

LEFT RELATIVELY CONVEX SUBGROUPS

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ABSTRACT. Let G be a group and H be a subgroup of G . We say that H is left relatively convex in G if the left G -set G/H has at least one G -invariant order; when G is left orderable, this holds if and only if H is convex in G under some left ordering of G .

We give a criterion for H to be left relatively convex in G that generalizes a famous theorem of Burns and Hale and has essentially the same proof. We show that all maximal cyclic subgroups are left relatively convex in free groups, in right-angled Artin groups, and in surface groups that are not the Klein-bottle group. The free-group case extends a result of Duncan and Howie.

More generally, every maximal m -generated subgroup in a free group is left relatively convex. The same result is valid, with some exceptions, for compact surface groups. Maximal m -generated abelian subgroups in right-angled Artin groups are left relatively convex.

If G is left orderable, then each free factor of G is left relatively convex in G . More generally, for any graph of groups, if each edge group is left relatively convex in each of its vertex groups, then each vertex group is left relatively convex in the fundamental group; this generalizes a result of Chiswell.

All maximal cyclic subgroups in locally residually torsion-free nilpotent groups are left relatively convex.

1. OUTLINE

Notation 1. Throughout this article, let G be a multiplicative group, and G_0 be a subgroup of G . For $x, y \in G$, $[x, y] := x^{-1}y^{-1}xy$, $x^y := y^{-1}xy$, and ${}^yx := yxy^{-1}$. For any subset X of G , $X^{\pm 1} := X \cup X^{-1}$, $\langle X \rangle$ denotes the subgroup of G generated by X , $\langle X^G \rangle$ denotes the normal subgroup of G generated by X , and $G/\triangleleft X \triangleright := G/\langle X^G \rangle$. When we write $A \subseteq B$ we mean that A is a subset of B , and when we write $A \subset B$ we mean that A is a proper subset of B .

In Section 2, we collect together some facts, several of which first arose in the proof of Theorem 28 of [Berg90]. If G is left orderable, Bergman calls G_0 ‘left relatively convex in G ’ if G_0 is convex in G under some left ordering of G , or, equivalently, the left G -set G/G_0 has some G -invariant order. Broadening the scope of his terminology, we shall say that G_0 is *left relatively convex in G* if the left G -set G/G_0 has some G -invariant order, even if G is not left orderable.

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We give a criterion for G_0 to be left relatively convex in G that generalizes a famous theorem of Burns and Hale [BH72] and has essentially the same proof. We deduce that if each noncyclic, finitely generated subgroup of G maps onto \mathbb{Z}^2 , then each maximal cyclic subgroup of G is left relatively convex in G . Thus, if F is a free group and C is a maximal cyclic subgroup of F , then F/C has an F -invariant order; this extends the result of Duncan and Howie [DH91] that a certain finite subset of F/C has an order that is respected by the partial F -action. Louder and Wilton [LW14] used the Duncan-Howie order to prove Wise's conjecture that, for subgroups H and K of a free group F , if H or K is a maximal cyclic subgroup of F , then $\sum_{HxK \in H \backslash F/K} \text{rank}(H^x \cap K) \leq \text{rank}(H) \text{rank}(K)$. They also gave a simple proof of the existence of a Duncan-Howie order; translating their argument from topological to algebraic language led us to the order on F/C . More generally, we introduce the concept of n -indicability and use it to show that each maximal m -generated subgroup of a free group is left relatively convex.

In Section 3, we find that the main result of [DS14] implies that, for any graph of groups, if each edge group is left relatively convex in each of its vertex groups, then each vertex group is left relatively convex in the fundamental group. This generalizes a result of Chiswell [Chi11]. In particular, in a left-orderable group, each free factor is left relatively convex.

One says that G is *discretely left orderable* if some infinite (maximal) cyclic subgroup of G is left relatively convex in G . Many examples of such groups are given in [LRR09]; for instance, it is seen that among free groups, braid groups, surface groups, and right-angled Artin groups, all the infinite ones are discretely left orderable. In Section 4 below, we show that all maximal cyclic subgroups are left relatively convex in right-angled Artin groups and in compact surface groups that are not the Klein-bottle group. More generally, we show that, with some exceptions, each maximal m -generated subgroup of a compact surface group is left relatively convex, and each maximal m -generated abelian subgroup of a right-angled Artin group is left relatively convex.

At the end, in Section 5, we show that all maximal cyclic subgroups in locally residually torsion-free nilpotent groups are left relatively convex.

2. LEFT RELATIVELY CONVEX SUBGROUPS

Definitions 2. Let X be a set and \mathcal{R} be a binary relation on X ; thus, \mathcal{R} is a subset of $X \times X$, and ' $x\mathcal{R}y$ ' means ' $(x, y) \in \mathcal{R}$ '. We say that \mathcal{R} is *transitive* when, for all $x, y, z \in X$, if $x\mathcal{R}y$ and $y\mathcal{R}z$, then $x\mathcal{R}z$, and here we write $x\mathcal{R}y\mathcal{R}z$ and say that y *fits between x and z with respect to \mathcal{R}* . We say that \mathcal{R} is *trichotomous* when, for all $x, y \in X$, exactly one of $x\mathcal{R}y$, $x = y$, and $y\mathcal{R}x$ holds, and here we say that the *sign* of the triple (x, \mathcal{R}, y) , denoted $\text{sign}(x, \mathcal{R}, y)$, is 1, 0, or -1 , respectively. A transitive, trichotomous binary relation is called an *order*. For any order $<$ on X , a subset Y of X is said to be *convex in X with respect to $<$* if no element of $X - Y$ fits between two elements of Y with respect to $<$.

Now suppose that X is a left G -set. The diagonal left G -action on $X \times X$ gives a left G -action on the set of binary relations on X . By a *binary G -relation on X* we mean a G -invariant binary relation on X , and by a *G -order on X* we mean a G -invariant order on X . If there exists at least one G -order on X , we say that X is *G -orderable*. If X is endowed with a G -order, we say that X is *G -ordered*. When X is G with the left multiplication action, we replace ' G ' with 'left', and write *left order*, *left orderable*, or *left ordered*, the latter two being hyphenated when they premodify a noun.

Analogous terminology applies for right G -sets.

Definitions 3. For $K \leq H \leq G$, we recall two mutually inverse operations. Let $x, y \in G$.

If $<$ is a G -order on G/K with respect to which H/K is convex in G/K , then we define an H -order $<_{\text{bottom}}$ on H/K and a G -order $<_{\text{top}}$ on G/H as follows. We take $<_{\text{bottom}}$ to be the restriction of $<$ to H/K . We define $xH <_{\text{top}} yH$ to mean $(\forall h_1, h_2 \in H)(xh_1K < yh_2K)$. This relation is trichotomous since $xH <_{\text{top}} yH$ if and only if $(xH \neq yH) \wedge (xK < yK)$; the former clearly implies the latter, and, when the latter holds, $K < x^{-1}yK$, and then, by the convexity of H/K in G/K , $h_1K < x^{-1}yK$, and then $y^{-1}xh_1K < K$, $y^{-1}xh_1K < h_2K$, and $xh_1K < yh_2K$. Thus, $<_{\text{top}}$ is a G -order on G/H .

Conversely, if $<_{\text{bottom}}$ is an H -order on H/K and $<_{\text{top}}$ is a G -order on G/H , we now define a G -order $<$ on G/K with respect to which H/K is convex in G/K . We define $xK < yK$ to mean

$$(xH <_{\text{top}} yH) \vee ((xH = yH) \wedge (K <_{\text{bottom}} x^{-1}yK)).$$

It is clear that $<$ is a well-defined G -order on G/K . Suppose $xK \in (G/K) - (H/K)$. Then $xH \neq H$. If $xH <_{\text{top}} H$, then $xK < hK$, for all $h \in H$, and similarly if $H <_{\text{top}} xH$. Thus, H/K is convex in G/K with respect to $<$.

In particular, G/K has some G -order with respect to which H/K is convex in G/K if and only if H/K is H -orderable and G/H is G -orderable. Taking $K = \{1\}$ and $H = G_0$, we find that the following are equivalent, as seen in the proof of Theorem 28 (vii) \Leftrightarrow (viii) of [Berg90].

- (3.1) G has some left order with respect to which G_0 is convex in G .
- (3.2) G_0 is left orderable, and G/G_0 is G -orderable.
- (3.3) G is left orderable, and G/G_0 is G -orderable.

This motivates the terminology introduced in the following definition, which presents an analysis similar to one given by Bergman in the proof of Theorem 28 in [Berg90]. Unlike Bergman, we do not require that the group G is left-ordered.

Definition 4. Let $\text{Ssg}(G)$ denote the set of all the subsemigroups of G , that is, subsets of G closed under the multiplication. We say that the subgroup G_0 of G is *left relatively convex in G* when any of the following equivalent conditions hold.

- (4.1) The left G -set G/G_0 is G -orderable.
- (4.2) The right G -set $G_0 \backslash G$ is G -orderable.
- (4.3) There exists some $G_+ \in \text{Ssg}(G)$ such that $G_+^{\pm 1} = G - G_0$; in this event, $G_+ \cap G_+^{-1} = \emptyset$ and $G_0G_+ = G_+G_0 = G_0G_+G_0 = G_+$.
- (4.4) For each finite subset X of $G - G_0$, there exists $S \in \text{Ssg}(G)$ such that $X \subseteq S^{\pm 1} \subseteq G - G_0$.

We then say also that G_0 is a *left relatively convex subgroup of G* . One may also use ‘right’ in place of ‘left’.

Proof of equivalence. (4.1) \Rightarrow (4.3). Let $<$ be a G -order on G/G_0 , and set

$$G_+ := \{x \in G \mid G_0 < xG_0\};$$

then $G_+^{-1} = \{x \in G \mid G_0 < x^{-1}G_0\} = \{x \in G \mid xG_0 < G_0\}$ and $G_0 = \{x \in G \mid G_0 = xG_0\}$. Hence, $G_+^{\pm 1} = G - G_0$. If $x, y \in G_+$, then $G_0 < xG_0$, $G_0 < yG_0$ and $G_0 < xG_0 < xyG_0$; thus $xy \in G_+$. Hence, $G_+ \in \text{Ssg}(G)$.

Now consider any $G_+ \in \text{Ssg}(G)$ such that $G_+^{\pm 1} = G - G_0$. Then $G_+ \cap G_+^{-1} = \emptyset$, since G_+ is a subsemigroup which does not contain 1. Also, $G_0G_+ \cap G_0 = \emptyset$, since $G_+ \cap G_0^{-1}G_0 = \emptyset$,

while $G_0G_+ \cap G_+^{-1} = \emptyset$, since $G_0 \cap G_+^{-1}G_+^{-1} = \emptyset$. Thus $G_0G_+ \subseteq G_+$, and equality must hold. Similarly, $G_+G_0 = G_+$.

(4.3) \Rightarrow (4.1). Let $x, y, z \in G$. We define $xG_0 < yG_0$ to mean $(xG_0)^{-1}(yG_0) \subseteq G_+$, or, equivalently, $x^{-1}y \in G_+$. Then $<$ is a well-defined binary G -relation on G/G_0 . Since $x^{-1}y$ belongs to exactly one of G_+ , G_0 , and G_+^{-1} , we see that $<$ is trichotomous. If $xG_0 < yG_0$ and $yG_0 < zG_0$, then G_+ contains $x^{-1}y, y^{-1}z$, and their product, which shows that $xG_0 < zG_0$. Thus $<$ is a G -order on G/G_0 .

(4.2) \Leftrightarrow (4.3) is the left-right dual of (4.1) \Leftrightarrow (4.3).

(4.3) \Rightarrow (4.4) with $S = G_+$.

(4.4) \Rightarrow (4.3). Bergman [Berg90] observes that an implication of this type follows easily from the Compactness Theorem of Model Theory; here, one could equally well use the quasi-compactness of $\{-1, 1\}^{G-G_0}$, which holds by a famous theorem of Tychonoff [Tych30]. The case of this implication where $G_0 = \{1\}$ was first stated by Conrad [Con59], who gave a short argument designed to be read in conjunction with a short argument of Ohnishi [Ohn52]. Let us show that a streamlined form of the Conrad-Ohnishi proof gives the general case comparatively easily.

Let 2^{G-G_0} denote the set of all subsets of $G-G_0$. For each $W \in 2^{G-G_0}$, let $\text{Fin}(W)$ denote the set of finite subsets of W , and $\langle\langle W \rangle\rangle$ denote the subsemigroup of G generated by W . For each $\varphi \in \{-1, 1\}^{G-G_0}$ and $x \in G-G_0$, set $\tilde{\varphi}(x) := x^{\varphi(x)} \in \{x, x^{-1}\}$. Set

$$\mathfrak{W} := \left\{ W \in 2^{G-G_0} \mid (\forall W' \in \text{Fin}(W)) (\forall X \in \text{Fin}(G-G_0)) \right. \\ \left. (\exists \varphi \in \{-1, 1\}^{G-G_0}) (G_0 \cap \langle\langle W' \cup \tilde{\varphi}(X) \rangle\rangle = \emptyset) \right\}.$$

It is not difficult to see that (4.4) says precisely that $\emptyset \in \mathfrak{W}$. Also, it is clear that

$$(\forall W \in 2^{G-G_0}) ((W \in \mathfrak{W}) \Leftrightarrow (\text{Fin}(W) \subseteq \mathfrak{W})).$$

It follows that \mathfrak{W} is closed in 2^{G-G_0} under the operation of taking unions of chains. By Zorn's Lemma, there exists some maximal element W of \mathfrak{W} .

We shall prove that $\langle\langle W \rangle\rangle^{\pm 1} = G-G_0$, and thus (4.3) holds. By taking $X = \emptyset$ in the definition of ' $W \in \mathfrak{W}$ ', we see that $\langle\langle W \rangle\rangle \subseteq G-G_0$, and thus $W^{\pm 1} \subseteq \langle\langle W \rangle\rangle^{\pm 1} \subseteq G-G_0$. It remains to show that $G-G_0 \subseteq W^{\pm 1}$. Since W is maximal in \mathfrak{W} , it suffices to show that

$$(\forall x \in G-G_0) ((W \cup \{x\} \in \mathfrak{W}) \vee (W \cup \{x^{-1}\} \in \mathfrak{W})).$$

Suppose then $W \cup \{x\} \notin \mathfrak{W}$; thus, we may fix $W_x \in \text{Fin}(W)$ and $X_x \in \text{Fin}(G-G_0)$ such that

$$(\forall \varphi \in \{-1, 1\}^{G-G_0}) (G_0 \cap \langle\langle W_x \cup \{x\} \cup \tilde{\varphi}(X_x) \rangle\rangle \neq \emptyset).$$

Let $W' \in \text{Fin}(W)$ and $X \in \text{Fin}(G-G_0)$. As $W \in \mathfrak{W}$, there exists $\varphi \in \{-1, 1\}^{G-G_0}$ such that

$$G_0 \cap \langle\langle W_x \cup W' \cup \tilde{\varphi}(\{x\} \cup X_x \cup X) \rangle\rangle = \emptyset.$$

Clearly, $\tilde{\varphi}(x) \neq x$. Thus, $\tilde{\varphi}(x) = x^{-1}$ and

$$G_0 \cap \langle\langle W' \cup \{x^{-1}\} \cup \tilde{\varphi}(X) \rangle\rangle = \emptyset.$$

This shows that $W \cup \{x^{-1}\} \in \mathfrak{W}$, as desired. \square

The Burns-Hale theorem [BH72, Theorem 2] says that if each nontrivial, finitely generated subgroup of G maps onto some nontrivial, left-orderable group, then G is left orderable. The following result, using a streamlined version of their proof, generalizes the Burns-Hale theorem in two ways. Namely, the scope is increased by stating the result for an arbitrary subgroup G_0 (in their case G_0 is trivial) and by imposing a weaker condition (in their case $\langle X \rangle$ is required to map onto a left-orderable group).

Theorem 5. *If, for each nonempty, finite subset X of $G-G_0$, there exists a proper, left relatively convex subgroup of $\langle X \rangle$ that includes $\langle X \rangle \cap G_0$, then G_0 is left relatively convex in G .*

Proof. For each finite subset X of $G-G_0$, we shall construct an $S_X \in \text{Ssg}(\langle X \rangle)$ such that $X \subseteq S_X^{\pm 1} \subseteq G-G_0$, and then (4.4) above will hold. We set $S_\emptyset := \emptyset$. We now assume that $X \neq \emptyset$. Let us write $H := \langle X \rangle$. By hypothesis, we have an H_0 such that $H \cap G_0 \leq H_0 < H$ and H_0 is left relatively convex in H . Notice that $H-H_0 \subseteq H-(H \cap G_0) \subseteq G-G_0$ and $X \cap H_0 \subset X$, since $X \not\subseteq H_0$. By induction on $|X|$, we have an $S_{X \cap H_0} \in \text{Ssg}(\langle X \cap H_0 \rangle)$ such that $X \cap H_0 \subseteq S_{X \cap H_0}^{\pm 1} \subseteq G-G_0$. By (4.3) above, since H_0 is left relatively convex in H , we have an $H_+ \in \text{Ssg}(H)$ such that $H_0 H_+ H_0 = H_+$ and $H_+^{\pm 1} = H-H_0$. We set $S_X := S_{X \cap H_0} \cup H_+$. Then $S_X \in \text{Ssg}(H)$, since $S_{X \cap H_0} \subseteq H_0$ and $H_0 H_+ H_0 = H_+$. Also,

$$X = (X \cap H_0) \cup (X-H_0) \subseteq S_{X \cap H_0}^{\pm 1} \cup (H-H_0) = S_X^{\pm 1} \subseteq G-G_0. \quad \square$$

Remark 6. Theorem 5 above has a variety of corollaries. For example, for any subset X of G , we have a sequence of successively weaker conditions: $\langle X \cup G_0 \rangle / \triangleleft G_0 \triangleright$ maps onto \mathbb{Z} ; $\langle X \cup G_0 \rangle / \triangleleft G_0 \triangleright$ maps onto a nontrivial, left-orderable group; there exists a proper, left relatively convex subgroup of $\langle X \cup G_0 \rangle$ that includes G_0 ; and, there exists a proper, left relatively convex subgroup of $\langle X \rangle$ that includes $\langle X \rangle \cap G_0$. The last implication follows from the following fact. If A and B are subgroups of G and A is left relatively convex in G , then $A \cap B$ is left relatively convex in B .

Definition 7. A group G is said to be n -indicible, where n is a positive integer, if it can be generated by fewer than n elements or it admits a surjective homomorphism onto \mathbb{Z}^n .

A group G is *locally n -indicible* if every finitely generated subgroup of G is n -indicible.

Note that some authors require in the definition of indicability that G admits a surjective homomorphism onto \mathbb{Z} , while here 1-indicible means that G is trivial or maps onto \mathbb{Z} , 2-indicible means that G is cyclic or maps onto \mathbb{Z}^2 , and so on.

Example 8. Free abelian groups of any rank and free groups of any rank are locally n -indicible for every n .

The notion of n -indicability is related to left relative convexity through the following corollary of Theorem 5.

Corollary 9. *Let $n \geq 2$. If G is locally n -indicible group then each maximal $(n-1)$ -generated subgroup of G is left relatively convex in G .*

In particular, in a free group, each maximal cyclic subgroup is left relatively convex.

Proof. If the subgroup G_0 is maximal $(n-1)$ -generated subgroup of G , then, for any nonempty, finite subset X of $G-G_0$, $\langle X \cup G_0 \rangle$ maps onto \mathbb{Z}^n , and $\langle X \cup G_0 \rangle / \triangleleft G_0 \triangleright$ maps onto \mathbb{Z} . \square

The idea of Corollary 9 can be used to show that certain maximal κ -generated abelian subgroups are left relatively convex, where κ is some cardinal.

Definition 10. A group G is *nasmof* if it is torsion-free and every nonabelian subgroup of G admits a surjective homomorphism onto $\mathbb{Z} * \mathbb{Z}$.

Example 11. The class of nasmof groups contains free and free abelian groups and it is closed under taking subgroups and direct products. Residually nasmof groups are nasmof,

and in particular residually free groups are nasmof. Every nasmof group G is 2-locally indicable, and by Corollary 9, maximal cyclic subgroups are left relatively convex.

Corollary 12. *Let κ be a cardinal. If G is a nasmof group then each maximal κ -generated abelian subgroup of G is left relatively convex in G .*

In particular, in a residually free group, each maximal κ -generated abelian subgroup is left relatively convex.

Proof. Let G_0 be maximal κ -generated abelian subgroup of G and X a nonempty finite subset of $G - G_0$. By maximality, if $\langle X \cup G_0 \rangle$ is abelian, then it is not κ -generated and κ must be a finite cardinal. In this case, $\langle X \cup G_0 \rangle$ is a finitely generated, torsion-free abelian group of rank greater than κ . If $\langle X \cup G_0 \rangle$ is nonabelian, then it maps onto $\mathbb{Z} * \mathbb{Z}$. In both cases, $\langle X \cup G_0 \rangle / \triangleleft G_0 \triangleright$ maps onto \mathbb{Z} . \square

3. GRAPHS OF GROUPS

Definitions 13. By a *graph*, we mean a quadruple (Γ, V, ι, τ) such that Γ is a set, V is a subset of Γ , and ι and τ are maps from $\Gamma - V$ to V . Here, we let Γ denote the graph as well as the set, and we write $V\Gamma := V$ and $E\Gamma := \Gamma - V$, called the *vertex-set* and *edge-set*, respectively. We then define *vertex*, *edge* $\iota e \xrightarrow{e} \tau e$, *inverse edge* $\tau e \xrightarrow{e^{-1}} \iota e$, *path*

$$(13.1) \quad v_0 \xrightarrow{e_1^{\varepsilon_1}} v_1 \xrightarrow{e_2^{\varepsilon_2}} v_2 \xrightarrow{e_3^{\varepsilon_3}} \cdots \xrightarrow{e_{n-2}^{\varepsilon_{n-2}}} v_{n-2} \xrightarrow{e_{n-1}^{\varepsilon_{n-1}}} v_{n-1} \xrightarrow{e_n^{\varepsilon_n}} v_n, \quad n \geq 0,$$

reduced path, and *connected graph* in the usual way. We say that Γ is a *tree* if $V \neq \emptyset$ and, for each $(v, w) \in V \times V$, there exists a unique reduced path from v to w . The *barycentric subdivision* of Γ is the graph $\Gamma^{(l)}$ such that $V\Gamma^{(l)} = \Gamma$ and $E\Gamma^{(l)} = E\Gamma \times \{\iota, \tau\}$, with $e \xrightarrow{(e, \iota)} \iota e$ and $e \xrightarrow{(e, \tau)} \tau e$.

We say that Γ is a *left G -graph* if Γ is a left G -set, V is a G -subset of Γ , and ι and τ are G -maps. For $\gamma \in \Gamma$, we let G_γ denote the G -stabilizer of γ .

Let T be a tree. A *local order on T* is a family $(\langle_v \mid v \in VT)$ such that, for each $v \in VT$, \langle_v is an order on $\text{link}_T(v) := \{e \in E\Gamma \mid v \in \{\iota e, \tau e\}\}$. By Theorem 3 of [DS14], for each local order $(\langle_v \mid v \in VT)$ on T , there exists a unique order \langle_T on VT such that, for each reduced T -path expressed as in (13.1) above,

$$\text{sign}(v_0, \langle_T, v_n) = \text{sign}\left(0, \langle_{\mathbb{Z}}, \sum_{i=1}^n \varepsilon_i + \sum_{i=1}^{n-1} \text{sign}(e_i, \langle_{v_i}, e_{i+1})\right),$$

where the sign notation is as in Definitions 2 above. We then call \langle_T the *associated order*, $\sum_{i=1}^n \varepsilon_i$ the *orientation-sum*, and $\sum_{i=1}^{n-1} \text{sign}(e_i, \langle_{v_i}, e_{i+1})$ the *turn-sum*. If T is a left G -tree, then, for any G -invariant local order on T , the associated order on VT is easily seen to be a G -order.

Theorem 14. *Suppose that T is a left G -tree such that, for each T -edge e , G_e is left relatively convex in $G_{\iota e}$ and in $G_{\tau e}$. Then, for each $t \in T$, G_t is left relatively convex in G . If there exists some $t \in T$ such that G_t is left orderable, then G is left orderable. Moreover, if the input orders are given effectively, then the output orders are given effectively,*

Proof. We choose one representative from each G -orbit in VT . For each representative v_0 , we choose an arbitrary order on the set of G_{v_0} -orbits $G_{v_0} \setminus \text{link}_T(v_0)$, and, within each G_{v_0} -orbit, we choose one representative e_0 and a G_{v_0} -order on G_{v_0}/G_{e_0} , which exists by (4.1) above;

since our G_{v_0} -orbit $G_{v_0}e_0$ may be identified with G_{v_0}/G_{e_0} , we then have a G_{v_0} -order on $G_{v_0}e_0$, and then on all of $\text{link}_T(v_0)$ by our order on $G_{v_0} \setminus \text{link}_T(v_0)$. We then use G -translates to obtain a G -invariant local order on T . This in turn gives the associated G -order on VT as in Definitions 13 above. In particular, for each T -vertex v , we have G -orders on Gv and G/G_v . By (4.1) above, G_v is then left relatively convex in G . For each T -edge e , G_e is left relatively convex in $G_{\iota e}$ by hypothesis, and then G_e is left relatively convex in G by Definitions 3 above. Thus, for each $t \in T$, G_t is left relatively convex in G .

By (3.2) \Rightarrow (3.3) above, if there exists some $t \in T$ such that G_t is left orderable, then G is left orderable. \square

Example 15. Let F be a free group and X be a free-generating set of F . The left Cayley graph of F with respect to X is a left F -tree on which F acts freely. Thus, the fact that free groups are left orderable can be deduced from Theorem 14 above; see [DS14].

Bearing in mind that intersections of left relatively convex subgroups is left relatively convex, we can generalize the previous example to the case that a group acts freely on some orbit of n -tuples of elements of T .

Corollary 16. *Suppose that T is a left G -tree such that, for each T -edge e , G_e is left relatively convex in $G_{\iota e}$ and in $G_{\tau e}$. Suppose that there exists a finite subset S of T with $\bigcap_{s \in S} G_s = \{1\}$, then G is left orderable.*

Definitions 17. By a *graph of groups* (\mathfrak{G}, Γ) , we mean a graph with vertex-set a family of groups $(\mathfrak{G}(v') \mid v' \in V \Gamma^{(l)})$ and edge-set a family of injective group homomorphisms $(\mathfrak{G}(e) \xrightarrow{\mathfrak{G}(e')} \mathfrak{G}(v) \mid e \xrightarrow{e'} v \in E \Gamma^{(l)})$, where Γ is a nonempty, connected graph and $\Gamma^{(l)}$ is its barycentric subdivision. For $\gamma \in \Gamma^{(l)}$, we call $\mathfrak{G}(\gamma)$ a *vertex group*, *edge group*, or *edge map* if γ belongs to VT , $E\Gamma$, or $E\Gamma^{(l)}$, respectively. One may think of (\mathfrak{G}, Γ) as a nonempty, connected graph, of groups and injective group homomorphisms, in which every vertex is either a sink, called a vertex group, or a source of valence two, called an edge group. We shall use the *fundamental group* and the *Bass-Serre tree* of (\mathfrak{G}, Γ) as defined in [Ser77] and [DD89].

Bass-Serre theory translates Theorem 14 above into the following form.

Theorem 18. *Suppose that G is the fundamental group of a graph of groups (\mathfrak{G}, Γ) such that the image of each edge map $\mathfrak{G}(e) \xrightarrow{\mathfrak{G}(e')} \mathfrak{G}(v)$ is left relatively convex in its vertex group, $\mathfrak{G}(v)$. Then each vertex group is left relatively convex in G . If some vertex group is left orderable, then G is left orderable. Moreover, if the input orders are given effectively, then the output orders are given effectively.* \square

Remarks 19. Theorem 18 above generalizes the result of Chiswell that a group is left orderable if it is the fundamental group of a graph of groups such that each vertex group is left ordered and each edge group is convex in each of its vertex groups; see Corollary 3.5 of [Chi11].

The result of Chiswell is a consequence of Corollary 3.4 of [Chi11], which shows that a group is left orderable if it is the fundamental group of a graph of groups such that each edge group is left orderable and each of its left orders extends to a left order on each of its vertex groups. (If, moreover, each edge group and vertex group is left ordered, and the maps from edge groups to vertex groups respect the orders, then the fundamental group has a left order

such that the maps from the vertex groups to the fundamental group respect the orders.) This applies to the case of cyclic edge groups and left-orderable vertex groups.

Corollary 3.4 of [Chi11] is, in turn, a consequence of Chiswell's necessary and sufficient conditions for the fundamental group of a graph of groups to be left orderable. As his proof involved ultraproducts, his orders were not constructed effectively.

Example 20. Let A and B be groups, C be a subgroup of A , and $x : C \rightarrow B$, $c \mapsto c^x$, be an injective homomorphism. The graph of groups $A \leftarrow C \rightarrow B$, where the maps are the inclusion map and x , has as fundamental group $A *_C B := A * B / \langle \{c^{-1} \cdot c^x \mid c \in C\} \rangle$, called the *free product with amalgamation* with vertex groups A and B , edge group C , and edge map x . We then view A and B as subgroups of $A *_C B$. In particular, $c^x = c$.

If C is left relatively convex in each of A and B , then A and B are left relatively convex in $A *_C B$, by Theorem 18 above.

In detail, suppose that $G = A *_C B$, that $<_A$ is an A -order on A/C , and that $<_B$ is a B -order on B/C . The Bass-Serre left G -tree T for $A \leftarrow C \rightarrow B$ has vertex-set $G/A \dot{\cup} G/B$ (where $\dot{\cup}$ denotes the disjoint union) and edge-set G/C , with $gA \xrightarrow{gC} gB$. Then $<_A$ and $<_B$ determine a G -invariant local order on T , and we have the associated G -order $<_T$ on VT , as in Definitions 13 above. Let us describe the G -order $<_T$ on G/A . Consider any $gA \in G/A$, and write $gA = a_1 b_1 a_2 b_2 \cdots a_n b_n A$, $n \geq 0$, where $a_1 \in A$, $a_2, \dots, a_n \in A-C$, $b_1, b_2, \dots, b_n \in B-C$. We then have a reduced T -path

$$\begin{aligned} A \xrightarrow{a_1 C} a_1 B \xrightarrow{(a_1 b_1 C)^{-1}} a_1 b_1 A \xrightarrow{a_1 b_1 a_2 C} a_1 b_1 a_2 B \xrightarrow{(a_1 b_1 a_2 b_2 C)^{-1}} \dots \\ \dots \xrightarrow{a_1 b_1 a_2 b_2 \cdots a_n C} a_1 b_1 a_2 b_2 \cdots a_n B \xrightarrow{(a_1 b_1 a_2 b_2 \cdots a_n b_n C)^{-1}} a_1 b_1 a_2 b_2 \cdots a_n b_n A = gA. \end{aligned}$$

The orientation-sum equals zero, and we have only the turn-sum, which simplifies by the G -invariance of the local order to give

$$\text{sign}(A, <_T, gA) = \text{sign}\left(0, <_Z, \sum_{i=1}^n \text{sign}(C, <_B, b_i C) + \sum_{i=2}^n \text{sign}(C, <_A, a_i C)\right).$$

We record the case where $C = \{1\}$.

Corollary 21. *In a left-orderable group, every free factor is left relatively convex.* \square

Example 22. In a free group, every free factor is left relatively convex, by Example 15 above.

Example 23. Suppose that A and B are free groups, or, more generally, groups all of whose maximal cyclic subgroups are left relatively convex; see Corollary 9 above. If C is a maximal cyclic subgroup in both A and B , then A and B are left relatively convex in $A *_C B$, by Example 20 above.

Example 24. Let A be a group, C be a subgroup of A , and $x : C \rightarrow A$, $c \mapsto c^x$, be an injective homomorphism. The graph of groups $C \rightrightarrows A$, where the maps are the inclusion map and x , has as fundamental group $A *_C x := A * \langle x \mid \emptyset \rangle / \langle \{x^{-1} \cdot c^{-1} \cdot x \cdot c^x \mid c \in C\} \rangle$, called the *HNN extension* with vertex group A , edge group C , and edge map x . We then view A and $\langle x \mid \emptyset \rangle$ as subgroups of $A *_C x$. In particular, $c^x = x^{-1} c x$.

If C and C^x are left relatively convex in A , then A is left relatively convex in $A *_C x$, by Theorem 18 above.

If $G = A *_C x$, then the Bass-Serre left G -tree T for $C \rightrightarrows A$ has vertex-set G/A and edge-set G/C , with $gA \xrightarrow{gC} gxA$.

4. SURFACE GROUPS AND RAAGS

The following applies to all noncyclic compact surface groups.

Example 25. Let $G = \langle \{x\} \dot{\cup} \{y\} \dot{\cup} Z \mid x^{-1}y^\varepsilon xyw \rangle$ with $\varepsilon \in \{-1, 1\}$ and $w \in \langle Z \mid \emptyset \rangle$. By Example 22 above, both $\langle y \rangle$ and $\langle yw \rangle$ are left relatively convex in $\langle \{y\} \dot{\cup} Z \mid \emptyset \rangle$, which in turn is left relatively convex in the HNN extension G , by Example 24 above. Here the Bass-Serre left G -tree T has vertex-set $G/\langle \{y\} \cup Z \rangle$ and edge-set $G/\langle y \rangle$.

Notice that $\langle \{x\} \cup Z \rangle$ is not left relatively convex in $\langle \{x\} \dot{\cup} \{y\} \dot{\cup} Z \mid (xy)^2 = x^2w^{-1} \rangle$.

Proposition 26. (a) *Every compact orientable surface group G of genus $g \geq 1$ is locally n -indicible, for every $n \geq 1$.*
(b) *Every compact non-orientable surface group G of genus $g \geq 2$ is locally n -indicible, for $1 \leq n \leq g - 1$.*

Proof. (a) Let G be a compact orientable surface group of genus $g \geq 1$, $n \geq 1$, and let H be an m -generated subgroup of G that cannot be generated by fewer than m elements, for some $m \geq n$. If H is of infinite index in G , then H is free of rank m , and if it is of finite index, then it is a compact orientable surface group of genus $m/2 \geq g$. In both cases, H admits a homomorphism onto \mathbb{Z}^m , and therefore also onto \mathbb{Z}^n .

(b) Let G be a compact non-orientable surface group of genus $g \geq 2$, $n \geq 1$, and let H be an m -generated subgroup of G that cannot be generated by fewer than m elements, for some $m \geq n$. If H is of infinite index in G , then H is free of rank m , and if it is of finite index, then H is a compact non-orientable surface group of genus $m \geq g \geq n + 1$. In the former case, H admits a homomorphism onto \mathbb{Z}^m , and therefore also onto \mathbb{Z}^n . In the latter case, H admits a homomorphism onto \mathbb{Z}^{m-1} and, since in this case $m - 1 \geq n$, H admits a homomorphism onto \mathbb{Z}^n . \square

The following applies to all noncyclic surface groups except the Klein-bottle group.

Corollary 27.

(a) *Let G be a compact orientable surface group G of genus $g \geq 1$. Every maximal m -generated subgroup of G is left relatively convex in G .*
(b) *Let G be a compact non-orientable surface group G of genus $g \geq 3$. Every maximal m -generated subgroup of G , for $1 \leq m \leq g - 2$ is left relatively convex in G .*

Proof. Follows from Proposition 26 and Corollary 9. \square

Definitions 28. Let X be a set, R be a subset of $[X, X]$ in $\langle X \mid \emptyset \rangle$, and $G = \langle X \mid R \rangle$. We say that G is a *right-angled Artin group*, or *raag* for short. For example, free groups and free abelian groups are raags.

Let Y be a subset of X . The map $X \rightarrow G$ which acts as the identity map on Y and sends $X - Y$ to $\{1\}$ induces well-defined homomorphisms $G \rightarrow G$ and $G/\langle X - Y \rangle \rightarrow G$. Moreover, the natural composite $G/\langle X - Y \rangle \rightarrow G \rightarrow G/\langle X - Y \rangle$ is the identity map, since it acts as such on the generating set Y . Thus we may identify $G/\langle X - Y \rangle$ with its image $\langle Y \rangle$ in G . It follows that $\langle Y \rangle$ is a raag. We let $\pi_{\langle X \rangle \rightarrow \langle Y \rangle}$ denote the map $G \rightarrow G/\langle X - Y \rangle = \langle Y \rangle$.

For each $x \in X$, $G = A *_C x$ where $A = \langle X - \{x\} \rangle$, $C = \langle \{y \in X - \{x\} \mid [x, y] \in R^{\pm 1}\} \rangle$, and $x : C \rightarrow A$, $c \mapsto c^x$, is the inclusion map. In essence, this was noted by Bergman [Berg76].

It is not difficult to show that $\langle Y \rangle$ is left relatively convex in G ; since (4.4) above is a local condition, it suffices to verify this for X finite, and here it holds by induction on $|X|$ and Example 24 above. In particular, G is left orderable and, hence, torsion-free.

By [AM15, Corollary 1.6], raags are nasmof and therefore locally 2-indicable. By Corollary 12, we have the following.

Corollary 29. *Let G be a subgroup of a right-angled Artin group and κ a cardinal. Every maximal κ -generated abelian subgroup of G is left relatively convex in G .*

In particular, maximal abelian subgroups of G are left relatively convex in G .

Example 30. There exists an example, attributed by Ashot Minasyan [Min12] to Martin Bridson, of a subgroup G of finite index in the right-angled Artin group $F_2 \times F_2$ such that G/G' is not torsion-free. This implies that there exists $n \geq 2$ such that the right-angled Artin group $F_2 \times F_2$ is not locally n -indicible.

We do not know if raags have the property that their maximal n -generated subgroups are left relatively convex.

5. RESIDUALLY TORSION-FREE NILPOTENT GROUPS AND LEFT RELATIVE CONVEXITY

Corollary 9, combined with the next few observations, provides many examples of left relatively convex cyclic subgroups.

Proposition 31. *If G is a finitely generated, nilpotent group with torsion-free center, then G is 2-indicable.*

Proof. Let G be any group (not necessarily nilpotent or with torsion free center), Z_1 be its center, and Z_2 be its second center, that is, Z_2/Z_1 is the center of G/Z_1 .

For $g \in G$ and $a \in Z_2$, the commutator $[a, g]$ is in Z_1 . From the identity $[ab, g] = [a, g]^b [b, g]$, we obtain, for $a, b \in Z_2$, $[ab, g] = [a, g][b, g]$. Therefore, for any element $g \in G$, $a \mapsto [a, g]$ is a homomorphism from Z_2 to Z_1 , and $a \mapsto ([a, g])_{g \in G}$ is a homomorphism from Z_2 to $\prod_{g \in G} Z_1$ with kernel Z_1 , which implies that Z_2/Z_1 embeds into a power of Z_1 .

We now let G be a finitely generated, nilpotent group with torsion free center and we argue by induction on the nilpotency class c of G .

If $c = 0$, then G is trivial, and hence 2-indicable. Assume that $c \geq 1$. Since Z_2/Z_1 embeds into a power of Z_1 , which is a torsion-free group, Z_2/Z_1 itself is a torsion-free group. Therefore G/Z_1 is a finitely generated, nilpotent group of class $c - 1$ with torsion-free center Z_2/Z_1 . By the inductive hypothesis, G/Z_1 is 2-indicable. If G/Z_1 is noncyclic, then G/Z_1 maps onto \mathbb{Z}^2 , and so does G ; thus we may assume that G/Z_1 is cyclic. In that case, G/Z_1 is trivial, G is a free abelian group, and, hence, G is 2-indicable. \square

Remark 32. Note that, under the assumption that Z_1 is torsion free, the observation that Z_2/Z_1 embeds into some power of Z_1 yields that Z_2/Z_1 , the center of G/Z_1 , is itself torsion-free. Inductive arguments then quickly yield that each upper central series factor Z_{i+1}/Z_i , for $i \geq 0$, is torsion-free, each quotient Z_j/Z_i , for $j > i \geq 0$, is torsion free, and under the additional assumption that G is nilpotent, each quotient G/Z_i , for $i \geq 0$, is torsion-free; these are well-known results of Mal'cev [Mal49] and we could use them to skip

the first part in the proof of Proposition 31 and move directly to the inductive part of the proof.

Proposition 31 also follows from Mal'cev's result on quotients, together with Lemma 13 in [BBEŠ12], which states that every finitely generated, nilpotent group that is not virtually cyclic maps onto \mathbb{Z}^2 (the proof of this result relies on the fact that torsion-free, virtually abelian, nilpotent groups are abelian, which easily follows from the uniqueness of roots in torsion-free nilpotent groups; another result of Mal'cev from [Mal49]).

With all these choices before us, we still opted for the proof of Proposition 31 provided above, because it is short and self-contained.

Proposition 33. *Every locally residually torsion-free nilpotent group is locally 2-indicable.*

Proof. Let G be a locally residually torsion-free nilpotent group and H a finitely generated subgroup of G . Then H is residually torsion-free nilpotent group. If H has a noncyclic, torsion-free, nilpotent quotient, then this quotient maps to \mathbb{Z}^2 by Proposition 31, and so does H . Otherwise, H is residually- \mathbb{Z} , which implies that it is abelian. Since H is finitely generated and torsion-free, it is free abelian, hence 2-indicable (in fact, H is cyclic in this case, since we already excluded the possibility of noncyclic quotients). \square

Remark 34. Note that if G is residually torsion-free nilpotent then it is also locally residually torsion-free nilpotent. In particular, for finitely generated groups there is no difference between being residually torsion-free nilpotent or being locally residually torsion-free nilpotent.

Example 35. If G is a

- residually free group [Mag35],
- right-angled Artin group or a subgroup of a right-angled Artin group [Dro83],
- 1-relator group with presentation

$$\langle X, a, b \mid [a, b] = w \rangle,$$

where $a, b \notin X$ and w is a group word over X , including fundamental groups of all compact surfaces other than the sphere, the projective plane, and the Klein bottle [Bau62, Fre63, Bau10],

- free group in any polynilpotent variety, including free solvable groups of any given class [Gru57], or
- pure braid group [FR88],
- 1-relator groups with presentation

$$\langle x_1, \dots, x_m, y_1, \dots, y_n \mid u = v \rangle,$$

where $v \in \langle y_1, \dots, y_n \rangle$, $v \neq 1$, $u \in A = \langle x_1, \dots, x_m \rangle$, $u \in \gamma_d(A)$ for some d such that u is not a proper power modulo $\gamma_{d+1}(A)$, where $\gamma_k(A)$ is the k th term of the lower central series of A [Lab15].

then G is a residually torsion-free nilpotent group.

By Proposition 33, such a group G is locally 2-indicable and, by Corollary 9, each maximal cyclic subgroup of G is left relatively convex.

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