac{s}{x}\right)^\lambda\right)\mathrm{d}x\right| &\leqslant M_2 s^{\lambda(n+2)-1}\int_s^1 x^{1-\lambda(n+2)}\mathrm{d}x \\ &= -M_2 s\omega(s; \lambda(n+2)-1).\end{aligned}$$

This upper bound, again on account of remark 2.6, tends to 0 as $s \to 0$ uniformly on K because $\lambda(n+2) - 1 > 0$ for all $\nu \in K$.

In brief, collecting all the functions which do not satisfy the good remainder property, we have proved that if $\nu = (\lambda, \mu) \in (1/n, 2/n) \times W$ then

$$\begin{aligned}V(s;\nu) = &\int_0^1 u^{\lambda n} G(0, u^\lambda)\frac{\mathrm{d}u}{u} + \left(\int_0^1 u^{\lambda n-1} G_1(0, u^\lambda)\frac{\mathrm{d}u}{u}\right)s \\ &+ \left(\int_0^1 u^{1-\lambda n} G_2(u,0)\,\mathrm{d}u - \frac{G(0,0)}{\lambda n} + \frac{G_1(0,0)}{1-\lambda n}\right)s^{\lambda n} + f_1(s;\nu),\end{aligned}$$

where $f_1(s;\nu)/s \in \mathcal{I}((1/(n+1), 2/n) \times W)$. Now it is easy to check that $V(s;\nu) - f_1(s;\nu)$ can be rewritten by means of the monomials $\{1, s\omega(s;\lambda n), s\}$ and the coefficients given in the statement. This is left to the reader, it is a straightforward calculation using that

$$G_2(x,y) = \frac{G_1(x,y) - G_1(0,y)}{x} \qquad \text{and} \qquad G_1(0,y) = G_x(0,y)$$

and that $s^{\lambda n} = (\lambda n - 1)s\omega(s;\lambda n) + s$. It is important to point out that the coefficients of the monomials $\{1, s, s^{\lambda n}\}$ above are only well defined in case that $\lambda \in (1/n, 2/n)$, and that in contrast the coefficients of the new monomials $\{1, s\omega(s;\lambda n), s\}$ are well defined for all $\lambda \in (1/(n+1), 2/n)$.

Let us turn now to the case $(\lambda,\mu) \in (1/(n+1), 1/n) \times W$. The first step in the process is common in all the cases. Hence, we obtain

$$V(s) = \int_0^1 u^{\lambda n} G(0, u^\lambda)\frac{\mathrm{d}u}{u} + h_1(s) + V_1(s),$$

where h_1 and V_1 are the functions defined before. Recall in particular that $h_1(s) + (G(0,0)/\lambda n)s^{\lambda n}$ verifies the good remainder property. In this case, in order to study V_1 we consider the analytic function $\tilde{G}_2$ such that $G_1(x,y) = G_1(x,0) + y\tilde{G}_2(x,y)$. Thus it follows that

$$V_1(s) = \int_{\mathcal{C}_s} xy^n G_1(x,0)\frac{\mathrm{d}x}{x} + \tilde{V}_2(s), \text{ where } \tilde{V}_2(s) := \int_{\mathcal{C}_s} xy^{n+1}\tilde{G}_2(x,y)\frac{\mathrm{d}x}{x}.$$

Since $x = s/y^{1/\lambda}$, one can verify that the change $u = s^\lambda/y$ gives

$$V_1(s) = \frac{s^{\lambda n}}{\lambda}\int_0^1 u^{1/\lambda - n} G_1(u^{1/\lambda}, 0)\frac{\mathrm{d}u}{u} + \tilde{h}_2(s) + \tilde{V}_2(s),$$

where

$$\tilde{h}_2(s) := -\frac{s^{\lambda n}}{\lambda}\int_0^{s^\lambda} u^{1/\lambda - n} G_1(u^{1/\lambda}, 0)\,\frac{\mathrm{d}u}{u}.$$

Then, taking into account that $G_1(x,0) = G_1(0,0) + xG_2(x,0)$, we obtain

$$\tilde{h}_2(s) = \frac{G_1(0,0)}{\lambda n - 1}\,s - \frac{s^{\lambda n}}{\lambda}\int_0^{s^\lambda} u^{2/\lambda - n} G_2(u^{1/\lambda}, 0)\frac{\mathrm{d}u}{u}$$

and one can show exactly in the same way as before that the second function in the above expression satisfies the good remainder property. We study next the function $\tilde{V}_2$. To this end we consider the analytic function $\tilde{G}_3$ which verifies $\tilde{G}_2(x,y) = \tilde{G}_2(0,y) + x\tilde{G}_3(x,y)$ and this enables us to write

$$\tilde{V}_2(s) = \int_{\mathcal{C}_s} xy^{n+1}\tilde{G}_2(0,y)\frac{\mathrm{d}x}{x} + \tilde{V}_3(s), \text{ where } \tilde{V}_3(s) := \int_{\mathcal{C}_s} x^2y^{n+1}\tilde{G}_3(x,y)\frac{\mathrm{d}x}{x}.$$

Once again, the substitution $y = (s/x)^\lambda$ and the change $u = s/x$ shows that

$$\tilde{V}_2(s) = s\int_0^1 u^{\lambda(n+1)-1}\tilde{G}_2(0, u^\lambda)\frac{\mathrm{d}u}{u} + \tilde{h}_3(s) + \tilde{V}_3(s),$$

where

$$\tilde{h}_3(s) := -s\int_0^s u^{\lambda(n+1)-1}\tilde{G}_2(0, u^\lambda)\frac{\mathrm{d}u}{u}.$$

Now one can prove that $\tilde{h}_3$ and $\tilde{V}_3$ verify the good remainder property. This fact follows exactly in the same way as in the preceding case and for the sake of brevity it is not included here. Therefore, collecting the terms not verifying the good remainder property, we can assert that if $(\lambda,\mu) \in (1/(n+1), 1/n) \times W$ then

$$\begin{aligned} V(s;\nu) = &\int_0^1 u^{\lambda n} G(0, u^\lambda)\frac{\mathrm{d}u}{u} + \left(\frac{1}{\lambda}\int_0^1 u^{1/\lambda - n} G_1(u^{1/\lambda}, 0)\frac{\mathrm{d}u}{u} - \frac{G(0,0)}{\lambda n}\right)s^{\lambda n} \\ &+ \left(\int_0^1 u^{\lambda(n+1)-1}\tilde{G}_2(0, u^\lambda)\frac{\mathrm{d}u}{u} + \frac{G_1(0,0)}{\lambda n - 1}\right)s + f_2(s;\nu), \end{aligned}$$

where $f_2(s;\nu)/s \in \mathcal{I}((1/(n+1), 2/n) \times W)$. At this point one can translate the monomials $\{1, s^{\lambda n}, s\}$ of $V(s;\nu) - f_2(s;\nu)$ into $\{1, s\omega(s;\lambda n), s\}$ and then elementary manipulations show that the new coefficients are precisely the ones appearing in the statement.

We study finally the case $\lambda = 1/n$. In this case, we claim that for all $\mu \in W$ we have

$$V\left(s;\left(\frac{1}{n},\mu\right)\right) = b_0\left(\left(\frac{1}{n},\mu\right)\right) + a_1\left(\left(\frac{1}{n},\mu\right)\right)s\omega(s;1) + a_2\left(\left(\frac{1}{n},\mu\right)\right)s + f_3(s;\mu), \tag{9}$$

where the function f_3 satisfies that

$$\lim_{s\to 0}\frac{f_3(s;\mu)}{s} = 0 \qquad \text{and} \qquad \lim_{s\to 0} f_3'(s;\mu) = 0 \text{ uniformly on compacts of } W. \tag{10}$$

The proof of (a) will follow once we show this claim. Indeed, the function

$$f(s;\nu) := \begin{cases} f_1(s;\nu) & \text{if } \lambda \in \left(\frac{1}{n}, \frac{2}{n}\right), \\ f_3(s;\mu) & \text{if } \lambda = \frac{1}{n}, \\ f_2(s;\nu) & \text{if } \lambda \in \left(\frac{1}{n+1}, \frac{1}{n}\right) \end{cases}$$

will verify the good remainder property and we will have proved that

$$V(s;\nu) = b_0(\nu) + a_1(\nu)s\omega(s;\lambda n) + a_2(\nu)s + f(s;\nu) \qquad \text{for all } \nu \in \left(\frac{1}{n+1}, \frac{2}{n}\right) \times W.$$

The fact that $f(s;\nu)/s \in \mathcal{I}((1/(n+1), 2/n) \times W)$ is very simple to show and it is left to the reader. It follows using that f_1 and f_2 verify the good remainder property and that f_3 satisfies (10).

So let us prove the claim. Since the first step of the process is the same for any λ, in case that $\lambda = 1/n$ we have that

$$V(s) = \int_0^1 G(0, u^{1/n})\mathrm{d}u + h_1(s) + V_1(s),$$

where

$$h_1(s) = -G(0,0)s - \int_0^s u^{1/n} g_1(u^{1/n})\mathrm{d}u \;\text{ and }\; V_1(s) = \int_{\mathcal{C}_s} xy^n G_1(x,y)\frac{\mathrm{d}x}{x}.$$

One can easily check that the function $s \longmapsto h_1(s) + G(0,0)s$ verifies (10). In order to study V_1, we consider the functions r_1, r_2 and R satisfying $G_1(x,y) = G_1(0,0) + xr_1(x) + yr_2(y) + xyR(x,y)$. Then, since $xy^n = s$, it turns out that

$$\begin{aligned} V_1(s) = &- G_1(0,0)s\log s + s\int_s^1 r_1(x)\mathrm{d}x + s\int_s^1 u^{1/n} r_2(u^{1/n})\frac{\mathrm{d}u}{u} \\ &+ s^{1+1/n}\int_s^1 x^{1-1/n} R\left(x, \left(\frac{s}{x}\right)^{1/n}\right)\frac{\mathrm{d}x}{x}. \end{aligned}$$

For the sake of brevity let us denote the second, third and fourth terms above by $\phi_2(s)$, $\phi_3(s)$ and $\phi_4(s)$, respectively. Easy computations show the functions

$$s \longmapsto \phi_2(s) - s\int_0^1 r_1(x)\mathrm{d}x \;\text{ and }\; s \longmapsto \phi_3(s) - s\int_0^1 u^{1/n} r_2(u^{1/n})\frac{\mathrm{d}u}{u}$$

verify (10). Let us prove that ϕ_4 also satisfies it. So fix a compact set K of W. Then there exists a positive constant M_1 such that $|R(x, y; \mu)| \leqslant M_1$ for all $(x, y) \in [0, 1]^2$ and $\mu \in K$. Therefore,

$$\left|\frac{\phi_4(s; \mu)}{s}\right| \leqslant M_1 s^{1/n} \int_s^1 x^{-1/n} \mathrm{d}x = -M_1 s\omega\left(s; \frac{1}{n}\right) \to 0 \text{ as } s \to 0.$$

On the other hand,

$$\phi_4'(s; \mu) = \frac{\phi_4(s; \mu)}{s} - sR(s, 1) + \frac{s^{2/n}}{n} \int_s^1 x^{-2/n} R_y\left(x, \left(\frac{s}{x}\right)^{1/n}\right) \mathrm{d}x$$

and taking $M_2 > 0$ so that $|R_y(x, y; \mu)| \leqslant M_2$ for all $(x, y) \in [0, 1]^2$ and $\mu \in K$ then

$$\left|s^{2/n} \int_s^1 x^{-2/n} R_y\left(x, \left(\frac{s}{x}\right)^{1/n}\right) \mathrm{d}x\right| \leqslant M_2\, s^{2/n} \int_s^1 x^{-2/n} \mathrm{d}x = -M_2 s\omega\left(s; \frac{2}{n}\right) \to 0$$

as $s \to 0$. Consequently, we can assert that

$$\begin{aligned} V\left(s; \left(\frac{1}{n}, \mu\right)\right) &= \int_0^1 G(0, u^{1/n}) \mathrm{d}u - G_1(0, 0) s \log s \\ &\quad + \left(\int_0^1 r_1(u) \mathrm{d}u + \int_0^1 u^{1/n} r_2(u^{1/n}) \frac{\mathrm{d}u}{u} - G(0, 0)\right) s + f_3(s; \mu), \end{aligned}$$

where f_3 satisfies (10). This, taking into account that $s \log s = s\omega(s; 1)$ and that

$$r_1(x) = \frac{G_1(x, 0) - G_1(0, 0)}{x} \quad \text{and} \quad r_2(y) = \frac{G_1(0, y) - G_1(0, 0)}{y}$$

shows the validity of (9) and completes the proof of the claim.

The proofs of (b) and (c) are simpler than that of (a) and follow using the same method. Let us show for instance (c). So assume that $\nu \in (0, 1/n) \times W$ and let us use the following convention. In the proof of (c) we shall say that a function $\psi(s; \nu)$ satisfies the good remainder property if $\psi(s; \nu)/s^{\lambda n} \in \mathcal{I}((0, 1/n) \times W)$. This is equivalent to require that

$$\lim_{s \to 0} \frac{\psi(s; \nu)}{s^{\lambda n}} = 0 \qquad \text{and} \qquad \lim_{s \to 0} \frac{\psi'(s; \nu)}{s^{\lambda n - 1}} = 0$$

uniformly on every compact subset of $(0, 1/n) \times W$. Once again we can write

$$V(s) = \int_0^1 u^{\lambda n} G(0, u^{\lambda}) \frac{\mathrm{d}u}{u} + h_1(s) + V_1(s),$$

where

$$V_1(s) = \int_{\mathcal{C}_s} x y^n G_1(x, y) \frac{\mathrm{d}x}{x} \qquad \text{and} \qquad h_1(s) = -\frac{G(0, 0)}{\lambda n} s^{\lambda n} - \int_0^s u^{\lambda(n+1)} g_1(u^{\lambda}) \frac{\mathrm{d}u}{u}.$$

We claim that the second function in h_1 satisfies the good remainder property. To prove this, let us fix a compact subset $K = [\kappa_1, \kappa_2] \times S$ of $(0, 1/n) \times W$. Then again there exits a positive constant M such that $|g_1(x; \mu)| \leqslant M$ for all $x \in [0, 1]$ and $\mu \in K$, and consequently,

$$\left|\frac{1}{s^{\lambda n}} \int_0^s u^{\lambda(n+1)} g_1(u^{\lambda}) \frac{\mathrm{d}u}{u}\right| \leqslant \frac{M}{s^{\lambda n}} \int_0^s u^{\lambda(n+1)} \frac{\mathrm{d}u}{u} = M \frac{s^{\lambda}}{\lambda(n+1)} \leqslant M \frac{s^{\kappa_1}}{\kappa_1(n+1)}.$$

This uniform upper bound clearly tends to 0 as $s \to 0$ since $\kappa_1 > 0$. Moreover,

$$\left|\frac{1}{s^{\lambda n - 1}} \frac{\mathrm{d}}{\mathrm{d}s} \left(\int_0^s u^{\lambda(n+1)} g_1(u^{\lambda}) \frac{\mathrm{d}u}{u}\right)\right| = \left|s^{\lambda} g_1(s^{\lambda})\right| \leqslant M s^{\kappa_1} \to 0 \qquad \text{as } s \to 0$$

and hence the claim is true. So it only remains to study V_1 and in this case we use $\tilde{G}_2$, which is defined by means of $G_1(x, y) = G_1(x, 0) + y\tilde{G}_2(x, y)$. We thus obtain

$$V_1(s) = \frac{s^{\lambda n}}{\lambda}\int_0^1 u^{1/\lambda - n} G_1(u^{1/\lambda}, 0)\frac{\mathrm{d}u}{u} + \tilde{h}_2(s) + \tilde{V}_2(s),$$

where

$$\tilde{V}_2(s) = \int_{\mathcal{C}_s} xy^{n+1}\tilde{G}_2(x, y)\frac{\mathrm{d}x}{x} \qquad \text{and} \qquad \tilde{h}_2(s) = -\frac{s^{\lambda n}}{\lambda}\int_0^{s^\lambda} u^{1/\lambda - n} G_1(u^{1/\lambda}, 0)\frac{\mathrm{d}u}{u}.$$

One can show that $\tilde{h}_2$ and $\tilde{V}_2$ verify the good remainder property. The proof of this fact is not included here for the sake of brevity. Therefore,

$$V(s;\mu) = \int_0^1 u^{\lambda n} G(0, u^\lambda)\frac{\mathrm{d}u}{u} + \left(\frac{1}{\lambda}\int_0^1 u^{1/\lambda - n} G_1(u^{1/\lambda}, 0)\frac{\mathrm{d}u}{u} - \frac{G(0,0)}{\lambda n}\right)s^{\lambda n} + s^{\lambda n} f(s;\nu)$$

with $f \in \mathcal{I}((0, 1/n) \times W)$, and this completes the proof of (c). □

Concerning the proof of theorem 3.3, it is to be mentioned that to obtain the expression of the first nontrivial coefficient of $V(s;\mu)$ we follow the approach of Saavedra in [21]. The author studies in that paper the asymptotic development of the time function associated to a vector field. Theorem 3.3 deals with a family of vector fields, and therefore we must use a 'good' unfolding of the scale appearing in the development of Saavedra. That is, a scale such that the order of the terms is preserved in a small neighbourhood of $(s, \mu) = (0, \mu_0)$ in $(0, \varepsilon) \times W$ (see [9]).

We conclude this section with the following result, which will be used in the proof of theorem A to study the remainder terms. For the sake of brevity, in its statement we denote $\mathcal{I}(W)$ and $\mathcal{I}_0(W)$ by $\mathcal{I}$ and $\mathcal{I}_0$, respectively.

Lemma 3.4. *Assume that $a(\mu)$, $k(\mu)$ and $r(\mu)$ are positive analytic functions.*

(a) If $g(s;\mu)$ and $f(s;\mu)$ belong to $\mathcal{I}_0$ and $\mathcal{I}$ respectively then $g \circ f \in \mathcal{I}$.
(b) If $f(s;\mu)$ belongs to $\mathcal{I}$ (respectively $\mathcal{I}_0$) and $\varphi := s^r(a + f)$ then $s^k \circ \varphi - a^k s^{kr}$ belongs to $s^{kr}\mathcal{I}$ (respectively $s^{kr}\mathcal{I}_0$).
(c) If $f(s;\mu)$ and $g(s;\mu)$ belong to $\mathcal{I}$ and $\varphi := s^r(a + f)$ then $(s^k g) \circ \varphi$ belongs to $s^{kr}\mathcal{I}$.
(d) If $g(s;\mu)$ belongs to $\mathcal{I}_0$ then $g\omega(s;r) \in \mathcal{I}$.
(e) If $g(s;\mu)$ belongs to $\mathcal{I}_0$ then $(s\omega(s;r)) \circ (s(a + g)) = a^r s\omega(s;r) + a\omega(a;r)s + \Psi$ with $\Psi \in s\mathcal{I}$.

Proof. To show (a) note first that if $g \in \mathcal{I}_0$ then $g = sh$ where h is an analytic function on $s = 0$. Thus $g \circ f = (sh) \circ f = f\,(h \circ f)$, and so it is clear that $g \circ f$ tends to zero as $s \to 0$ uniformly on μ. On the other hand,

$$s(g \circ f)' = s(f\,(h \circ f))' = sf'((h \circ f) + f\,(h' \circ f)) \to 0, \quad \text{as } s \to 0$$

uniformly on μ since $f \in \mathcal{I}$ and h is analytic on $s = 0$. This proves (a). To show (b) notice that

$$s^k \circ \varphi - a^k s^{kr} = s^{kr}((a + f)^k - a^k) = s^{kr}((a + s)^k - a^k) \circ f).$$

Hence, since $(a + s)^k - a^k \in \mathcal{I}_0$, it is clear that if $f \in \mathcal{I}$, then by applying (a) we can assert that $s^k \circ \varphi - a^k s^{kr} \in s^{kr}\mathcal{I}$. It is obvious moreover that this function belongs to $s^{kr}\mathcal{I}_0$ in case that $f \in \mathcal{I}_0$. Let us turn next to the assertion in (c). Since $(s^k g) \circ \varphi = s^{kr}(a + f)^k(g \circ \varphi)$, we must prove that $(a + f)^k(g \circ \varphi) \in \mathcal{I}$. The fact that this function tends to zero as $s \to 0$ uniformly on μ is clear. On the other hand a computation shows that

$$s((a + f)^k(g \circ \varphi))' = k(a + f)^{k-1}sf'(g \circ \varphi) + s(a + f)^k(g' \circ \varphi)\varphi'.$$

The first term in the addition above tends uniformly to zero as $s \to 0$ because so do sf' and $g \circ \varphi$. Let us show that the same occurs with the second one. Indeed, one can check that

$$s(a+f)^k(g' \circ \varphi)\varphi' = (a+f)^k(g' \circ \varphi)\varphi\left(s\frac{\varphi'}{\varphi}\right) = (a+f)^k(g' \circ \varphi)\varphi\left(r + \frac{sf'}{a+f}\right)$$

and this function tends to zero as $s \to 0$ uniformly on μ because $f, g \in \mathcal{I}$ and $\varphi \to 0$ as $s \to 0$. We conclude therefore that $(a+f)^k(g \circ \varphi) \in \mathcal{I}$ as desired. In order to prove (d), we shall use that, on account of remark 2.6, $s\omega(s; r) \in \mathcal{I}$ and that, since $g \in \mathcal{I}_0$, $g = sh$, where h is an analytic function on $s = 0$. Taking these two facts into account it follows easily that if $s \to 0$ then $g\omega(s; r) = hs\omega(s; r) \to 0$ and $s(g\omega(s; r))' = sh's\omega(s; r) + hs(s\omega(s; r))' \to 0$ uniformly on μ. Finally, to prove (e), let us define $\Psi := (s\omega(s; r)) \circ (s(a+g)) - (a^r s\omega(s; r) + a\omega(a; r)s)$. Hence we must show that $\Psi \in s\mathcal{I}$. One can easily verify that the relation

$$\omega(\psi\phi; r) = \psi^{r-1}\omega(\phi; r) + \omega(\psi; r)$$

holds for any $\psi(s; \mu)$ and $\phi(s; \mu)$. By using this equality with $\psi = a + g$ and $\phi = s$ it follows that

$$(s\omega(s; r)) \circ (s(a+g)) = (a+g)s\omega(s(a+g); r) = (a+g)^r s\omega(s; r) + (a+g)s\omega(a+g; r).$$

The substitution of this expression in Ψ and elementary manipulations show that

$$\Psi = (((a+s)^r - a^r) \circ g)s\omega(s; r) + ((s\omega(a+s; r)) \circ g)s + as((\omega(a+s; r) - \omega(a; r)) \circ g).$$

Since the functions $(a+s)^r - a^r$, $s\omega(a+s; r)$ and $\omega(a+s; r) - \omega(a; r)$ belong to $\mathcal{I}_0$, and so does g by assumption, from the above equality we can assert that $\Psi \in s\omega(s; r)\mathcal{I}_0 + s\mathcal{I}_0$. This, on account of (d), shows that $\Psi \in s\mathcal{I}$ and concludes the proof of the result. $\square$

4. Proof of the main result

Proof of the theorem A. For the sake of simplicity in the formulae we shall omit the parameter dependence when there is no risk of ambiguity.

Let δ and ε be small enough so that the points $(0, \delta)$ and $(\varepsilon, 0)$ belong to the linearizing domain U (see definition 2.1). Thus, taking advantage of the linearizing local diffeomorphism Φ (see figure 3), we define two auxiliary transverse sections Σ_δ and Σ_ε to X parametrized by $s \longmapsto \Phi(s, \delta)$ and $s \longmapsto \Phi(\varepsilon, s)$, respectively. Next we consider the Dulac and time mappings between Σ_σ and Σ_δ. To this end we use the parametrization of the

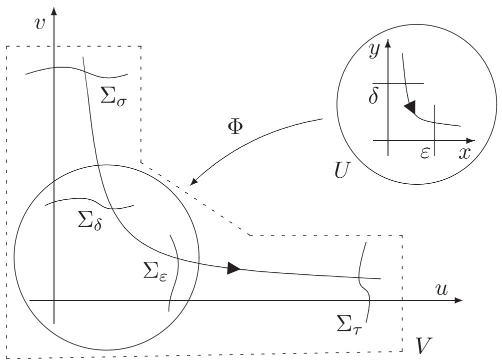

Figure 3. Auxiliary sections in the proof of theorem A.

corresponding transverse sections. More precisely, if $\varphi(t, (x_0, y_0); \mu)$ denotes the solution of X_μ passing through (x_0, y_0) at $t = 0$, we define $R_1(s; \mu)$ and $T_1(s; \mu)$ by means of the relation

$$\varphi(T_1(s; \mu), \sigma(s)) = \Phi(R_1(s; \mu), \delta).$$

We consider also the mappings between Σ_δ and Σ_ε, say $R_2(s; \mu)$ and $T_2(s; \mu)$, and the ones between Σ_ε and Σ_τ, say $R_3(s; \mu)$ and $T_3(s; \mu)$. Exactly as before, these mappings are defined by means of

$$\varphi(T_2(s; \mu), \Phi(s, \delta)) = \Phi(\varepsilon, R_2(s; \mu)) \qquad \text{and} \qquad \varphi(T_3(s; \mu), \Phi(\varepsilon, s)) = \tau(R_3(s; \mu)).$$

Note that the hypothesis FLP implies that $R_2(s) = \delta(s/\varepsilon)^\lambda$. Now, according to these definitions, we can split up $R(s)$ and $T(s)$ as

$$R(s; \mu) = R_3(R_2(R_1(s))) \qquad \text{and} \qquad T(s; \mu) = T_1(s) + T_2(R_1(s)) + T_3(R_2(R_1(s))).$$

It is to be pointed out that $T_i(s)$ and $R_i(s)$ depend on δ and ε, but that $T(s)$ and $R(s)$ as a whole do not. This will be the key point in order to compute their first nontrivial coefficients.

Now we shall use lemma 3.2 to study the asymptotic expansions of $R_1(s; \mu)$ and $T_1(s; \mu)$. In order to achieve the assumptions of this lemma it is necessary to perform the change of coordinates given by $(x, y) \longmapsto (y, x)$. We thus apply it to

$$\frac{1}{y^0}(f(x, y; \mu)\partial_x + yg(x, y; \mu)\partial_y),$$

with

$$f(x, y) = \frac{Q(y, x)}{x^{n-1}} \text{ and } g(x, y) = \frac{P(y, x)}{x^n}.$$

The parametrizations of the transverse sections that we consider are

$$\xi(s) = (\sigma_2(s), \sigma_1(s)) \qquad \text{and} \qquad \zeta(s) = (\Phi_2(s, \delta), \Phi_1(s, \delta)) = (\delta\psi_2(s, \delta), s\psi_1(s, \delta)).$$

Here we took remark 2.2 into account. Hence by applying lemma 3.2 we can assert that

$$T_1(s; \mu) = A_0(\mu) + A_1(\mu)s + sh_1(s; \mu) \qquad \text{and} \qquad R_1(s, \mu) = B(\mu)s + sh_2(s; \mu) \tag{11}$$

with $h_i \in \mathcal{I}_0(W)$ and where

$$B(\mu) = \frac{\sigma_1'(0)}{\psi_1(0, \delta)} \exp\left(\int_{\sigma_2(0)}^{\delta\psi_2(0,\delta)} \frac{P(0, y)}{Q(0, y)} \frac{\mathrm{d}y}{y}\right),$$

$$A_0(\mu) = \int_{\sigma_2(0)}^{\delta\psi_2(0,\delta)} \frac{x^{n-1}}{Q(0, x)}\,\mathrm{d}x,$$

$$A_1(\mu) = \frac{\delta^n \psi_2(0, \delta)^{n-1}(\psi_2)_x(0, \delta)}{Q(0, \delta\psi_2(0, \delta))} B(\mu) - \frac{\sigma_2'(0)\,\sigma_2(0)^{n-1}}{Q(0, \sigma_2(0))} - \sigma_1'(0) \int_{\sigma_2(0)}^{\delta\psi_2(0,\delta)} \frac{Q_u(0, v)v^{n-1}}{Q(0, v)^2} \exp\left(\int_{\sigma_2(0)}^{v} \frac{P(0, y)}{Q(0, y)} \frac{\mathrm{d}y}{y}\right) \mathrm{d}v.$$

Let us turn now to study $T_3(s; \mu)$ and $R_3(s; \mu)$. To this end we apply lemma 3.2 to

$$\frac{1}{y^n}(f(x, y; \mu)\partial_x + yg(x, y; \mu)\partial_y) \qquad \text{with } f(x, y) = xP(x, y) \text{ and } g(x, y) = Q(x, y).$$

In this case the parametrizations of the transverse sections that we consider are $\zeta(s) = \tau(s)$ and $\xi(s) = \Phi(\varepsilon, s) = (\varepsilon\psi_1(\varepsilon, s), s\psi_2(\varepsilon, s))$. We thus obtain

$$T_3(s; \mu) = C(\mu)s^n + s^n h_3(s; \mu) \text{ and } R_3(s, \mu) = D(\mu)s + sh_4(s; \mu) \tag{12}$$

with $h_i \in \mathcal{I}_0(W)$ and

$$C(\mu) = \psi_2(\varepsilon,0)^n \int_{\varepsilon\psi_1(\varepsilon,0)}^{\tau_1(0)} \exp\left(n\int_{\varepsilon\psi_1(\varepsilon,0)}^{x} \frac{Q(u,0)}{P(u,0)}\frac{\mathrm{d}u}{u}\right)\frac{\mathrm{d}x}{xP(x,0)},$$

$$D(\mu) = \frac{\psi_2(\varepsilon,0)}{\tau_2'(0)} \exp\left(\int_{\varepsilon\psi_1(\varepsilon,0)}^{\tau_1(0)} \frac{Q(u,0)}{P(u,0)}\frac{\mathrm{d}u}{u}\right).$$

Hence, it remains to study $T_2(s;\mu)$ because recall that $R_2(s;\mu) = \delta(s/\varepsilon)^\lambda$. This will be done by means of theorem 3.3. Since in this result the transverse sections are assumed to be on $\{y = 1\}$ and $\{x = 1\}$, we must compose the linearizing diffeomorphism Φ with $(x, y) \longmapsto (\varepsilon x, \delta y)$. We thus consider $\tilde{\Phi}(x, y) := \Phi(\varepsilon x, \delta y)$, and then from remark 2.2 it follows that

$$X_\mu = \tilde{\Phi}_*\left(\frac{1}{y^n G(x,y)}(x\partial_x - \lambda y\partial_y)\right) \qquad \text{with } G(x,y) := \delta^n g(\varepsilon x, \delta y).$$

Let us recall that the existence of g is a consequence of definition 2.1. As we will see, this function can be given explicitly by means of Φ and X_μ but this is not necessary right now. Consequently we have that $T_2(s;\mu) = V(s/\varepsilon)$, where $V(s;\mu)$ is the function considered in theorem 3.3 taking $G(x, y)$ as above. Applying theorem 3.3 we obtain

$$V(s) = b_0(\mu) + b_1(\mu)s + sr_1(s;\mu) \qquad \text{if } \mu \in W_1, \tag{13}$$

$$V(s) = b_0(\mu) + b_2(\mu)s^{\lambda n} + s^{\lambda n} r_2(s;\mu) \qquad \text{if } \mu \in W_2, \tag{14}$$

$$V(s) = b_0(\mu) + a_1(\mu)s\omega(s;\lambda n) + a_2(\mu)s + sr_3(s;\mu) \qquad \text{if } \mu \in W_3, \tag{15}$$

with $r_i \in \mathcal{I}(W_i)$ and where

$$b_0(\mu) = \delta^n \int_0^1 u^{\lambda n} g(0, \delta^\lambda u^\lambda)\frac{\mathrm{d}u}{u},$$

$$b_1(\mu) = \delta^n \varepsilon \int_0^1 u^{\lambda n-1} g_x(0, \delta^\lambda u^\lambda)\frac{\mathrm{d}u}{u},$$

$$b_2(\mu) = \frac{\delta^n}{nQ(0,0)} + \delta^n\varepsilon\int_0^1 \left(\frac{g(\varepsilon u,0)-g(0,0)}{\varepsilon u}\right)\frac{\mathrm{d}u}{u^{\lambda n}},$$

$$a_1(\mu) = -\delta^n \varepsilon g_x(0,0) \qquad \text{if } \lambda(\mu) = \frac{1}{n}.$$

It is worthwhile making the following observation concerning the functions r_i above. To be precise, for instance in case that $\mu \in W_1$, by applying theorem 3.3 we obtain a remainder term of the form $s\tilde{r}_1(s; (\lambda,\mu))$ with $\tilde{r}_1 \in \mathcal{I}((1/n, +\infty) \times W_1)$. It is clear however, on account of the definition of W_1, that $r_1(s;\mu) := \tilde{r}_1(s; (\lambda(\mu),\mu)) \in \mathcal{I}(W_1)$.

At this point, we have all the necessary ingredients in order to prove the result. The rest of the proof is carried out in two steps. The first one will be to prove the assertions concerning the remainder terms in the asymptotic expansion of $R(s;\mu)$ and $T(s;\mu)$. The second one will be to compute the explicit expression of each coefficient in these expansions.

We begin with $R(s;\mu)$. Recall that $R(s) = R_3(R_2(R_1(s)))$ where $R_1(s)$ and $R_3(s)$ are given respectively in (11) and (12) and where $R_2(s) = \delta(s/\varepsilon)^\lambda$. So, by applying lemma 3.4, we can assert that $R_2(R_1(s)) = s^\lambda(\varepsilon^{-\lambda}\delta B^\lambda + g_1)$ with $g_1 \in \mathcal{I}_0(W)$. Then, from lemma 3.4 again, it follows that

$$R(s) = R_3(s^\lambda(\varepsilon^{-\lambda}\delta B^\lambda + g_1)) = \rho s^\lambda + s^\lambda f_0(s;\mu), \text{ with } \rho := \varepsilon^{-\lambda}\delta B^\lambda D \text{ and } f_0 \in \mathcal{I}(W). \tag{16}$$

Let us turn next to the study of $T(s;\mu)$. To this end notice first that

$$\begin{aligned} T(s;\mu) &= T_1(s) + T_2(R_1(s)) + T_3(R_2(R_1(s))) \\ &= T_1(s) + T_2(Bs + sh_2) + T_3(s^{\lambda}(\varepsilon^{-\lambda}\delta B^{\lambda} + g_1)), \end{aligned} \tag{17}$$

where $T_1(s;\mu)$ and $T_3(s;\mu)$ are given respectively in (11) and (12). In particular, from lemma 3.4 it follows that

$$T_3(s^{\lambda}(\varepsilon^{-\lambda}\delta B^{\lambda} + g_1)) = (C\varepsilon^{-\lambda n}\delta^n B^{\lambda n})s^{\lambda n} + s^{\lambda n} g_2(s;\mu), \qquad \text{with } g_2 \in \mathcal{I}(W). \tag{18}$$

Hence, taking also into account the expression of $T_1(s;\mu)$, we conclude from (17) that

$$T(s;\mu) = A_0 + A_1 s + sh_1 + T_2(Bs + sh_2) + (C\varepsilon^{-\lambda n}\delta^n B^{\lambda n})s^{\lambda n} + s^{\lambda n} g_2, \tag{19}$$

where $h_1, h_2 \in \mathcal{I}_0(W)$ and $g_2 \in \mathcal{I}(W)$. At this point, since we shall later need the concrete expression of g_2, let us explain in detail how lemma 3.4 shows that $g_2 \in \mathcal{I}(W)$. Thus, from (12) and setting $\tilde{B} := \varepsilon^{-\lambda}\delta B^{\lambda}$ for sake of brevity, we have that

$$T_3(s^{\lambda}(\tilde{B} + g_1)) = C(s^{\lambda}(\tilde{B} + g_1))^n + (s^{\lambda}(\tilde{B} + g_1))^n h_3(s^{\lambda}(\tilde{B} + g_1)).$$

Due to $g_1 \in \mathcal{I}_0(W)$, from (b) in lemma 3.4 we obtain that $C(s^{\lambda}(\tilde{B} + g_1))^n = (C\tilde{B}^n)s^{\lambda n} + s^{\lambda n}\tilde{g}_2$ with $\tilde{g}_2 \in \mathcal{I}_0(W)$. Therefore, the remainder term in (18) is given by

$$g_2 := \tilde{g}_2 + (\tilde{B} + g_1)^n h_3(s^{\lambda}(\tilde{B} + g_1)), \text{ where } g_1, h_3, \tilde{g}_2 \in \mathcal{I}_0(W). \tag{20}$$

Since $h_3 \in \mathcal{I}_0(W)$ and $s^{\lambda}(\tilde{B}+g_1) \in \mathcal{I}(W)$, (a) in lemma 3.4 shows that $h_3(s^{\lambda}(\tilde{B}+g_1)) \in \mathcal{I}(W)$. Consequently, on account of $\mathcal{I}_0(W) + \mathcal{I}(W) \subset \mathcal{I}(W)$, this proves that $g_2 \in \mathcal{I}(W)$ as claimed.

On the other hand recall that $T_2(s;\mu) = V(s/\varepsilon)$, where $V(s)$ is given by (13), (13) or (14) depending on which set W_i is μ. Let us assume first that $\mu \in W_1$. Then, from (13) and applying lemma 3.4 again, it follows that

$$T_2(Bs + sh_2) = b_0 + (\varepsilon^{-1}b_1 B)s + sg_3(s;\mu), \qquad \text{with } g_3 \in \mathcal{I}(W_1).$$

Thus, collecting the terms in (19) that we obtain after the substitution of this expression, one gets

$$T(s) = \underbrace{(A_0 + b_0)}_{\Delta_0} + \underbrace{(A_1 + b_1\varepsilon^{-1}B)}_{\Delta_1} s + sf_1(s;\mu), \qquad \text{with } f_1 \in \mathcal{I}(W_1).$$

Here, to collect the remainder terms, we used that $s^{\lambda n-1}g_2 \in \mathcal{I}(W_1)$ and this is so because, on account of $\mu \in W_1$, we have that $\lambda n - 1 > 0$. Assume next that $\mu \in W_2$. In this case, from (13) and applying lemma 3.4 once again, it turns out that

$$T_2(Bs + sh_2) = b_0 + (\varepsilon^{-\lambda n}b_2 B^{\lambda n})s^{\lambda n} + s^{\lambda n}g_4(s;\mu), \qquad \text{with } g_4 \in \mathcal{I}(W_2).$$

Therefore, from (19) it follows that

$$T(s) = \underbrace{(A_0 + b_0)}_{\Delta_0} + \underbrace{(b_2\varepsilon^{-\lambda n}B^{\lambda n} + C\varepsilon^{-\lambda n}\delta^n B^{\lambda n})}_{\Delta_2} s^{\lambda n} + s^{\lambda n} f_2(s;\mu), \qquad \text{with } f_2 \in \mathcal{I}(W_2).$$

In this case to gather the remainder terms we used that $s^{1-\lambda n}h_1 \in \mathcal{I}(W_2)$. Let us consider finally the case $\mu \in W_3$. Now, from (15) and on account of lemma 3.4, we can assert that

$$\begin{aligned} T_2(Bs + sh_2) = b_0 &+ (a_1\varepsilon^{-1}B\omega(\varepsilon^{-1}B;\lambda n) + a_2\varepsilon^{-1}B)s \\ &+ (a_1\varepsilon^{-\lambda n}B^{\lambda n})s\omega(s;\lambda n) + sg_5(s;\mu), \qquad \text{with } g_5 \in \mathcal{I}(W_3). \end{aligned}$$

Thus, using that $s^{\lambda n} = (\lambda n - 1)s\omega(s;\lambda n) + s$, the substitution of this expression in (19) yields

$$\begin{aligned} T(s) = &\underbrace{(A_0 + b_0)}_{\Delta_0} + \underbrace{\left(a_1\varepsilon^{-\lambda n}B^{\lambda n} + (\lambda n - 1)C\varepsilon^{-\lambda n}\delta^n B^{\lambda n}\right)}_{\Delta_3} s\omega(s;\lambda n) \\ &+ \underbrace{(A_1 + a_1\varepsilon^{-1}B\omega(\varepsilon^{-1}B;\lambda n) + a_2\varepsilon^{-1}B + C\varepsilon^{-\lambda n}\delta^n B^{\lambda n})}_{\Delta_4} s + sf_3(s;\mu), \end{aligned}$$

with $f_3 \in \mathcal{I}(W_3)$.

It is to be noted that the gathering of the remainder term above has been a little more delicate than the preceding cases because it does not follow directly from lemma 3.4. Notice that this remainder term is given by $sh_1 + s^{\lambda n} g_2 + sg_5$ with $h_1 \in \mathcal{I}_0(W)$, $g_2 \in \mathcal{I}(W)$ and $g_5 \in \mathcal{I}(W_3)$. The fact that $sh_1 + sg_5 \in s\mathcal{I}(W_3)$ is obvious, and so it remains to check that $s^{\lambda n} g_2 \in s\mathcal{I}(W_3)$. This follows from the particular expression of g_2, which is given in (20). Indeed, since $h_3, \tilde{g}_2 \in \mathcal{I}_0(W)$, we can assert that $h_3 = s\hat{h}_3$ and $\tilde{g}_2 = s\hat{g}_2$, where $\hat{h}_3$ and $\hat{g}_2$ are analytic on $s = 0$. Then, from (20),

$$s^{\lambda n} g_2 = s\big(s^{\lambda n}\hat{g}_2 + s^{\lambda(n+1)-1}(\tilde{B} + g_1)^{n+1}\hat{h}_3(s^{\lambda}(\tilde{B} + g_1))\big)$$

and this, since $\lambda(n+1) - 1 > 0$ due to $\mu \in W_3$, shows that $s^{\lambda n} g_2 \in s\mathcal{I}(W_3)$, as desired.

This concludes the part of the proof concerning the remainder terms. Let us turn now to the computation of the coefficients ρ and Δ_i. Notice however that at this point we can already assert that these coefficients depend analytically on μ. The key point to compute them explicitly is that in fact they do not depend on δ nor ε. Therefore, in order to obtain simpler expressions we will take limits when both parameters tend to zero. To this end we first rewrite the functions $A_1(\mu)$, $B(\mu)$, $C(\mu)$ and $D(\mu)$ in terms of $L(u)$ and $M(u)$ (which were given before the statement of the result) because then it is easier to take limits when δ and ε tend to zero. Thus, some computations show that

$$\begin{aligned}
B(\mu) &= \delta^{-1/\lambda}\frac{\sigma_1'(0)\sigma_2(0)^{1/\lambda}}{\psi_1(0,\delta)\psi_2(0,\delta)^{1/\lambda}}L(\delta\psi_2(0,\delta)),\\
A_1(\mu) &= \delta^{n-1/\lambda}\frac{\sigma_1'(0)\sigma_2(0)^{1/\lambda}(\psi_2)_x(0,\delta)}{\psi_1(0,\delta)\psi_2(0,\delta)^{1/\lambda-n+1}}\frac{L(\delta\psi_2(0,\delta))}{Q(0,\delta\psi_2(0,\delta))} - \frac{\sigma_2'(0)\sigma_2(0)^{n-1}}{Q(0,\sigma_2(0))}\\
&\quad - \sigma_1'(0)\sigma_2(0)^{1/\lambda}\int_{\sigma_2(0)}^{\delta\psi_2(0,\delta)}\frac{Q_u(0,v)L(v)v^{n-1/\lambda}}{Q(0,v)^2}\frac{\mathrm{d}v}{v},\\
C(\mu) &= \varepsilon^{\lambda n}\frac{\psi_1(\varepsilon,0)^{\lambda n}\psi_2(\varepsilon,0)^n}{M(\varepsilon)^n}\int_{\varepsilon\psi_1(\varepsilon,0)}^{\tau_1(0)}\frac{M(u)^n}{P(u,0)}\frac{\mathrm{d}u}{u^{1+\lambda n}},\\
D(\mu) &= \varepsilon^{\lambda}\frac{\psi_2(\varepsilon,0)\psi_1(\varepsilon,0)^{\lambda}}{\tau_2'(0)\tau_1(0)^{\lambda}}\frac{M(\tau_1(0))}{M(\varepsilon)}.
\end{aligned}$$

We begin with the coefficient $\rho(\mu)$ of the Dulac map $R(s;\mu)$, which is given in (16). We obtain

$$\rho(\mu) = \lim_{(\varepsilon,\delta)\to(0,0)}\varepsilon^{-\lambda}D\delta B^{\lambda} = \lim_{\delta\to 0}(\delta B^{\lambda})\lim_{\varepsilon\to 0}(\varepsilon^{-\lambda}D) = \sigma_1'(0)^{\lambda}\sigma_2(0)L(0)^{\lambda}\frac{M\big(\tau_1(0)\big)}{\tau_2'(0)\tau_1(0)^{\lambda}}.$$

Next, we proceed with the coefficient $\Delta_0(\mu)$. In this case, it turns out that

$$\Delta_0 = A_0 + b_0 = \int_{\sigma_2(0)}^{\delta\psi_2(0,\delta)}\frac{x^{n-1}}{Q(0,x)}\,\mathrm{d}x + \delta^n\int_0^1 u^{\lambda n}g(0,\delta^{\lambda}u^{\lambda})\frac{\mathrm{d}u}{u} \to \int_{\sigma_2(0)}^{0}\frac{x^{n-1}}{Q(0,x)}\,\mathrm{d}x,$$
$$\text{as } \delta\to 0.$$

Let us turn now to the computation of Δ_1. First of all notice that when we deal with this coefficient we are assuming that $\mu \in W_1$, which corresponds to $\lambda n - 1 > 0$. Taking this into account,

$$\begin{aligned}
\Delta_1 &= \lim_{\delta\to 0}(A_1 + b_1\varepsilon^{-1}B) = \lim_{\delta\to 0}(A_1)\\
&= -\frac{\sigma_2'(0)\sigma_2(0)^{n-1}}{Q(0,\sigma_2(0))}\\
&\quad - \sigma_1'(0)\sigma_2(0)^{1/\lambda}\int_{\sigma_2(0)}^{0}\frac{Q_u(0,v)L(v)v^{n-1/\lambda}}{Q(0,v)^2}\frac{\mathrm{d}v}{v}.
\end{aligned}$$

As before, when computing Δ_2 we must take into account that $\mu \in W_2$ (i.e. $\lambda n - 1 < 0$). Thus,

$$\Delta_2 = \lim_{(\varepsilon,\delta)\to(0,0)} (\varepsilon^{-\lambda n} b_2 B^{\lambda n} + C\varepsilon^{-\lambda n}\delta^n B^{\lambda n}) = \lim_{\delta\to 0} \underbrace{(\delta^n B^{\lambda n})}_{\kappa_1} \lim_{\varepsilon\to 0} \underbrace{(\varepsilon^{-\lambda n}\delta^{-n} b_2 + C\varepsilon^{-\lambda n})}_{\kappa_2}.$$

Here, we are using that in fact κ_1 does not depend on ε and that κ_2 does not depend on δ. One can easily check that

$$\lim_{\delta\to 0} \kappa_1 = \sigma_1'(0)^{\lambda n} \sigma_2(0)^n L(0)^{\lambda n}. \tag{21}$$

The computation of $\lim_{\varepsilon\to 0} \kappa_2$ is a little more delicate. This is so because the two terms in κ_2, namely $\varepsilon^{-\lambda n}\delta^{-n} b_2$ and $C\varepsilon^{-\lambda n}$, are divergent as $\varepsilon \to 0$. To prove that in fact these divergences compensate when we consider the two terms together it is necessary to expand $C(\mu)$. This is the reason why we introduce the function

$$N(u) := \begin{cases} \dfrac{1}{u}\left(\dfrac{M(u)^n}{P(u,0)} - \dfrac{1}{P(0,0)}\right) & \text{if } u \neq 0, \\ \dfrac{\mathrm{d}}{\mathrm{d}u}\left(\dfrac{M(u)^n}{P(u,0)}\right)\Bigg|_{u=0} & \text{if} u = 0. \end{cases}$$

Then one can check that

$$C(\mu) = \frac{\psi_2(\varepsilon,0)}{\lambda n P(0,0) M(\varepsilon)^n} - \varepsilon^{\lambda n} \frac{\psi_1(\varepsilon,0)^{\lambda n}\psi_2(\varepsilon,0)^n}{M(\varepsilon)^n} \frac{\tau_1(0)^{-\lambda}}{n\lambda P(0,0)} \\ + \varepsilon^{\lambda n} \frac{\psi_1(\varepsilon,0)^{\lambda n}\psi_2(\varepsilon,0)^n}{M(\varepsilon)^n} \int_{\varepsilon\psi_1(\varepsilon,0)}^{\tau_1(0)} \frac{N(u)}{u^{\lambda n}}\,\mathrm{d}u,$$

and consequently,

$$\kappa_2 = \frac{\varepsilon^{-\lambda n}}{nQ(0,0)} + \varepsilon^{-\lambda n} \frac{\psi_2(\varepsilon,0)}{\lambda n P(0,0) M(\varepsilon)^n} + \varepsilon^{1-\lambda n} \int_0^1 \left(\frac{g(\varepsilon u,0) - g(0,0)}{\varepsilon u}\right) \frac{\mathrm{d}u}{u^{\lambda n}} \\ - \frac{\psi_1(\varepsilon,0)^{\lambda n}\psi_2(\varepsilon,0)^n}{M(\varepsilon)^n} \frac{\tau_1(0)^{-\lambda n}}{n\lambda P(0,0)} + \frac{\psi_1(\varepsilon,0)^{\lambda n}\psi_2(\varepsilon,0)^n}{M(\varepsilon)^n} \int_{\varepsilon\psi_1(\varepsilon,0)}^{\tau_1(0)} \frac{N(u)}{u^{\lambda n}}\,\mathrm{d}u.$$

Note at this point that the two first terms in κ_2 can be collected as $\varepsilon^{-\lambda n} h(\varepsilon)$, where

$$h(\varepsilon) := \frac{1}{nQ(0,0)} + \frac{\psi_2(\varepsilon,0)}{\lambda n P(0,0) M(\varepsilon)^n}.$$

Then, since one can verify that $h(0) = 0$, there exists an analytic function $\tilde{h}$ such that $h(\varepsilon) = \varepsilon\tilde{h}(\varepsilon)$. This shows, on account of $1 - \lambda n > 0$, that $\lim_{\varepsilon\to 0} \varepsilon^{-\lambda n} h(\varepsilon) = 0$. Note, on the other hand, that the functions $N(u)$ and $(g(u,0) - g(0,0))/u$ are continuous at $u = 0$. Thus, since $u^{-\lambda n}$ is integrable near $u = 0$ (once again due to $1 - \lambda n > 0$), by applying the dominated convergence theorem to the integral terms in κ_2 we obtain

$$\lim_{\varepsilon\to 0} \kappa_2 = \frac{\tau_1(0)^{-\lambda n}}{nQ(0,0)} + \int_0^{\tau_1(0)} \frac{N(u)}{u^{\lambda n}}\,\mathrm{d}u.$$

Consequently,

$$\Delta_2 = \lim_{\delta\to 0} \kappa_1 \lim_{\varepsilon\to 0} \kappa_2 = \sigma_1'(0)^{\lambda n} \sigma_2(0)^n L(0)^{\lambda n} \left(\frac{\tau_1(0)^{-\lambda n}}{nQ(0,0)} + \int_0^{\tau_1(0)} \frac{N(u)}{u^{\lambda n}}\,\mathrm{d}u\right).$$

Finally, if $\mu_0 \in W_3$ is such that $\lambda(\mu_0) = 1/n$ then $\Delta_3(\mu_0) = a_1 B/\varepsilon = -g_x(0,0)\delta^n B$. Thus, since $\kappa_1 = \delta^n B^{\lambda n}$, from (21) it follows that

$$\Delta_3(\mu_0) = -g_x(0,0) \lim_{\delta\to 0} \kappa_1 = -g_x(0,0)\sigma_1'(0)\sigma_2(0)^n L(0).$$

We claim that $g_x(0,0) = n(Q_u(0,0)/P(0,0)^2)$ and notice that the result will follow once we prove this. To this end note first that, according to definition 2.1 and remark 2.2, we have

$$X_\mu = \Phi_* \left(\frac{1}{y^n g(x,y)} (x\partial_x - \lambda y \partial_y) \right).$$

This provides two equalities, namely

$$y^n g(x,y)\Phi_1(x,y)P(\Phi(x,y)) = \Phi_2(x,y)^n (x\Phi_{1x}(x,y) - \lambda y \Phi_{1y}(x,y)) \tag{22}$$

and

$$y^n g(x,y)\Phi_2(x,y)Q(\Phi(x,y)) = \Phi_2(x,y)^n (x\Phi_{2x}(x,y) - \lambda y \Phi_{2y}(x,y)). \tag{23}$$

Since $\Phi_1(x,y) = x\psi_1(x,y)$ and $\Phi_2(x,y) = y\psi_2(x,y)$, from (22) it follows easily that

$$g(x,y) = \frac{\psi_2(x,y)^n}{P(\Phi(x,y))} \left(1 + \frac{x\psi_{1x}(x,y) - \lambda y \psi_{1y}(x,y)}{\psi_1(x,y)} \right).$$

Therefore, on account of $\psi_i(0,0) = 1$, a computation shows that

$$g_x(0,0) = \frac{\psi_{1x}(0,0) + n\psi_{2x}(0,0)}{P(0,0)} - \frac{P_u(0,0)}{P(0,0)^2}. \tag{24}$$

On the other hand, by the substitution of $g(x,y)$ in (23) we obtain

$$\frac{\psi_1(x,y)}{\psi_2(x,y)} \frac{x\psi_{2x}(x,y) - \lambda(\psi_2(x,y) + y\psi_{2y}(x,y))}{\psi_1(x,y) + x\psi_{1x}(x,y) - \lambda y \psi_{1y}(x,y)} = \frac{Q(\Phi(x,y))}{P(\Phi(x,y))}$$

and, taking $\lambda = 1/n$ into account, the derivative of this equality with respect to x yields

$$\psi_{1x}(0,0) + n\psi_{2x}(0,0) = n \frac{\mathrm{d}}{\mathrm{d}u} \left(\frac{Q(u,0)}{P(u,0)} \right)_{u=0}.$$

Now, the claim follows from the substitution of the above expression in (24) and using once again that $\lambda := -Q(0,0)/P(0,0) = 1/n$. This concludes the proof of the result. □

5. An application to Loud systems

Consider now a family $\{X_\mu, \mu \in W\}$ of polynomial vector fields having a centre at the origin. In order to completely understand the qualitative properties of the period function in this family it is necessary to study the bifurcation of critical periods from the boundary of the period annulus, say $\mathcal{P}_\mu$, as the parameter μ varies. To this end we must choose local parametrizations for the set of periodic orbits near the boundary of $\mathcal{P}_\mu$.

In case that the centre at the origin is non-degenerate for all $\mu \in W$, there is a natural way to parametrize the set of periodic orbits near the inner boundary. For instance, using that $\{(x,0),\ x \in (0,\delta)\}$ is a transverse section for $\delta > 0$ small enough. Then, for each $s \gtrsim 0$, one can denote by $P(s;\mu)$ the period of the periodic orbit of X_μ passing through the point $(s,0)$. In this situation (see [5]), it is said that k critical periods bifurcate from the inner boundary of the period annulus corresponding to the parameter value $\mu_0 \in W$ if for every $\varepsilon > 0$ and every neighbourhood U of μ_0 there exists $\mu_1 \in U$ such that $P_s(s;\mu_1) = 0$ has k solutions on $(0,\varepsilon)$.

In this section, we study the bifurcation of critical periods from the outer boundary. As before, to define this notion we must parametrize the set of periodic orbits near the outer boundary of $\mathcal{P}_\mu$. More precisely, if $\mu_0 \in W$ is the parameter value to study, we consider a neighbourhood U of μ_0 and an analytic map $\sigma : [0,\delta) \times U \to \Sigma_\mu$ such that, for each $\mu \in U$, we have the following properties:

1. $\Sigma_\mu \subset \bar{\mathcal{P}}_\mu$,
2. $\sigma(0;\mu)$ belongs to the outer boundary of $\mathcal{P}_\mu$,
3. Σ_μ is transverse to X_μ.

Then, if $P(s;\mu)$ denotes now the period of the periodic orbit of X_μ passing through the point $\sigma(s;\mu)\in\mathcal{P}_\mu$, we define the bifurcation from the outer boundary exactly as before. Let us remark that the number of critical periods that bifurcate does not depend on the particular parametrization considered.

For the family of quadratic vector fields having a centre, Chicone and Jacobs [5] solved completely the problem of the bifurcation of critical periods from the inner boundary. The fact that these centres are non-degenerate allows to consider the Taylor development of $P(s;\mu)$ at $s=0$ and to study its coefficients, the so-called period constants. This is the key point in [5]. In this section, as an application of theorem A, we investigate the bifurcation of critical periods from the outer boundary. The difficulty in this study lies in the fact that the polycycle $\partial\mathcal{P}_\mu$ undergoes a qualitative bifurcation. This forces us to take different parametrizations for the set of periodic orbits. In addition, once such a parametrization is chosen, the function $s\longmapsto P(s;\mu)$ is not analytic at $s=0$.

Denoting $\mu:=(D,F)$, we study the subfamily of quadratic vector fields

$$X_\mu := (-y+xy)\partial_x + (x+Dx^2+Fy^2)\partial_y$$

in case that the parameter μ belongs to $W:=\{(D,F)\in\mathbb{R}^2 : D\in(-1,0),\ F\in(0,1)\}$. It is well known (see [26, 29] for instance) that, for any $\mu\in W$, the critical point of X_μ at the origin is a centre with $\mathcal{P}_\mu=\{(x,y)\in\mathbb{R}^2 : x<1\}$. Setting

$$S_1 := \left\{\mu\in W : D=-\tfrac{1}{2},\, F\in\left(\tfrac{1}{2},1\right)\right\} \text{ and } S_2 := \left\{\mu\in W : F=\tfrac{1}{2}\right\},$$

in this section we prove the following result:

Theorem 5.1. *Consider the period function of the centre at the origin of* $\{X_\mu, \mu\in W\}$*. If* $\mu\notin S_1\cup S_2$ *then no critical period bifurcates from the outer boundary of the period annulus.*

Since one can check that X_μ is transverse to $\{(x,0),\ 0<x<1\}$, we have a global parametrization of the set of periodic orbits in $\mathcal{P}_\mu$ by the value of x. Thus, we denote by $P(s;\mu)$ the period of the periodic orbit of X_μ passing through the point $(1-s,0)$. Next, in order to study the period function $s\longmapsto P(s;\mu)$, we shall take advantage of the fact that the transformation $(x,y,t)\longmapsto(x,-y,-t)$ preserves the Loud normal form. To this end, let us denote by $\varphi(t,(x_0,y_0);\mu)$ the solution of X_μ passing through (x_0,y_0) at $t=0$. Then, for each $s\in(0,1)$, we define $T(s;\mu)$ as the minimum positive number so that $\varphi_2(T(s;\mu),(1-s,0);\mu)=0$. It is clear therefore that $P(s;\mu)=2T(s;\mu)$.

The next result gives the first nontrivial term of the asymptotic development of $T(s;\mu)$ at $s=0$ in case that $\mu\notin S_2$ (if $\mu\in S_2$ then X_μ does not have a Darboux first integral and remark 2.3 can not be applied). Note that $W\backslash S_2=W_1\cup W_2$, where

$$W_1 := \left\{\mu\in W : \tfrac{1}{2}<F<1\right\} \text{ and } W_2 := \left\{\mu\in W : 0<F<\tfrac{1}{2}\right\}.$$

Proposition 5.2. *Denote* $\Delta_0(\mu)=\pi/\left(2\sqrt{F(D+1)}\right)$ *and* $\lambda(\mu)=F/(1-F)$*. Then the following holds:*

(a) If $\mu\in W_1$ *then* $T(s;\mu)=\Delta_0(\mu)+\Delta_1(\mu)s+sf_1(s;\mu)$*, where* $f_1\in\mathcal{I}(W_1)$ *and*

$$\Delta_1(\mu)=\frac{\sqrt{\pi}\,(2D+1)}{2\sqrt{F(D+1)^3}}\frac{\Gamma((2\lambda-1)/(2\lambda))}{\Gamma((3\lambda-1)/(2\lambda))}.$$

(b) If $\mu\in W_2$ *then* $T(s;\mu)=\Delta_0(\mu)+\Delta_2(\mu)\,s^\lambda+s^\lambda f_2(s;\mu)$*, where* $f_2\in\mathcal{I}(W_2)$ *and*

$$\Delta_2(\mu)=\frac{1}{1-F}\left(\frac{F}{D+1}\right)^{(\lambda+1)/2}\int_0^{+\infty}\left(\left(\frac{D}{F-1}u^2+1\right)^{(\lambda-1)/2}-1\right)\frac{\mathrm{d}u}{u^{\lambda+1}}.$$

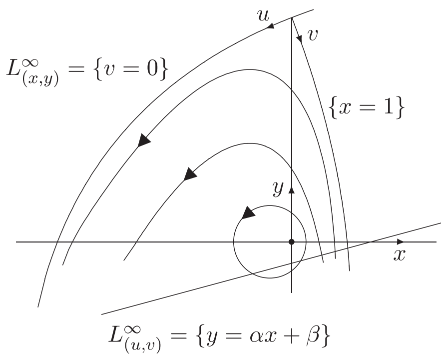

Figure 4. The projective change of coordinates.

Proof. Note first that, for $s \gtrsim 0$, $s \longmapsto (1-s, 0)$ and $s \longmapsto (-1/s, 0)$ are transverse sections to X_μ. Denote them by Σ_1 and Σ_2, respectively. It is also clear that $T(s;\mu)$ is precisely the time between Σ_1 and Σ_2. In order to apply theorem A we must first perform a suitable projective change of coordinates. To this end (see figure 4) we choose any straight line $y = \alpha x + \beta$ not intersecting $\mathcal{P}_\mu \cap \{y \geqslant 0\}$, which corresponds to requiring that $\alpha > 0$ and $\alpha + \beta < 0$, and we consider the coordinates

$$(u, v) = \phi(x, y) := \left(\frac{1-x}{y-\alpha x-\beta}, \frac{1}{y-\alpha x-\beta}\right).$$

A computation shows that this change of coordinates brings X_μ to

$$\tilde{X}_\mu = \frac{1}{v}(u\,P(u, v;\mu)\,\partial_u + v\,Q(u, v;\mu)\,\partial_v),$$

where

$$P(u, v) = (1-\alpha u)^2 + (\alpha+\beta)v - v^2 + (1-\alpha^2-\alpha\beta)uv - F(1+\alpha(v-u)+\beta v)^2 - D(u-v)^2$$

and

$$Q(u, v) = \alpha^2 u^2 + (1-\alpha\beta-\alpha^2)uv - \alpha u - v^2 - F\big(1+\alpha(v-u)+\beta v\big)^2 - D(u-v)^2.$$

One can also check that if $\mu \notin S_2$, then $H(x, y) = (1-x)^{-2F}(\frac{1}{2}y^2 - ax^2 - bx - c)$, with

$$a = \frac{D}{2(1-F)}, \qquad b = \frac{D-F+1}{(1-F)(1-2F)} \qquad \text{and} \qquad c = \frac{F-D-1}{2F(1-F)(1-2F)},$$

is a first integral of X_μ. Thus it turns out that $H(\phi^{-1}(u, v))$ is a Darboux first integral of $\tilde{X}_\mu$ in case that $\mu \notin S_2$. According to remark 2.3, this fact guarantees that $\{\tilde{X}_\mu,\ \mu \in W\backslash S_2\}$ is a family of vector fields verifying FLP.

We shall apply theorem A to study the time function between $\Sigma_\sigma := \phi(\Sigma_1)$ and $\Sigma_\tau := \phi(\Sigma_2)$. Notice that *a priori* this function depends also on α and β. We point out however that in fact it does not, and this is so because by construction this function is precisely $T(s;\mu)$. This will be the key point of the proof. A computation shows that Σ_σ and Σ_τ are given respectively by

$$\sigma(s) = \left(\frac{s}{\alpha s-\alpha-\beta}, \frac{1}{\alpha s-\alpha-\beta}\right) \text{ and } \tau(s) = \left(\frac{s+1}{\alpha-\beta s}, \frac{s}{\alpha-\beta s}\right).$$

Following the notation of theorem A, since $\lambda := -(Q(0,0)/P(0,0)) = F/(1-F)$, it turns out that

$$W_1 = \left\{\mu \in W : \tfrac{1}{2} < F < 1\right\} \text{ and } W_2 = \left\{\mu \in W : 0 < F < \tfrac{1}{2}\right\}.$$

Notice then that $W \setminus S_2 = W_1 \cup W_2$. In addition

$$\Delta_0(\mu) = \int_{\sigma_2(0)}^{0} \frac{\mathrm{d}v}{Q(0,v)} = \int_0^{-1/(\alpha+\beta)} \frac{\mathrm{d}v}{(D+1)v^2 + F(1+(\alpha+\beta)v)^2} = \frac{\pi}{2\sqrt{F(D+1)}}.$$

Let us study first the case in which $\mu \in W_1$. According to theorem A, in this situation we can assert that $T(s;\mu) = \Delta_0(\mu) + \Delta_1(\mu)s + sf_1(s;\mu)$, where $f_1 \in \mathcal{I}(W_1)$ and

$$\Delta_1(\mu) = -\frac{\sigma_2'(0)}{Q(0,\sigma_2(0))} + \sigma_1'(0)\sigma_2(0)^{1/\lambda} \int_0^{\sigma_2(0)} \frac{Q_u(0,v)L(v)}{Q(0,v)^2} \frac{\mathrm{d}v}{v^{1/\lambda}}.$$

Some computations show that

$$L(v) := \exp\left(\int_{\sigma_2(0)}^{v} \left(\frac{P(0,y)}{Q(0,y)} + \frac{1}{\lambda}\right)\frac{\mathrm{d}y}{y}\right) = (-\alpha-\beta)^{1/F}\left(v^2 + \frac{F}{D+1}(1+(\alpha+\beta)\,v)^2\right)^{1/(2F)}$$

and that

$$\frac{Q_u(0,v)}{Q(0,v)^2} = \frac{\alpha(2F-1)((\alpha+\beta)v+1) + (2D+1)v}{((1+D)v^2 + F(1+(\alpha+\beta)v)^2)^2}.$$

Consequently, since one can check that

$$\sigma_1'(0)\sigma_2(0)^{1/\lambda} = \left(\frac{-1}{\alpha+\beta}\right)^{1/F} \qquad \text{and} \qquad \frac{\sigma_2'(0)}{Q(0,\sigma_2(0))} = \frac{\alpha}{D+1},$$

we conclude that

$$\Delta_1(\mu) = -\frac{\alpha}{D+1} + \frac{1}{(D+1)^{1/(2F)}} \int_0^{-1/(\alpha+\beta)} \frac{\alpha(2F-1)((\alpha+\beta)v+1) + (2D+1)v}{((D+1)v^2 + F(1+(\alpha+\beta)\,v)^2)^{2-(1/2F)}} \frac{\mathrm{d}v}{v^{1/\lambda}}.$$

At this point we take advantage of the fact $\Delta_1(\mu)$ does not depend on α and β. Thus, by making $\alpha \searrow 0$ and $(\alpha+\beta) \nearrow 0$ in the expression above, it follows that

$$\Delta_1(\mu) = \frac{2D+1}{(D+1)^2} \int_0^{+\infty} \left(v^2 + \frac{F}{D+1}\right)^{1/(2F)-2} \frac{\mathrm{d}v}{v^{1/\lambda-1}}.$$

Here we used the dominated convergence theorem. Finally, direct integration shows that

$$\Delta_1(\mu) = \frac{\sqrt{\pi}\,(2D+1)}{2\sqrt{F(D+1)^3}} \frac{\Gamma((2\lambda-1)/(2\lambda))}{\Gamma((3\lambda-1)/(2\lambda))}.$$

Let us consider now the case in which $\mu \in W_2$. In this case by applying theorem A it turns out that $T(s;\mu) = \Delta_0(\mu) + \Delta_2(\mu)s^\lambda + s^\lambda f_2(s;\mu)$, where $f_2 \in \mathcal{I}(W_2)$ and

$$\Delta_2(\mu) = \sigma_1'(0)^\lambda \sigma_2(0) L(0)^\lambda \left\{\frac{\tau_1(0)^{-\lambda}}{Q(0,0)} + \int_0^{\tau_1(0)} \left(\frac{M(u)}{P(u,0)} - \frac{M(0)}{P(0,0)}\right)\frac{\mathrm{d}u}{u^{\lambda+1}}\right\}.$$

One can verify that

$$M(u) := \exp\left(\int_0^u \left(\frac{Q(x,0)}{P(x,0)} + \lambda\right)\frac{\mathrm{d}x}{x}\right) = \left(\frac{D}{F-1}\,u^2 + (\alpha u - 1)^2\right)^{1/(2-2F)}$$

and that

$$P(u,0) = (1-F)\left(\frac{D}{F-1}\,u^2 + (\alpha u - 1)^2\right).$$

Therefore, taking also into account that $\tau_1(0) = 1/\alpha$, $Q(0,0) = -F$ and

$$\sigma_1'(0)^\lambda \sigma_2(0) L(0)^\lambda = \left(\frac{F}{D+1}\right)^{1/(2-2F)},$$

we obtain

$$\Delta_2(\mu) = \left(\frac{F}{D+1}\right)^{1/(2-2F)} \times \left\{ -\frac{\alpha^\lambda}{F} + \frac{1}{1-F}\int_0^{1/\alpha} \left(\left(\frac{D}{F-1}\,u^2 + (\alpha u - 1)^2 \right)^{(2F-1)/(2-2F)} - 1 \right) \frac{\mathrm{d}u}{u^{\lambda+1}} \right\}.$$

We use now that in fact $\Delta_2(\mu)$ does not depend on α, and that consequently we can make $\alpha \searrow 0$ to compute it. Thus, by means the dominated convergence theorem it is easy to show that

$$\Delta_2(\mu) = \frac{1}{1-F}\left(\frac{F}{D+1}\right)^{1/(2-2F)} \int_0^{+\infty} \left(\left(\frac{D}{F-1}\,u^2 + 1 \right)^{(2F-1)/(2-2F)} - 1 \right) \frac{\mathrm{d}u}{u^{\lambda+1}}$$

and this concludes the proof of the proposition. □

Proof of theorem 5.1. Fix some $\mu^\star \in W \setminus (S_1 \cup S_2)$ and notice that then $\mu^\star \in W_1$ or $\mu^\star \in W_2$.

Assume first that $\mu^\star \in W_1$. In this case, by applying proposition 5.2 and taking definition 2.4 into account, it follows that $T_s(s;\mu) \to \Delta_1(\mu^\star)$, as $(s,\mu) \to (0,\mu^\star)$. In addition, on account of $\mu^\star \notin S_1$, one can verify that $\Delta_1(\mu^\star) \neq 0$. Consequently, we can assert that there exists a neighbourhood $U^\star$ of $\mu^\star$ and $\varepsilon > 0$ such that $T_s(s;\mu) \neq 0$ for all $s \in (0,\varepsilon)$ and $\mu \in U^\star$. Since $P(s;\mu) = 2T(s;\mu)$, this shows that no critical period bifurcates from the outer boundary.

Let us study next the case $\mu^\star \in W_2$. By applying proposition 5.2 to this case we obtain that, in a neighbourhood of $\mu^\star$, $T_s(s;\mu) = s^{\lambda-1}(\lambda\Delta_2(\mu) + \lambda f_2(s;\mu) + s f_{2s}(s;\mu))$ with $f_2 \in \mathcal{I}(W_2)$. Therefore, taking definition 2.4 into account, it turns out that

$$\frac{T_s(s;\mu)}{\lambda s^{\lambda-1}} \to \Delta_2(\mu^\star) \qquad \text{as } (s,\mu) \to (0,\mu^\star).$$

Since one can easily check that $\Delta_2(\mu) \neq 0$ for all $\mu \in W_2$, this proves that neither in this case are there critical periods bifurcating from the outer boundary. This completes the proof of the result. □

We conclude this section with a result, namely corollary 5.3, that guarantees, for a given vector field X_μ, the existence of at least one critical period. The idea is very simple. By means of proposition 5.2, we can decompose the parameter space as

$$W \setminus (S_1 \cup S_2) = A_+ \cup A_-,$$

so that if μ belongs to A_+ (respectively A_-) then the period function of X_μ is monotonically increasing (respectively decreasing) near the outer boundary of $\mathcal{P}_\mu$. This decomposition follows from computing the signum of $\Delta_1(\mu)$ and $\Delta_2(\mu)$, and one can easily verify that

$$A_- = \left\{\mu : D \in \left(-\tfrac{1}{2}, 0\right), F \in \left(\tfrac{1}{2}, 1\right)\right\} \quad \text{and} \quad A_+ = W \setminus (S_1 \cup S_2 \cup A_-).$$

On the other hand, using the first period constant, we can do the same near the inner boundary. The first period constant for the quadratic centres can be found in [5]. For the subfamily that we study it is given by

$$P_2(D,F) = 10D^2 + 10DF - D + 4F^2 - 5F + 1.$$

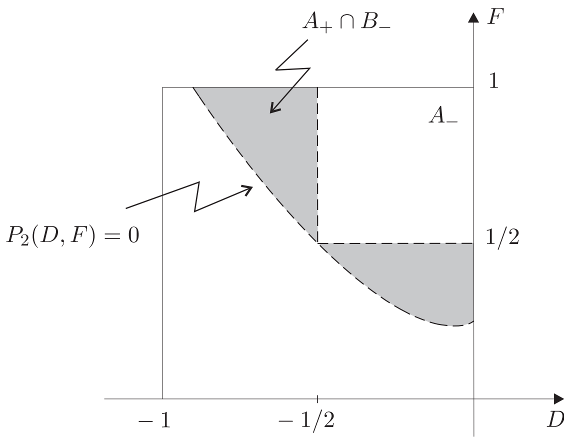

Figure 5. Regions with at least one critical period.

Hence, it turns out that $W\backslash\{P_2 = 0\} = B_+ \cup B_-$, where if μ belongs to the set B_+ (respectively B_-) then the period function of X_μ is monotonically increasing (respectively decreasing) near the inner boundary of $\mathcal{P}_\mu$.

Consequently, by Bolzano's theorem, if μ belongs to $A_+ \cap B_-$ or $A_- \cap B_+$ then we can assert that the period function of X_μ has at least one critical period. The set $A_- \cap B_+$ is empty but $A_+ \cap B_-$ is not, and so we have proved (see figure 5) the following result.

Corollary 5.3. *If $\mu \in A_+ \cap B_-$ then the period function of the centre at the origin of X_μ has at least one critical period.*

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