CLASSIFYING SPACES FOR PROPER ACTIONS OF LOCALLY FINITE GROUPS

Warren Dicks, Peter H. Kropholler, Ian J. Leary and Simon Thomas

April 16, 2002

For each finite ordinal n, and each locally finite group G of cardinality $\aleph_n$, we construct an $(n+1)$-dimensional, contractible CW-complex on which G acts with finite stabilizers. We use the complex to obtain information about cohomology with induced coefficients. Our techniques also give information about the location of some large free abelian groups in the hierarchy $\mathbf{H}\mathfrak{F}$.

Throughout, let G be a group, and let A be a $\mathbb{Z}$-module with trivial G-action. We let AG denote the induced $\mathbb{Z}G$-module $\mathbb{Z}G \otimes_{\mathbb{Z}} A$. Ring-actions on modules and group-actions on sets are tacitly understood to be on the left, if not specified otherwise.

1. Holt's Conjectures

The purpose of this section is to describe our main algebraic results, Theorems 3.10 and 5.4, and place them in the context of prior work.

1.1 Notation. Let $\mathrm{rank}(G)$ denote the smallest cardinal κ such that there exists some set of generators of G of cardinality κ.

If G is *not* finitely generated, then $\mathrm{rank}(G) = |G|$, and we define $\aleph$-$\mathrm{rank}(G)$ to be the ordinal α such that $\mathrm{rank}(G) = \aleph_\alpha$; if G *is* finitely generated, then $\mathrm{rank}(G) < |G|$, and we set $\aleph$-$\mathrm{rank}(G) = -1$.

Recall that, for each ordinal α, ω_α denotes the least ordinal of cardinality $\aleph_\alpha$.

We find it convenient to set $\aleph_{-1} = 1$.

Throughout this section, let $n \in \mathbb{N}\,(= \omega_0)$. □

D. F. Holt proposed the following description of the cohomology with induced coefficients, for locally finite groups.

1.2 Conjecture (Holt [6]). *If G is locally finite, then*

$$|\,\mathrm{H}^n(G, AG)| = |A|^{\aleph_{n-1}} \quad \textit{if } \ n = \aleph\text{-rank}(G) + 1,$$
$$\mathrm{H}^n(G, AG) = 0 \quad \textit{if } \ n \neq \aleph\text{-rank}(G) + 1.$$

Commentary. In Examples 3.3 we recall that, for any group G,

$$\mathrm{H}^n(G, AG) = \begin{cases} A & \text{if } G \text{ is finite and } n = 0, \\ 0 & \text{if } G \text{ is finite or } n = 0, \text{ but not both.} \end{cases} \tag{1.3}$$

1991 *Mathematics Subject Classification.* Primary 20F50; Secondary 20J06, 20K20, 57M07.

Thus the conjecture really concerns the cases where $n \geq 1$ and G is infinite, and the notation has been artificially contrived to embrace the trivial marginal cases.

For any infinite group G, the set of cocycles for G with coefficients in AG is of cardinality $|A|^{|G|}$, so $1 \leq |\,\mathrm{H}^n(G, AG)| \leq |A|^{|G|}$. If G is infinite and locally finite, then the conjecture implies that only the extreme values can be achieved. □

We now briefly state the cases which are known, including those obtained in this paper.

1.4 Notation. We say that G has the *finite extension property for proper subgroups* if each proper subgroup of G is a proper subgroup of finite index in some subgroup of G. For example, abelian torsion groups have this property.

Let us say that A is $o(G)$*-inverting* if, for every finite subgroup H of G, multiplication by $|H|$ gives an automorphism of A; equivalently, for each $g \in G$ whose order $o(g)$ is finite, multiplication by $o(g)$ gives an automorphism of A.

If R is a ring (associative, with 1), then R is $o(G)$-inverting, as $\mathbb{Z}$-module, if and only if the order of each finite subgroup of G is a unit in R. If R is not $o(G)$-inverting, then it is easy to show that $\mathrm{cd}_R\, G$, the cohomological dimension of G with respect to R, is ∞, a value which we shall think of as $\omega_0 = \aleph_0$. □

1.5 Known cases of Conjecture 1.2. *Let G be a locally finite group.*

(1) $\mathrm{H}^n(G, AG) = 0$ *if* $n > \aleph\text{-rank}(G) + 1$.
(2) *If G has the finite extension property for proper subgroups, then* $\mathrm{H}^n(G, AG) = 0$ *if* $n \neq \aleph\text{-rank}(G) + 1$.
(3) *For* $n \in \{0, 1\}$, $\mathrm{H}^n(G, AG) = 0$ *if* $n \neq \aleph\text{-rank}(G) + 1$.
(4) *For* $n \in \{0, 1, 2\}$, $|\,\mathrm{H}^n(G, AG)| = |A|^{\aleph_{n-1}}$ *if* $n = \aleph\text{-rank}(G) + 1$.
(5) *It is consistent with* ZFC *that*
$|\,\mathrm{H}^n(G, AG)| \geq 2^{\aleph_{n-1}}$ *if* $n = \aleph\text{-rank}(G) + 1$ *and A is nonzero.*
Hence, it is consistent with ZFC *that*
$|\,\mathrm{H}^n(G, AG)| = |A|^{\aleph_{n-1}}$ *if* $n = \aleph\text{-rank}(G) + 1$ *and* $|A| \leq \aleph_{n-1}$.

Commentary. (1), the "easy" part of Conjecture 1.2, is proved in Theorem 3.10. It was proved in [11] for the case where A is $o(G)$-inverting, and, before that, in [4], [5] for the case where A is $o(G)$-inverting and torsion.

(2) was proved by Holt [5]. We give another proof of the abelian case in Corollary 6.10.

(3). By (1.3), this holds for $n = 0$. It was proved by Holt [6] for $n = 1$; see Theorem 6.4.

(4). By (1.3), this holds for $n = 0$. It is well known for $n = 1$; see Theorem 4.5. In Theorem 5.4, we prove it for $n = 2$; Holt [6] had previously shown this was consistent with ZFC, see [14, Section 1].

(5). Suppose that $n = \aleph\text{-rank}(G) + 1$ and that A is nonzero.

We shall now see that it is consistent with ZFC that $|\,\mathrm{H}^n(G, AG)| \geq 2^{\aleph_{n-1}}$.

For each prime p, we write $\mathbb{Z}(p^\infty) := \lim\limits_{m \to \infty} \mathbb{Z}/p^m\mathbb{Z}$, where, for $m \in \mathbb{N}$, the map $\mathbb{Z}/p^m\mathbb{Z} \to \mathbb{Z}/p^{m+1}\mathbb{Z}$ is given by multiplication by p.

We claim that there exists a $\mathbb{Z}$-module k, and a $\mathbb{Z}$-submodule A' of A, such that the quotient A/A' is isomorphic to k, and either $k = \mathbb{Q}$, or there exists a prime p such that $k = \mathbb{Z}/p\mathbb{Z}$ or $k = \mathbb{Z}(p^\infty)$.

Consider first the case where A is *not* divisible, so there exists a prime p such that A/pA is nonzero. But A/pA is a direct sum of $\mathbb{Z}$-submodules each of which is isomorphic to $\mathbb{Z}/p\mathbb{Z}$. Hence A/pA projects onto any such summand.

If A *is* divisible, then A is a direct sum of $\mathbb{Z}$-submodules each of which is isomorphic to $\mathbb{Q}$ or to $\mathbb{Z}(p^\infty)$ for some prime p; see, for example, [3, Theorem IV.23.1]. Hence A projects onto any such summand.

In all cases, we can find A', k as claimed.

Now there is a long exact sequence in cohomology which contains the subsequence

$$\mathrm{H}^n(G, AG) \to \mathrm{H}^n(G, kG) \to \mathrm{H}^{n+1}(G, A'G).$$

By (1), $\mathrm{H}^{n+1}(G, A'G) = 0$, so $|\,\mathrm{H}^n(G, AG)| \geq |\,\mathrm{H}^n(G, kG)|$.

Thus, for the first part, it remains to show that it is consistent with ZFC that $|\,\mathrm{H}^n(G, kG)| \geq 2^{\aleph_{n-1}}$.

It is proved in [11] that, if k is an $o(G)$-inverting prime field, then it is consistent with ZFC that $|\,\mathrm{H}^n(G, kG)| \geq 2^{\aleph_{n-1}}$. However on carefully reading that proof, one sees that all applications of the $o(G)$-inverting hypothesis can be replaced with applications of (1), so, in fact, it is proved that it is consistent with ZFC that if k is a prime field then $|\,\mathrm{H}^n(G, kG)| \geq 2^{\aleph_{n-1}}$.

It remains to consider the case where $k = \mathbb{Z}(p^\infty)$ for some prime p. Again, it is not difficult to show that the argument in [11] can be further modified to cover this case by interpreting $\dim k^m := m$ for all $m \in \mathbb{N}$. If M is a finitely generated (free) $\mathbb{Z}$-submodule of $\mathbb{Z}G$, and kM denotes the image of the natural map $k \otimes_{\mathbb{Z}} M \to k \otimes_{\mathbb{Z}} \mathbb{Z}G = kG$, then one can show $\dim kM = \operatorname{rank} M$, since k is divisible. Moreover, if M' is a $\mathbb{Z}$-submodule of M, then $\dim kM' \leq \dim kM$, and, if equality holds, then $kM' = kM$, since k is divisible. Using these observations, one can verify that the argument in [11] applies with $k = \mathbb{Z}(p^\infty)$.

It follows that, in all cases, it is consistent with ZFC that $|\,\mathrm{H}^n(G, kG)| \geq 2^{\aleph_{n-1}}$.

Now suppose that $|A| \leq \aleph_{n-1}$. Hence $|A|^{\aleph_{n-1}} = 2^{\aleph_{n-1}}$; see [7, p.49] for the case where $n \geq 1$. Thus, it is consistent with ZFC that $|\,\mathrm{H}^n(G, AG)| \geq |A|^{\aleph_{n-1}}$.

We previously observed that $|\,\mathrm{H}^n(G, AG)| \leq |A|^{\aleph_{n-1}}$, so it is consistent with ZFC that $|\,\mathrm{H}^n(G, AG)| = |A|^{\aleph_{n-1}}$.

This proves (5). It had previously been proved by Holt [5] in the case where G has the finite extension property for proper subgroups; see [14, Section 1]. □

We wish to refine part of Conjecture 1.2.

1.6 Conjecture. *If G is locally finite, and $n = \aleph$-rank$(G) + 1$, then $\mathrm{H}^n(G, AG)$ contains a $\mathbb{Z}$-submodule isomorphic to $A^{\aleph_{n-1}}$, and hence $|\,\mathrm{H}^n(G, AG)| = |A|^{\aleph_{n-1}}$.*

Commentary. By (1.3), this holds for $n = 0$. It is probably well known for $n = 1$; see Theorem 4.5. In Theorem 5.4, we prove it for $n = 2$. □

Conjecture 1.2 was preceded by, and motivated by, an earlier proposal, concerning the cohomological dimension of locally finite groups.

1.7 Conjecture. *If G is a locally finite group, and R is a nonzero, $o(G)$-inverting ring, then* $\mathrm{cd}_R\, G = \min\{\aleph\text{-rank}(G) + 1, \infty\}$.

Commentary. Holt [4] proposed this conjecture with the additional hypothesis that R is a field of prime order, and, in [11], the additional hypothesis was weakened to R being commutative.

Notice that $\min\{\aleph\text{-rank}(G) + 1, \infty\}$ can be expressed as $\inf\{n \in \mathbb{N} \mid \aleph_n > |G|\}$, where the infimum of the empty set is taken to be ∞.

The inequality $\mathrm{cd}_R\, G \leq \aleph\text{-rank}(G) + 1$ follows from a classic result of Auslander [1, Proposition 3]; see [12, Lemma 3.7] or Theorem 3.10 below.

Cohomological dimension cannot increase on passing to a subgroup, so we may assume that $\aleph\text{-rank}(G) < \omega_0$, and let $n = \aleph\text{-rank}(G) + 1$. The conjecture now amounts to the claim that $\mathrm{H}^n(G, M) \neq 0$ for some RG-module M. Notice that, on RG-modules, $\mathrm{H}^{n+1}(G, -)$ vanishes and (hence) $\mathrm{H}^n(G, -)$ is right exact; also, M is a quotient of some free RG-module. The conjecture is therefore equivalent to the claim that $\mathrm{H}^n(G, AG) \neq 0$ for some free R-module A. This claim is implied by the claim that $\mathrm{H}^n(G, AG) \neq 0$ for some vector space A over the prime subfield of some simple quotient ring of R. This means that we may assume that R is a prime field.

The foregoing claims are known to be consistent with ZFC [11], and have been proved in various cases. The case where G is abelian was proved by N. Chen; see [12, Corollary 7.6] or Corollary 6.10 below. Chen's result has been extended in two directions. Osofsky [12, Corollary 7.5] settled the case where G is generated by finite groups whose pairwise products are subgroups; in particular, G has the finite extension property for proper subgroups. Holt [5] settled the case where G has the finite extension property for proper subgroups and R is a finite prime field. □

2. G-complexes

In this section we construct finite-dimensional contractible spaces with locally-finite groups acting on them.

2.1 Definitions. A map $f\colon X_1 \to X_2$ between CW-complexes is *cellular* if it carries the d-skeleton of X_1 to the d-skeleton of X_2 for all $d \in \mathbb{N}$.

A *G-CW-complex*, or *G-complex* for short, is a CW-complex X with a G-action such that each element of G acts continuously on X, permuting the open cells, and fixing only those cells which it fixes pointwise. It follows that G acts cellularly.

If X is a G-complex then, for each $H \leq G$, the set X^H, consisting of points fixed by all of H, is a CW-subcomplex of X, and the set X/H consisting of the H-orbits is a quotient CW-complex.

Let $\mathfrak{sub}(G)$ denote the set of all subgroups of G. A subset $\mathfrak{X}$ of $\mathfrak{sub}(G)$ is a *subgroup-closed G-family* if each subgroup of each element of $\mathfrak{X}$ belongs to $\mathfrak{X}$ and, moreover, $\mathfrak{X}$ is closed under taking conjugates by elements of G.

If $\mathfrak{X}$ is a subgroup-closed G-family, then by *a space of type* $\mathrm{E}(G, \mathfrak{X})$ we mean a G-complex X with the properties that, for each $H \in \mathfrak{X}$, X^H is contractible, and for each $H \in \mathfrak{sub}(G) - \mathfrak{X}$, X^H is empty. In this event, X is also said to be a *classifying space for G-actions with stabilizers in $\mathfrak{X}$*.

If $\mathfrak{X}$ is a class of groups, and $\mathfrak{X} \cap \mathfrak{sub}(G)$ is a subgroup-closed G-family, then by *a space of type* $\mathrm{E}(G, \mathfrak{X})$ we mean a space of type $\mathrm{E}(G, \mathfrak{X} \cap \mathfrak{sub}(G))$.

We let $\mathfrak{F}$ denote the class of finite groups. Notice that $\mathfrak{F} \cap \mathfrak{sub}(G)$ is a subgroup-closed G-family. A space of type $\mathrm{E}(G, \mathfrak{F})$ is called *an* $\underline{\mathrm{E}}G$. (It is also called a classifying space for *proper* G-actions, that is, G-actions with finite stabilizers.) □

The following is well known.

2.2 Proposition. *If $\mathfrak{X}$ is a subgroup-closed G-family, then there exists a space of type* $\mathrm{E}(G, \mathfrak{X})$*, and any G-map between two spaces of type* $\mathrm{E}(G, \mathfrak{X})$ *is a G-homotopy equivalence.*

Proof. The first part can be seen by Milnor's construction. Thus, let Δ be any G-set such that $\mathfrak{X}$ is precisely the set of subgroups of G which fix at least one point of Δ. Let $X = \Delta * \Delta * \Delta * \cdots$, the union of iterated joins of Δ. Then X is a space of type $\mathrm{E}(G, \mathfrak{X})$.

For the second part, see, for example, [15, Proposition II.2.7]. □

2.3 Corollary. *If $\mathfrak{X}_1 \subseteq \mathfrak{X}_2$ are subgroup-closed G-families, and X_1 (resp. X_2) is a space of type* $\mathrm{E}(G, \mathfrak{X}_1)$ *(resp.* $\mathrm{E}(G, \mathfrak{X}_2)$*), then there exists a cellular G-map $X_1 \to X_2$.*

Proof. The join $X_1 * X_2$ is a space of type $\mathrm{E}(G, \mathfrak{X}_2)$, and the inclusions

$$\iota_1 \colon X_1 \to X_1 * X_2, \quad \iota_2 \colon X_2 \to X_1 * X_2$$

are G-maps. By Proposition 2.2, ι_2 is a G-homotopy equivalence, and the homotopy inverse $X_1 * X_2 \to X_2$ composed with ι_1 gives a G-map $X_1 \to X_2$. This is then G-homotopic to a cellular G-map $X_1 \to X_2$; see, for example, [15, Theorem II.2.1]. □

One could give a dual proof, using the projection maps from the Cartesian product $X_1 \times X_2$, which is a space of type $\mathrm{E}(G, \mathfrak{X}_1)$.

The following is a topological analogue of a classic result of Auslander [1, Proposition 3].

2.4 Theorem. *Let β be a limit ordinal, let $(G_\alpha \mid \alpha \leq \beta)$ be a continuous chain of subgroups of G, and let $(\mathfrak{X}_\alpha \mid \alpha \leq \beta)$ be a continuous chain of subsets of $\mathfrak{sub}(G)$ such that, for each $\alpha \leq \beta$, $\mathfrak{X}_\alpha$ is a subgroup-closed G_α-family.*

Let $n \in \mathbb{N}$, and suppose that, for each $\alpha < \beta$, there exists an n-dimensional space Y_α of type $\mathrm{E}(G_\alpha, \mathfrak{X}_\alpha)$*. Then there exists an $(n+1)$-dimensional space of type* $\mathrm{E}(G_\beta, \mathfrak{X}_\beta)$*.*

Proof. For each $\alpha < \beta$, $\mathfrak{X}_\alpha \subseteq \mathfrak{X}_{\alpha+1} \cap \mathfrak{sub}(G_\alpha)$ are subgroup-closed G_α-families. Also, Y_α is a space of type $\mathrm{E}(G_\alpha, \mathfrak{X}_\alpha)$, and $Y_{\alpha+1}$ can be viewed as a space of type $\mathrm{E}(G_\alpha, \mathfrak{X}_{\alpha+1} \cap \mathfrak{sub}(G_\alpha))$. By Corollary 2.3, there exists a cellular G_α-map $Y_\alpha \to Y_{\alpha+1}$, and hence a cellular $G_{\alpha+1}$-map $f_\alpha \colon G_{\alpha+1} \times_{G_\alpha} Y_\alpha \to Y_{\alpha+1}$. Let M_α denote the mapping cylinder of f_α. Since f_α is cellular, M_α has the structure of a CW-complex, and is a space of type $\mathrm{E}(G_{\alpha+1}, \mathfrak{X}_{\alpha+1})$. Notice that $\dim M_\alpha = n+1$, since $\dim Y_\alpha = \dim Y_{\alpha+1} = n$.

We recursively construct a continuous chain $(X_\alpha \mid \alpha \leq \beta)$ where X_α is a space of type $\mathrm{E}(G_\alpha, \mathfrak{X}_\alpha)$, and, for $\alpha \geq 1$, $\dim X_\alpha = n+1$.

We take $X_0 = Y_0$, and at limit ordinals we take directed unions.

Suppose $\alpha < \beta$ and that X_α has been constructed.

By Proposition 2.2, since X_α and Y_α are of type $\mathrm{E}(G_\alpha, \mathfrak{X}_\alpha)$ there exists a cellular G_α-map $Y_\alpha \to X_\alpha$, and hence a cellular $G_{\alpha+1}$-map $G_{\alpha+1} \times_{G_\alpha} Y_\alpha \to G_{\alpha+1} \times_{G_\alpha} X_\alpha$. Take $X_{\alpha+1}$ to be the identification space, or pushout,

$$\begin{array}{ccc} G_{\alpha+1} \times_{G_\alpha} Y_\alpha & \longrightarrow & M_\alpha \\ \big\downarrow & & \big\downarrow \\ G_{\alpha+1} \times_{G_\alpha} X_\alpha & \longrightarrow & X_{\alpha+1}. \end{array}$$

Notice that $\dim X_{\alpha+1} = n+1$, since $\dim Y_\alpha = n$, $\dim M_\alpha = n+1$, and $\dim X_\alpha \le n+1$. It is not difficult to check that $X_{\alpha+1}$ is of type $\mathrm{E}(G_{\alpha+1}, \mathfrak{X}_{\alpha+1})$.

This completes the proof. $\square$

2.5 Remark. For $n = 0$ and $\beta = \omega_0$, the construction in the above proof gives the Bass-Serre tree of the graph of groups corresponding to the countable ascending chain $(G_\alpha \mid \alpha < \omega_0)$. Here, $\mathfrak{X}_{\omega_0} = \bigcup_{\alpha<\omega_0} \mathfrak{sub}(G_\alpha)$. $\square$

2.6 Theorem. *If $n \in \mathbb{N}$, and G is a locally finite group with $\aleph$-rank$(G) < n$, then there exists an n-dimensional $\underline{\mathrm{E}}G$.*

Proof. We argue by induction on n.

If $n = 0$, then G is finite. Here a single point with trivial G-action is a 0-dimensional $\underline{\mathrm{E}}G$.

Thus we may assume that $n \ge 1$, and that the result holds for smaller n.

We can choose a continuous chain $(G_\alpha \mid \alpha \le \omega_n)$ of subgroups of G such that $G_{\omega_n} = G$, and, for each $\alpha < \omega_n$, $\aleph$-rank$(G_\alpha) < n-1$, so, by the induction hypothesis, there exists an $(n-1)$-dimensional $\underline{\mathrm{E}}G_\alpha$. Thus, by Theorem 2.4, there exists an n-dimensional $\underline{\mathrm{E}}G$.

This completes the proof. $\square$

2.7 Remarks. The foregoing construction applies in greater generality.

Suppose that, for every group H, there is specified a subgroup-closed H-family $\mathfrak{Y}(H)$ satisfying the following three conditions:

Any group isomorphism $H_1 \to H_2$ induces a bijection $\mathfrak{Y}(H_1) \to \mathfrak{Y}(H_2)$.

If $H_1 \le H_2$, then $\mathfrak{Y}(H_1) \subseteq \mathfrak{Y}(H_2)$.

If H is the union of a well-ordered chain of subgroups H_α, then $\mathfrak{Y}(H)$ is the union of the $\mathfrak{Y}(H_\alpha)$.

Let $\mathfrak{G}_0$ denote the class consisting of those groups H such that $\mathfrak{Y}(H) = \mathfrak{sub}(H)$. For $n < \omega_0$, recursively define $\mathfrak{G}_{n+1}$ to be the class consisting of those groups which can be expressed as the union of a well-ordered chain of subgroups which lie in $\mathfrak{G}_n$.

The above argument then shows that if $G \in \mathfrak{G}_n$, then there exists an n-dimensional $\mathrm{E}(G, \mathfrak{Y}(G))$.

For any $n > 0$ and any $G \in \mathfrak{G}_n$, it can be arranged that all of the spaces involved in the construction of $\mathrm{E}(G, \mathfrak{Y}(G))$ have distinguished contractible subcomplexes which are transversals for the group actions. In particular, the quotient complex $\mathrm{E}(G, \mathfrak{Y}(G))/G$ is contractible, although we will not use this information. $\square$

We record one example.

2.8 Theorem. *If $n \in \mathbb{N}$, and $\aleph$-rank$(G) < n$, and $\mathfrak{X}$ is the set of all subgroups of all finitely generated subgroups of G, then there exists an n-dimensional* $\mathrm{E}(G, \mathfrak{X})$.

Proof. For each group H, let $\mathfrak{Y}(H)$ be the set consisting of the subgroups of the finitely generated subgroups of H. It is easy to see that $\mathfrak{Y}(-)$ respects isomorphisms, inclusions and well-ordered unions.

It can be shown, by induction on n, that $G \in \mathfrak{G}_n$, in the notation of the previous remark, so, by that remark, there exists an n-dimensional $\mathrm{E}(G, \mathfrak{Y}(G))$. □

2.9 Example. In the foregoing theorem, if G is abelian (or locally finite), then $\mathfrak{X}$ is the set of finitely generated subgroups of G. □

3. Eventual vanishing of cohomology with induced coefficients

In this section, we recall how $\mathrm{H}^*(G, -)$ can be computed using an $\underline{\mathrm{E}}G$, and apply the method in the case where G is locally finite.

3.1 Definitions. Let M be a $\mathbb{Z}G$-module.

We say that M is *G-acyclic* if $\mathrm{H}^n(G, M) = 0$ for all $n \geq 1$.

Any $\mathbb{Z}G$-summand of a G-acyclic $\mathbb{Z}G$-module is again G-acyclic.

If $\cdots \to P_1 \to P_0 \to \mathbb{Z} \to 0$ is a $\mathbb{Z}G$-projective resolution of $\mathbb{Z}$, then it is easy to see that M is G-acyclic if and only if the sequence

$$0 \to \mathrm{Hom}_{\mathbb{Z}G}(\mathbb{Z}, M) \to \mathrm{Hom}_{\mathbb{Z}G}(P_0, M) \to \mathrm{Hom}_{\mathbb{Z}G}(P_1, M) \to \cdots$$

is exact. When this holds, we say that $\mathrm{Hom}_{\mathbb{Z}G}(-, M)$ *carries augmented $\mathbb{Z}G$-projective resolutions of $\mathbb{Z}$ to exact sequences.*

For any set Δ, we let $A[[\Delta]]$ denote the set of all functions from Δ to A, and such a function $x \mapsto a_x$ will be written as a formal sum $\sum_{x \in \Delta} a_x.x$, and its *support* is

$$\mathrm{supp}(\sum_{x \in \Delta} a_x.x) := \{x \in \Delta \mid a_x \neq 0\}.$$

The $\mathbb{Z}$-module structure of $A[[\Delta]]$ is defined in the obvious way. We let $A[\Delta]$, or $A\Delta$ denote the $\mathbb{Z}$-submodule of $A[[\Delta]]$ consisting of the elements with finite support. If Δ is a G-set, we view $A[[\Delta]]$ as a $\mathbb{Z}G$-module, with

$$g(\sum_{x \in \Delta} a_x.x) = \sum_{x \in \Delta} a_x.gx = \sum_{x \in \Delta} a_{g^{-1}x}.x.$$

(Notice that the action of G on the set of all functions from Δ to A is traditionally, and more naturally, on the right, with

$$(x \mapsto a_x)g = (x \mapsto a_{gx}),$$

and our action of g on the left corresponds to the usual action of g^{-1} on the right. For our purposes, it is convenient to have actions on the left, wherever possible.) Then $A\Delta$ is a $\mathbb{Z}G$-submodule of $A[[\Delta]]$. In the case where $\Delta = G$, this notation is consistent with the notation for the induced $\mathbb{Z}G$-module AG. Here $A[[G]]$ is called a *coinduced* $\mathbb{Z}G$-module.

There is a natural bijection

$$\mathrm{Hom}_{\mathbb{Z}}(M, A) \to \mathrm{Hom}_{\mathbb{Z}G}(M, A[[G]]), \quad \psi \mapsto (m \mapsto \sum_{x \in G} \psi(x^{-1}m).x). \tag{3.2}$$

Since $\mathrm{Hom}_{\mathbb{Z}}(-, A)$ carries $\mathbb{Z}$-split exact sequences of $\mathbb{Z}G$-modules to $\mathbb{Z}G$-split exact sequences of $\mathbb{Z}G$-modules, we see that co-induced $\mathbb{Z}G$-modules are G-acyclic. □

3.3 Examples. Let G be a finite group.

Here, $AG = A[[G]]$, so induced $\mathbb{Z}G$-modules are co-induced, and hence G-acyclic.

Suppose that M is an $o(G)$-inverting $\mathbb{Z}G$-module. Then the multiplication map

$$M[[G]] \to M, \quad \sum_{g \in G} m_g.g \mapsto \sum_{g \in G} g m_g,$$

is $\mathbb{Z}G$-split with right inverse $m \mapsto \frac{1}{|G|} \sum\limits_{g \in G} g^{-1}m.g$. Here, M is a $\mathbb{Z}G$-summand of an induced $\mathbb{Z}G$-module, so M is G-acyclic. $\square$

In the following, G acts on tensor products over $\mathbb{Z}$ via the diagonal action.

3.4 Lemma. *Let M be a $\mathbb{Z}G$-module.*

(1) *The functor* $\mathrm{Hom}_{\mathbb{Z}G}(-\otimes_{\mathbb{Z}} \mathbb{Z}G, M)$ *carries $\mathbb{Z}$-split exact sequences of $\mathbb{Z}G$-modules to exact sequences.*

(2) *Let H be a subgroup of G. If M is H-acyclic, then the functor*

$$\mathrm{Hom}_{\mathbb{Z}G}(\mathbb{Z}[G/H] \otimes_{\mathbb{Z}} -, M)$$

carries augmented $\mathbb{Z}G$-projective resolutions of $\mathbb{Z}$ to exact sequences.

Proof. (2). Let L be a $\mathbb{Z}G$-module. There is a natural identification of $\mathbb{Z}G$-modules,

$$\mathbb{Z}[G/H] \otimes_{\mathbb{Z}} L = \mathbb{Z}G \otimes_{\mathbb{Z}H} L,$$

with $gH \otimes \ell$ corresponding to $g \otimes g^{-1}\ell$.

It follows that we can identify

$$\mathrm{Hom}_{\mathbb{Z}G}(\mathbb{Z}[G/H] \otimes_{\mathbb{Z}} -, M) = \mathrm{Hom}_{\mathbb{Z}G}(\mathbb{Z}G \otimes_{\mathbb{Z}H} -, M) = \mathrm{Hom}_{\mathbb{Z}H}(-, M)$$

as functors on $\mathbb{Z}G$-modules. Since M is H-acyclic, this functor carries augmented $\mathbb{Z}G$-projective resolutions of $\mathbb{Z}$ to exact sequences.

(1) is proved similarly. $\square$

3.5 Notation. Let X be a G-complex.

We shall treat X as the G-set whose elements are the open cells of X. The cellular chain complex of X is then the permutation module $\mathbb{Z}X$, with the structure of a differential graded $\mathbb{Z}G$-module, with differential ∂ of degree -1. Here the grading is that determined by the dimensions of the cells, so the nth component $\mathrm{C}_n(\mathbb{Z}X)$ has as $\mathbb{Z}$-basis the cells of dimension n.

We let $\eta\colon X \times X \to \mathbb{Z}$ denote the function such that $\partial x = \sum\limits_{y \in X} \eta(x,y).y$ for each $x \in X$. Thus, if x is an n-cell, then $\eta(x,y) = 0$ unless y is one of the finitely many $(n-1)$-cells incident to x. $\square$

The following is a degenerate case of the equivariant cohomology spectral sequence; see, for example, [2, VII.7.10(7.10)].

3.6 Theorem. *Let X be an acyclic G-complex. If M is a $\mathbb{Z}G$-module which is G_x-acyclic for each $x \in X$, then $\mathrm{H}^*(\mathcal{H}\mathrm{om}_{\mathbb{Z}G}(\mathbb{Z}X, M)) \simeq \mathrm{H}^*(G, M)$, as graded abelian groups.*

Recall that $\mathcal{H}\mathrm{om}_{\mathbb{Z}G}(\mathbb{Z}X, M)$ denotes the differential graded abelian group with nth component $\mathrm{C}^n(\mathcal{H}\mathrm{om}_{\mathbb{Z}G}(\mathbb{Z}X, M)) = \mathrm{Hom}_{\mathbb{Z}G}(\mathrm{C}_n(\mathbb{Z}X), M)$.

Proof. The homology of $(\mathbb{Z}X, \partial)$ is $\mathbb{Z}$, concentrated in degree zero.

We choose a free $\mathbb{Z}G$-resolution of $\mathbb{Z}$, and write it as $(\mathbb{Z}Y, \partial)$ for some G-free G-set Y; for example, we could take Y to be an $\mathrm{E}G$, and $\mathbb{Z}Y$ its cellular chain complex. Then $\mathcal{H}\mathrm{om}_{\mathbb{Z}G}(\mathbb{Z}Y, M)$ is an additive abelian differential graded group, and its cohomology is $\mathrm{H}^*(G, M)$.

We consider the double complex $\mathbb{Z}X \otimes_{\mathbb{Z}} \mathbb{Z}Y$ with diagonal G-action, and the double complex $\mathcal{H}\mathrm{om}_{\mathbb{Z}G}(\mathbb{Z}X \otimes_{\mathbb{Z}} \mathbb{Z}Y, M)$. We get a fourth-quadrant commuting diagram which can be schematically represented as

$$\begin{array}{ccccc} & & & & 0 \\ & & & & \downarrow \\ & & & & \mathcal{H}\mathrm{om}_{\mathbb{Z}G}(\mathbb{Z}X, M) \\ & & & & \downarrow \\ 0 & \longrightarrow & \mathcal{H}\mathrm{om}_{\mathbb{Z}G}(\mathbb{Z}Y, M) & \longrightarrow & \mathcal{H}\mathrm{om}_{\mathbb{Z}G}(\mathbb{Z}X \otimes_{\mathbb{Z}} \mathbb{Z}Y, M). \end{array} \tag{3.7}$$

To show that the cohomology group of the outer row, $\mathcal{H}\mathrm{om}_{\mathbb{Z}G}(\mathbb{Z}X, M)$, is isomorphic to the cohomology group of the outer column, $\mathcal{H}\mathrm{om}_{\mathbb{Z}G}(\mathbb{Z}Y, M)$, it suffices to show that the remaining, or inner, rows and columns of (3.7) are exact. Each inner column is exact because $\mathcal{H}\mathrm{om}_{\mathbb{Z}G}(\mathbb{Z}X \otimes_{\mathbb{Z}} -, M)$ is exact on augmented projective $\mathbb{Z}G$-resolutions of $\mathbb{Z}$, by Lemma 3.4(2). Similarly, each inner row is exact because $\mathcal{H}\mathrm{om}_{\mathbb{Z}G}(- \otimes_{\mathbb{Z}} \mathbb{Z}Y, M)$ is exact on $\mathbb{Z}$-split exact sequences of $\mathbb{Z}G$-modules, by Lemma 3.4(1). □

3.8 Corollary. *Let M be a $\mathbb{Z}G$-module, and let X be a finite-dimensional acyclic G-complex. If M is G_x-acyclic for each $x \in X$, then $\mathrm{H}^n(G, M) = 0$ for all $n > \dim X$.* □

We record the case of finite stabilizers.

3.9 Corollary. *Let M be a $\mathbb{Z}G$-module, and suppose that M is H-acyclic for each finite subgroup H of G; for example, this holds if $M = AG$, or if M is $o(G)$-inverting. Let X be an acyclic G-complex with finite stabilizers; for example, this holds if X is an $\underline{\mathrm{E}}G$. Then $\mathrm{H}^n(G, M) = 0$ for all $n > \dim X$.* □

Here we can apply Theorem 2.6.

3.10 Theorem. *Let G be a locally finite group, and M a $\mathbb{Z}G$-module which is H-acyclic for each finite subgroup H of G; for example, this holds if $M = AG$, or if M is $o(G)$-inverting. Then $\mathrm{H}^n(G, M) = 0$ for all $n > \aleph\text{-rank}(G) + 1$.* □

3.11 Remark. Theorem 3.10 can also be proved using the argument of the first paragraph of [11, Section 1]. □

4. Locally finite groups of cardinality $\aleph_0$

In this section, we recall how $\mathrm{H}^*(G, AG)$ can be computed using an $\underline{\mathrm{E}}G$, and apply the method in the one-dimensional case.

4.1 Definitions. Let X be an $\underline{\mathrm{E}}G$, or, more generally, any acyclic G-complex in which all cell stabilizers are finite, and let Notation 3.5 apply.

We have natural identifications

$$A[[X]] = A^X = \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}X, A). \tag{4.2}$$

For simplicity, let us suppose that X is finite dimensional.

Then $A[[X]]$ has the structure of a differential graded $\mathbb{Z}G$-module, in which the differential ∂^* has degree $+1$, and is given by

$$\partial^*(\sum_{x\in X} a_x.x) = \sum_{x\in X}(\sum_{y\in X} \eta(x,y)a_y).x.$$

The cohomology of $(A[[X]], \partial^*)$ is A concentrated in degree zero.

Let

$$A_G[[X]] := \{\sum_{x\in X} a_x.x \in A[[X]] \mid \{g \in G \mid a_{gx} \neq 0\} \text{ is finite, for all } x \in X\}.$$

Since G-stabilizers are finite, we see that $A_G[[X]]$ consists of all functions from X to A with finite support in each G-orbit.

It is straightforward to check that $A_G[[X]]$ is a differential graded $\mathbb{Z}G$-submodule of $A[[X]]$.

We write $\mathrm{C}^n(A_G[[X]])$, $\mathrm{B}^n(A_G[[X]])$, and $\mathrm{Z}^n(A_G[[X]])$ for the n-cochains, n-coboundaries, and n-cocycles, respectively. □

Sometimes the notation $\mathcal{H}\mathrm{om}_c(\mathbb{Z}[X], A)$ is used to denote $A_G[[X]]$; see, for example, [2, Lemma VIII.7.4].

The following is a variation on the usual "compact supports" cohomology; see, for example, [2, Proposition VIII.7.5]. It is particularly useful in the study of ends of groups.

4.3 Theorem. *If X is a finite-dimensional acyclic G-complex with finite stabilizers, then there is a natural isomorphism* $\mathrm{H}^*(A_G[[X]]) \simeq \mathrm{H}^*(G, AG)$ *of graded abelian groups.*

Proof. There is a natural identification of $\mathrm{Hom}_{\mathbb{Z}G}(\mathbb{Z}X, A[[G]])$ with $\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}X, A)$; see (3.2). There is also a natural identification of $\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}X, A)$ with $A[[X]]$; see (4.2). It is easy to show that under these identifications, $\mathcal{H}\mathrm{om}_{\mathbb{Z}G}(\mathbb{Z}X, AG)$ corresponds to $A_G[[X]]$. Hence $\mathrm{H}^*(A_G[[X]]) \simeq \mathrm{H}^*(\mathcal{H}\mathrm{om}_{\mathbb{Z}G}(\mathbb{Z}X, AG))$. Finally, $\mathrm{H}^*(\mathcal{H}\mathrm{om}_{\mathbb{Z}G}(\mathbb{Z}X, AG)) \simeq \mathrm{H}^*(G, AG)$, by Theorem 3.6. □

There is a natural right G-action on $\mathrm{H}^*(G, AG)$, arising from the $\mathbb{Z}G$-bimodule structure on AG. This agrees with the natural right G-action on $A_G[[X]]$ which we have transformed into a left G-action.

Let us illustrate how Theorem 4.3 can be used to study $\mathrm{H}^*(G, AG)$ when G is locally finite of cardinality $\aleph_0$. To do this we now construct a standardized $\underline{\mathrm{E}}G$, as in Remark 2.5.

4.4 Definition. Let G be a locally finite group of cardinality $\aleph_0$.

Index the elements of G with ω_0, so $G = \{h_\alpha \mid \alpha < \omega_0\}$. For each $\beta \leq \omega_0$, let $H_\beta := \langle h_\alpha \mid \alpha < \beta \rangle$. Let $(G_\alpha \mid \alpha \leq \omega_0)$ be the subsequence of $(H_\alpha \mid \alpha \leq \omega_0)$ obtained by omitting each term which is equal to an earlier term.

Notice that $G_{\omega_0} = G$ and, for each $\alpha < \omega_0$, $|G_\alpha| < \aleph_0$. Moreover, $G_0 = 1$, and, for each $\alpha < \omega_0$, $G_{\alpha+1} = \langle G_\alpha, g_\alpha \rangle$, where $g_\alpha = h_{\alpha'}$ and α' is the least ordinal such that $h_{\alpha'} \notin G_\alpha$.

We define the *line*, denoted $\mathbb{R}$, to be the tree with vertices $(v_n \mid n \in \mathbb{Z})$, (oriented) edges $(e_n \mid n \in \mathbb{Z})$, and incidence relations $\iota e_n = v_n$, $\tau e_n = v_{n+1}$, for all $n \in \mathbb{Z}$.

We define the *half-line*, denoted $\mathbb{R}^+$, to be the subtree of $\mathbb{R}$ with vertices $(v_\alpha \mid \alpha < \omega_0)$, and edges $(e_\alpha \mid \alpha < \omega_0)$.

There is a G-tree X with G-transversal $\mathbb{R}^+$ such that the G-stabilizer of v_α, and of e_α, is G_α, for all $\alpha < \omega_0$. This completely specifies X. It is not difficult to see that X is a locally finite tree with one end.

We denote the set of vertices (resp. edges) of X by VX (resp. EX). □

The following is fairly standard, but we do not know an explicit reference.

4.5 Theorem. *If G is a locally finite group of cardinality $\aleph_0$, then $\mathrm{H}^*(G, AG)$ is concentrated in degree 1, $\mathrm{H}^1(G, AG)$ contains a $\mathbb{Z}$-submodule isomorphic to $A^{\aleph_0}$, and $|\mathrm{H}^1(G, AG)| = |A|^{\aleph_0} = |A|^{|G|}$.*

Proof. By (1.3) and Theorem 3.10, $\mathrm{H}^*(G, AG)$ is concentrated in degree 1, so it remains to study $\mathrm{H}^1(G, AG)$.

Let X be as in Definition 4.4.

By Theorem 4.3, $\mathrm{H}^1(G, AG) = \mathrm{H}^1(A_G[[X]])$.

Let Q denote $\{e_\alpha \mid \alpha < \omega_0\}$, so we can view

$$A[[Q]] \subseteq A_G[[EX]] = \mathrm{C}^1(A_G[[X]]) = \mathrm{Z}^1(A_G[[X]]).$$

Let $\mathcal{P}$ be a partition of ω_0 into $\aleph_0$ subsets, each of cardinality $\aleph_0$. We denote the map $\omega_0 \to \mathcal{P}$ by $\alpha \mapsto [\alpha]$.

Consider any $\phi \in A^{\mathcal{P}}$, $[\alpha] \mapsto \phi[\alpha]$.

Define

$$\phi^\dagger := \sum_{\alpha<\omega_0} \phi[\alpha].e_\alpha \in A[[Q]] \subseteq \mathrm{Z}^1(A_G[[X]]).$$

We claim that if $\phi^\dagger \in \mathrm{B}^1(A_G[[X]])$ then $\phi = 0$.

Suppose not, so there exists $\psi \in \mathrm{C}^0(A_G[[X]]) = A_G[[VX]]$ such that $\partial^*\psi = \phi^\dagger$, and there exists $p \in \mathcal{P}$ such that $\phi(p) \neq 0$.

There exists $\mu < \omega_0$ such that $\mathrm{supp}(\psi) \cap Gv_0 \subseteq G_\mu v_0$.

There exists $\alpha \in p$ such that $\alpha > \mu$. Notice $\phi[\alpha] = \phi(p) \neq 0$.

In $\mathbb{Z}X$, $\partial(\sum\limits_{i=0}^{\alpha} e_i) = v_{\alpha+1} - v_0$. Hence $\partial((g_\alpha - g_{\alpha-1})\sum\limits_{i=0}^{\alpha} e_i) = -(g_\alpha - g_{\alpha-1})v_0$, since $g_\alpha, g_{\alpha-1} \in G_{\alpha+1} = G_{v_{\alpha+1}}$.

We can view ψ as an additive map $\mathbb{Z}X \to A$, and apply it to the foregoing to get $\phi^\dagger((g_\alpha - g_{\alpha-1})\sum\limits_{i=0}^{\alpha} e_i) = \psi(-(g_\alpha - g_{\alpha-1})v_0) = 0$, since $g_\alpha, g_{\alpha-1} \notin G_{\alpha-1} \supseteq G_\mu$. Thus

$$0 = \phi^\dagger((g_\alpha - g_{\alpha-1})\sum_{i=0}^{\alpha} e_i) = \phi^\dagger(\sum_{i=0}^{\alpha} g_\alpha e_i) - \phi^\dagger(\sum_{i=0}^{\alpha-1} g_{\alpha-1} e_i) - \phi^\dagger(e_\alpha) = 0 - 0 - \phi(p).$$

Hence $\phi(p) = 0$, which is a contradiction.

This proves that the composition

$$A^{\mathcal{P}} \to A[[Q]] \to \mathrm{H}^1(A_G[[X]]) = \mathrm{H}^1(G, AG)$$

is injective. Since $|\mathcal{P}| = \aleph_0$, $A^{\aleph_0}$ embeds in $\mathrm{H}^1(G, AG)$. □

5. Locally finite groups of cardinality $\aleph_1$

In this section we study $\mathrm{H}^*(G, AG)$ when G is locally finite with $\aleph$-rank$(G) = 1$, topologizing and refining results of D. F. Holt.

We begin by constructing a standardized $\underline{\mathrm{E}}G$.

5.1 Definitions. Let G be a locally finite group of cardinality $\aleph_1$.

Let ω_1' denote the set of limit ordinals less than ω_1.

As in Definition 4.4, we start by indexing the elements of G, $G = \{h_\alpha \mid \alpha < \omega_1\}$, set $H_\beta := \langle h_\alpha \mid \alpha < \beta\rangle$ for each $\beta \leq \omega_1$, and let $(G_\alpha \mid \alpha \leq \omega_1)$ be the subsequence of $(H_\alpha \mid \alpha \leq \omega_1)$ obtained by omitting each term which is either finite, or equal to an earlier term.

Notice $(G_\alpha \mid \alpha \leq \omega_1)$ is a continuous chain of subgroups of G, $G_{\omega_1} = G$, and, for each $\alpha < \omega_1$, $|G_\alpha| = \aleph_0$ and $G_{\alpha+1} = \langle G_\alpha, g_\alpha\rangle$, where $g_\alpha = h_{\alpha'}$, and α' is the least ordinal such that $h_{\alpha'} \notin G_\alpha$.

For each subgroup H of G, we set $H^\bullet := G_0 \cap H$.

We shall now construct a family $(G_{\alpha,n} \mid \alpha < \omega_1, n < \omega_0)$ of finite subgroups of G such that the following hold:

(1) for each $\alpha < \omega_1$, $(G_{\alpha,n} \mid n < \omega_0)$ is an increasing chain with union G_α;
(2) for each $\alpha < \omega_1$, $n < \omega_0$, $G_{\alpha+1,n} = \langle G_{\alpha,n}, g_\alpha\rangle$;
(3) for each $\alpha \in \omega_1'$, $n < \omega_0$, there is a distinguished element

$$g_{\alpha,n} \in G^\bullet_{\alpha,n+1} - G^\bullet_{\alpha,n} (= G_0 \cap (G_{\alpha,n+1} - G_{\alpha,n}));$$

(4) for each $\alpha \in \omega_1'$, $n < \omega_0$, $G_{\alpha+1,n} \cap G_\alpha = G_{\alpha,n}$.

We proceed as follows.

First, choose an arbitrary chain $(G_{0,n} \mid n < \omega_0)$ of finite subgroups with union G_0.

For $\alpha \in \omega'$, choose an arbitrary chain $(\tilde{G}_{\alpha,m} \mid m < \omega_0)$ of finite subgroups with union G_α. For each $m < \omega_0$, define $\tilde{G}_{\alpha+1,m} := \langle \tilde{G}_{\alpha,m}, g_\alpha\rangle$. Choose any increasing function $\omega_0 \to \omega_0$, $n \mapsto m_n$, such that the chain $(\tilde{G}^\bullet_{\alpha+1,m_n} \mid n < \omega_0)$ of finite subgroups of G_0 with union G_0 is *strictly* increasing. For $n < \omega_0$, define

$$G_{\alpha+1,n} := \tilde{G}_{\alpha+1,m_n} \text{ and } G_{\alpha,n} := G_\alpha \cap G_{\alpha+1,n}.$$

Notice that (4) holds, and that we may assume that (3) holds. Also $G_{\alpha,n} \supseteq \tilde{G}_{\alpha,m_n}$, so

$$G_{\alpha+1,n} = \tilde{G}_{\alpha+1,m_n} = \langle \tilde{G}_{\alpha,m_n}, g_\alpha\rangle \subseteq \langle G_{\alpha,n}, g_\alpha\rangle \subseteq G_{\alpha+1,n}.$$

For $\alpha < \omega_1$, and $n < \omega_0$, if $G_{\alpha,n}$ is defined, set $G_{\alpha+1,n} := \langle G_{\alpha,n}, g_\alpha\rangle$; this completes the recursive definition of the family $(G_{\alpha,n} \mid \alpha < \omega_1, n < \omega_0)$, and we see that (1) and (2) also hold.

For $\alpha < \omega_1$, let Y_α denote the G_α-tree of Definition 4.4 corresponding to the chain $(G_{\alpha,n} \mid n < \omega_0)$.

We define the *plane*, denoted $\mathbb{R}^2$, to be the two-dimensional CW-complex with vertices $(v_{m,n} \mid (m,n) \in \mathbb{Z}^2)$, and edges $(x_{m,n}, y_{m,n} \mid (m,n) \in \mathbb{Z}^2)$, and faces $(f_{m,n} \mid (m,n) \in \mathbb{Z}^2)$, and incidence relations given by, for $(m,n) \in \mathbb{Z}^2$,

$$\iota x_{m,n} = \tau x_{m-1,n} = \iota y_{m,n} = \tau y_{m,n-1} = v_{m,n}$$

and $f_{m,n}$ is attached along the path $x_{m,n}, y_{m+1,n}, x^{-1}_{m,n+1}, y^{-1}_{m,n}$.

We define the *semi-infinite strip*, denoted $[0,1] \times \mathbb{R}^+$, to be the subcomplex of $\mathbb{R}^2$ with vertices $(v_{0,n}, v_{1,n} \mid n < \omega_0)$, edges $(x_{0,n}, y_{0,n}, y_{1,n} \mid n < \omega_0)$, and faces $(f_{0,n} \mid n < \omega_0)$.

Notice there are two distinguished subcomplexes of $[0,1] \times \mathbb{R}^+$ which are isomorphic to $\mathbb{R}^+$, and will be denoted $\{0\} \times \mathbb{R}^+$ and $\{1\} \times \mathbb{R}^+$.

Let $\alpha < \omega_1$. We construct a two-dimensional $G_{\alpha+1}$-space M_α, for which $[0,1] \times \mathbb{R}^+$ is a $G_{\alpha+1}$-transversal, and the $G_{\alpha+1}$-stabilizer of $v_{0,n}$, $x_{0,n}$, $y_{0,n}$, and $f_{0,n}$ is $G_{\alpha,n}$, while the $G_{\alpha+1}$-stabilizer of $v_{1,n}$ and $y_{1,n}$ is $G_{\alpha+1,n}$. This completely specifies M_α. Notice that M_α is the mapping cylinder of a $G_{\alpha+1}$-map

$$G_{\alpha+1} \times_{G_\alpha} Y_\alpha \to Y_{\alpha+1}.$$

If $\alpha \in \omega_1'$, then Y_α is a G_α-subtree of $Y_{\alpha+1}$.

To construct X we glue together

$$(G \times_{G_{\alpha+1}} M_\alpha \mid \alpha < \omega_1)$$

amalgamating

$$(G \times_{G_\alpha} Y_\alpha \mid \alpha < \omega_1),$$

as in the proof of Theorem 2.4. The image in $X_{\alpha+1} \subseteq X$ of $f_{0,n} \in M_\alpha$ will be denoted $f_{\alpha,n}$.

We denote the set of vertices (resp. edges, resp. faces) of X by VX (resp. EX, resp. FX).

Let us recall the notion of a club (= **cl**osed **u**n**b**ounded subset). Thus, an ω_1-*club* is any subset S of ω_1 such that the set of the least upper bounds of the nonempty subsets of S is precisely $S \cup \{\omega_1\}$. If S is an ω_1-club, then so is $S \cap \omega_1'$; recall that ω_1' denotes the set of limit ordinals in ω_1.

For $\phi \in A_G[[X]]$ and $\alpha \leq \omega_1$, if $\mathrm{supp}(\phi) \cap GX_\alpha \subseteq X_\alpha$, we say that ϕ *respects* α. It is straightforward to show that the set of ordinals in ω_1 respected by ϕ is an ω_1-club. □

5.2 Lemma (Holt [6]). *Let G be a periodic group, and let H, K be proper subgroups of G which generate G. Let X be the G-graph with vertex set the disjoint union of G/H and G/K, and edge set $G/(H \cap K)$, with $g(H \cap K)$ joining gH to gK, for each $g \in G$. Then X is connected, and deleting the two vertices H and K, and the one edge $H \cap K$, leaves a connected space.*

Proof. Collapsing all the edges of X leaves a transitive G-set, and one of the points is fixed by H and K. Since H and K generate G, this point is fixed by G, so forms a G-orbit. Thus we have only one point, so X is connected.

Let Y be the subgraph of X obtained by deleting the vertices H and K and all their incident edges. It suffices to show that Y is a connected graph. Since X is connected, it suffices to show that each X-neighbour of H is Y-connected to each X-neighbour of K. Thus, let $h \in H - K$, and $k \in K - H$; it suffices to show that the vertices hK and kH are Y-connected.

Let $L = \langle hk^{-1} \rangle = \langle kh^{-1} \rangle$. Since G is periodic, L is finite.

We consider the action of L on X. Let m and n denote the orders of the L-orbits of the vertices H and K, respectively. By symmetry, we may assume that $m \geq n$. Notice that L is not contained in K, so $n \geq 2$.

Let $g = kh^{-1}$. In X, there is an edge $H \cap K$ joining H to K, and an edge $k(H \cap K)$ joining $kH = gH$ to $kK = K$. Applying powers of g to these, we get a path in X with vertices

$$H, K, gH, gK, g^2H, g^2K, \dots, g^{n-1}H, g^{n-1}K.$$

By the definition of m and n, these $2n$ vertices are all distinct, so, on deleting the first two, we get a path in Y connecting $gH = kH$ to $g^{n-1}K = g^{-1}K = hK$. □

5.3 Theorem (Holt [6]). *If G is locally finite, and $|G| = \aleph_1$, then* $\mathrm{H}^1(G, AG) = 0$.

Proof. Let X be as in Definitions 5.1.

Consider any $\phi \in \mathrm{Z}^1(A_G[[X]])$. Thus $\mathrm{supp}(\phi)$ is a collection of edges of X, with only finitely many in each G-orbit. A subset of X which meets (that is, has nonempty intersection with) $\mathrm{supp}(\phi)$ is said to be *broken by* ϕ. Since ϕ is a 1-cocycle, we get 0 if we sum, in A, the ϕ-labels, with the appropriate signs, around any face, or along any closed path in the 1-skeleton, since X is simply connected. Thus there is a well-defined *ϕ-sum* from any vertex to any other vertex.

Consider any $\alpha \in \omega_1'$ such that ϕ respects α as in the last paragraph of Definitions 5.1.

From Definitions 5.1, there is a cellular $G_{\alpha+1}$-map $M_\alpha \to X$, so ϕ induces an element $\phi_{\alpha+1} \in A[[M_\alpha]]$. Since the G-stabilizers for X are finite, $\phi_{\alpha+1}$ lies in $A_{G_{\alpha+1}}[[M_\alpha]]$, Moreover, $\phi_{\alpha+1}$ respects α in the obvious sense, since Y_α is mapped to X_α, by construction.

There exists $n_0 < \omega_0$ such that $\mathrm{supp}(\phi_{\alpha+1}) \cap G_{\alpha+1}x_{0,0} \subseteq G_{\alpha+1,n_0}x_{0,0}$. This means that, for $g \in G_{\alpha+1}$, if $\phi_{\alpha+1}$ breaks $gx_{0,0}$, then the terminal vertex of $gx_{0,0}$ lies in $G_{\alpha+1,n_0}v_{1,0}$.

Now consider any n such that $n_0 < n < \omega_0$.

Let p_n denote the reduced open path in the tree $\{0\} \times \mathbb{R}^+$ from $v_{0,0}$ to $v_{0,n}$, and $e \cdot p_n$ the open path obtained by concatenating $e := x_{0,0}^{-1}$ and p_n. We are interested in the $G_{\alpha+1,n}$-graph Z generated by the closure $\overline{e \cdot p_n}$.

Consider any $g \in G_{\alpha+1,n}$.

If $\phi_{\alpha+1}$ breaks gp_n then $g \in G_\alpha$, since $\phi_{\alpha+1}$ respects α, and hence $g \in G_{\alpha,n}$, so $gv_{0,n} = v_{0,n}$. That is, if $\phi_{\alpha+1}$ breaks gp_n, then the terminal vertex of gp_n is $v_{0,n}$.

We apply Lemma 5.2 to the graph Y' obtained by taking $H^- = G_{\alpha+1,0}$ and $K^- = G_{\alpha,n_0}$, so $\langle H^-, K^- \rangle = \langle G_{\alpha,n_0}, g_\alpha \rangle = G_{\alpha+1,n_0}$. By Lemma 5.2, deleting $g^{-1}K^-$ from Y' leaves a connected space containing $G_{\alpha+1,n_0}/H^-$, where we include the trivial case where $g^{-1}K^-$ does not belong to Y'.

Let Y be the $G_{\alpha+1,n_0}$-subspace of M_α generated by $\overline{e \cdot p_n}$, and consider the $G_{\alpha+1,n_0}$-map from Y to Y' which assigns $v_{1,0}$ to H^-, $v_{0,n}$ to K^-, and $e \cdot p_n$ to

$H^- \cap K^-$. It follows that deleting $g^{-1}K^-\overline{p_n}$ from Y leaves a connected space containing $G_{\alpha+1,n_0}v_{1,0}$.

Suppose $g \notin G_{\alpha+1,n_0}$, so $g \in G_{\alpha+1,n} - G_{\alpha+1,n_0}$. Then $\operatorname{supp}(\phi_{\alpha+1}) \cap gY \subseteq K^- p_n$. Hence, on deleting $\operatorname{supp}(\phi_{\alpha+1})$ from the 1-skeleton of M_α, one of the resulting components contains $gG_{\alpha+1,n_0}v_{1,0}$.

Next, we apply Lemma 5.2 to the graph Z' obtained by taking $H = G_{\alpha+1,n_0}$ and $K = G_{\alpha,n}$, so $\langle H, K\rangle = G_{\alpha+1,n}$. We conclude that

$$Z' - (\{H\} \cup \{K\} \cup (H \cup K)/(H \cap K))$$

is a connected graph.

Let Z be the $G_{\alpha+1,n}$-subspace of M_α generated by $\overline{e \cdot p_n}$, and consider the map of $G_{\alpha+1,n}$-spaces from Z to Z' which assigns $v_{1,0}$ to H, $v_{0,n}$ to K, and $e \cdot p_n$ to $H \cap K$. There is induced a surjective map

$$Z - (\{v_{1,0}\} \cup \{v_{0,n}\} \cup (H \cup K)(e \cdot p_n)) \to Z' - (\{H\} \cup \{K\} \cup (H \cup K)/(H \cap K)).$$

Notice that $\phi_{\alpha+1}$ breaks only edges of Z which lie in $He \cup Kp_n$, so there is a map from $Z - (\{v_{1,0}\} \cup \{v_{0,n}\} \cup (H \cup K)(e \cdot p_n))$ to the set of components of the 1-skeleton of $M_\alpha - \operatorname{supp}(\phi_{\alpha+1})$. Moreover, we have seen that each subset $gHv_{1,0}$ maps to a component of the 1-skeleton of $M_\alpha - \operatorname{supp}(\phi_{\alpha+1})$. Thus the map factors through $Z' - (\{H\} \cup \{K\} \cup (H \cup K)/(H \cap K))$, which is connected, so maps to a single component. Hence some component X' of the 1-skeleton of $M_\alpha - \operatorname{supp}(\phi_{\alpha+1})$ contains $(\langle H, K\rangle - H)v_{1,0}$, that is, $(G_{\alpha+1,n} - G_{\alpha+1,n_0})v_{1,0}$.

Since $n > n_0$ was arbitrary, all of $(G_{\alpha+1} - G_{\alpha+1,n_0})v_{1,0}$ is contained in X'. Thus, for any path between any two elements of $(G_{\alpha+1} - G_{\alpha+1,n_0})v_{1,0}$, the $\phi_{\alpha+1}$-sum, and hence the ϕ-sum, is 0.

Let $\psi_{\alpha+1} \in \mathrm{C}^0(A[[X_{\alpha+1}]])$ be defined on each vertex v as the ϕ-sum along any path from any vertex of $(G_{\alpha+1} - G_{\alpha+1,n_0})v_{1,0}$ to v. Then $\psi_{\alpha+1} \in \mathrm{C}^0(A_{G_{\alpha+1}}[[X_{\alpha+1}]])$ and $\phi|_{X_{\alpha+1}} = \partial^*(\psi_{\alpha+1})$.

The $\alpha < \omega_1$ which are respected by ϕ converge to ω_1, and it follows that the corresponding (unique) $\psi_{\alpha+1}$ converge to an element $\psi \in \mathrm{C}^0(A_G[[X]])$ such that $\phi = \partial^*(\psi)$, so $\phi \in \mathrm{B}^1(A_G[[X]])$.

Hence $\mathrm{H}^1(A_G[[X]]) = 0$, so $\mathrm{H}^1(G, AG) = 0$. □

Up until now, in this section, we have given a straightforward topological translation of [6], which we felt illuminated the arguments. We now come to a new result.

5.4 Theorem. *If G is locally finite, and $|G| = \aleph_1$, then $\mathrm{H}^2(G, AG)$ has a subgroup isomorphic to $A^{\aleph_1}$, and $|\mathrm{H}^2(G, AG)| = |A^G| = |A|^{\aleph_1}$.*

Proof. Let X be as in Definitions 5.1.

Let $\alpha \in \omega_1'$.

Let M'_α denote the G_α-space with G_α-transversal $[-1, 0] \times \mathbb{R}^+$, where the G_α-stabilizers of $v_{-1,n}$, $y_{-1,n}$, $x_{-1,n}$ and $f_{-1,n}$ are $G^\bullet_{\alpha,n} (= G_0 \cap G_{\alpha,n})$, while the G_α-stabilizers of $v_{0,n}$ and $y_{0,n}$ are $G_{\alpha,n}$. This completely specifies M'_α.

Notice that M'_α is the mapping cylinder of an injective G_α-map $G_\alpha \times_{G_0} Y^\bullet_\alpha \to Y_\alpha$, where $Y^\bullet_\alpha$ is the G_0-tree of Definition 4.4 corresponding to the chain $(G^\bullet_{\alpha,n} \mid n < \omega_0)$, a G_0-subtree of Y_α.

We have a cellular G_α-map $Y_\alpha \to X_\alpha$, and, similarly, by Corollary 2.3, we can construct a cellular G_0-map $Y_\alpha^\bullet \to Y_0$ between spaces of type $\underline{E}G_0$. These two maps can be extended to a G_α-map $M'_\alpha \to X_\alpha$.

Notice that Y_α is contained in both M_α and M'_α, and the map $Y_\alpha \to X_\alpha$ has been extended to $M_\alpha \to X_{\alpha+1}$ and to $M'_\alpha \to X_\alpha$.

For each $n < \omega_0$, let $\lambda'_{\alpha,n}$ denote the least ordinal such that $X_{\lambda'_{\alpha,n}+1}$ contains the image of the face $f_{-1,n}$, under the map $M'_\alpha \to X_\alpha$. Notice that $\lambda'_{\alpha,n} + 1 < \alpha$, since α is a limit ordinal. Choose a strictly ascending sequence $(\lambda_{\alpha,n} < \alpha \mid n < \omega_0)$ with limit α, such that $\lambda_{\alpha,n} > \max\{\lambda'_{\alpha,i} \mid 0 \le i \le n\}$. Let

$$h_{\alpha,n} := g_{\lambda_{\alpha,n}} \in G_{\lambda_{\alpha,n}+1} - G_{\lambda_{\alpha,n}}.$$

Let $Q = \{h_{\alpha,n} f_{\alpha,n} \mid (\alpha, n) \in \omega'_1 \times \mathbb{N}\}$, so

$$A[[Q]] \subseteq A_G[[FX]] = \mathrm{C}^2(A_G[[X]]) = \mathrm{Z}^2(A_G[[X]]).$$

Recall that a subset S of ω_1 is said to be ω_1-*stationary* if S meets each ω_1-club. Let $\mathcal{P}$ be a partition of ω'_1 into $\aleph_1$ subsets, each being ω_1-stationary; see, for example, [7, Lemma 7.6, p.59].

Consider any $\phi \in A^{\mathcal{P}}$, $[\alpha] \mapsto \phi[\alpha]$.

Define

$$\phi^\dagger := \sum_{(\alpha,n)\in\omega'_1\times\mathbb{N}} \phi[\alpha].h_{\alpha,n} f_{\alpha,n} \in A[[Q]] \subseteq \mathrm{Z}^2(A_G[[X]]).$$

It is easy to see that $\phi^\dagger$ respects all the ordinals in ω_1.

We claim that if $\phi^\dagger \in \mathrm{B}^2(A_G[[X]])$, then $\phi = 0$.

Suppose not, so there exists $\psi \in \mathrm{C}^1(A_G[[X]]) = A_G[[EX]]$, such that $\partial^*\psi = \phi^\dagger$, and there exists $p \in \mathcal{P}$ such that $\phi(p) \neq 0$. We shall obtain a contradiction.

Since Y_0 is countable, there exists $\mu < \omega_1$ such that $\mathrm{supp}(\psi) \cap GY_0 \subseteq G_\mu Y_0$.

Since p is ω_1-stationary, and the set of ordinals respected by ψ is an ω_1-club, there exists $\alpha \in p$ such that $\alpha > \mu$ and ψ respects α. Notice $\phi[\alpha] = \phi(p) \neq 0$.

Now $\phi^\dagger$ and ψ induce elements $\phi_{\alpha+1}$ and $\psi_{\alpha+1}$, respectively, in $A_{G_{\alpha+1}}[[M_\alpha]]$, and, moreover, $\partial^*\psi_{\alpha+1} = \phi_{\alpha+1}$ in $A_{G_{\alpha+1}}[[M_\alpha]]$. Notice $\phi_{\alpha+1} = \sum\limits_{n<\omega_0} \phi(p).h_{\alpha,n} f_{0,n}$.

Let $\xi := -\sum\limits_{n<\omega_0} \phi(p).h_{\alpha,n} y_{0,n} \in A_{G_\alpha}[[EY_\alpha]] \subseteq A_{G_{\alpha+1}}[[M_\alpha]]$. Then $\partial^*(\xi) = \phi_{\alpha+1}$ in $A_{G_{\alpha+1}}[[M_\alpha]]$. Thus $\partial^*(\psi_{\alpha+1} - \xi) = 0$. Moreover, $\psi_{\alpha+1} - \xi$ respects α, since both ξ and $\psi_{\alpha+1}$ respect α. By the proof of Theorem 5.3, there exists $\nu < \alpha$ such that the $(\psi_{\alpha+1} - \xi)$-sum along any path in Y_α between any two vertices in $(G_\alpha - G_\nu)v_{0,0}$ is zero.

Now $\phi^\dagger$ and ψ induce elements ϕ_α and ψ_α in $A_{G_\alpha}[[M'_\alpha]]$; moreover, $\partial^*\psi_\alpha = \phi_\alpha$ in $A_{G_\alpha}[[M'_\alpha]]$, and $\psi_{\alpha+1}$ and ψ_α agree on Y_α. Also, ξ can be viewed as an element of $A_{G_\alpha}[[M'_\alpha]]$, so the $(\psi_\alpha - \xi)$-sum along any path in Y_α between any two vertices in $(G_\alpha - G_\nu)v_{0,0}$ is zero.

There exists $\kappa < \alpha$ such that ψ_α vanishes on $(G_\alpha - G_\kappa)x_{-1,0}$.

Choose $n < \omega_0$ such that $\lambda_{\alpha,n}$ is greater than μ, ν and κ. Choose $g \in G^\bullet_{\alpha,n+1} - G^\bullet_{\alpha,n}$. Thus $g \in G_0$, g fixes $x_{-1,n+1}$, and g moves $y_{0,n}$.

In $\mathbb{Z}[M_\alpha]$,

$$\partial(\sum_{i=0}^{n} f_{-1,i}) = x_{-1,0} - x_{-1,n+1} + \sum_{i=0}^{n}(y_{0,i} - y_{-1,i}).$$

Hence

$$\partial(h_{\alpha,n}(1-g)\sum_{i=0}^{n} f_{-1,i}) = h_{\alpha,n}(1-g)(x_{-1,0} + \sum_{i=0}^{n}(y_{0,i} - y_{-1,i})).$$

We can view ψ_α as an additive map $\mathbb{Z}[EM_\alpha] \to A$, and apply it to the foregoing equation, to get

$$\begin{aligned}\phi_\alpha(h_{\alpha,n}(1-g)\sum_{i=0}^{n} f_{-1,i})\\ =&\psi_\alpha(h_{\alpha,n}(1-g)(x_{-1,0} + \sum_{i=0}^{n}(y_{0,i} - y_{-1,i}))).\end{aligned}$$

Notice that $h_{\alpha,n}G_{\lambda_{\alpha,n}} \cap G_{\lambda_{\alpha,n}} = \emptyset$.

Since $h_{\alpha,n}, h_{\alpha,n}g \notin G_{\lambda'_{\alpha,i}}$, for $0 \le i \le n$, we see $\phi_\alpha(h_{\alpha,n}(1-g)\sum\limits_{i=0}^{n} f_{-1,i}) = 0$.
Also, $h_{\alpha,n}, h_{\alpha,n}g \notin G_\mu$, so $\psi_\alpha(h_{\alpha,n}(1-g)y_{-1,i}) = 0$.
Also, $h_{\alpha,n}, h_{\alpha,n}g \notin G_\kappa$, so $\psi_\alpha(h_{\alpha,n}(1-g)x_{-1,0}) = 0$.
It follows that $\psi_\alpha(h_{\alpha,n}(1-g)\sum\limits_{i=0}^{n} y_{0,i}) = 0$.

But $h_{\alpha,n}, h_{\alpha,n}g \notin G_\nu$, so $(\psi_\alpha - \xi)(h_{\alpha,n}(1-g)\sum\limits_{i=0}^{n} y_{0,i}) = 0$. Hence

$$\xi(\sum_{i=0}^{n} h_{\alpha,n}(1-g)y_{0,i}) = 0.$$

Now $\xi(h_{\alpha,n}y_{0,n}) = -\phi(p)$, and ξ vanishes on all other summands because $h_{\alpha,n}G_0 \cap G_{\lambda_{\alpha,i}+1} = \emptyset$ for $0 \le i \le n-1$, and $gy_{0,n} \ne y_{0,n}$. Hence $\phi(p) = 0$, which is a contradiction.

Since $|\mathcal{P}| = \aleph_1$, we have an embedding of $A^{\aleph_1}$ in $\mathrm{H}^2(G, AG)$. □

6. Cohomology of directed unions

In this section, we recall some known results about cohomology for well-ordered directed unions, with special emphasis on abelian groups.

6.1 Notation. We let $(\mathrm{P}(G), \partial)$ denote the bar resolution for G, and let $\mathrm{P}_n(G)$ denote its nth component, for each $n \in \mathbb{Z}$. Thus $(\mathrm{P}(G), \partial)$ is a free $\mathbb{Z}G$-resolution of $\mathbb{Z}$, and, for $n \ge 0$, $\mathrm{P}_n(G)$ has as $\mathbb{Z}$-basis the Cartesian power G^{n+1}, with G acting by left multiplication on the first coordinate, and, for $n \ge 1$,

$$\begin{aligned}&\partial_n(g_0, \dots, g_n) :=\\ &\quad\sum_{i=0}^{n-1}(-1)^i(g_0, \dots, g_{i-1}, g_i g_{i+1}, g_{i+2}, \dots, g_n) + (-1)^n(g_0, \dots, g_{n-1}).\end{aligned}$$

As usual, if $n \le -1$, $\mathrm{P}_n(G) = 0$, and, if $n \le 0$, $\partial_n = 0$. □

The following is a degenerate case of the cohomology spectral sequence for well-ordered directed unions; see, for example [13, Section 3].

6.2 Lemma (Robinson [13, Proposition 1]). *Let $n \in \mathbb{N}$, let M be a $\mathbb{Z}G$-module, let β be a limit ordinal, and let $(G_\alpha \mid \alpha \leq \beta)$ be a continuous chain of subgroups of G. If $\mathrm{H}^{n-1}(G_\alpha, M) = 0$ for all $\alpha < \beta$, then $\mathrm{H}^n(G_\beta, M) = \varprojlim_{\alpha<\beta} \mathrm{H}^n(G_\alpha, M)$.*

Proof. For each $\alpha \leq \beta$, view the bar resolution $\mathrm{P}(G_\alpha)$ as a $\mathbb{Z}$-subcomplex of $\mathrm{P}(G_\beta)$, so $(\mathrm{P}_n(G_\alpha) \mid \alpha \leq \beta)$ is a continuous chain.

We want to show that the natural map

$$\mathrm{H}^n(G_\beta, M) \to \varprojlim_{\alpha<\beta} \mathrm{H}^n(G_\alpha, M) \tag{6.3}$$

is bijective.

We begin by showing it is injective.

Consider any element ξ of the kernel of (6.3). Then ξ is represented by an n-cocycle $\phi_\beta\colon \mathrm{P}_n(G_\beta) \to M$, so ϕ_β is $\mathbb{Z}G_\beta$-linear, and $\phi_\beta \circ \partial_{n+1} = 0$.

Consider any $\alpha \leq \beta$. Let $\phi_{\beta,\alpha}$ denote the restriction of ϕ_β to $\mathrm{P}_n(G_\alpha)$. We shall construct, transfinitely, a continuous directed system of maps

$$(\psi_{\beta,\alpha}\colon \mathrm{P}_{n-1}(G_\alpha) \to M \mid \alpha \leq \beta)$$

such that $\psi_{\beta,\alpha}$ is $\mathbb{Z}G_\alpha$-linear, and $\psi_{\beta,\alpha} \circ \partial_n = \phi_{\beta,\alpha}$. It will then follow that $\phi_\beta = \phi_{\beta,\beta} = \psi_{\beta,\beta} \circ \partial_n$ is a coboundary, that $\xi = 0$, and that (6.3) is injective.

If $\alpha < \beta$, then $\phi_{\beta,\alpha}$ represents an element ξ_α of $\mathrm{H}^n(G_\alpha, M)$, and, since ξ lies in the kernel of (6.3), $\xi_\alpha = 0$, so there exists a $\mathbb{Z}G_\alpha$-linear map

$$\psi_\alpha\colon \mathrm{P}_{n-1}(G_\alpha) \to M$$

such that $\psi_\alpha \circ \partial_n = \phi_{\beta,\alpha}$.

Let $\psi_{\beta,0} = \psi_0$.

If $\alpha < \beta$ and we have constructed $\psi_{\beta,\alpha}$, then we construct $\psi_{\beta,\alpha+1}$ as follows. Let $\psi_{\alpha+1,\alpha}$ denote the restriction of $\psi_{\alpha+1}$ to $P_{n-1}(G_\alpha)$. Since $\psi_{\alpha+1} \circ \partial_n = \phi_{\beta,\alpha+1}$, we see that $\psi_{\alpha+1,\alpha} \circ \partial_n = \phi_{\beta,\alpha} = \psi_{\beta,\alpha} \circ \partial_n$. Thus $\psi_{\beta,\alpha} - \psi_{\alpha+1,\alpha}$ is an $(n-1)$-cocycle, so represents an element of $\mathrm{H}^{n-1}(G_\alpha, M)$. By hypothesis, this element is zero, so there exists a $\mathbb{Z}G_\alpha$-linear map $\mu_\alpha\colon \mathrm{P}_{n-2}(G_\alpha) \to M$ such that $\psi_{\beta,\alpha} - \psi_{\alpha+1,\alpha} = \mu_\alpha \circ \partial_{n-1}$. Let $\mu_{\alpha,\alpha+1}\colon \mathrm{P}_{n-2}(G_{\alpha+1}) \to M$ denote the unique $\mathbb{Z}G_{\alpha+1}$-linear map which is μ_α on $\mathrm{P}_{n-2}(G_\alpha)$, and is zero on $G_{\alpha+1}^{n-1} - (G_{\alpha+1} \times G_\alpha^{n-2})$. Now define

$$\psi_{\beta,\alpha+1} := \psi_{\alpha+1} + \mu_{\alpha,\alpha+1} \circ \partial_{n-1}.$$

By construction, $\psi_{\beta,\alpha+1}$ acts on $\mathrm{P}_{n-1}(G_\alpha)$ as $\psi_{\alpha+1,\alpha} + \mu_\alpha \circ \partial_{n-1} = \psi_{\beta,\alpha}$. Also $\psi_{\beta,\alpha+1} \circ \partial_n = \psi_{\alpha+1}\partial_n + 0 = \phi_{\beta,\alpha+1}$.

If α is a limit ordinal and we have constructed $(\psi_{\beta,\alpha'} \mid \alpha' < \alpha)$, then the latter has a direct limit which we take to be $\psi_{\beta,\alpha}$.

This completes the construction, so (6.3) is injective.

A similar, but easier, argument shows that (6.3) is surjective, and here the hypothesis on H^{n-1} is not needed. $\square$

A special case of this result appeared in the penultimate paragraph of the proof of Theorem 5.3. We can now say even more.

6.4 Theorem (Holt [6]). *If G is locally finite, and $\aleph$-rank$(G) \neq 0$, then* $\mathrm{H}^1(G, AG) = 0$.

Proof. By (1.3), we may assume that $\aleph$-rank$(G) \geq 1$, and we proceed by induction on $\aleph$-rank(G). If $\aleph$-rank$(G) = 1$, the assertion holds by Theorem 5.3. Thus we may assume that $\aleph$-rank$(G) \geq 2$ and the result holds for smaller groups. There exists an ordinal β and a continuous chain $(G_\alpha \mid \alpha \leq \beta)$ of subgroups of G with $G_\beta = G$, and $1 \leq \aleph$-rank$(G_\alpha) < \aleph$-rank(G) for all $\alpha < \beta$.

By (1.3) and the induction hypothesis, $\mathrm{H}^0(G_\alpha, AG) = \mathrm{H}^1(G_\alpha, AG) = 0$ for all $\alpha < \beta$, since AG is an induced $\mathbb{Z}G_\alpha$-module. By Lemma 6.2, $\mathrm{H}^1(G, AG) = 0$. □

6.5 Remark. In light of Theorem 3.10, the method of proof of Theorem 6.4 shows that

$$\text{“If } n \in \mathbb{N},\ G \text{ is locally finite, and } n \neq \aleph\text{-rank}(G) + 1, \text{ then } \mathrm{H}^n(G, \mathbb{Z}G \otimes_{\mathbb{Z}} -) = 0,\text{”}$$

is equivalent to

$$\text{“If } n \in \mathbb{N},\ G \text{ is locally finite, and } n = \aleph\text{-rank}(G), \text{ then } \mathrm{H}^n(G, \mathbb{Z}G \otimes_{\mathbb{Z}} -) = 0.\text{”} \quad \square$$

6.6 Definitions. Let H be a subgroup of G, and let L, M be $\mathbb{Z}G$-modules.

Let $N_G(H)$ denote the normalizer of H in G, and let $C_G(H)$ denote the centralizer of H in G.

There is a natural action of $N_G(H)$ on $\mathrm{Ext}^*_{\mathbb{Z}H}(L, M)$, where L, M are viewed as $\mathbb{Z}H$-modules by restriction. Perhaps the simplest way to define this action is as follows. Let (P, ∂) be a projective $\mathbb{Z}G$-resolution of L, and view (P, ∂) as a projective $\mathbb{Z}H$-resolution of L. Then any $g \in N_G(H)$ gives rise to an action on $\mathrm{Hom}_{\mathbb{Z}H}(\mathrm{P}, M)$, $\phi \mapsto \phi^g$, where $\phi^g(p) = g^{-1}(\phi(gp))$. Since g normalizes H, we see that ϕ^g is $\mathbb{Z}H$-linear. The $N_G(H)$-action respects cocycles and coboundaries, so induces an action on the cohomology, $\mathrm{Ext}^*_{\mathbb{Z}H}(L, M)$.

In the foregoing, if ϕ is $\mathbb{Z}G$-linear, then $\phi^g = \phi$. It follows that the image of the restriction map

$$\mathrm{Ext}^*_{\mathbb{Z}G}(L, M) \to \mathrm{Ext}^*_{\mathbb{Z}H}(L, M)$$

is contained in the set $\mathrm{Ext}^*_{\mathbb{Z}H}(L, M)^{N_G(H)}$ of points fixed by $N_G(H)$.

Let g be an element of $C_G(H)$. Left multiplication by g determines a $\mathbb{Z}H$-linear endomorphism of M and we denote it by $g\cdot|_M$. Similarly, $g\cdot|_{\mathrm{P}} \in \mathrm{End}_{\mathbb{Z}H}(\mathrm{P})$ is a lift of $g\cdot|_L \in \mathrm{End}_{\mathbb{Z}H}(L)$. Now $\mathrm{Ext}^*_{\mathbb{Z}H}(L, M)$ is a bimodule over $\mathrm{End}_{\mathbb{Z}H}(M)$ and $\mathrm{End}_{\mathbb{Z}H}(L)$, and we see that the action of g on $\mathrm{Ext}^*_{\mathbb{Z}H}(L, M)$ is given by $\eta \mapsto (g\cdot|_M)^{-1} \circ \eta \circ (g\cdot|_L)$.

If we restrict to the case where $L = \mathbb{Z}$ with trivial G-action, we see that the image of the restriction map $\mathrm{H}^*(G, M) \to \mathrm{H}^*(H, M)$ is contained in $\mathrm{H}^*(H, M)^{N_G(H)}$, and hence in $\mathrm{H}^*(H, M)^{C_G(H)}$, and here $C_G(H)$ acts by left multiplication on M. □

The following is now an immediate consequence of Lemma 6.2

6.7 Corollary. *Let $n \in \mathbb{N}$, let M be a $\mathbb{Z}G$-module, let β be a limit ordinal, and let $(G_\alpha \mid \alpha \leq \beta)$ be a continuous chain of subgroups of G with $G_\beta = G$. If, for each $\alpha < \beta$, $\mathrm{H}^n(G_\alpha, M)^{N_G(G_\alpha)} = 0$ and $\mathrm{H}^{n-1}(G_\alpha, M) = 0$, then $\mathrm{H}^n(G, M) = 0$.* □

We record consequences for abelian groups which seem to be new.

6.8 Theorem. *Let G be an abelian group, λ an ordinal, and Δ a G-set with stabilizers of $\aleph$-rank strictly less than λ.*

(1) *For each $n \in \mathbb{N}$, if* $\aleph\text{-rank}(G) \geq \lambda + n$ *then* $\mathrm{H}^n(G, A\Delta) = 0$.
(2) *If* $\aleph\text{-rank}(G) \geq \lambda + \omega_0$ *then* $\mathrm{H}^*(G, A\Delta) = 0$.

Proof. (1). We argue by induction on n.

If $n = 0$, then all G-stabilizers of elements of Δ have infinite index in G, so Δ has no finite G-orbits. Thus $(A\Delta)^G = 0$, that is, $\mathrm{H}^0(G, A\Delta) = 0$.

Now suppose that $n \geq 1$, and that the result holds for smaller n. Let β denote the least ordinal of cardinality $\mathrm{rank}(G)$, so β is a limit ordinal. Moreover, there exists a continuous chain of subgroups $(G_\alpha \mid \alpha \leq \beta)$ such that $G_\beta = G$, and, for each $\alpha < \beta$,

$$\lambda + n - 1 \leq \aleph\text{-rank}(G_\alpha) < \aleph\text{-rank}(G).$$

Consider $\alpha < \beta$.

By the induction hypothesis, $\mathrm{H}^{n-1}(G_\alpha, A\Delta) = 0$, so, by Corollary 6.7, it remains to show that $\mathrm{H}^n(G_\alpha, A\Delta)^{N_G(G_\alpha)} = 0$. Here $N_G(G_\alpha) = C_G(G_\alpha) = G$, since G is abelian. Thus, we want to show that $\mathrm{H}^n(G_\alpha, A\Delta)^G = 0$ where G is acting via multiplication on $A\Delta$.

Consider any element $\zeta \in \mathrm{H}^n(G_\alpha, A\Delta)$. We can use the bar resolution $\mathrm{P}(G_\alpha)$, and represent ζ by a $\mathbb{Z}G_\alpha$-linear map $\phi\colon \mathrm{P}_n(G_\alpha) \to A\Delta$. Since $|G_\alpha| < |G|$, we see there is a G_α-subset Δ' of Δ such that $A\Delta'$ contains the image of ϕ, and $|\Delta'| < |G|$. Let Δ'' be the complement of Δ' in Δ. Then $A\Delta = A\Delta' \oplus A\Delta''$, so

$$\mathrm{H}^n(G_\alpha, A\Delta) = \mathrm{H}^n(G_\alpha, A\Delta') \oplus \mathrm{H}^n(G_\alpha, A\Delta''),$$

and, in the corresponding expression $\zeta = (\zeta', \zeta'')$, we have $\zeta'' = 0$.

Now

$$|\{g \in G \mid g\Delta' \cap \Delta' \neq \emptyset\}| < |G|$$

because the elements of Δ have G-stabilizers of cardinality strictly less than $|G|$. Hence there exists $g \in G$ such that $g\Delta' \cap \Delta' = \emptyset$, that is, $g\Delta' \subseteq \Delta''$, so

$$\mathrm{H}^n(G_\alpha, A\Delta')^g \subseteq \mathrm{H}^n(G_\alpha, A\Delta'').$$

It follows that if $\zeta^g = \zeta$ then $\zeta = 0$. This proves that $\mathrm{H}^n(G_\alpha, A\Delta)^G = 0$, and (1) is proved.

(2) follows from (1). □

The case where $\Delta = G$ is of interest; here we can take $\lambda = 0$.

6.9 Corollary. *Let $n \in \mathbb{N}$, and let G be an abelian group. Then,* $\mathrm{H}^n(G, AG) = 0$ *if* $n < \aleph\text{-rank}(G) + 1$. □

6.10 Corollary. *Let $n \in \mathbb{N}$, and let G be an abelian, torsion group.*

(1) (Holt [5]). $\mathrm{H}^n(G, AG) = 0$ *if* $n \neq \aleph\text{-rank}(G) + 1$.
(2) (Chen [12, Corollary 7.6]). *If R is a nonzero, $o(G)$-inverting ring, then* $\mathrm{cd}_R\, G = n$ *if and only if* $\aleph\text{-rank}(G) = n - 1$, *that is,*

$$\mathrm{cd}_R\, G = \min\{\aleph\text{-rank}(G) + 1, \infty\}. \quad \square$$

Proof. (1) follows from Theorem 3.10 and Corollary 6.9. (2) follows from the fact that, if $\mathrm{cd}_R(G) < \infty$, then $\mathrm{H}^m(G, \mathbb{Z}G \otimes_{\mathbb{Z}} -) \neq 0$ for $m = \mathrm{cd}_R(G)$; see the Commentary on Conjecture 1.7. □

7. Cardinals, free abelian groups, and $\mathbf{H}\,\mathfrak{F}$

We now recall the hierarchies introduced in [9]; see [10] for more details.

7.1 Notation. Let $\mathfrak{X}$ denote a class of groups.

All the classes of groups that we consider are closed under isomorphism, for example, the class $\mathfrak{F}$ of all finite groups.

We let $\mathbf{L}\,\mathfrak{X}$ denote the class of groups whose finitely generated subgroups all lie in $\mathfrak{X}$. For example, if $\mathfrak{X}$ contains all finitely generated abelian groups, then $\mathbf{L}\,\mathfrak{X}$ contains all abelian groups.

We let $\mathbf{H}_1\,\mathfrak{X}$ denote the class of all groups G for which there exists a finite-dimensional contractible G-complex with all stabilizers lying in $\mathfrak{X}$. For example, $\mathbf{H}_1\,\mathfrak{F}$ contains all finitely generated abelian groups, since, if G is finitely generated and abelian, then G has a finite subgroup N such that G/N is isomorphic to $\mathbb{Z}^n$ for some $n \in \mathbb{N}$, and thus G/N acts freely on $\mathbb{R}^n$ preserving a CW-structure.

If $\mathbf{H}_1\,\mathfrak{X} = \mathfrak{X}$, then $\mathfrak{X}$ is said to be $\mathbf{H}_1$-*closed.*

We let $\mathbf{H}\,\mathfrak{X}$ denote the smallest $\mathbf{H}_1$-closed class of groups which contains $\mathfrak{X}$. This class has a hierarchy indexed by the ordinals, where for each ordinal β, we define $\mathbf{H}_\beta\,\mathfrak{X}$ recursively, by setting

$$\begin{aligned}
\mathbf{H}_0\,\mathfrak{X} &:= \mathfrak{X},\\
\mathbf{H}_\beta\,\mathfrak{X} &:= \mathbf{H}_1\,\mathbf{H}_{\beta-1}\,\mathfrak{X} \text{ if } \beta \text{ is a successor ordinal,}\\
\mathbf{H}_\beta\,\mathfrak{X} &:= \bigcup_{\alpha<\beta} \mathbf{H}_\alpha\,\mathfrak{X} \text{ if } \beta \text{ is a limit ordinal.} \quad \Box
\end{aligned}$$

We can use Theorem 2.8 to get new sufficient conditions for membership in $\mathbf{H}\,\mathfrak{X}$.

7.2 Theorem. *Let $\mathfrak{X}$ be a subgroup-closed class of groups, and let $G \in \mathbf{L}\,\mathfrak{X}$.*

(1) *If $\aleph$-rank$(G) = -1$ then $G \in \mathbf{H}_0\,\mathfrak{X}$.*
(2) *If $\aleph$-rank$(G) < \omega_0$ then $G \in \mathbf{H}_1\,\mathfrak{X}$.*
(3) *If $\aleph$-rank$(G) = \omega_0$ then $G \in \mathbf{H}_2\,\mathfrak{X}$.*

Proof. (1). If $\aleph$-rank$(G) = -1$ then G is a finitely generated element of $\mathbf{L}\,\mathfrak{X}$, so lies in $\mathfrak{X} = \mathbf{H}_0\,\mathfrak{X}$.

(2). If $\aleph$-rank$(G) = n$ for some $n \in \mathbb{N}$ then, by Theorem 2.8, G acts on a contractible $(n+1)$-dimensional CW-complex with stabilizers contained in finitely generated subgroups of G. Since G lies in $\mathbf{L}\,\mathfrak{X}$, and $\mathfrak{X}$ is subgroup closed, we see that these stabilizers lie in $\mathfrak{X}$. Hence $G \in \mathbf{H}_1\,\mathfrak{X}$.

(3). If $\aleph$-rank$(G) = \omega_0$, then we can write G as the union of an ascending chain $(G_n \mid n \in \mathbb{N})$ of subgroups, such that, for each $n \in \mathbb{N}$, $\aleph$-rank$(G_n) = n$. As in Remark 2.5, there exists a G-tree with each cell stabilizer contained in G_n, for some $n \in \mathbb{N}$, so lying in $\mathbf{H}_1\,\mathfrak{X}$ by (2). Hence, $G \in \mathbf{H}_2\,\mathfrak{X}$. $\Box$

We next consider some necessary conditions for membership in $\mathbf{H}\,\mathfrak{X}$, which are natural generalizations of [8, Lemma 1].

7.3 Lemma. *Let $n \in \mathbb{N}$, let R be a ring, let*

$$0 \to M_n \to M_{n-1} \to \cdots \to M_1 \to M_0 \to M_{-1} \to 0$$

be an exact sequence of R-modules, and let L be an R-module.

Suppose that $\mathrm{Ext}^i_R(L, M_i) = 0$ for $i = 0, \ldots, n$. Then $\mathrm{Ext}^0_R(L, M_{-1}) = 0$.

Proof. Clearly the result holds for $n = 0$. Thus we may assume that $n \geq 1$, and that the result holds with $n-1$ in place of n.

Let M'_{n-1} denote the cokernel of the map $M_n \to M_{n-1}$, so we have exact sequences

$$0 \to M_n \to M_{n-1} \to M'_{n-1} \to 0 \tag{7.4}$$

$$0 \to M'_{n-1} \to M_{n-2} \to \cdots \to M_0 \to M_{-1} \to 0 \tag{7.5}$$

Now (7.4) gives rise to a long exact sequence which contains the segment

$$\mathrm{Ext}^{n-1}_R(L, M_{n-1}) \to \mathrm{Ext}^{n-1}_R(L, M'_{n-1}) \to \mathrm{Ext}^n_R(L, M_n).$$

Here the outer terms are zero, by hypothesis, so the inner term is zero. The induction hypothesis can now be applied to (7.5), and we see that $\mathrm{Ext}^0_R(L, M_{-1}) = 0$.

The result follows. $\square$

We record the contrapositive of the case where $R = \mathbb{Z}G$, and $M_{-1} = L = \mathbb{Z}$ with trivial G-action.

7.6 Corollary. *If $n \in \mathbb{N}$, and*

$$0 \to M_n \to M_{n-1} \to \cdots \to M_1 \to M_0 \to \mathbb{Z} \to 0$$

is an exact sequence of $\mathbb{Z}G$-modules, then there exists i such that $0 \leq i \leq n$ and $\mathrm{H}^i(G, M_i) \neq 0$. $\square$

7.7 Proposition. *If $\mathfrak{X}$ is a class of groups, and $\mathrm{H}^*(G, \mathbb{Z}\Delta) = 0$ for every G-set Δ for which all stabilizers lie in $\mathfrak{X}$, then $G \notin \mathbf{H}_1\,\mathfrak{X}$.*

Proof. Suppose that $G \in \mathbf{H}_1\,\mathfrak{X}$, so there exists a finite-dimensional, contractible CW-complex X on which G acts with all stabilizers lying in $\mathfrak{X}$. Let n denote the dimension of X. The augmented cellular chain complex of X,

$$0 \to \mathrm{C}_n(X) \to \mathrm{C}_{n-1}(X) \to \cdots \to \mathrm{C}_1(X) \to \mathrm{C}_0(X) \to \mathbb{Z} \to 0,$$

is an exact sequence of $\mathbb{Z}G$-modules, so, by Corollary 7.6, there exists i such that $0 \leq i \leq n$ and $\mathrm{H}^i(G, C_i(X)) \neq 0$. Thus $\mathrm{H}^*(G, C_i(X)) \neq 0$. But we can write $C_i(X) = \mathbb{Z}\Delta$, where Δ is the set of i-dimensional open cells of X, so is a G-set with all stabilizers in $\mathfrak{X}$. This contradicts the hypothesis. $\square$

Combining Theorem 6.8(2) and Proposition 7.7, we get the following.

7.8 Corollary. *Let $\mathfrak{X}$ be a class of groups and λ be an ordinal. Suppose that G is an abelian group in $\mathbf{H}_1\,\mathfrak{X}$ such that every subgroup of G which lies in $\mathfrak{X}$ has $\aleph$-rank strictly less than λ. Then $\aleph\text{-rank}(G) < \lambda + \omega_0$.* $\square$

7.9 Theorem. *If β is any ordinal, then every abelian group in $\mathbf{H}_\beta\,\mathfrak{F}$ has $\aleph$-rank strictly less than $\omega_0\beta$.*

Proof. We argue by induction on β.

The result holds for $\beta = 0$ by definition of $\mathfrak{F}$.

Thus we may assume that $\beta > 0$, and that the result holds for smaller ordinals.

Consider the case where β is a limit ordinal. Here, each abelian group in $\mathbf{H}_\beta\,\mathfrak{F}$ lies in $\mathbf{H}_\alpha\,\mathfrak{F}$ for some $\alpha < \beta$, so, by the induction hypothesis, is of cardinality strictly less than $\aleph_{\omega_0\alpha} \leq \aleph_{\omega_0\beta}$.

Now consider the case where β is a successor ordinal, and write $\beta = \alpha + 1$. By the induction hypothesis, every abelian group in $\mathbf{H}_\alpha\,\mathfrak{F}$ is of cardinality strictly less than $\aleph_{\omega_0\alpha}$. If we apply Corollary 7.8, with $\mathfrak{X} = \mathbf{H}_\alpha\,\mathfrak{F}$, we see that every abelian group in $\mathbf{H}_1\,\mathfrak{X} = \mathbf{H}_\beta\,\mathfrak{F}$ is of cardinality strictly less than $\aleph_{\omega_0\alpha+\omega_0} = \aleph_{\omega_0\beta}$. □

We can now make six statements, of varying profundity and novelty, about how some *free* abelian groups fit into the hierarchy $\mathbf{H}\,\mathfrak{F}$.

7.10 Theorem. *For each cardinal κ, let $\mathbb{A}_\kappa$ denote the free abelian group of rank κ.*

(1) $\mathbb{A}_0 \in \mathbf{H}_0\,\mathfrak{F}$.
(2) $(\mathbb{A}_\kappa \mid 1 \leq \kappa < \aleph_0) \subseteq \mathbf{H}_1\,\mathfrak{F} - \mathbf{H}_0\,\mathfrak{F}$.
(3) $(\mathbb{A}_\kappa \mid \aleph_0 \leq \kappa < \aleph_{\omega_0}) \subseteq \mathbf{H}_2\,\mathfrak{F} - \mathbf{H}_1\,\mathfrak{F}$.
(4) $\mathbb{A}_{\aleph_{\omega_0}} \in \mathbf{H}_3\,\mathfrak{F} - \mathbf{H}_2\,\mathfrak{F}$.
(5) *For each finite ordinal n, $\mathbb{A}_{\aleph_{\omega_0 n}} \notin \mathbf{H}_{n+1}\,\mathfrak{F}$; equivalently, every free abelian group in $\mathbf{H}_{n+1}\,\mathfrak{F}$ has $\aleph$-rank strictly less than $\omega_0 n$.*
(6) *For each infinite ordinal β, $\mathbb{A}_{\aleph_{\omega_0\beta}} \notin \mathbf{H}_\beta\,\mathfrak{F}$.*

Proof. (6) is a special case of Theorem 7.9.

(1) is clear.

(2). Consider $n \in \mathbb{N}$. Then $\mathbb{A}_n$ acts freely on an n-dimensional CW-complex with underlying space $\mathbb{R}^n$, so $\mathbb{A}_n \in \mathbf{H}_1\,\mathfrak{F}$. It is clear that, if $n \geq 1$, then $\mathbb{A}_n \notin \mathbf{H}_0\,\mathfrak{F}$.

(5). It is well known that, for each $n \in \mathbb{N}$, $\mathrm{H}^n(\mathbb{A}_n, \mathbb{Z}) = \mathbb{Z}$. This can be seen by induction, using a long exact sequence in cohomology which gives a short exact sequence of graded groups

$$0 \to \mathrm{H}^*(\mathbb{A}_{n-1}, \mathbb{Z}) \to \mathrm{H}^{*+1}(\mathbb{A}_n, \mathbb{Z}) \to \mathrm{H}^{*+1}(\mathbb{A}_{n-1}, \mathbb{Z}) \to 0;$$

it can also be seen from the fact that the n-torus is a $\mathrm{K}(\mathbb{A}_n, 1)$.

Suppose that $\mathbb{A}_{\aleph_0} \in \mathbf{H}_1\,\mathfrak{F}$, so, for some $n \in \mathbb{N}$, $\mathbb{A}_{\aleph_0}$ acts freely on a contractible n-dimensional CW-complex X. Hence $\mathbb{A}_{n+1}$ acts freely on X, so $\mathrm{H}^{n+1}(\mathbb{A}_{n+1}, -) = 0$, which contradicts the fact that $\mathrm{H}^{n+1}(\mathbb{A}_{n+1}, \mathbb{Z}) = \mathbb{Z}$. Thus $\mathbb{A}_{\aleph_0} \notin \mathbf{H}_1\,\mathfrak{F}$, so every free abelian group in $\mathbf{H}_1\,\mathfrak{F}$ has finite rank, that is, has $\aleph$-rank equal to -1.

Now suppose that $n \geq 1$, and that every free abelian group in $\mathbf{H}_n\,\mathfrak{F}$ has $\aleph$-rank strictly less than $\omega_0(n-1)$. Corollary 7.8, with $\mathfrak{X} = \mathbf{H}_n\,\mathfrak{F}$, and $\lambda = \omega_0(n-1)$, implies that every free abelian group in $\mathbf{H}_{n+1}\,\mathfrak{F}$ has $\aleph$-rank strictly less than $\omega_0 n$.

Now (5) has been proved by induction.

(3) and (4). By (2), $\mathbf{H}_1\,\mathfrak{F}$ includes all finitely generated free abelian groups. Hence every free abelian group lies in $\mathbf{L}\,\mathbf{H}_1\,\mathfrak{F}$. If we apply Theorem 7.2(2) and (3), with $\mathfrak{X} = \mathbf{H}_1\,\mathfrak{F}$, we find that the free abelian groups of $\aleph$-rank strictly less than ω_0 lie in $\mathbf{H}_2\,\mathfrak{F}$, and that the free abelian group of $\aleph$-rank ω_0 lies in $\mathbf{H}_3\,\mathfrak{F}$. Combined with (5), these imply (3) and (4). □

7.11 Remark. Theorem 7.10 gives an interesting new proof that

$$\mathbf{H}_0\,\mathfrak{F} \neq \mathbf{H}_1\,\mathfrak{F} \neq \mathbf{H}_2\,\mathfrak{F} \neq \mathbf{H}_3\,\mathfrak{F}. \quad \square$$

7.12 Conjecture. $\mathbb{A}_{\aleph_{\omega_0+1}} \notin \mathbf{H}_3\,\mathfrak{F}$; equivalently, $\mathbb{A}_{\aleph_{\omega_0+1}} \notin \mathbf{H}\,\mathfrak{F}$. $\square$

7.13 Conjecture. $\mathbf{H}_3\,\mathfrak{F} \neq \mathbf{H}\,\mathfrak{F}$. $\square$

7.14 Conjecture. There exists an ordinal α such that $\mathbf{H}_\alpha\,\mathfrak{F} = \mathbf{H}\,\mathfrak{F}$. $\square$

Acknowledgments. W. Dicks was partially supported by the DGES and the DGI through grants PB96-1152 and BFM2000-0354, respectively. I. Leary was partially supported by the EPSRC. S. Thomas was partially supported by NSF grants.

I. Leary and P. Kropholler thank the Centre de Recerca Matemàtica of the Institut d'Estudis Catalans for the hospitality they received, and S. Thomas thanks the Departament de Matemàtiques of the Universitat Autònoma de Barcelona for its hospitality.

References

1. M. Auslander, *On the dimension of modules and algebras, III*, Nagoya Math. J. **9** (1955), 67–77.
2. Kenneth S. Brown, *Cohomology of groups*, GTM **87**, Springer-Verlag, Berlin, 1982.
3. László Fuchs, *Infinite abelian groups, Vol. I*, Academic Press, New York, 1970.
4. D. F. Holt, *On the cohomology of locally finite groups*, Quart. J. Math. Oxford **32** (1981), 165–172.
5. D. F. Holt, *The cohomological dimensions of locally finite groups*, J. London Math. Soc. **24** (1981), 129-134.
6. D. F. Holt, *Uncountable locally finite groups have one end*, Bull. London Math. Soc. **13** (1981), 557-560.
7. T. Jech, *Set theory*, Academic Press, New York, 1978.
8. P. H. Kropholler, *Soluble groups of type* FP_∞ *have finite torsion-free rank*, Bull. London Math. Soc. **25** (1993), 558–566.
9. P. H. Kropholler, *On groups of type* FP_∞, J. Pure Appl. Algebra **90** (1993), 55–67.
10. P. H. Kropholler and G. Mislin, *Groups acting on finite dimensional spaces with finite stabilizers*, Comment. Math. Helv. **73** (1998), 122–136.
11. P. H. Kropholler and S. Thomas, *The consistency of Holt's conjectures on cohomological dimension of locally finite groups*, J. London Math. Soc. **55** (1997), 76–86.
12. B. Osofsky, *Projective dimensions of "nice" directed unions*, J. Pure Appl. Algebra **13** (1978), 179–219.
13. D. J. S. Robinson, *Vanishing theorems for cohomology of locally nilpotent groups,* pp. 120–129 in: Group Theory, Proceedings, Brixen/Bressanone 1986, (Eds. O.H. Kegel, F. Mengazzo, G. Zacher), LNM **1281**, Springer–Verlag, Berlin, 1987.
14. Simon Thomas, *An independence result in group cohomology*, Bull. London Math. Soc. **28** (1996), 264–268.
15. Tammo tom Dieck, *Transformation groups*, Studies in Mathematics **8**, de Gruyter, Berlin, 1987.

Departament de Matemàtiques, Universitat Autònoma de Barcelona, E-08193 Bellaterra (Barcelona), Spain

E-mail address: `dicks@mat.uab.es`

School of Mathematical Sciences, Queen Mary, University of London, Mile End Road, London E1 4NS, United Kingdom

E-mail address: `P.H.Kropholler@qmw.ac.uk`

Faculty of Mathematical Studies, University of Southampton, Southampton SO17 1BJ, United Kingdom
E-mail address: `I.J.Leary@maths.soton.ac.uk`

Department of Mathematics, Rutgers University, New Brunswick, N. J. 08903, USA
E-mail address: `sthomas@math.rutgers.edu`