LECTURE NOTES ON McCOOL'S PRESENTATIONS FOR STABILIZERS

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Abstract. Let F be any free group with a finite basis, let S be any finite set of conjugacy classes of elements of F, and let $\mathrm{Aut}(F, S)$ denote the group of all automorphisms of F which carry S to itself. In 1975, McCool described a finite presentation for $\mathrm{Aut}(F, S)$; even the fact that $\mathrm{Aut}(F, S)$ is finitely generable had not been noted previously. McCool's proof has some subtle points, and the standard treatments leave some details to the reader. We give a self-contained, detailed proof of a slight generalization of McCool's result. We also give proofs of all the background results of Dyck, Dehn, Nielsen, Reidemeister, Schreier, Gersten, Higgins & Lyndon, Whitehead, and Rapaport.

Our viewpoint is mainly graph-theoretic. We lift Higgins & Lyndon's arguments from outer automorphisms to automorphisms by using graph-theoretic techniques due to Gersten, as opposed to using Rapaport's technique of adding a new variable. We lift McCool's arguments about finite, two-dimensional CW-complexes to arguments about groups acting on trees, where they may be made rigorous.

1. Introduction

1.1. **Definitions.** Let G be a group. By a *straight word in* G, we mean an element of G. By a *cyclic word in* G, we mean the G-conjugacy class of an element of G. By a *word in* G, we mean a straight-or-cyclic word in G. Let $\mathrm{Aut}\,G$ denote the group of all automorphisms of G, acting on the right, written as exponents. In a natural way, $\mathrm{Aut}\,G$ acts on the set of all words in G, and on the set of all sets of words in G. If S is any set of words in G, then we define $\mathrm{Aut}(G, S) := \{\varphi \in \mathrm{Aut}\,G : S^{\varphi} = S\}$, which is a subgroup of $\mathrm{Aut}\,G$ called the $(\mathrm{Aut}\,G)$-*stabilizer of* S. □

1.2. **Historical notes.** Let F be a free group with a finite basis (free-generating set), and S and S' be finite sets of words in F.

Nielsen(1919) used an elegant algebraic argument to obtain a finite generating set for $\mathrm{Aut}\,F$. Later, Nielsen(1924) used a very difficult algebraic argument to obtain a finite presentation for $\mathrm{Aut}\,F$. His proof used a rewriting algorithm and a Dehn tree; although we shall not discuss Nielsen's proof, we shall see that both of these tools are still very important in the theory. Chandler & Magnus(1982) wrote that 'an unsystematic poll taken by Magnus in 1970 seems to indicate that for about a decade after the death of Nielsen in 1959 there existed no living mathematician who had read Nielsen's paper in detail or would have been able to derive his result. This situation changed dramatically' when McCool(1974 and 1975b) obtained a new finite presentation for $\mathrm{Aut}\,F$ and a new proof of Nielsen's finite presentation for $\mathrm{Aut}\,F$.

2010 *Mathematics Subject Classification.* Primary: Secondary:
Key words and phrases.
*Research supported by MINECO (Spain), project numbers MTM2014-53644-P, MTM2017-83487-P.

Whitehead(1936) associated with S a finite graph $\mathbf{\Gamma}(S)$ (described in Historical notes 9.3 below), and gave a difficult topological proof that $\mathbf{\Gamma}(S)$ is connected, from which it follows that $\mathbf{\Gamma}(S)$ may be constructed by using a simple algorithm. By comparing $\mathbf{\Gamma}(S)$ and $\mathbf{\Gamma}(S')$, one sees whether or not the set $\{\varphi \in \operatorname{Aut} F : S^\varphi = S'\}$ is empty, and one finds an element if any exist. Rapapport(1958) introduced a rewriting algorithm, and gave a very difficult algebraic proof that $\mathbf{\Gamma}(S)$ is connected. Higgins & Lyndon(1962) introduced a more powerful rewriting algorithm, and gave a relatively simple algebraic proof that $\mathbf{\Gamma}(S)$ is connected. These efforts to understand Whitehead's work were rewarded with an unexpected result when McCool(1975a) refined the Higgins & Lyndon algorithm and showed how to read off a finite presentation for $\operatorname{Aut}(F,S)$ from $\mathbf{\Gamma}(S)$; even the fact that $\operatorname{Aut}(F,S)$ is finitely generable had not been noted previously. □

The purpose of these notes is to present a self-contained proof, with all details checked, of an explicit finite presentation for $\operatorname{Aut}(F,S)$. The proof has some subtle points, and the standard treatments leave several details to the reader. We use most of the results already mentioned, together with later techniques introduced by Serre(1977), Hoare(1979), and Gersten(1984a). Let us now outline the course.

In Section 2, we collect together much of the basic vocabulary and notation we shall be using. In Section 3, we formally explain free groups, bases, and presentations. In Section 4, we formally explain graphs and Serre's definition of their fundamental groups; this section produces free groups from graphs, while the next section produces trees from free groups. In Section 5, we explain Schreier graphs, Dehn trees, Dyck normal forms, and the length of a finite set of words in a free group. In Section 6, we use the Schreier-Serre technique to prove the Nielsen-Schreier theorem that subgroups of free groups are free, and then we describe Reidemeister-Schreier presentations for subgroups.

In the remainder of the course, we study $\operatorname{Aut}(F)$ for a free group F with a finite basis. In Section 7, we give Nielsen's finite generating set for $\operatorname{Aut}(F)$ and Nielsen's original proof. In Section 8, we present Whitehead's larger finite generating family for $\operatorname{Aut}(F)$, which has the advantage of behaving extraordinarily well with respect to lengths of finite sets of words. In Section 9, we list McCool's relators, state McCool's explicit finite presentation of $\operatorname{Aut} F$, and state explicitly McCool's finite presentation for $\operatorname{Aut}(F,S)$. We then finally know where we are headed. In Section 10, we give Gersten's graph-theoretic description of Whitehead's generating family, and start to accumulate some of its properties. In Section 11, we state and prove six rules for Whitehead's generators which encapsulate the set of rules given by Higgins & Lyndon, which in turn greatly simplified the set of rules given by Rapaport. In Section 12, we record the consequences of five of the six rules; in particular, the relators given in Section 9 are relators. In Section 13, we introduce Rapaport's important concept of decreasing a peak, here expressed in terms of groups acting on trees. In Section 14, we perform the most strenuous part which is to prove that the McCool peaks are Rapaport decreasable via the Higgins & Lyndon rewriting rules. In Section 15, we record Rapaport's consequence of this result. In Section 16, we give proofs for the presentations stated in Section 9.

1.3. **Historical notes.** Possibly the most famous example of $\operatorname{Aut}(F,S)$ is the *algebraic n-string-braid group* of Artin(1925 and 1947), wherein F is a free group with basis $\{t_1, t_2, \ldots, t_n\}$ and S is the set $\{t_1{\cdot}t_2{\cdot}\cdots t_n, t_1^{\circlearrowleft}, t_2^{\circlearrowleft}, \ldots, t_n^{\circlearrowleft}\}$; here, we are writing $t_i^{\circlearrowleft}$ to denote the cyclic word which is the F-conjugacy class of t_i. One of the many important results that Artin obtained in 1925 was a finite presentation for $\mathrm{Aut}(F,S)$. His proof used algebra and topology. In 1947, he wrote the following about his 1925 paper: "Most of the proofs are entirely intuitive...It is possible to correct the proofs. The difficulties that one encounters if one tries to do so come from the fact that projection of the braid, which is an excellent tool for intuitive investigations, is a very clumsy one for rigorous proofs.". Magnus(1934) gave an algebraic proof of Artin's 1925 presentation by applying a procedure which had been invented in 1927 by Artin's colleagues Reidemeister and Schreier. □

McCool(1975a) wrote that 'the idea of exhibiting $\mathrm{Aut}(F,S)$ as the fundamental group of a finite complex was shown to me by Lyndon. My original proof...was much more tedious.'. The straightforward translation from fundamental groups to groups acting on trees does not seem to me to add tedium, and it is the only language that I have found in which I could write out rigorous proofs.

Although there are some innovations in these notes, we have not explained anything alien to the period 1882–1984, we have explained only a small portion of what was learned about free groups in that period, and we have explained nothing at all about the massive activity in the reseach on free groups since 1984. We believe that at least the information we have given about McCool's presentations is up-to-date.

2. Basic vocabulary and notation

2.1. **Definitions.** Following Bourbaki, we let $\mathbb{N}$ denote the set of finite cardinals; thus, $\mathbb{N} := \{0, 1, 2, \ldots, \}$.

For any sets A and B, we denote the disjoint union of A and B by $A \sqcup B$.

For any set S and subsets A and B of S, we write $A{-}B := \{a \in A : a \notin B\}$.

Let A be any set. We denote the cardinal of A by $|A|$. For each $\ell \in \mathbb{N}$, we let $A^{\times \ell}$ denote the ℓth Cartesian power of A; thus, the elements of $A^{\times \ell}$ are ℓ-tuples of elements of A, $(a_1, a_2, \ldots, a_\ell)$. An *$A$-sequence* is an element of $A^{\times \ell}$ for some $\ell \in \mathbb{N}$, and ℓ is then called the *length* of the A-sequence. A set S is said to be *a family of elements of A* if there is specified a set map $S \to A$; by abuse of notation, we then treat each element of S as if it were equal to its image in A. □

2.2. **Definitions.** Let G be a group.

We denote the associative binary operation by $G^{\times 2} \to G$, $(h, g) \mapsto h{\cdot}g$. We denote the identity element by 1, and the inversion operation by $G \to G$, $g \mapsto g^{-1}$. (On the two occasions where we abelianize G, we shall change the notation for the operation to +.)

For all $g, h \in G$, we set $g^h := h^{-1}{\cdot}g{\cdot}h$ and ${}^h g := h{\cdot}g{\cdot}h^{-1}$.

For each $h \in G$, the set $h^G := \{h^g : g \in G\}$ is called the *G-conjugacy class of h*, which is a cyclic word in G.

For subsets A, B of G, we write $A{\cdot}B := \{a{\cdot}b \mid a \in A, b \in B\}$, ${}^A B := \{{}^a b \mid a \in A, b \in B\}$, $A^B := \{a^b \mid a \in A, b \in B\}$, $A^{-1} := \{a^{-1} \mid a \in A\}$, and $A^{\pm 1} := A \cup A^{-1}$. For an element g of G, we write $A{\cdot}g := A{\cdot}\{g\}$, and similarly for $g{\cdot}A$.

For each subset S of G, we set $\langle S \rangle := \{s_1{\cdot}s_2{\cdot}\cdots{\cdot}s_\ell : (s_1, s_2, \ldots, s_\ell) \text{ is an } S^{\pm 1}\text{-sequence}\}$. Then $\langle S \rangle$ is the inclusion-smallest subgroup of G which includes S. We call $\langle S \rangle$ the *subgroup of G generated by S*. If $\langle S \rangle = G$, we say that S is a *generating set* for G. We say that G is *finitely generable* if there exists some finite generating set for G. By the *normal closure of S in G*, we mean $\langle {}^G S \rangle$, which is the inclusion-smallest normal subgroup of G which includes S. Analogous terminology is used with families of elements of G in place of subsets of G.

We set $G^{(\cdot 2)} := \langle\{g^2 \mid g \in G\}\rangle$, which is a normal subgroup of G. For all $x, y \in G$, $x{\cdot}y{\cdot}x^{-1}{\cdot}y^{-1} = x^2{\cdot}(x^{-1}{\cdot}y)^2{\cdot}(y^{-1})^2 \in G^{(\cdot 2)}$. The quotient group $G/G^{(\cdot 2)}$ may be viewed as a vector space over the field $\mathbb{F}_2$ of two elements. We set $\operatorname{rank}(G;\mathbb{F}_2) := \dim_{\mathbb{F}_2}(G/G^{(\cdot 2)})$.

Consider any G-sequence $\sigma = (g_1, g_2, \ldots, g_\ell)$. We say that σ is *reduced* if $g_{i+1} \neq g_i^{-1}$ for each $i \in \{1, 2, \ldots, \ell{-}1\}$, and σ is *cyclically reduced* if $(g_1, g_2, \ldots, g_\ell, g_1, g_2, \ldots, g_\ell)$ is reduced. For each $g \in G$, we say that σ is *for* g if $g_1{\cdot}g_2{\cdots}g_\ell = g$. For each cyclic word x in G, we say that σ is *for* x if $g_1{\cdot}g_2{\cdots}g_\ell \in x$.

By a *right-multiplicative* G*-set* A, we mean a set A given with a map $A{\times}G \to A$ denoted $(a, g) \mapsto a{\cdot}g$ such that the following hold: for each $a \in A$, we have $a{\cdot}1 = a$; for all $a \in A$ and g, $h \in G$, we have $a{\cdot}(g{\cdot}h) = (a{\cdot}g){\cdot}h$. For each $a \in A$, we write $G_a := \{g \in G : a{\cdot}g = a\}$, which is a subgroup of G called the G*-stabilizer of* a. If the map $A{\times}G \to A$ were denoted $(a, g) \mapsto a^g$, we would say that A is a *right-exponential* G*-set*. If A is either a right-multiplicative G-set or a right-exponential G-set, we say that A *is a right* G*-set*. We define left-multiplicative, left-exponential, and left G-sets analogously. If A is a left or right G-set, we say that A *is a* G*-set* and we write G_a to denote the G-stabilizer of a, for each $a \in A$.

If A and B are right-multiplicative G-sets, then a map $\varphi : A \to B$, $a \mapsto \varphi(a)$, is said to be a *right-multiplicative* G*-map* if $\varphi(a{\cdot}g) = (\varphi(a)){\cdot}g$ for all $(a, g) \in A{\times}G$. We define right-exponential G-maps, right G-maps, left-multiplicative G-maps, left-exponential G-maps, left G-maps, and G-maps analogously.

For each subgroup H of G, we write $H\backslash G := \{H{\cdot}g \mid g \in G\}$ and $G/H := \{g{\cdot}H \mid g \in G\}$; in a natural way, these are right-multiplicative and left-multiplicative G-sets respectively. $\square$

3. Free groups and presentations

3.1. **Definitions.** Let E be any set. Set $E^{\pm 1} := E{\times}\{1, -1\}$ and $A := \bigsqcup_{\ell \in \mathbb{N}} \big((E^{\pm 1})^{\times \ell}\big)$. Let the unique element of $(E^{\pm 1})^{\times 0}$ be denoted 1_A. We view E as the subset $E{\times}\{1\}$ of $E^{\pm 1}$, and, we view $E^{\pm 1}$ as the subset $(E^{\pm 1})^{\times 1}$ of A. On A, we have a unary operation called *inversion*,

$$A \to A,\ a \mapsto a^{-1}, \quad \text{where } (e_1, \ldots, e_\ell)^{-1} := (e_\ell^{-1}, \ldots, e_1^{-1}),$$

where, for each $(e, \epsilon) \in E^{\pm 1}$, $(e, \epsilon)^{-1} := (e, -\epsilon) \in E^{\pm 1}$. On A, we also have a binary operation called *concatenation*,

$$A^{\times 2} \to A,\ (a, b) \mapsto a \circledcirc b, \quad \text{where } (e_1, \ldots, e_\ell) \circledcirc (f_1, \ldots, f_m) := (e_1, \ldots, e_\ell, f_1, \ldots, f_m).$$

Then A is a *monoid*, in that concatenation is associative and 1_A is the identity element. Let $\mathfrak{A}$ denote the set of all those subsets S of $A^{\times 2}$ such that the following hold: S is an equivalence relation on A; for each $a \in A$, we have $(a^{-1} \circledcirc a, 1_A) \in S$; for all (a, a'), $(b, b') \in S$, we have $(a \circledcirc b, a' \circledcirc b') \in S$. For each $S \in \mathfrak{A}$, we let A/S denote the set of S-equivalence classes; it is straightforward to show that A/S has a unique group structure such that the quotient map $A \to A/S$ is an inversion-respecting monoid morphism. In $A^{\times 2}$, set $S_0 := \bigcap_{S \in \mathfrak{A}} S$. It is clear that $S_0 \in \mathfrak{A}$. We call A/S_0 *the free group on* E, and denote it by $\mathbf{F}(E)$. Thus, $\mathbf{F}(E)$ is a group with a specified set map $E \to \mathbf{F}(E)$; moreover, we claim that, for each group G, each set map $E \to G$ is the composite of our set map $E \to \mathbf{F}(E)$ with a unique group morphism $\mathbf{F}(E) \to G$. To verify this claim, we note that, first, the set map $E \to G$ is the composite of the inclusion map $E \to A$ with an inversion-respecting monoid morphism $A \to G$, which determines an element $S \in \mathfrak{A}$ in a natural way, and, secondly, we may then construct maps $\mathbf{F}(E) \to A/S \to G$, and the proof of the claim follows. Here, we say that the group morphism $\mathbf{F}(E) \to G$ is *induced* by the set map $E \to G$.

Exercise: The map $E^{\pm 1} \to \mathbf{F}(E)$ is injective.

Exercise: $\operatorname{rank}(\mathbf{F}(E); \mathbb{F}_2) = |E|$.

We shall often treat E as a subset of $\mathbf{F}(E)$, and here no ambiguity arises from the two definitions of $E^{\pm 1}$. □

3.2. **Definitions.** We say that a group G is a *free* group if there exists some set E such that G is isomorphic to $\mathbf{F}(E)$, or, equivalently, there exists some set map $E \to G$ such that the induced group morphism $\mathbf{F}(E) \to G$ is an isomorphism. In the latter event, we say that *the set map $E \to G$ is a G-basis*; it is necessarily injective. For any subset E of G, if the inclusion map $E \to G$ is a G-basis, then we say that *the subset E is a G-basis.* □

3.3. **Definitions.** Let E be any set, R be any family of elements of $\mathbf{F}(E)$, and N be the normal closure in $\mathbf{F}(E)$ of R. We write $\langle\, E \mid R \,\rangle$ to denote the group $\mathbf{F}(E)/N$.

For any group G, we write $G = \langle\, E \mid R \,\rangle$ when there is specified an isomorphism from $\langle\, E \mid R \,\rangle$ to G, or, equivalently, there is specified a set map $E \to G$ such that the induced group morphism $\mathbf{F}(E) \to G$ is surjective and has kernel N. By abuse of notation, we then say that *$\langle\, E \mid R \,\rangle$ is a presentation for G*; here, it is important to specify the set map $E \to G$. If, moreover, the sets E and R are finite, then we say that $\langle\, E \mid R \,\rangle$ is a *finite* presentation for G. We say that G is *finitely presentable* if there exists some finite presentation for G. □

4. Building free groups out of graphs

4.1. **Definitions** (Classic)**.** By a *graph*, we mean a quintuple $(\Gamma, \mathrm{V}\Gamma, \mathrm{E}\Gamma, \iota, \tau)$ such that Γ is a set, $\mathrm{V}\Gamma$ and $\mathrm{E}\Gamma$ are disjoint subsets of Γ whose union is Γ, and ι and τ are maps from $\mathrm{E}\Gamma$ to $\mathrm{V}\Gamma$. We use the same symbol Γ to denote both the graph and the set. We call $\mathrm{V}\Gamma$ and $\mathrm{E}\Gamma$ the *vertex-set* and *edge-set* of Γ respectively, and call their elements Γ-*vertices* and Γ-*edges* respectively. The maps ι and τ are called the *initial* and *terminal* incidence functions respectively.

For each subset Υ of Γ, we write $\mathrm{V}\Upsilon := \Upsilon \cap \mathrm{V}\Gamma$ and $\mathrm{E}\Upsilon := \Upsilon \cap \mathrm{E}\Gamma$. We say that Υ is a *subgraph* of Γ if $\mathrm{E}\Upsilon \subseteq \{e \in \mathrm{E}\Gamma : \{\iota(e), \tau(e)\} \subseteq \mathrm{V}\Upsilon\}$; if equality holds, we say that Υ is a *full* subgraph of Γ.

We set $\mathrm{E}^{\pm 1}\Gamma := (\mathrm{E}\Gamma)^{\pm 1} \subseteq \mathbf{F}(\mathrm{E}\Gamma)$. For $e \in \mathrm{E}\Gamma$, we set $\iota(e^{-1}) := \tau(e)$ and $\tau(e^{-1}) := \iota(e)$.

For each $v \in \mathrm{V}\Gamma$, we set $\operatorname{link}_\Gamma(v) := \{e \in \mathrm{E}^{\pm 1}\Gamma : \iota e = v\}$. By the Γ-*valence* of v, we mean $|\operatorname{link}_\Gamma(v)|$.

By a Γ-*path*, we mean a sequence of the form $p = (v_0, e_1, v_1, e_2, v_2, \ldots, v_{\ell-1}, e_\ell, v_\ell)$ where $\ell \in \mathbb{N}$ and, for each $i \in \{1, 2, \ldots, \ell\}$, $e_i \in \mathrm{E}^{\pm 1}\Gamma$, $v_{i-1} = \iota e_i$, and $v_i = \tau e_i$. We sometimes find it helpful to depict p as $v_0 \xrightarrow{e_1} v_1 \xrightarrow{e_2} v_2 \cdots v_{\ell-1} \xrightarrow{e_\ell} v_\ell$. We say p is a path *from v_0 to v_ℓ*, *p connects* v_0 to v_ℓ, *p initiates at* v_0, *p terminates at* v_ℓ, and *p has length* ℓ. If $v_\ell = v_0$, then we say that p is a *closed* path *based at* v_0. If $e_i \neq e_{i-1}^{-1}$ for each $i \in \{2, 3, \ldots, \ell\}$, then we say that p is a *reduced* path. We sometimes abbreviate p to $(e_1, e_2, \ldots, e_\ell)$, even if $\ell = 0$ when v_0 is specified. For each $e \in \mathrm{E}^{\pm 1}\Gamma$, by the *number of times p traverses e*, we mean $\left|\left\{i \in \{1, 2, \ldots, \ell\} : e_i \in \{e\}^{\pm 1}\right\}\right|$. We call the element $e_1 \cdot e_2 \cdots e_\ell$ of $\langle \mathrm{E}\Gamma \mid \emptyset \rangle$ the Γ-*label of* p. By a *subpath* of p, we mean any subsequence of p of the form $(v_i, e_{i+1}, v_{i+1}, \ldots, e_j, v_j)$ where i, $j \in \{0, 1, \ldots, \ell\}$ and $i \leqslant j$.

For $v, w \in \mathrm{V}\Gamma$, we let Γ-Paths(v,w) denote the set of all Γ-paths from v to w; we then have the *inversion* map

$$\Gamma\text{-Paths}(v,w) \to \Gamma\text{-Paths}(w,v), \quad p \mapsto p^{-1},$$

where $(e_1, e_2, \ldots, e_\ell)^{-1} := (e_\ell^{-1}, \ldots, e_2^{-1}, e_1^{-1})$. For $u, v, w \in \mathrm{V}\Gamma$, we have the *concatenation* map

$$\Gamma\text{-Paths}(u,v) \times \Gamma\text{-Paths}(v,w) \to \Gamma\text{-Paths}(u,w), \quad (p_1, p_2) \mapsto p_1 \circledcirc p_2,$$

where $(e_1, e_2, \ldots, e_\ell) \circledcirc (e'_1, e'_2, \ldots, e'_m) := (e_1, e_2, \ldots, e_\ell, e'_1, e'_2, \ldots, e'_m)$. The Γ-label of a concatenation of two Γ-paths equals the product of the Γ-labels of the two Γ-paths. If a Γ-path p is closed and $p \circledcirc p$ is reduced, we say that p is *cyclically reduced.* For any $v \in \mathrm{V}\Gamma$ and any subset W of $\mathrm{V}\Gamma$, we let Γ-Paths(v, W) denote the set of all Γ-paths which initiate at v and terminate at some element of W.

We say that Γ is a *tree* if $\mathrm{V}\Gamma \neq \emptyset$ and, for all $v, w \in \mathrm{V}\Gamma$, there exists a unique reduced Γ-path from v to w. We say that Γ is *connected* if, for all $v, w \in \mathrm{V}\Gamma$, there exists a Γ-path from v to w. By a *component* of Γ, we mean a maximal nonempty, connected subgraph of Γ. Thus, Γ equals the disjoint union of its components. We say that Γ is a *forest* if each component of Γ is a tree. It may be seen that Γ is a forest if and only if every reduced, closed Γ-path has length zero.

Let G be a group. We say that a graph is a (left or right) (multiplicative or exponential) *G-graph* if both the vertex-set and the edge-set are G-sets, and the initial and terminal incidence functions are G-maps.

4.2. **Definitions** (Schreier(1927), modified by Serre(1977))**.** Let Γ be a nonempty connected graph.

Let Γ_0 be a maximal subtree of Γ. By Zorn's lemma, $\mathrm{V}\Gamma_0 = \mathrm{V}\Gamma$. For any vertices u, v of Γ, we let $\Gamma_0[u,v]$ denote the Γ-label of the unique reduced Γ_0-path from u to v; notice that $\Gamma_0[v,v] = 1$, and that $\Gamma_0[v,u]{\cdot}\Gamma_0[u,w] = \Gamma_0[v,w]$, for each $u \in \mathrm{V}\Gamma$. Set $\mathrm{E}'\Gamma := \mathrm{E}\Gamma - \mathrm{E}\Gamma_0$.

Let v_0 be a Γ-vertex, chosen to serve as a basepoint. Let $\pi(\Gamma, v_0)$ denote the set consisting of the Γ-labels of the closed Γ-paths based at v_0. It is not difficult to see that $\pi(\Gamma, v_0)$ is a subgroup of $\mathbf{F}(\mathrm{E}\Gamma)$.

We have four group morphisms:

$\mathbf{F}(\mathrm{E}'\Gamma) \xrightarrow{\text{embed}} \mathbf{F}(\mathrm{E}\Gamma)$, $g \mapsto g^{\text{embed}}$, sends each $e \in \mathrm{E}'\Gamma$ to $e \in \mathrm{E}\Gamma$;

$\mathbf{F}(\mathrm{E}\Gamma) \xrightarrow{\text{deflate}} \mathbf{F}(\mathrm{E}'\Gamma)$, $g \mapsto g^{\text{deflate}}$, sends each $e \in \mathrm{E}\Gamma_0$ to 1, each $e \in \mathrm{E}'\Gamma$ to $e \in \mathrm{E}'\Gamma$;

$\pi(\Gamma, v_0) \xrightarrow{\text{include}} \mathbf{F}(\mathrm{E}\Gamma)$, $g \mapsto g^{\text{include}}$, is the inclusion map;

$\mathbf{F}(\mathrm{E}\Gamma) \xrightarrow{\text{inflate}} \pi(\Gamma, v_0)$, $g \mapsto g^{\text{inflate}}$, sends each $e \in \mathrm{E}\Gamma$ to $e^{\text{inflate}} := \Gamma_0[v_0, \iota e]{\cdot}e{\cdot}\Gamma_0[\tau e, v_0]$.

We shall show that the two composite maps

$$\pi(\Gamma, v_0) \xrightarrow{\text{include}} \mathbf{F}(\mathrm{E}\Gamma) \xrightarrow{\text{deflate}} \mathbf{F}(\mathrm{E}'\Gamma) \quad \text{and} \quad \mathbf{F}(\mathrm{E}'\Gamma) \xrightarrow{\text{embed}} \mathbf{F}(\mathrm{E}\Gamma) \xrightarrow{\text{inflate}} \pi(\Gamma, v_0)$$

are mutually inverse.

For each $e \in \mathrm{E}'\Gamma$,

$$e^{\text{deflate} \circ \text{embed} \circ \text{inflate}} = e^{\text{embed} \circ \text{inflate}} = \Gamma_0[v_0, \iota e]{\cdot}e{\cdot}\Gamma_0[\tau e, v_0],$$

and, also,

$$e^{\text{embed} \circ \text{inflate} \circ \text{include} \circ \text{deflate}} = (\Gamma_0[v_0, \iota e]{\cdot}e{\cdot}\Gamma_0[\tau e, v_0])^{\text{include} \circ \text{deflate}} = 1{\cdot}e{\cdot}1 = e,$$

which verifies one of the identity maps.

For each $e \in \mathrm{E}\Gamma_0$,

$$e^{\text{deflate} \circ \text{embed} \circ \text{inflate}} = 1^{\text{embed} \circ \text{inflate}} = 1 = \Gamma_0[v_0, \iota e]\cdot\Gamma_0[\iota e, \tau e]\cdot\Gamma_0[\tau e, v_0] = \Gamma_0[v_0, \iota e]\cdot e\cdot\Gamma_0[\tau e, v_0].$$

We have now seen that, for each $e \in \mathrm{E}\Gamma^{\pm 1}$, $e^{\text{deflate} \circ \text{embed} \circ \text{inflate}} = \Gamma_0[v_0, \iota e]\cdot e\cdot\Gamma_0[\tau e, v_0]$. An arbitrary element of $\pi(\Gamma, v_0)$ equals $e_1\cdot e_2 \cdots e_\ell$ for some Γ-path

$$v_0 \xrightarrow{e_1} v_1 \xrightarrow{e_2} \cdots \xrightarrow{e_\ell} v_\ell = v_0,$$

and then

$$\begin{aligned}(e_1\cdot e_2 \cdots e_\ell)^{\text{include} \circ \text{deflate} \circ \text{embed} \circ \text{inflate}} &= (e_1\cdot e_2 \cdots e_\ell)^{\text{deflate} \circ \text{embed} \circ \text{inflate}} \\ &= \Gamma_0[v_0, v_0]\cdot e_1\cdot\Gamma_0[v_1, v_0]\cdot\Gamma_0[v_0, v_1]\cdot e_2 \cdots e_\ell\cdot\Gamma_0[v_\ell, v_0] \\ &= 1\cdot e_1\cdot 1\cdot e_2 \cdots 1\cdot e_\ell = e_1\cdot e_2 \cdots e_\ell,\end{aligned}$$

which verifies the other identity map.

We have now proved that $\pi(\Gamma, v_0) = \langle\, \mathrm{E}'\Gamma | \emptyset \rangle = \langle\, \mathrm{E}\,\Gamma |\, \mathrm{E}\,\Gamma_0 \rangle$ with the map $\mathrm{E}\,\Gamma \to \pi(\Gamma, v_0)$, $e \mapsto \Gamma_0[v_0, \iota e]\cdot e\cdot\Gamma_0[\tau e, v_0]$. In particular, $\pi(\Gamma, v_0)$ is a free group. □

5. Building trees out of free groups

5.1. **Notation** (after Schreier(1927)). Let G be any group, E be any family of elements of G, and V be any right-multiplicative G-set. The *Schreier graph for V with respect to E*, denoted $V \curvearrowleft E$, is the graph with vertex-set V and edge-set $V \times E$, in which each edge (v, e) has initial vertex v and terminal vertex $v\cdot e$. The $(V \curvearrowleft E)$-paths $(v, (v, e), v\cdot e)$ and $(v\cdot e, (v, e)^{-1}, v)$ are depicted as $v \xrightarrow{\cdot e} v\cdot e$ and $v\cdot e \xrightarrow{\cdot e^{-1}} v$ respectively. We sometimes identify $\mathrm{E}^{\pm 1}(V \curvearrowleft E)$ with $V \times (E^{\pm 1})$.

A $(V \curvearrowleft E)$-path p will sometimes be depicted in the form

$$v_0 \xrightarrow{\cdot e_1} v_1 \xrightarrow{\cdot e_2} v_2 \rightarrow \cdots \rightarrow v_{\ell-1} \xrightarrow{\cdot e_\ell} v_\ell$$

for a unique $E^{\pm 1}$-sequence $(e_1, e_2, \ldots, e_\ell)$, called the *$E^{\pm 1}$-label* of p. Then p is a reduced $(V \curvearrowleft E)$-path if and only the right $E^{\pm 1}$-label of p is a reduced $E^{\pm 1}$-sequence. We define the *Schreier-label* of p to be $e_1\cdot e_2 \cdots e_\ell \in \langle E \mid \emptyset \rangle$. Notice that $v_0\cdot(e_1\cdot e_2 \cdots e_\ell) = v_\ell$. For each $v \in V$, $(V \curvearrowleft E)$-Paths(v, V) is mapped onto $\langle E \mid \emptyset \rangle$ by taking Schreier-labels, and a path initiating at v with Schreier-label g terminates at $v\cdot g$.

For any subset W of V, we write $W \curvearrowleft E$ to denote the *full subgraph of $V \curvearrowleft E$ with vertex-set W*; recall that this means that the edge-set of $W \curvearrowleft E$ consists of all $(V \curvearrowleft E)$-edges whose initial and terminal vertices lie in W.

For any subgroup H of F, one may form $(H \backslash F) \curvearrowleft E$. If $H \trianglelefteq F$, then $(H \backslash F) \curvearrowleft E$ is also called the *Cayley graph for the group $H \backslash F$ with respect to E*.

The Schreier-labels of $(F \backslash F) \curvearrowleft E$ agree with the $\big((F \backslash F) \curvearrowleft E\big)$-labels of Definitions 4.1, and taking Schreier-labels of $(V \curvearrowleft E)$-paths is interchangeable with applying the natural graph morphism from $V \curvearrowleft E$ to $(F \backslash F) \curvearrowleft E$. □

5.2. **Theorem** (Dehn). *For any free group F and any F-basis E, the graph $F \curvearrowleft E$ is a tree.*

Proof (Fox(1953), streamlined by Dicks(1980)). Set $T := F \curvearrowleft E$. To reduce the number of parentheses, for each $(v, e) \in F \times E = \mathrm{E}\,T$, let us set $v \otimes e := (v, e)$; thus, $\iota(v \otimes e) = v$ and $\tau(v \otimes e) = v\cdot e$.

Clearly, T is nonempty.

Let $\sim$ denote the inclusion-minimal equivalence relation on $\mathrm{V}T$ such that, for each T-edge $v{\otimes}e$, we have $\iota(v{\otimes}e) \sim \tau(v{\otimes}e)$. For each $v \in \mathrm{V}T$, let $[v]$ denote the $\sim$-equivalence class of v; then, for each $e \in E$, we have $[v] = [v{\cdot}e]$. There exists a left-multiplicative-F-set isomorphism between the set of components of T and the set of $\sim$-equivalence classes. For each $e \in E$, we have $[1] = [1{\cdot}e] = [e{\cdot}1] = e{\cdot}[1]$. Hence, the F-stabilizer of $[1]$ includes E, which generates F. Thus, for all $g \in F$, $[1] = g{\cdot}[1] = [g]$. Hence, $[1] = F$. Thus, T is connected.

For each set S, we let $\mathbb{Z}[S]$ denote the additive abelianization of $\mathbf{F}(S)$. The maps $\iota, \tau : \mathrm{E}T \to \mathrm{V}T$ induce group morphisms $\hat{\iota}, \hat{\tau} : \mathbb{Z}[\mathrm{E}T] \to \mathbb{Z}[\mathrm{V}T]$. If T were not a tree, there would exist some positive-length, reduced, closed T-path p; then p would have some positive-length, closed subpath p' which traverses each T-edge at most once; and then the abelianization map $\mathbf{F}(\mathrm{E}T) \to \mathbb{Z}[\mathrm{E}T]$ would carry the T-label of p' to a nonzero element of the kernel of $\hat{\tau}{-}\hat{\iota}$. Thus, to show that T is a tree, it suffices to show that $\hat{\tau}{-}\hat{\iota}$ is injective. In a natural way, $\mathbb{Z}[\mathrm{E}T]$ is a left-multiplicative F-set, and we may form the semi-direct-product group $\left(\begin{smallmatrix} F & \mathbb{Z}[\mathrm{E}T] \\ \{0\} & \{1\} \end{smallmatrix}\right)$ with matrix-style multiplication, wherein each element $\left(\begin{smallmatrix} a & b \\ 0 & 1 \end{smallmatrix}\right)$ is denoted $\lceil a, b \rceil$. Since E is an F-basis, there exists a unique group morphism $F \to \left(\begin{smallmatrix} F & \mathbb{Z}[\mathrm{E}T] \\ \{0\} & \{1\} \end{smallmatrix}\right)$, $g \mapsto \lceil \varphi g, \alpha g \rceil$, such that $\lceil \varphi e, \alpha e \rceil = \lceil e, 1{\otimes}e \rceil$ for each $e \in E$. For all v, $g \in F$,

$$\lceil \varphi(v{\cdot}g), \alpha(v{\cdot}g) \rceil = \lceil \varphi v, \alpha v \rceil {\cdot} \lceil \varphi g, \alpha g \rceil = \lceil (\varphi v){\cdot}(\varphi g), (\varphi v)(\alpha g) + \alpha v \rceil.$$

The map $\varphi : F \to F$ is an identity map, since $\varphi e = e$ and $\varphi(v{\cdot}g) = (\varphi v){\cdot}(\varphi g)$. The map $\alpha : F \to \mathbb{Z}[\mathrm{E}T]$ satisfies $\alpha e = 1{\otimes}e$ and $\alpha(v{\cdot}g) = (\varphi v){\cdot}(\alpha g) + \alpha v$. Thus, we have a map $\alpha : \mathrm{V}T \to \mathbb{Z}[\mathrm{E}T]$ such that, for each $v{\otimes}e \in \mathrm{E}T$,

$$\alpha\big(\tau(v{\otimes}e)\big) - \alpha\big(\iota(v{\otimes}e)\big) = \alpha(v{\cdot}e) - \alpha(v) = (\varphi v){\cdot}(\alpha e) = (v){\cdot}(1{\otimes}e) = v{\otimes}e.$$

Now α induces a $\mathbb{Z}$-module morphism $\hat{\alpha} : \mathbb{Z}[\mathrm{V}T] \to \mathbb{Z}[\mathrm{E}T]$, and the composite

$$\mathbb{Z}[\mathrm{E}T] \xrightarrow{\hat{\tau}-\hat{\iota}} \mathbb{Z}[\mathrm{V}T] \xrightarrow{\hat{\alpha}} \mathbb{Z}[\mathrm{E}T]$$

is the identity map on $\mathbb{Z}[\mathrm{E}T]$, since it carries each $v{\otimes}e \in \mathrm{E}T$ to itself. Hence, $\hat{\tau}{-}\hat{\iota}$ is injective, as desired. □

5.3. **Definitions** (Dyck(1882))**.** With Notation 5.1, for each straight word x in F, there exists some reduced $E^{\pm1}$-sequence $(e_1, e_2, \ldots, e_\ell)$ for x. Here,

$$1 \xrightarrow{\cdot e_1} e_1 \xrightarrow{\cdot e_2} e_1{\cdot}e_2 \rightarrow \cdots \rightarrow e_1{\cdot}e_2 \cdots e_{\ell-1} \xrightarrow{\cdot e_\ell} e_1{\cdot}e_2 \cdots e_{\ell-1}{\cdot}e_\ell = x$$

is a reduced $(F \curvearrowleft E)$-path from 1 to x, which is unique by Theorem 5.2. Thus, $(e_1, e_2, \ldots, e_\ell)$ is unique. We set E-length$(x) := \ell$. For each $e \in E^{\pm1}$, we set

$$\#(e \in x; E) := \big|\{ i \in \{1, 2, \ldots, \ell\} : e_i \in \{e\}^{\pm1} \}\big|.$$

For each positive integer i, we set $x[i;E] := e_i$ if $i \leqslant \ell$, and we set $x[i;E] := 1$ if $i > \ell$; thus, $x[i;E] \in \{1\} \cup E^{\pm1}$.

For each cyclic word x in F, there exists some cyclically reduced $E^{\pm1}$-sequence $(e_1, \ldots, e_\ell)$ for x. In Example 5.4 below, we shall see that $(e_1, \ldots, e_\ell)$ is then unique up to cyclic permutation. We set E-length$(x) := \ell$. For each $e \in E^{\pm1}$, we set

$$\#(e \in x; E) := \big|\{ i \in \{1, 2, \ldots, \ell\} : e_i \in \{e\}^{\pm1} \}\big|.$$

For each finite set X of words in F, we set E-length$(X) := \sum\limits_{x \in X} E$-length$(x)$ and, for each $e \in E$, we set $\#(e \in X; E) := \sum\limits_{x \in X} \#(e \in x; E)$.

Often, when X is either a word in F or a finite set or words in F, and E is clear from the context, we set $\#(e \in X) := \#(e \in X; E)$, and $x[i] := x[i; E]$. Then $\#(e^{-1} \in X) = \#(e \in X)$ and $E\text{-length}(X) := \sum_{e \in E} \#(e \in X)$. □

5.4. **Example.** Let F be any free group, E be any F-basis, and V be the right-exponential F-set which is the set F itself with the conjugation action $V \times F \to V$, $(v, g) \mapsto v^g = g^{-1} \cdot v \cdot g$. We wish to describe the graph $V{\curvearrowleft}E$.

Consider any $v \in V$.

Define $\hbar(v) := E$-length(v). We then have a map $\hbar : V \to \mathbb{N}$, and we envision it as a sort of topographical height-function on $V{\curvearrowleft}E$.

By *the v-component*, we shall mean the $(V{\curvearrowleft}E)$-component which contains v. The vertex-set of the v-component is the F-conjugacy class of v.

If $|E| = 0$, then $(V{\curvearrowleft}E)$ is a topological point, with exactly one vertex and no edges.

If $|E| = 1$, then the v-component is a topological circle, with exactly one vertex and one edge.

Let us now consider the case where $|E| \geqslant 2$. If $v = 1$, then the v-component has exactly one vertex and $|E|$ edges. Let us now consider the case where $v \neq 1$.

If the reduced $E^{\pm 1}$-sequence for v is cyclically reduced, then there exist exactly two length-one $(V{\curvearrowleft}E)$-paths initiating at v and terminating in a $(V{\curvearrowleft}E)$-vertex which is $\hbar$-equal to v; here, all the other length-one $(V{\curvearrowleft}E)$-paths initiating at v terminate in $(V{\curvearrowleft}E)$-neighbours which are $\hbar$-greater than v, by two.

If the reduced $E^{\pm 1}$-sequence for v is not cyclically reduced, then v has exactly one $\hbar$-smaller $(V{\curvearrowleft}E)$-neighbour, by two, and all the other $(V{\curvearrowleft}E)$-neighbours are $\hbar$-greater, by two.

It follows that the v-component consists of some trees attached to a topological circle whose vertex-set is given by the set of distinct cyclic permutations of some cyclically reduced $E^{\pm 1}$-sequence; all reduced paths leading out of the circle are $\hbar$-increasing.

One consequence is that, for the reduced $E^{\pm 1}$-sequences of the elements of a cyclic word in F, the ones that are cyclically reduced are all the same up to cyclic permutation.

By an *$\hbar$-valley*, we mean a $(V{\curvearrowleft}E)$-path which is the concatenation of three subpaths along which $\hbar$ is decreasing on the first, constant on the second, and increasing on the third; any of the three subpaths may have length zero. If two $(V{\curvearrowleft}E)$-vertices are joined by a $(V{\curvearrowleft}E)$-path, then they are joined by an $\hbar$-valley. Similar behaviour will be a recurring theme in these notes. □

5.5. **Historical notes** (excerpted from Chandler & Magnus(1982))**.** Dyck(1882) initiated the study of free groups, using the terminology *die allgemeinste Gruppe aus m erzeugenden Operationen*. He asserted that each element of a free group with basis E has a unique reduced $E^{\pm 1}$-sequence, and gave an intuitive argument; the first rigorous published proof was given by Artin(1926), that proof being joint work with Schreier. Max Dehn, who was a mentor of both Nielsen and Magnus, invented the terminology *freie*[1] *Gruppe*, and constructed the tree of Theorem 5.2 above, by using Dyck's result. Nielsen(1921 and 1924) introduced Dehn's terminology and tree into the literature. □

[1] Excerpted from the Oxford English Dictionary: The Teutonic word *frei* originally meant 'beloved', whence *Freund*, but its sense became altered when it was employed to distinguish family members from slaves in households. The converse process occurred in Latin where *liberi*, literally the 'free' members of the household, came to mean 'children'.

6. Subgroups of free groups

6.1. **Definitions** (Schreier(1927), modified by Serre(1977))**.** Recall Notation 5.1, and let F be a free group, E be an F-basis, W be a right-multiplicative F-set, and Γ be a connected full subgraph of $W\curvearrowleft E$. In particular, $(\mathrm{V}\Gamma)\curvearrowleft E = \Gamma$.

Let v_0 be a Γ-vertex chosen to be a basepoint of Γ, and let $\overline{\pi}(\Gamma, v_0)$ denote the set consisting of the Schreier-labels of the closed Γ-paths based at v_0. It is not difficult to see that $\overline{\pi}(\Gamma, v_0)$ is a subgroup of F. Let Γ_0 be a maximal subtree of Γ, and set $\mathrm{E}'\Gamma := \mathrm{E}\Gamma - \mathrm{E}\Gamma_0$. For each $v \in \mathrm{V}\Gamma$, denote by $\overrightarrow{v} \in F$ the Schreier-label of the reduced Γ_0-path from v_0 to v, and set $\overleftarrow{v} := \overrightarrow{v}^{-1}$, which is the Schreier-label of the reduced Γ_0-path from v to v_0. For each $(v, e) \in \mathrm{E}\,\Gamma \subseteq \mathrm{V}\Gamma \times E$, set $\overset{\to\gets}{(v, e)} := \overrightarrow{v} \cdot e \cdot \overleftarrow{v{\cdot}e} \in F$; as $\overset{\to\gets}{(v, e)}$ is the Schreier-label of a closed Γ-path based at v_0, we see that $\overset{\to\gets}{(v, e)} \in \overline{\pi}(\Gamma, v_0)$. If $(v, e) \in \mathrm{E}\,\Gamma_0$, then $\overset{\to\gets}{(v, e)}$ is the Schreier-label of a closed Γ_0-path based at v_0, and, hence, is trivial. Set

$$\overrightarrow{\mathrm{V}\Gamma} := \{\overrightarrow{v} : v \in \mathrm{V}\Gamma\} \subseteq F \quad \text{and} \quad \overset{\to\gets}{\mathrm{E}'\Gamma} := \{\, \overset{\to\gets}{(v, e)} : (v, e) \in \mathrm{E}'\Gamma\} \subseteq \overline{\pi}(\Gamma, v_0) \leqslant F.$$

It is not difficult to see that the map $\mathrm{V}\Gamma \to \overrightarrow{\mathrm{V}}\Gamma$ is bijective, and that $\overline{\pi}(\Gamma, v_0){\cdot}\overrightarrow{\mathrm{V}\Gamma}$ is the set consisting of the Schreier-labels of the Γ-paths initiating at v_0.

We claim that $\overset{\to\gets}{\mathrm{E}'\Gamma}$ is a basis of $\overline{\pi}(\Gamma, v_0)$, and that the map $\mathrm{E}'\Gamma \to \overset{\to\gets}{\mathrm{E}'\Gamma}$ is bijective. Here $\mathrm{E}\Gamma \subseteq \mathrm{V}\Gamma \times E$, and the second-coordinate map $\mathrm{E}\Gamma \to E$ induces a group morphism $\langle\, \mathrm{E}\Gamma \mid \emptyset\rangle \to F$, $g \mapsto \overline{g}$, called the *Schreier-label map*. Recall from Definitions 4.2 that $\pi(\Gamma, v_0)$ is a free subgroup of $\langle\, \mathrm{E}\Gamma \mid \emptyset\rangle$ with basis the family $(\,\Gamma_0[v_0, \iota e]{\cdot}e{\cdot}\Gamma_0[\tau e, v_0] : e \in \mathrm{E}'\Gamma)$. Each reduced Γ-path may be reconstructed from its Γ-label in $\langle\, \mathrm{E}\Gamma \mid \emptyset\rangle$, and also from its initial vertex together with its Schreier-label in F. It follows that the map $\pi(\Gamma, v_0) \to \overline{\pi}(\Gamma, v_0)$, $g \mapsto \overline{g}$, is an isomorphism, and the claim is proved.

The free subgroup $\overline{\pi}(\Gamma, v_0)$ of F will be very important in our study. In the case where Γ is finite, the algorithm for constructing a $\overline{\pi}(\Gamma, v_0)$-basis amounts to choosing a maximal subtree and reading the Schreier-label of the path that travels in the subtree from the basepoint to the initial vertex of each edge outside the tree, traverses said edge, and travels in the subtree back to the basepoint.

We now wish to obtain more information about the free subgroup $\overline{\pi}(\Gamma, v_0)$ of F. Recall that $F_{v_0} := \{g \in F : v_0{\cdot}g = v_0\}$. Set $\widetilde{\mathrm{V}\Gamma} := \{g \in F : v_0{\cdot}g \in \mathrm{V}\Gamma\}$; then $\widetilde{\mathrm{V}\Gamma}$ is a left-multiplicative F_{v_0}-subset of F which contains 1. Thus, $\widetilde{\mathrm{V}\Gamma}\curvearrowleft E$ is a left-multiplicative F_{v_0}-subforest of the left-multiplicative F-tree $F\curvearrowleft E$. Let T denote the component of $\widetilde{\mathrm{V}\Gamma}\curvearrowleft E$ which contains the vertex 1. Recall that $F_{(v_0,T)} := \{g \in F_{v_0} : g{\cdot}T = T\}$. We wish to show that $\overline{\pi}(\Gamma, v_0) = F_{(v_0,T)}$. Since $1 \in \mathrm{V}T$, we have $F_{(v_0,T)} \subseteq \{g \in F_{v_0} : g \in \mathrm{V}T\}$. Since the action of F_{v_0} permutes the components of $\widetilde{\mathrm{V}\Gamma}\curvearrowleft E$, we see that $F_{(v_0,T)} \supseteq \{g \in F_{v_0} : g \in \mathrm{V}T\}$, and then equality holds. The map $\widetilde{\mathrm{V}\Gamma} \to \mathrm{V}\Gamma$, $g \mapsto v_0{\cdot}g$, induces a graph map $\widetilde{\mathrm{V}\Gamma}\curvearrowleft E \to (\mathrm{V}\Gamma)\curvearrowleft E = \Gamma$. For each $g \in F$, we have $v_0{\cdot}g \in \mathrm{V}\Gamma$ if and only if $g \in \widetilde{\mathrm{V}\Gamma}$; further, for each $e \in E^{\pm 1}$, $v_0{\cdot}g{\cdot}e \in \mathrm{V}\Gamma$ if and only if $g{\cdot}e \in \widetilde{\mathrm{V}\Gamma}$. Thus the induced map $\mathrm{link}_{\widetilde{\mathrm{V}\Gamma}\curvearrowleft E}(g) \to \mathrm{link}_{\Gamma}(v_0{\cdot}g)$ is bijective. Hence, the set of Schreier-labels of the set of reduced Γ-paths which initiate at v_0 equals the set of Schreier-labels of the set of reduced T-paths which initiate at 1; in other words, $\overline{\pi}(\Gamma, v_0){\cdot}\overrightarrow{\mathrm{V}\Gamma} = \mathrm{V}T$. For each $g \in F$, we have $g \in \overline{\pi}(\Gamma, v_0)$ if and only if g is the Schreier-label of a reduced Γ-path from v_0 to v_0, and this happens if and only if $v_0{\cdot}g = v_0$ and $g \in \mathrm{V}T$. It follows that $\overline{\pi}(\Gamma, v_0) = F_{(v_0,T)}$, as claimed.

Consider any $g \in \mathrm{V}T$. We have $v_0{\cdot}g \in \mathrm{V}\Gamma$ and $\overrightarrow{v_0{\cdot}g} \in F_{v_0}{\cdot}g \subseteq \mathrm{V}T$. The element $\overrightarrow{v_0{\cdot}g}$ of $F_{v_0}{\cdot}g$ is called the *Schreier representative of* $F_{v_0}{\cdot}g$. Let $(e_1, e_2, \ldots, e_\ell)$ be the reduced $E^{\pm 1}$-expression for g. Since $\Gamma = \mathrm{V}\Gamma \curvearrowleft E$, the reduced Γ-path initiating at v_0 and having Schreier-label $(e_1, e_2, \ldots, e_\ell)$ is

$$v_0 \xrightarrow{(v_0,e_1)} v_0{\cdot}e_1 \xrightarrow{(v_0\cdot e_1,e_2)} v_0{\cdot}e_1{\cdot}e_2 \rightarrow \cdots \rightarrow v_0{\cdot}g.$$

For each $i \in \{1, 2, \ldots, \ell\}$, set $v_i := v_0{\cdot}e_1{\cdot}e_2{\cdot}e_\ell$. We have then written our Γ-path as

$$v_0 \xrightarrow{(v_0,e_1)} v_1 \xrightarrow{(v_1,e_2)} v_2 \rightarrow \cdots \rightarrow v_\ell,$$

and $v_\ell = v_0$ if and only if $g \in F_{v_0}$. The foregoing may be viewed as an algorithm for deciding if a given element of $\mathrm{V}T$ lies in F_{v_0}. If $g \in F_{g_0}$, then, in F,

$$\overset{\rightarrow\leftarrow}{(v_0, e_1)} \cdot \overset{\rightarrow\leftarrow}{(v_1, e_2)} \cdots \overset{\rightarrow\leftarrow}{(v_{\ell-1}, e_\ell)} = (\overrightarrow{v_0}{\cdot}e_1{\cdot}\overleftarrow{v_1}){\cdot}(\overrightarrow{v_1}{\cdot}e_2{\cdot}\overleftarrow{v_2})\cdots(\overrightarrow{v_{\ell-1}}{\cdot}e_\ell{\cdot}\overleftarrow{v_\ell}) = \overrightarrow{v_0}{\cdot}e_1{\cdot}e_2\cdots e_\ell{\cdot}\overleftarrow{v_\ell} = 1{\cdot}g{\cdot}1.$$

On omitting those $\overset{\rightarrow\leftarrow}{(v_{i-1}, e_i)}$ such that $(v_{i-1}, e_i) \in \mathrm{E}\Gamma_0$, we obtain the reduced $\overset{\rightarrow\leftarrow}{\mathrm{E}'\Gamma}^{\pm 1}$-sequence for g. Converting the $E^{\pm 1}$-sequence of an element of $\mathrm{V}T$ into its $\overset{\rightarrow\leftarrow}{\mathrm{E}'\Gamma}^{\pm 1}$-sequence is called *Schreier rewriting.* □

6.2. **Schreier's Theorem.** *Let F be any free group, E be any F-basis, and H be any subgroup of F. Set $\Gamma := (H\backslash F) \curvearrowleft E$ and $v_0 := H{\cdot}1 \in \mathrm{V}\Gamma$. Then $\overline{\pi}(\Gamma, v_0) = H$. In particular, H is a free group, and, for each maximal subtree Γ_0 of Γ, there exists an H-basis $\mathrm{E}\Gamma - \mathrm{E}\Gamma_0 \to H$.*

Proof. With Definitions 6.1, we take $W := H\backslash F$. Then $\widetilde{\mathrm{V}\Gamma} = F$, $T = \widetilde{\mathrm{V}\Gamma} \curvearrowleft E = F \curvearrowleft E$, and $\overline{\pi}(\Gamma, v_0) = F_{v_0,T} = F_{v_0} = H$. □

In particular, we have the following; see Historical notes 6.5 below.

6.3. **The Nielsen-Schreier Theorem.** *Subgroups of free groups are free.* □

6.4. **Definitions.** Let G be any group, $\langle\, E \mid R \,\rangle$ be any presentation for G, and H be any subgroup of G. We shall now describe the Reidemeister-Schreier presentation for H.

Set $F := \mathbf{F}(E)$, let N denote the kernel of the implicitly given map $\varphi : F \to G$, and let us identify $G = \langle\, E \mid R \,\rangle = F/N$. Here, $N = \langle\, {}^{F}R \,\rangle \leqslant F$. Since $G = N\backslash F$, we see that G and $H\backslash G$ are right-multiplicative F-sets. Set $\Gamma := (H\backslash G) \curvearrowleft E$ and $v_0 := H{\cdot}1 \in \mathrm{V}\Gamma$, let Γ_0 be any maximal subtree of Γ, and set $\mathrm{E}'\Gamma := \mathrm{E}\Gamma - \mathrm{E}\Gamma_0$.

Using Definitions 6.1, we now proceed to prove that $H = \langle\, \overset{\rightarrow\leftarrow}{\mathrm{E}'\Gamma} \mid {}^{\overrightarrow{\mathrm{V}\Gamma}}R \,\rangle$. This is called *the Reidemeister-Schreier presentation for H corresponding to the presentation $\langle\, E \mid R \,\rangle$ for G.* It depends on the choice of Γ_0.

Set $\widetilde{H} := \{f \in F : \varphi(f) \in H\}$; thus, $\widetilde{H}$ is a subgroup of F which includes N, and $\widetilde{H}/N = H$. As right-multiplicative F-sets, $\widetilde{H}\backslash F \simeq (\widetilde{H}/N)\backslash(F/N) = H\backslash G$. Thus, we may identify Γ with $(\widetilde{H}\backslash F) \curvearrowleft E$. By Theorem 6.2, $\overset{\rightarrow\leftarrow}{\mathrm{E}'\Gamma}$ is an $\widetilde{H}$-basis.

We claim that $\widetilde{H}{\cdot}\overrightarrow{\mathrm{V}\Gamma} = F$. Consider any $f \in F$, and set $v := \widetilde{H}{\cdot}f \in \mathrm{V}\Gamma$. Since $\overrightarrow{v}$ is the Schreier-label of the Γ_0-path from $\widetilde{H}{\cdot}1$ to $\widetilde{H}{\cdot}f$, we have $(\widetilde{H}{\cdot}1){\cdot}\overrightarrow{v} = \widetilde{H}{\cdot}f$. Now $f \in \widetilde{H}{\cdot}f = \widetilde{H}{\cdot}\overrightarrow{v} \subseteq \widetilde{H}{\cdot}\overrightarrow{\mathrm{V}\Gamma}$, as desired.

Since $N = \langle {}^{F}R\rangle$ and $F = \widetilde{H}\cdot\overrightarrow{\mathrm{V}\Gamma}$, we see that the normal closure of ${}^{\overrightarrow{\mathrm{V}\Gamma}}R$ in $\widetilde{H}$ is N. Hence, $H = \widetilde{H}/N = \langle\, \overset{\rightarrow\leftarrow}{\mathrm{E}'\Gamma} \mid {}^{\overrightarrow{\mathrm{V}\Gamma}}R\,\rangle$, as desired. By Definitions 6.1, Schreier rewriting converts the $E^{\pm1}$-sequence of each element of ${}^{\overrightarrow{\mathrm{V}\Gamma}}R$ into its $\overset{\rightarrow\leftarrow}{\mathrm{E}'\Gamma}^{\,\pm1}$-sequence, and this is an important aspect of a presentation.

Notice that in the case where $\langle\, E \mid R\,\rangle$ is a finite presentation for G and H is a finite-index subgroup of G, the graph Γ is finite, and $\langle\, \overset{\rightarrow\leftarrow}{\mathrm{E}'\Gamma} \mid {}^{\overrightarrow{\mathrm{V}\Gamma}}R\,\rangle$ is a finite presentation for H. □

6.5. **Historical notes.** Nielsen(1921) proved that finitely generated subgroups of free groups are free, by giving a practical algebraic algorithm for passing from a finite generating set to a basis; the algorithm is visible in the proof of Theorem 7.2 below. Stallings(1983) gave a practical graph-theoretic algorithm for passing from a finite generating set to the finite, connected, basepointed graph from which Schreier read off a basis.

Let G be any group, $\langle\, E \mid R\,\rangle$ be any presentation for G, and H be any subgroup of G. In connection with his seminal work on knot theory, Reidemeister(1927) produced by direct calculation a finite presentation for H in the case where H is a finite-index, normal subgroup of G, and both E and R are finite. Schreier(1927) used Dehn trees to refine the argument, and gave an explicit presentation for H in the general case. He called the resulting rewriting procedure 'das Reidemeistersche Verfahren', while Reidemeister(1932) called it 'der Ersetzungsvorschriften nach Schreier'. The first consequence that Schreier noticed was that subgroups of free groups are free. According to Chandler & Magnus(1982), Dehn said that he had always known this result because connected subgraphs of trees are trees. □

7. Nielsen's generating set for Aut F

7.1. **Notation.** Let E be a *finite* set and write $F := \mathbf{F}(E)$.

When we write, say, $(e_1 \leftrightarrow e_2,\ e_3 \mapsto e_1^{-1}\cdot e_3\cdot e_4)\mathrm{wrt}E$ with $\{e_1, e_2, e_3, e_4\} \subseteq E^{\pm1}$, we shall mean the unique element of $\mathrm{Aut}\,F$ which sends e_1 to e_2, e_2 to e_1, e_3 to $e_1^{-1}\cdot e_3\cdot e_4$, and each element of $E-\{e_1, e_2, e_3\}^{\pm1}$ to itself; it will be implicit that such an automorphism exists. The letters 'wrt' are to be read as 'with respect to'. Similar notation will be used for any free group and any basis thereof.

Let $\mathrm{Sym}(E^{\pm1})$ denote the group of all permutations of $E^{\pm1}$, acting right-exponentially. Let $\mathrm{Sym}_{\pm}(E^{\pm1})$ denote the centralizer of the inversion permutation in $\mathrm{Sym}(E^{\pm1})$; thus, any $\varphi \in \mathrm{Sym}(E^{\pm1})$ will lie in $\mathrm{Sym}_{\pm}(E^{\pm1})$ if and only if $(e^{-1})^{\varphi} = (e^{\varphi})^{-1}$ for all $e \in E^{\pm1}$. Each element of $\mathrm{Sym}_{\pm}(E^{\pm1})$ extends to a unique element of $\mathrm{Aut}\,F$, and we view the resulting group morphism as an embedding; that is, $\mathrm{Sym}_{\pm}(E^{\pm1})$ will be viewed as a subgroup of $\mathrm{Aut}\,F$. It is well known that $\{(e \leftrightarrow f)\mathrm{wrt}E : e, f \in E^{\pm1}, e \neq f\}$ is a generating set for $\mathrm{Sym}_{\pm}(E^{\pm1})$. □

7.2. **Theorem** (Nielsen(1919))**.** *With* Notation 7.1*, set*

$$\mathfrak{E}_0 := \big\{(e{\leftrightarrow}f)\mathrm{wrt}E \mid e, f \in E^{\pm1}, e \neq f\big\},$$
$$\mathfrak{E}_1 := \big\{(e \mapsto e\cdot f)\mathrm{wrt}E \mid e, f \in E^{\pm1}, \{e\}^{\pm1} \neq \{f\}^{\pm1}\big\},$$

and $\mathfrak{E} := \mathfrak{E}_0 \cup \mathfrak{E}_1$*. Then* $\mathfrak{E}$ *generates* $\mathrm{Aut}\,F$*.*

Proof (Nielsen, essentially)*.* We use the notation of Definitions 5.3, and when X is an element of F or a finite subset of F, we set $||X|| := E\text{-length}(X)$. For a finite subset X of F, a positive integer i, and an element e of $E^{\pm1}$, we write $\#(e,X,i) := \big|\{x\in X : x[i]=e\}\big|$. We fix an arbitrary total order $\succ$ on $E^{\pm1}$, and, for $e_1,\, e_2\in E^{\pm1}$, we say that e_1 *is* $\succ$*-greater than* e_2 if $e_1\succ e_2$. For finite subsets X and Y of F, we say that X *is length-lexicographically greater than* Y, and write $X\sqsupset Y$, if either $||X||>||Y||$ or all of the following hold: $||X||=||Y||$; there exist some positive integer i and some $e\in E^{\pm1}$ such that $\#(e,X,i)\ \neq\ \#(e,Y,i)$, and, if i_0 denotes the least such i and e_0 denotes the $\succ$-greatest e such that $\#(e,X,i_0)\ \neq\ \#(e,Y,i_0)$, then $\#(e_0,X,i_0)\ >\ \#(e_0,Y,i_0)$. If X is not length-lexicographically greater than Y, we write $X\not\sqsupset Y$. It is straightforward to check that the relation $\sqsupset$ is transitive; it is clearly anti-reflexive.

Let φ be an arbitrary element of $\operatorname{Aut}F$, and set $X := (E^{\pm1})^{\varphi}$.

For each X-sequence (x,y) such that $\{x\}^{\pm1}\neq\{y\}^{\pm1}$ and $\{x\}^{\pm1}\sqsupset\{x{\cdot}y\}^{\pm1}$, we say that the set $Y := (X{-}\{x\}^{\pm1})\cup\{x{\cdot}y\}^{\pm1}$ is *a Nielsen successor of* X. Notice that $X\sqsupset Y$; also, if we set $e := x^{\varphi^{-1}}$ and $f := y^{\varphi^{-1}}$, then $e,f\in E^{\pm1}$, $\{e\}^{\pm1}\neq\{f\}^{\pm1}$, and

$$Y = \big((E^{\pm1}{-}\{e\}^{\pm1})\cup\{e{\cdot}f\}^{\pm1}\big)^{\varphi} = (E^{\pm1})^{\left((e\mapsto e\cdot f)\,\mathrm{wrt}\,E\right)\circ\varphi}.$$

Since $\mathfrak{E}_0$ generates $\operatorname{Sym}_{\pm}(E^{\pm1})$, to show that $\mathfrak{E}$ generates $\operatorname{Aut}F$ it now suffices to show that there exists some finite sequence $X_0,X_1,\ldots,X_\ell$ such that $X_0=X$, $X_\ell=E^{\pm1}$, and, for each $i\in\{0,1,\ldots,\ell{-}1\}$, X_{i+1} is a Nielsen successor of X_i.

Consider any finite sequence $X=X_0,X_1,\ldots,X_\ell$ such that, for each $i\in\{0,1,\ldots,\ell{-}1\}$, X_{i+1} is a Nielsen successor of X_i. Since $X_0\sqsupset X_1\sqsupset\cdots\sqsupset X_\ell$, the X_i are all distinct and $||X||=||X_0||\geqslant||X_1||\geqslant\cdots\geqslant||X_\ell||$. Thus, there are only finitely many possible options for ℓ, and we may assume that our ℓ is the greatest possible option. By replacing X with X_ℓ, we may assume that X has no Nielsen successors, and it suffices to show that $X=E^{\pm1}$. Now, for each X-sequence (x,y), if $\{x\}^{\pm1}\neq\{y\}^{\pm1}$, then $\{x\}^{\pm1}\not\sqsupset\{x{\cdot}y\}^{\pm1}$; also, $\{x\}^{\pm1}\not\sqsupset\{y{\cdot}x\}^{\pm1}$, since we may replace (x,y) with (x^{-1},y^{-1}).

For any F-sequence $(g_1,g_2,\ldots,g_\ell)$, we define $g_1{*}g_2{*}\cdots{*}g_\ell\in F\sqcup\{0\}$ by

$$g_1{*}g_2{*}\cdots{*}g_\ell := \begin{cases} g_1{\cdot}g_2{\cdot}\cdots{\cdot}g_\ell & \text{if } ||g_1{\cdot}g_2{\cdot}\cdots{\cdot}g_\ell|| = ||g_1||+||g_2||+\cdots+||g_\ell||,\\ 0 & \text{otherwise.}\end{cases}$$

Suppose that (x,y) is some length-two, reduced X-sequence. There exists a unique length-three F-sequence (a,b,c) such that $x=a{*}b^{-1}$, $y=b{*}c^{-1}$, and $x{\cdot}y=a{*}c^{-1}$. We call b the $E^{\pm1}$*-cancellation in* (x,y). We claim that $||a||\geqslant||b||$. Suppose not; that is, $||b||>||a||$, and, hence, $||y||>||x{\cdot}y||$. It is not difficult to see that $x\neq y$. Now $\{y\}^{\pm1}\sqsupset\{x{\cdot}y\}^{\pm1}$, and this is a contradiction, as desired. Hence, $||a||\geqslant\frac12||x||\geqslant||b||$ and $a\neq1$. With (y^{-1},x^{-1}) in place of (x,y), this says that $||c||\geqslant\frac12||y||\geqslant||b||$ and $c\neq1$.

Suppose that (x,y,z) is some length-three, reduced X-sequence. Let b and c denote the $E^{\pm1}$-cancellation in (x,y) and (y,z) respectively, and set $y' := b^{-1}{\cdot}y{\cdot}c$. We claim that $y=b{*}y'{*}c^{-1}$ and $y'\neq1$. Set $a := x{\cdot}b$ and $d := z^{-1}{\cdot}c$. Then

$$x=a{*}b^{-1},\ \ y=b{*}(y'{\cdot}c^{-1})=(b{\cdot}y'){*}c^{-1},\ \ z=c{*}d^{-1},\ \ x{\cdot}y=a{*}(y'{\cdot}c^{-1}),\ \ y{\cdot}z=(b{\cdot}y'){*}d^{-1},$$

and, by the previous paragraph, $||y'{\cdot}c^{-1}||\geqslant\frac12||y||\geqslant||c||$. It follows that $y'{\cdot}c^{-1}=y'{*}c^{-1}$. Hence, $y=b{*}y'{*}c^{-1}$. It remains to show that $y'\neq1$. We suppose that $y'=1$, and derive a contradiction. We now have

$$x=a{*}b^{-1},\quad y=b{*}c^{-1},\ \ z=c{*}d^{-1},\quad x{\cdot}y=a{*}c^{-1},\quad y{\cdot}z=b{*}d^{-1},$$

and, by the previous paragraph, $||a||\geqslant||b||\leqslant||c||$ and $||b||\geqslant||c||\leqslant||d||$. In particular, $||b||=||c||$. Since $y\neq1$, we have $b\neq c$, and then $y\neq x$ and $y\neq z$. Since $||a||\geqslant||b||=||c||$,

it is not difficult to see that if $\{b\} \sqsupset \{c\}$, then
$$\{x\}^{\pm 1} = \{a{*}b^{-1}, b{*}a^{-1}\} \sqsupset \{a{*}c^{-1}, c{*}a^{-1}\} = \{x{\cdot}y\}^{\pm 1}.$$
This is a contradiction. Hence, $\{c\} \sqsupset \{b\}$. Since $||d|| \geqslant ||b|| = ||c||$, we see that
$$\{z\}^{\pm 1} = \{c{*}d^{-1}, d{*}c^{-1}\} \sqsupset \{b{*}d^{-1}, d{*}b^{-1}\} = \{y{\cdot}z\}^{\pm 1}.$$
This is a contradiction, and the claim is proved.

Consider any $e \in E^{\pm 1}$. There exists a (unique) reduced X-sequence $(x_1, x_2, \ldots, x_\ell)$ for e. Here, $\ell \geqslant 1$. Set $b_0 := b_\ell := 1$, and, for each $i \in \{1, 2, \ldots, \ell{-}1\}$, let b_i denote the $E^{\pm 1}$-cancellation in (x_i, x_{i+1}). For each $i \in \{1, 2, \ldots, \ell\}$, set $x'_i := b_{i-1}^{-1}{\cdot}x_i{\cdot}b_i$. For each $i \in \{2, 3, \ldots, \ell{-}1\}$, the previous paragraph applies to (x_{i-1}, x_i, x_{i+1}) and shows that $x_i = b_{i-1}{*}x'_i{*}b_i^{-1}$ and $x'_i \neq 1$. These assertions hold also for $i = 1$ and for $i = \ell$, by the penultimate paragraph. Now $e = x_1 x_2 \cdots x_\ell = x'_1{*}x'_2{*}\cdots{*}x'_\ell$. We see that $1 \leqslant \ell \leqslant ||x'_1{*}x'_2{*}\cdots{*}x'_\ell|| = ||e|| = 1$. Hence, $\ell = 1$ and $e = x_1 \in X$. Thus, $E^{\pm 1} \subseteq X$, as desired. □

7.3. **Historical note.** Chandler & Magnus(1982) discuss some proofs in the literature prior to Nielsen's that are not rigorous. □

We shall not use the following interesting result, and we leave its proof as an exercise.

7.4. **Corollary** (Nielsen). *For any e_1, $e_2 \in E$ with $e_1 \neq e_2$,*
$$\big\{(e_1{\leftrightarrow}f)\mathrm{wrt}E : f \in \big(E{-}\{e_1\}\big) \cup \{e_1^{-1}\}\big\} \;\cup\; \big\{(e_2 \leftrightarrow e_2^{-1}, e_1 \mapsto e_1{\cdot}e_2)\mathrm{wrt}E\big\}$$
is a set of $|E|{+}1$ order-two automorphisms of F which generates $\mathrm{Aut}\,F$. □

Of course, if $|E| \leqslant 1$, then $|E|$ order-two automorphisms suffice.

8. Whitehead's generating family for Aut F

8.1. **Notation.** We use Notation 7.1. Whitehead(1936) expanded on Nielsen's study of $\mathrm{Aut}\,F$, and we change his starting point into a finite generating family for $\mathrm{Aut}\,F$. Set
$$\mathcal{E}_0 := \{(d, f) \in E^{\pm 1}{\times}E^{\pm 1} : d \neq f\},$$
$$\mathcal{E}_1 := \{(\mathbf{v}, e) : \mathbf{v} \subseteq \{1\} \cup E^{\pm 1} \text{ and } e \in \mathbf{v}{-}\mathbf{v}^{-1}\},$$
$$\mathcal{E} := \mathcal{E}_0 \sqcup \mathcal{E}_1 \text{ and } \mathcal{F} := \langle \mathcal{E} \mid \emptyset \rangle.$$
Throughout, each $(d, f) \in \mathcal{E}_0$ will be denoted $(d \leftrightarrow f)$, and each $(\mathbf{v}, e) \in \mathcal{E}_1$ will be denoted $\left[\begin{smallmatrix}\mathbf{v}\\ e\end{smallmatrix}\right]$. For each subset $\mathbf{v}$ of $\{1\} \cup E^{\pm 1}$, we write $\mathbf{v}^* := (\{1\} \cup E^{\pm 1}) - \mathbf{v}$. We now let each elemen of $\mathcal{E}$ act right-exponentially on F as an automorphism as follows. We let each $(d \leftrightarrow f) \in \mathcal{E}_0$ act as $(d \leftrightarrow f)\mathrm{wrt}E$. For each $\left[\begin{smallmatrix}\mathbf{v}\\ e\end{smallmatrix}\right] \in \mathcal{E}_1$, we take $(\mathbf{v}_0, e_0) \in \big\{(\mathbf{v}, e),\ (\mathbf{v}^*, e^{-1})\big\}$ such that $1 \in \mathbf{v}_0$ and, for each $f \in E^{\pm 1}$, we set
$$(8.1.1)\quad f^{\left[\begin{smallmatrix}\mathbf{v}\\ e\end{smallmatrix}\right]} := \begin{cases} f^{-1} & \text{if } f \in \{e\}^{\pm 1},\\ e_0^{|\{f\}-\mathbf{v}_0|}{\cdot}f{\cdot}e_0^{-|\{f^{-1}\}-\mathbf{v}_0|} & \text{if } f \in E^{\pm 1}{-}\{e\}^{\pm 1}.\end{cases}$$
In this way, F becomes a right-exponential $\mathcal{F}$-set, and we have a group morphism $\mathcal{F} \to \mathrm{Aut}\,F$; we let $\mathcal{N}$ denote its kernel, and view the quotient $\mathcal{F}/\mathcal{N}$ as a subgroup of $\mathrm{Aut}\,F$. For elements φ_1 and φ_2 of $\mathcal{F}$, we write $\varphi_1 \equiv \varphi_2$ to mean that φ_1 and φ_2 have the same action on F, or, equivalently, $\varphi_1{\cdot}\mathcal{N} = \varphi_2{\cdot}\mathcal{N}$. For any $e, f \in E^{\pm 1}$, such that $\{e\}^{\pm 1} \neq \{f\}^{\pm 1}$, we see from (8.1.1) that $\left[\begin{smallmatrix}\{e,f\}\\ e\end{smallmatrix}\right]{\cdot}\mathcal{N} = (e \leftrightarrow e^{-1},\ f \mapsto e^{-1}{\cdot}f)\mathrm{wrt}E$. It follows from Nielsen's Theorem 7.2 that
$$(8.1.2)\quad \mathcal{F}/\mathcal{N} = \mathrm{Aut}\,F. \qquad \square$$

8.2. **Historical notes.** The action of $\mathcal{E}_1$ on F is presented differently by different authors. Rapaport, clarifying Whitehead's exposition, spoke of automorphisms in which, for some $e \in E^{\pm 1}$, each $f \in E$ is carried to an element of $\{f,\ f{\cdot}e,\ e^{-1}{\cdot}f,\ e^{-1}{\cdot}f{\cdot}e\}$. For an element e of $E^{\pm 1}$ and a subset $\mathbf{v}$ of $E^{\pm 1}$, Higgins & Lyndon denoted by $(\mathbf{v}, e)$ the *unique* automorphism which sends each $f \in \mathbf{v}$ to an element of $\{f{\cdot}e,\ e^{-1}{\cdot}f{\cdot}e\}$ and sends each $f \in E^{\pm 1} - \mathbf{v}$ to an element of $\{f,\ e^{-1}{\cdot}f\}$. Hoare composed $(\mathbf{v}, e)$ with $(e \leftrightarrow e^{-1})wrtE$, and denoted the result by $(\mathbf{v}, e)$; it is Hoare's automorphisms which we use, with different notation.

One of Rapaport's innovations was to use results about cyclic words to obtain results about straight words, by adding a new variable to the basis. This method has been used to this day, but a quick computation revealed that, in our case, the resulting action on E is that given by (8.1.1) above, where the new variable has now been reduced to a marker called 1. This is in keeping with the spirit of Whitehead's careful analysis of basepoints. We shall use a result of Gersten(1984a) to obtain all the advantages that Rapaport obtained by adding a variable and passing to cyclic words. □

9. Statements of McCool's finite presentations

9.1. **Notation.** We use Notation 8.1. In particular,

$$\begin{aligned}
&\mathcal{E}_0 := \{(d \leftrightarrow f) : d,\ f \in E^{\pm 1},\ d \neq f\},\\
&\mathcal{E}_1 := \{\left[\begin{smallmatrix}\mathbf{v}\\ e\end{smallmatrix}\right] : \mathbf{v} \subseteq \{1\} \cup E^{\pm 1} \text{ and } e \in \mathbf{v} - \mathbf{v}^{-1}\},\\
&\mathcal{E} := \mathcal{E}_0 \sqcup \mathcal{E}_1 \text{ and } \mathcal{F} := \langle \mathcal{E} \mid \emptyset \rangle.
\end{aligned}$$

For each $n \in \mathbb{N}$, we write $\mathcal{P}_n(E)$ to denote the set of all n-tuples $(\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n)$ such that $\mathbf{v}_1 \cup \mathbf{v}_2 \cup\ \cdots \cup \mathbf{v}_n = \{1\} \cup E^{\pm 1}$ and $\mathbf{v}_i \cap \mathbf{v}_j = \emptyset$ for all $i,\ j \in \{1, 2, \ldots, n\}$ with $i \neq j$; thus, the set of those $\mathbf{v}_i$ which are nonempty is a partition of $\{1\} \cup E^{\pm 1}$. To simplify the exposition, we shall abuse notation and use a phrase such as "for all $(\mathbf{v}_1 \overset{e_1}{\rightsquigarrow} \mathbf{v}_2,\ \mathbf{v}_3 \overset{e_4}{\longleftarrow} \mathbf{v}_4) \in \mathcal{P}_4(E)$" to mean "for all $(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4) \in \mathcal{P}_4(E)$, all $e_1 \in \mathbf{v}_1 \cap \mathbf{v}_2^{-1}$, and all $e_4 \in \mathbf{v}_4 \cap \mathbf{v}_3^{-1}$". For each $\left[\begin{smallmatrix}\mathbf{v}\\ e\end{smallmatrix}\right] \in \mathcal{E}_1$, we may thus write $(\mathbf{v} \overset{e}{\rightsquigarrow} \mathbf{v}^*) \in \mathcal{P}_2(E)$. The elements of $\mathcal{P}_2(E)$, $\mathcal{P}_3(E)$, and $\mathcal{P}_4(E)$ give the bulk of our generators, relators, and proofs for $\operatorname{Aut} F$, respectively.

The image of $\mathcal{E}_0$ in $\operatorname{Aut} F$ is a generating set for the finite subgroup $\operatorname{Sym}_{\pm}(E^{\pm 1})$. Let $\mathcal{R}_0$ be any finite subset of $\mathbf{F}(\mathcal{E}_0)$ such that $\mathcal{R}_0 \supseteq \{(d \leftrightarrow f)^2 : (d \leftrightarrow f) \in \mathcal{E}_0\}$ and such that $\langle \mathcal{E}_0 \mid \mathcal{R}_0 \rangle = \operatorname{Sym}_{\pm}(E^{\pm 1})$ with the natural map $\mathcal{E}_0 \to \operatorname{Sym}_{\pm}(E^{\pm 1})$. Set

$$\begin{aligned}
\mathcal{R}_1 &:= \{\left[\begin{smallmatrix}\mathbf{v}\\ e\end{smallmatrix}\right]^2 : \left[\begin{smallmatrix}\mathbf{v}\\ e\end{smallmatrix}\right] \in \mathcal{E}_1\},\\
\mathcal{R}_2 &:= \{\left[\begin{smallmatrix}\mathbf{v}^*\\ e^{-1}\end{smallmatrix}\right]{\cdot}\left[\begin{smallmatrix}\mathbf{v}\\ e\end{smallmatrix}\right]^{-1} : \left[\begin{smallmatrix}\mathbf{v}\\ e\end{smallmatrix}\right] \in \mathcal{E}_1\},\\
\mathcal{R}_3 &:= \{\left[\begin{smallmatrix}\mathbf{v}\\ e\end{smallmatrix}\right]{\cdot}(d{\leftrightarrow}f){\cdot}\left[\begin{smallmatrix}\mathbf{v}^{(d\leftrightarrow f)}\\ e^{(d\leftrightarrow f)}\end{smallmatrix}\right]{\cdot}(d{\leftrightarrow}f)^{-1} : (d{\leftrightarrow}f) \in \mathcal{E}_0,\ \left[\begin{smallmatrix}\mathbf{v}\\ e\end{smallmatrix}\right] \in \mathcal{E}_1\},\\
\mathcal{R}_4 &:= \{\left[\begin{smallmatrix}\mathbf{v}\\ e\end{smallmatrix}\right]{\cdot}(e \leftrightarrow e'){\cdot}\left[\begin{smallmatrix}\mathbf{v}\\ e\end{smallmatrix}\right]{\cdot}\left[\begin{smallmatrix}\mathbf{v}\\ e'\end{smallmatrix}\right]^{-1} : \left[\begin{smallmatrix}\mathbf{v}\\ e\end{smallmatrix}\right], \left[\begin{smallmatrix}\mathbf{v}\\ e'\end{smallmatrix}\right] \in \mathcal{E}_1, e' \neq e\},\\
\mathcal{R}_5 &:= \{\left[\begin{smallmatrix}\mathbf{u}\\ d\end{smallmatrix}\right]{\cdot}(d{\leftrightarrow}d^{-1}){\cdot}\left[\begin{smallmatrix}\mathbf{w} \cup \{d\}\\ d\end{smallmatrix}\right]{\cdot}\left[\begin{smallmatrix}\mathbf{v}\\ d^{-1}\end{smallmatrix}\right]^{-1} : (\mathbf{u} \overset{d}{\rightsquigarrow} \mathbf{v},\ \mathbf{w}) \in \mathcal{P}_3(E)\},\\
\mathcal{R}_6 &:= \{\left[\begin{smallmatrix}\mathbf{v}\\ e\end{smallmatrix}\right]{\cdot}\left[\begin{smallmatrix}\mathbf{u}\\ d\end{smallmatrix}\right]{\cdot}\left[\begin{smallmatrix}\mathbf{w}\cup\{d^{-1}\}\\ e^{-1}\end{smallmatrix}\right]{\cdot}\left[\begin{smallmatrix}\mathbf{u}\\ d\end{smallmatrix}\right]^{-1} : (\mathbf{u} \overset{d}{\rightsquigarrow} \mathbf{v} \overset{e}{\rightsquigarrow} \mathbf{w}) \in \mathcal{P}_3(E)\},\\
\mathcal{R}_7 &:= \{\left[\begin{smallmatrix}\mathbf{w}\\ f\end{smallmatrix}\right]{\cdot}\left[\begin{smallmatrix}\mathbf{u}\\ d\end{smallmatrix}\right]{\cdot}\left[\begin{smallmatrix}\mathbf{w}\\ f\end{smallmatrix}\right]{\cdot}\left[\begin{smallmatrix}\mathbf{u}\\ d\end{smallmatrix}\right]^{-1} : (\mathbf{u} \overset{d}{\rightsquigarrow} \mathbf{v} \overset{f}{\longleftarrow} \mathbf{w}) \in \mathcal{P}_3(E)\},\\
\mathcal{R}\ &:= \bigcup_{i=0}^{7} \mathcal{R}_i.
\end{aligned}$$

McCool(1974) proved that $\langle \mathcal{E} \mid \mathcal{R}\rangle = \operatorname{Aut} F$ with the natural map $\mathcal{E} \to \operatorname{Aut} F$; we shall give a proof over the course of these notes. $\square$

9.2. **Notation.** We use Notation 9.1. Let $\mathbb{W}$ denote the set of all finite sets of words in F, and consider an arbitrary $S \in \mathbb{W}$. Set $\mathbb{W}_{\leqslant S} := \{W \in \mathbb{W} : E\text{-length}(W) \leqslant E\text{-length}(S)\}$; thus, $\mathbb{W}_{\leqslant S}$ is a finite set which contains S. As F is a right-exponential $\mathcal{F}$-set, it follows that $\mathbb{W}$ is also a right-exponential $\mathcal{F}$-set in a natural way, and, as in Notation 5.1, we may form the graph $\mathbb{W}\curvearrowleft\mathcal{E}$ and its finite full subgraph $\mathbb{W}_{\leqslant S}\curvearrowleft\mathcal{E}$. Let $\mathbf{\Gamma}(S)$ denote that component of $\mathbb{W}_{\leqslant S}\curvearrowleft\mathcal{E}$ which contains S. Set $\Gamma := \mathbf{\Gamma}(S)$ and $v_0 := S \in \mathrm{V}\Gamma$. Then Γ is a finite, connected, basepointed graph which may be constructed algorithmically in theory - if rarely in practice. Let Γ_0 be any maximal subtree of Γ, and set $\mathrm{E}'\Gamma := \mathrm{E}\Gamma - \mathrm{E}\Gamma_0$. With Definitions 6.1, we then have subsets $\overrightarrow{\mathrm{V}\Gamma}$ and $\overset{\rightarrow\leftarrow}{\mathrm{E}'\Gamma}$ of $\mathcal{F}$. McCool(1975a) gave a description which we shall express as the finite presentation $\operatorname{Aut}(F,S) = \big\langle\, \overset{\rightarrow\leftarrow}{\mathrm{E}'\Gamma} \mid {}^{\overrightarrow{\mathrm{V}\Gamma}}\mathcal{R} \cap \langle\, \overset{\rightarrow\leftarrow}{\mathrm{E}'\Gamma}\,\rangle \,\big\rangle$ with the natural map $\overset{\rightarrow\leftarrow}{\mathrm{E}'\Gamma} \to \operatorname{Aut} F$. In Section 16 below, we prove this with Γ replaced with a subgraph that is slightly more complicated to explain. By Definitions 6.1, Schreier rewriting converts the $E^{\pm 1}$-sequence of each element of ${}^{\overrightarrow{\mathrm{V}\Gamma}}\mathcal{R} \cap \langle\, \overset{\rightarrow\leftarrow}{\mathrm{E}'\Gamma}\,\rangle$ into its $\overset{\rightarrow\leftarrow}{\mathrm{E}'\Gamma}^{\pm 1}$-sequence, and this is an important aspect of the presentation.

For each Γ-vertex v and each element ρ of $\mathcal{R}$, the element $\vec{v}\cdot\rho\cdot\overleftarrow{v}$ of ${}^{\overrightarrow{\mathrm{V}\Gamma}}\mathcal{R}$ lies in $\langle\, \overset{\rightarrow\leftarrow}{\mathrm{E}'\Gamma}\,\rangle$ if and only if the closed $(\mathbb{W}\curvearrowleft\mathcal{E})$-path based at v with Schreier-label ρ is a Γ-path. Algorithmically, we initiate at v and determine whether or not reading ρ as a Schreier-label gives a path that keeps the E-length below E-length(S); if not, we reach a point where the next $\mathcal{E}^{\pm 1}$-label takes the E-length too high. When $\rho \in \mathcal{R}_0$, the E-length is kept constant. Each element of $\mathcal{R}-\mathcal{R}_0$ has $\mathcal{E}$-length 2 or 4.

In the case where $S = \emptyset$, we have $\Gamma = \{S\}\curvearrowleft\mathcal{E}$, $\Gamma_0 = \{S\}$, $\overrightarrow{\mathrm{V}\Gamma} = \{1\}$, and $\overset{\rightarrow\leftarrow}{\mathrm{E}'\Gamma} = \{1\}\cdot\mathcal{E}\cdot\{1\}$. Here the presentation is $\operatorname{Aut}(F,\emptyset) = \langle\, \mathcal{E} \mid \mathcal{R}\,\rangle$, which is the presentation stated in Notation 9.1. $\square$

9.3. **Historical notes.** The finite presentation $\operatorname{Aut}(F,S) = \big\langle\, \overset{\rightarrow\leftarrow}{\mathrm{E}'\Gamma} \mid {}^{\overrightarrow{\mathrm{V}\Gamma}}\mathcal{R} \cap \langle\, \overset{\rightarrow\leftarrow}{\mathrm{E}'\Gamma}\,\rangle \,\big\rangle$ evolved through work of Nielsen(1919), Whitehead(1936), Rapaport(1958), Higgins & Lyndon(1962), McCool(1974, 1975a), and Hoare(1979).

One may replace $\mathbb{W}$ with any right-exponential $\mathcal{F}$-set $\mathbb{W}'$ of a certain type. All one needs is that each element W of $\mathbb{W}'$ has an *underlying set* $U(W) \in \mathbb{W}$ such that the map $\mathbb{W}' \to \mathbb{W}$, $W \mapsto U(W)$, is finite-to-one, and satisfies $U(W^{\varphi}) = \big(U(W)\big)^{\varphi}$ for each $\varphi \in \mathcal{F}$. We define E-length$(W) := E$-length$\big(U(W)\big)$, and all of our arguments apply to $\mathbb{W}'$. To obtain a finite presentation for $\operatorname{Aut}(F,W)$, $W \in \mathbb{W}'$, one may alternatively view $\operatorname{Aut}(F,W)$ as a finite-index subgroup of $\operatorname{Aut}(F,U(W))$ and compute its Reidemeister-Schreier presentation.

Let us define an *E-valley* to mean a $(\mathbb{W}'\curvearrowleft\mathcal{E})$-path which is the concatenation of three subpaths along which the E-length is decreasing on the first, constant on the second, and increasing on the third; any of the three subpaths may have length zero.

Whitehead(1936) took each element of $\mathbb{W}'$ to be a finite sequence of straight words in F or a finite sequence of cyclic words in F, and his Theorem 3 shows that if two given elements of $\mathbb{W}'$ are joined by some $(\mathbb{W}'\curvearrowleft\mathcal{E})$-path, then they are joined by some E-valley. He gave a difficult topological proof. His result easily gives his algorithm which, with input a length-two $\mathbb{W}'$-sequence (W_1, W_2), decides whether or not the finite graph $\mathbf{\Gamma}(W_1) \cup \mathbf{\Gamma}(W_2)$ is connected, decides whether or not there exists some $\varphi \in \operatorname{Aut} F$ such that $W_1^{\varphi} = W_2$, and constructs such a φ if one exists. The graph denoted $\mathbf{\Gamma}(S)$ in Historical notes 1.2 above is the full subgraph of $\mathbb{W}\curvearrowleft\mathcal{E}$ with vertex-set $\{S^{\varphi} : \varphi \in \operatorname{Aut} F,\ E\text{-length}(S^{\varphi}) \leqslant E\text{-length}(S)\}$; in light of Whitehead's Theorem 3, this graph equals the graph denoted $\mathbf{\Gamma}(S)$ in Notation 9.2 above.

Rapaport(1958) took each element of $\mathbb{W}'$ to be a cyclic word in F or a finite sequence of straight words in F, and showed that if two possibly equal elements of $\mathbb{W}'$ are joined by some $(\mathbb{W}'\curvearrowleft\mathcal{E})$-path with Schreier-label $\varphi \in \mathcal{F}$, then they are joined by some E-valley with Schreier-label $\varphi' \in \varphi\cdot\mathcal{N}$. She gave a very difficult algebraic proof, using expressions involving 53 types of syllables. Rapaport's result easily implies that $\operatorname{Aut}(F, W)$ is finitely generable for each $W \in \mathbb{W}'$, a property that was first mentioned in the literature by McCool(1975a), who remarks that Lyndon had independently discovered it.

Higgins & Lyndon(1962) took each element of $\mathbb{W}'$ to be a finite set (possibly meaning sequence) of cyclic words in F, implicitly proved that $\mathcal{R} \subseteq \mathcal{N}$, and explicitly gave a relatively simple algebraic proof that if two given elements of $\mathbb{W}'$ are joined by some $(\mathbb{W}'\curvearrowleft\mathcal{E})$-path, then they are joined by some E-valley.

McCool(1974) took each element of $\mathbb{W}'$ to be a cyclic word in F or a finite sequence of straight words in F, refined the argument of Higgins & Lyndon, and proved that if two possibly equal elements of $\mathbb{W}'$ are joined by some $(\mathbb{W}'\curvearrowleft\mathcal{E})$-path with Schreier-label $\varphi \in \mathcal{F}$, then they are joined by some E-valley with Schreier-label $\varphi' \in \varphi\cdot\langle {}^{\mathcal{F}}\mathcal{R}\rangle$. From this, he deduced the finite presentation $\operatorname{Aut} F = \langle \mathcal{E} | \mathcal{R}\rangle$. Then McCool(1975b) deduced the finite presentation of $\operatorname{Aut} F$ that had been obtained by Nielsen(1924). Gersten(1984b) simplified McCool's presentation of $\operatorname{Aut} F$.

McCool(1975a) took each element of $\mathbb{W}'$ to be a finite sequence of cyclic words in F or a finite set of cyclic words in F, proved that, for $W \in \mathbb{W}'$, $\operatorname{Aut}(F, W)$ is finitely presentable, and described a finite presentation in terms of a finite two-dimensional CW-complex.

Hoare(1979) generalized McCool's results to the case of finite sequences of finite sets of words in F, and simplified the proofs. $\square$

10. Gersten's graph-theoretic description of the action of $\mathcal{E}_1$ on F

10.1. **Notation.** Let E, F, $\mathcal{E}$, and $\mathcal{F}$ be as in Notation 9.1, and let S be a finite set of words in F. Consider any $\left[\begin{smallmatrix}\mathbf{v}\\ e\end{smallmatrix}\right] \in \mathcal{E}_1$. Let t be a new variable. Set $\widetilde{E} := E \sqcup \{t\}$, $\widetilde{F} := \langle \widetilde{E} \mid \emptyset\rangle$, and $\Gamma_{\mathbf{v}} := \big\{\mathbf{v} \cup \{t\}, \mathbf{v}^* \cup \{t^{-1}\}\big\}$. For each $f \in \{1\} \cup \widetilde{E}^{\pm 1}$, let $[f]$ denote the element of $\Gamma_{\mathbf{v}}$ which contains f.

We view the two-element partition $\Gamma_{\mathbf{v}}$ of $\{1\} \cup \widetilde{E}^{\pm 1}$ as endowed with the structure of a two-vertex, basepointed, connected graph with edge-set $(\widetilde{E} - \{e^{-1}\}) \cup \{e\}$ and a distinguished maximal subtree, as follows. The two vertices are the elements of $\Gamma_{\mathbf{v}}$, the basepoint is $[1]$, each $\Gamma_{\mathbf{v}}$-edge $f \in (\widetilde{E} - \{e^{-1}\}) \cup \{e\}$ has initial vertex $[f]$ and terminal vertex $[f^{-1}]$, and the distinguished maximal tree has edge-set $\{t\}$. We sketch $\Gamma_{\mathbf{v}}$ as $\mathbf{v} \cup \{t\} \overset{t,e}{\rightsquigarrow} \mathbf{v}^* \cup \{t^{-1}\}$; the labels on the squiggly arrow indicate how two of the edges are to be attached. We set $\pi\Gamma_{\mathbf{v}} := \pi(\Gamma_{\mathbf{v}}, [1]) \leqslant \widetilde{F}$. Let deflate : $\widetilde{F} \to F$, $g \mapsto g^{\text{deflate}}$, denote the epimorphism which sends t to 1 and sends each element of E to itself. As $\{t\}$ is the edge-set of a maximal subtree of $\Gamma_{\mathbf{v}}$, the epimorphism deflate : $\widetilde{F} \to F$ restricts to an isomorphism $\pi\Gamma_{\mathbf{v}} \xrightarrow{\sim} F$. The inverse $F \xrightarrow{\sim} \pi\Gamma_{\mathbf{v}}$ extends to a monomorphism inflate : $F \to \widetilde{F}$, $g \mapsto g^{\text{inflate}}$. We shall now verify the graph-theoretic description given by Gersten(1984a) for the action of $\left[\begin{smallmatrix}\mathbf{v}\\ e\end{smallmatrix}\right]$:

(10.1.1) $\left[\begin{smallmatrix}\mathbf{v}\\ e\end{smallmatrix}\right]\cdot\mathcal{N} = \text{inflate}\circ(e\leftrightarrow t)\text{wrt}\widetilde{E}\circ\text{deflate}\,.$

Proof of (10.1.1). It is clear that the automorphism $(e\leftrightarrow t)\text{wrt}\widetilde{E}$ of $\widetilde{F}$ induces an automorphism on the subgroup $\pi\Gamma_{\mathbf{v}}$ of $\widetilde{F}$; hence, $\text{inflate}\circ(e\leftrightarrow t)\text{wrt}\widetilde{E}\circ\text{deflate}\in\operatorname{Aut}F$.

For each $f\in E^{\pm1}$, we have a $\Gamma_{\mathbf{v}}$-path

$$[1]\overset{t^{\eta}}{\rightsquigarrow}[f]\overset{f}{\rightsquigarrow}[f^{-1}]\overset{t^{\nu}}{\rightsquigarrow}[1]$$

for unique $\eta,\nu\in\{-1,0,1\}$; we then have $f^{\,\text{inflate}}=t^{\eta}\cdot f\cdot t^{\nu}$. Now

$$f^{\,\text{inflate}\circ(e\leftrightarrow t)\text{wrt}\widetilde{E}\circ\text{deflate}}=(t^{\eta}\cdot f\cdot t^{\nu})^{(e\leftrightarrow t)\text{wrt}\widetilde{E}\circ\text{deflate}}=e^{\eta}\cdot f^{(e\leftrightarrow t)\text{wrt}\widetilde{E}\circ\text{deflate}}\cdot e^{\nu}.$$

In the case where $f=e$, we have $e^{\,\text{inflate}\circ(e\leftrightarrow t)\text{wrt}\widetilde{E}\circ\text{deflate}}=e^{\eta+\nu}$.

We wish to show that (10.1.1) agrees with (8.1.1).

Consider first the case where $1\in\mathbf{v}$. Then $\mathbf{v}_0=\mathbf{v}$, $e_0=e$, $[1]=\mathbf{v}\cup\{t\}$, and either $f\in\mathbf{v}$, giving $\eta=0$, or $f\in\mathbf{v}^*$, giving $\eta=1$; in each case, $\eta=|\{f\}-\mathbf{v}_0|$. Similarly, $\nu=-|\{f^{-1}\}-\mathbf{v}_0|$. In particular, $e^{\,\text{inflate}\circ(e\leftrightarrow t)\text{wrt}\widetilde{E}\circ\text{deflate}}=e^{0+(-1)}$.

Consider next the case where $1\in\mathbf{v}^*$. Then $\mathbf{v}_0=\mathbf{v}^*$, $e_0=e^{-1}$, $[1]=\mathbf{v}^*\cup\{t^{-1}\}$, and either $f\in\mathbf{v}$ giving $\eta=-1$, or $f\in\mathbf{v}^*$ giving $\eta=0$; in each case, $\eta=-|\{f\}-\mathbf{v}_0|$. Similarly, $\nu=|\{f^{-1}\}-\mathbf{v}_0|$. In particular, $e^{\,\text{inflate}\circ(e\leftrightarrow t)\text{wrt}\widetilde{E}\circ\text{deflate}}=e^{-1+0}$. □

Notice that

(10.1.2) $\left[\begin{smallmatrix}\mathbf{v}\\ e\end{smallmatrix}\right]^2\equiv 1,$

since $\text{deflate}\circ\text{inflate}$ sends each element of $\pi\Gamma_{\mathbf{v}}$ to itself, and $(e\leftrightarrow t)^2\text{wrt}\widetilde{E}=1$. Thus each element of $\mathcal{E}$ acts as an order-two automorphism of F. Notice also that

(10.1.3) $\left[\begin{smallmatrix}\mathbf{v}^*\\ e^{-1}\end{smallmatrix}\right]\equiv\left[\begin{smallmatrix}\mathbf{v}\\ e\end{smallmatrix}\right]$

by (8.1.1); graph-theoretically, when we construct $\left[\begin{smallmatrix}\mathbf{v}^*\\ e^{-1}\end{smallmatrix}\right]$, we may take the name of the new symbol to be t^{-1}, and thereby arrange that $\Gamma_{\mathbf{v}^*}=\Gamma_{\mathbf{v}}$.

(10.1.4) For all $e'\in\mathbf{v}-\mathbf{v}^{-1}$, if $e'\neq e$, then $\left[\begin{smallmatrix}\mathbf{v}\\ e\end{smallmatrix}\right]^{\left[\begin{smallmatrix}\mathbf{v}\\ e'\end{smallmatrix}\right]}\equiv(e\leftrightarrow e')$,

since $(e\leftrightarrow t)^{(e'\leftrightarrow t)}\text{wrt}\widetilde{E}=(e\leftrightarrow e')\text{wrt}\widetilde{E}$.

For each $\varphi\in\mathcal{F}$, we define

(10.1.5) $$\Delta_{S,E}(\varphi):=E\text{-length}(S^{\varphi})-E\text{-length}(S).$$

For each straight word w in F, w^{inflate} is the $\Gamma_{\mathbf{v}}$-label of a reduced closed $\Gamma_{\mathbf{v}}$-path at $[1]$. For each cyclic word w in F, the image w^{inflate} in $\widetilde{F}$ is a subset of a unique cyclic word in $\widetilde{F}$, and, by convention, we redefine w^{inflate} to mean this cyclic word in $\widetilde{F}$; it is important for our purposes that w^{inflate} then contains the $\Gamma_{\mathbf{v}}$-label of a cyclically reduced closed $\Gamma_{\mathbf{v}}$-path. It is straightforward to see the following.

$$\begin{aligned}
\Delta_{S,E}(\left[\begin{smallmatrix}\mathbf{v}\\ e\end{smallmatrix}\right]) &\overset{(10.1.5)}{=} -E\text{-length}(S)+E\text{-length}(S^{\left[\begin{smallmatrix}\mathbf{v}\\ e\end{smallmatrix}\right]})\\
&= -E\text{-length}(S)+E\text{-length}\big(S^{\,\text{inflate}\circ(e\leftrightarrow t)\text{wrt}\widetilde{E}\circ\,\text{deflate}}\big)\\
&= -E\text{-length}(S)+\widetilde{E}\text{-length}\big(S^{\,\text{inflate}\circ(e\leftrightarrow t)\text{wrt}\widetilde{E}}\big)-\#(t\in S^{\,\text{inflate}\circ(e\leftrightarrow t)\text{wrt}\widetilde{E}};\widetilde{E})\\
&= -E\text{-length}(S)+\widetilde{E}\text{-length}\big(S^{\,\text{inflate}}\big)-\#(e\in S^{\,\text{inflate}};\widetilde{E})\\
&= \#(t\in S^{\,\text{inflate}};\widetilde{E})-\#(e\in S;E).
\end{aligned}$$

We define the important value $\delta_{S,E}(\mathbf{v}) := \#(t \in S^{\text{inflate}}; \widetilde{E})$, and obtain Whitehead's formula

(10.1.6) $\Delta_{S,E}(\left[\begin{smallmatrix}\mathbf{v}\\ e\end{smallmatrix}\right]) = \delta_{S,E}(\mathbf{v}) - \#(e \in S; E)$.

For each $f \in E^{\pm 1} - \{e\}^{\pm 1}$, it is clear from (8.1.1) that $\#(f \in S^{\left[\begin{smallmatrix}\mathbf{v}\\ e\end{smallmatrix}\right]}; E) \leqslant \#(f \in S; E)$. Of course, this inequality remains valid if S is replaced with $S^{\left[\begin{smallmatrix}\mathbf{v}\\ e\end{smallmatrix}\right]}$, and since $\left[\begin{smallmatrix}\mathbf{v}\\ e\end{smallmatrix}\right]^2 \overset{(10.1.2)}{\equiv} 1$, we see that equality holds.

(10.1.7) For all $f \in E^{\pm 1} - \{e\}^{\pm 1}$, $\#(f \in S^{\left[\begin{smallmatrix}\mathbf{v}\\ e\end{smallmatrix}\right]}; E) \;=\; \#(f \in S; E)$. $\square$

11. The six rules

The following is a slight variation on results of Higgins & Lyndon(1962).

11.1. **Theorem.** *With the notation of* Section 10 *and the abbreviations* $\Delta_S := \Delta_{S,E}$ *and* $\delta_S := \delta_{S,E}$*, the following hold.*

For all $(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4) \in \mathcal{P}_4(E)$,

(11.1.P1) $\delta_S(\mathbf{v}_1) + \delta_S(\mathbf{v}_2) \;\leqslant\; \delta_S(\mathbf{v}_1 \cup \mathbf{v}_3) + \delta_S(\mathbf{v}_1 \cup \mathbf{v}_4) \;\geqslant\; \delta_S(\mathbf{v}_3) + \delta_S(\mathbf{v}_4)$.

For all $(\mathbf{v}_1 \overset{e_1}{\rightsquigarrow} \mathbf{v}_2,\ \mathbf{v}_3,\ \mathbf{v}_4) \in \mathcal{P}_4(E)$,

(11.1.P2) $\left[\begin{smallmatrix}\mathbf{v}_3 \cup \mathbf{v}_1\\ e_1\end{smallmatrix}\right] \cdot \left[\begin{smallmatrix}\mathbf{v}_1\\ e_1\end{smallmatrix}\right] \cdot \left[\begin{smallmatrix}\mathbf{v}_1 \cup \mathbf{v}_4\\ e_1\end{smallmatrix}\right] \equiv \left[\begin{smallmatrix}\mathbf{v}_2\\ e_1^{-1}\end{smallmatrix}\right] \overset{(10.1.2)}{\equiv} \left[\begin{smallmatrix}\mathbf{v}_3 \cup \mathbf{v}_1 \cup \mathbf{v}_4\\ e_1\end{smallmatrix}\right]$.

For all $(\mathbf{v}_1 \overset{e_1}{\rightsquigarrow} \mathbf{v}_2,\ \mathbf{v}_3 \overset{e_3}{\rightsquigarrow} \mathbf{v}_4) \in \mathcal{P}_4(E)$*, the following hold.*

(11.1.P3) $\left[\begin{smallmatrix}\mathbf{v}_1\\ e_1\end{smallmatrix}\right]^{\left[\begin{smallmatrix}\mathbf{v}_3\\ e_3\end{smallmatrix}\right]} \equiv \left[\begin{smallmatrix}\mathbf{v}_1\\ e_1\end{smallmatrix}\right]$.

(11.1.P4) *If* $\mathbf{v}_4 = \{e_3^{-1}\}$*, then* $\left[\begin{smallmatrix}\mathbf{v}_1 \cup \mathbf{v}_3\\ e_1\end{smallmatrix}\right]^{\left[\begin{smallmatrix}\mathbf{v}_3\\ e_3\end{smallmatrix}\right]} \equiv \left[\begin{smallmatrix}\mathbf{v}_1 \cup \mathbf{v}_4\\ e_1\end{smallmatrix}\right]$.

(11.1.P5) $\Delta_S(\left[\begin{smallmatrix}\mathbf{v}_1\\ e_1\end{smallmatrix}\right] \cdot \left[\begin{smallmatrix}\mathbf{v}_3\\ e_3\end{smallmatrix}\right]) = \Delta_S(\left[\begin{smallmatrix}\mathbf{v}_1\\ e_1\end{smallmatrix}\right]) + \Delta_S(\left[\begin{smallmatrix}\mathbf{v}_3\\ e_3\end{smallmatrix}\right])$.

(11.1.P6) $\Delta_S\big(\left[\begin{smallmatrix}\mathbf{v}_1 \cup \mathbf{v}_3\\ e_1\end{smallmatrix}\right] \cdot \left[\begin{smallmatrix}\mathbf{v}_3\\ e_3\end{smallmatrix}\right]\big) = \Delta_S\big(\left[\begin{smallmatrix}\mathbf{v}_1 \cup \mathbf{v}_3\\ e_1\end{smallmatrix}\right]\big) + \Delta_S(\left[\begin{smallmatrix}\mathbf{v}_3\\ e_3\end{smallmatrix}\right])$.

This section is devoted to introducing notation used in the proof, and proving the six statements in order.

Fix $(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4) \in \mathcal{P}_4(E)$. Let t_1, t_2, t_3 be new symbols. Set $\widehat{F} := \langle E \cup \{t_1, t_2, t_3\} \mid \emptyset \rangle$, $t_{12} := t_1 \cdot t_2$, $t_{23} := t_2 \cdot t_3$, $t_{123} := t_1 \cdot t_2 \cdot t_3$, and

$$\Gamma := \Gamma_{(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4)} := \big\{\mathbf{v}_1 \cup \{t_1\},\ \mathbf{v}_2 \cup \{t_1^{-1}, t_2\},\ \mathbf{v}_3 \cup \{t_2^{-1}, t_3\},\ \mathbf{v}_4 \cup \{t_3^{-1}\}\big\}.$$

As in Notation 10.1, we give this four-element partition of $\{1\} \cup \big(E \cup \{t_1, t_2, t_3\}\big)^{\pm 1}$ the structure of a four-vertex, connected, basepointed graph with edge-set $E \cup \{t_1, t_2, t_3\}$, and distinguished maximal subtree with edge-set $\{t_1, t_2, t_3\}$. Again we have $\pi\Gamma \leqslant \widehat{F}$. One may depict Γ as

$$\mathbf{v}_1 \cup \{t_1\} \ \overset{t_1}{\rightsquigarrow}\ \mathbf{v}_2 \cup \{t_1^{-1}, t_2\} \ \overset{t_2}{\rightsquigarrow}\ \mathbf{v}_3 \cup \{t_2^{-1}, t_3\} \ \overset{t_3}{\rightsquigarrow}\ \mathbf{v}_4 \cup \{t_3^{-1}\};$$

the partition determines how the edges lying in E are attached. We shall sometimes write a meaningless expression such as

(11.1.7) $$\mathbf{v}_1 \ \overset{t_1}{\rightsquigarrow}\ \mathbf{v}_2 \ \overset{t_2}{\rightsquigarrow}\ \mathbf{v}_3 \ \overset{t_3}{\rightsquigarrow}\ \mathbf{v}_4,$$

and call it *a blueprint for* Γ, since it contains enough information to reconstruct Γ.

Let Deflate : $\widehat{F} \to F$, $g \mapsto g^{\text{Deflate}}$, denote the group morphism which sends each element of $\{t_1, t_2, t_3\}$ to 1 and each element of E to itself. As $\{t_1, t_2, t_3\}$ is the edge-set of a maximal subtree of Γ, the epimorphism $\mathrm{Deflate} : \widehat{F} \to F$ restricts to an isomorphism $\pi\Gamma \xrightarrow{\sim} F$. The inverse $F \xrightarrow{\sim} \pi\Gamma$ extends to a monomorphism $\mathrm{Inflate} : F \to \widehat{F}$, $g \mapsto g^{\mathrm{Inflate}}$.

Proof of (11.1.P1). Each element of S^{Inflate} may be represented by an alternating product of elements of $\{t_1, t_2, t_3, t_{12}, t_{23}, t_{123}\}^{\pm 1}$ and elements of $F - \{1\}$, where the alternating product is interpreted cyclically when representing cyclic words. For each subset J of $\{1, 2, 3, 12, 23, 123\}$, we write $t_J := \{t_j : j \in J\}$, we go through the elements of S^{Inflate} adding up the number of occurrences of elements of ${t_J}^{\pm 1}$, and we denote the resulting sum by $\#(t_J \in S^{\mathrm{Inflate}})$, which we then abbreviate to $\#t_J$.

Consider any subset I of $\{1, 2, 3, 4\}$, set $\mathbf{v} := \bigcup\limits_{i \in I} \mathbf{v}_i$, and let I' denote the set of those $j \in \{1, 2, 3, 12, 23, 123\}$ such that t_j joins $\mathbf{v}$ to $\mathbf{v}^*$ in the blueprint (11.1.7). It is not difficult to see that $\delta_S(\mathbf{v}) = \#t_{I'}$. We record the following.

(11.1.8) $\delta_S(\mathbf{v}_1) = \#t_{\{1,12,123\}}$. $\delta_S(\mathbf{v}_2) = \#t_{\{1,2,23\}}$. $\delta_S(\mathbf{v}_3) = \#t_{\{2,3,12\}}$. $\delta_S(\mathbf{v}_4) = \#t_{\{3,23,123\}}$.

(11.1.9) $\delta_S(\mathbf{v}_1 \cup \mathbf{v}_3) = \#t_{\{1,2,3,123\}}$. $\qquad \delta_S(\mathbf{v}_1 \cup \mathbf{v}_4) = \#t_{\{1,3,12,23\}}$.

Hence,

$$\begin{aligned} \delta_S(\mathbf{v}_1) + \delta_S(\mathbf{v}_2) &= \#t_{\{1,2,3,12,23,123\}} + \#t_{\{1\}} - \#t_{\{3\}}, \\ \delta_S(\mathbf{v}_3) + \delta_S(\mathbf{v}_4) &= \#t_{\{1,2,3,12,23,123\}} - \#t_{\{1\}} + \#t_{\{3\}}, \\ \delta_S(\mathbf{v}_1 \cup \mathbf{v}_3) + \delta_S(\mathbf{v}_1 \cup \mathbf{v}_4) &= \#t_{\{1,2,3,12,23,123\}} + \#t_{\{1\}} + \#t_{\{3\}}. \end{aligned}$$

Thus, (11.1.P1) holds. □

For the remainder of this section, we assume that we have $e_1 \in \mathbf{v}_1 \cap \mathbf{v}_2^{-1}$. Let Γ_2 denote the four-vertex, connected, basepointed graph which has as edge-set the $\widehat{F}$-basis $(E - \{e_1^{-1}\}) \cup \{e_1, t_1, t_2, t_3\}$ and has

$$\mathbf{v}_1 \overset{e_1,t_1}{\rightsquigarrow} \mathbf{v}_2 \overset{t_2}{\rightsquigarrow} \mathbf{v}_3 \overset{t_3}{\rightsquigarrow} \mathbf{v}_4$$

as a blueprint. Since the elements of the new basis correspond to Γ-paths, we see that $\pi\Gamma_2 = \pi\Gamma$. Set $t_0 := e_1^{-1} {\cdot} t_1$ and

$$\text{(11.1.10)} \qquad \begin{aligned} \widehat{\left[\begin{smallmatrix}\mathbf{v}_1\\ e_1\end{smallmatrix}\right]} :&= (e_1 \leftrightarrow t_1)\mathrm{wrt}(E \cup \{t_1, t_2, t_3\}) \\ &= (e_1 \leftrightarrow t_1)\mathrm{wrt}(E \cup \{t_1, t_2, t_{23}\}) \\ &= (t_0 \mapsto t_0^{-1})\mathrm{wrt}(E \cup \{t_0, t_3, t_{23}\}). \end{aligned}$$

Then $\widehat{\left[\begin{smallmatrix}\mathbf{v}_1\\ e_1\end{smallmatrix}\right]}$ comes from a graph automorphism of Γ_2, and hence induces an automorphism on the subgroup $\pi\Gamma_2$ of $\widehat{F}$. Since $\pi\Gamma_2 = \pi\Gamma$, we have $\mathrm{Inflate} \circ \widehat{\left[\begin{smallmatrix}\mathbf{v}_1\\ e_1\end{smallmatrix}\right]} \circ \mathrm{Deflate} \in \mathrm{Aut}\, F$. It may be seen that $\mathrm{Inflate} \circ \widehat{\left[\begin{smallmatrix}\mathbf{v}_1\\ e_1\end{smallmatrix}\right]} \circ \mathrm{Deflate} = \left[\begin{smallmatrix}\mathbf{v}_1\\ e_1\end{smallmatrix}\right] {\cdot} \mathcal{N}$, by deflating $\{t_2, t_3\}$ and setting $t := t_1$.

In the sequel, we shall tacitly understand that $\mathrm{Inflate} \circ \widehat{\left[\begin{smallmatrix}?\\ ?\end{smallmatrix}\right]} \circ \mathrm{Deflate} = \left[\begin{smallmatrix}?\\ ?\end{smallmatrix}\right] {\cdot} \mathcal{N}$ is true and easily verified.

We consider the blueprint $\mathbf{v}_2 \xleftarrow{e_1,t_1} \mathbf{v}_1 \overset{t_{12}}{\rightsquigarrow} \mathbf{v}_3 \overset{t_3}{\rightsquigarrow} \mathbf{v}_4$, and set

$$\text{(11.1.11)} \qquad \begin{aligned} \widehat{\left[\begin{smallmatrix}\mathbf{v}_2\\ e_1^{-1}\end{smallmatrix}\right]} :&= (e_1 \leftrightarrow t_1)\mathrm{wrt}\big(E \cup \{t_1, t_3, t_{12}\}\big) \\ &= (t_0 \mapsto t_0^{-1},\ t_2 \mapsto t_0 {\cdot} t_2,\ t_{23} \mapsto t_0 {\cdot} t_{23})\mathrm{wrt}(E \cup \{t_0, t_2, t_{23}\}). \end{aligned}$$

We consider the blueprint $\mathbf{v}_3 \xleftarrow{t_{12}} \mathbf{v}_1 \overset{e_1,t_1}{\rightsquigarrow} \mathbf{v}_2 \overset{t_{23}}{\rightsquigarrow} \mathbf{v}_4$, and set

$$(11.1.12)\qquad \left[\widehat{\begin{smallmatrix}\mathbf{v}_1\cup\mathbf{v}_3\\ e_1\end{smallmatrix}}\right] := (e_1 \leftrightarrow t_1)\mathrm{wrt}\big(E\cup\{t_1,\, t_{12},\, t_{23}\}\big)$$
$$= (t_0 \mapsto t_0^{-1},\;\, t_2 \mapsto t_0{\cdot}t_2)\mathrm{wrt}(E\cup\{t_0,\, t_2,\, t_{23}\}).$$

We consider the blueprint $\mathbf{v}_4 \xleftarrow{t_{123}} \mathbf{v}_1 \overset{e_1,t_1}{\rightsquigarrow} \mathbf{v}_2 \overset{t_2}{\rightsquigarrow} \mathbf{v}_3$, and set

$$(11.1.13)\qquad \left[\widehat{\begin{smallmatrix}\mathbf{v}_1\cup\mathbf{v}_4\\ e_1\end{smallmatrix}}\right] := (e_1 \leftrightarrow t_1)\mathrm{wrt}\big(E\cup\{t_1,\, t_2,\, t_{123}\}\big)$$
$$= (t_0 \mapsto t_0^{-1},\;\, t_{23} \mapsto t_0{\cdot}t_{23})\mathrm{wrt}(E\cup\{t_0, t_2, t_{23}\}),$$

Proof of (11.1.P2). By (11.1.12), (11.1.10), and (11.1.13),

$$\left[\widehat{\begin{smallmatrix}\mathbf{v}_1\cup\mathbf{v}_3\\ e_1\end{smallmatrix}}\right]\circ\left[\widehat{\begin{smallmatrix}\mathbf{v}_1\\ e_1\end{smallmatrix}}\right]\circ\left[\widehat{\begin{smallmatrix}\mathbf{v}_1\cup\mathbf{v}_4\\ e_1\end{smallmatrix}}\right]$$
$$= (t_0 \mapsto t_0^{-1},\;\, t_2 \mapsto t_0{\cdot}t_2)\circ(t_0 \mapsto t_0^{-1})\circ(t_0 \mapsto t_0^{-1}, t_{23} \mapsto t_0{\cdot}t_{23})\mathrm{wrt}\big(E\cup\{t_0,\, t_2,\, t_{23}\}\big)$$
$$= (t_0 \mapsto t_0^{-1},\;\, t_2 \mapsto t_0{\cdot}t_2,\;\, t_{23} \mapsto t_0{\cdot}t_{23})\mathrm{wrt}\big(E\cup\{t_0,\, t_2,\, t_{23}\}\big) \overset{(11.1.11)}{=} \left[\widehat{\begin{smallmatrix}\mathbf{v}_2\\ e_1^{-1}\end{smallmatrix}}\right].$$

Hence (11.1.P2) holds. □

We now assume that $e_3 \in \mathbf{v}_3 \cap \mathbf{v}_4^{-1}$. We consider the blueprint $\mathbf{v}_1 \overset{t_1}{\rightsquigarrow} \mathbf{v}_2 \overset{t_{23}}{\rightsquigarrow} \mathbf{v}_4 \xleftarrow{e_3,\,t_3} \mathbf{v}_3$, and set

$$(11.1.14)\qquad \left[\widehat{\begin{smallmatrix}\mathbf{v}_3\\ e_3\end{smallmatrix}}\right] := (e_3 \leftrightarrow t_3)\mathrm{wrt}\big(E\cup\{t_1,\, t_3,\, t_{23}\}\big)$$
$$= (e_3 \mapsto t_2^{-1}{\cdot}t_{23},\;\, t_2 \mapsto t_{23}{\cdot}e_3^{-1})\mathrm{wrt}\big(E\cup\{t_0,\, t_2,\, t_{23}\}\big).$$

Proof of (11.1.P3). In $\mathrm{Aut}\,\widehat{F}$, the elements $\left[\widehat{\begin{smallmatrix}\mathbf{v}_1\\ e_1\end{smallmatrix}}\right] \overset{(11.1.10)}{=} (e_1 \leftrightarrow t_1)\mathrm{wrt}(E\cup\{t_1, t_3, t_{23}\})$ and $\left[\widehat{\begin{smallmatrix}\mathbf{v}_3\\ e_3\end{smallmatrix}}\right] \overset{(11.1.14)}{=} (e_3 \leftrightarrow t_3)\mathrm{wrt}(E\cup\{t_1, t_3, t_{23}\})$ commute. Thus, (11.1.P3) holds. □

Proof of (11.1.P4). By (11.1.12) and (11.1.14), we have

$$\left[\widehat{\begin{smallmatrix}\mathbf{v}_1\cup\mathbf{v}_3\\ e_1\end{smallmatrix}}\right]^{\left[\widehat{\begin{smallmatrix}\mathbf{v}_3\\ e_3\end{smallmatrix}}\right]} = (t_0 \mapsto t_0^{-1},\;\, t_2 \mapsto t_0{\cdot}t_2)^{(e_3\mapsto t_2^{-1}\cdot t_{23},\;\, t_2\mapsto t_{23}\cdot e_3^{-1})}\mathrm{wrt}\big((E{-}\{e_3\}^{\pm1})\cup\{e_3, t_0,\, t_2,\, t_{23}\}\big)$$
$$= (t_0 \mapsto t_0^{-1},\;\, t_{23}{\cdot}e_3^{-1} \mapsto t_0{\cdot}t_{23}{\cdot}e_3^{-1})\mathrm{wrt}\big((E{-}\{e_3\}^{\pm1})\cup\{t_2^{-1}{\cdot}t_{23},\, t_0,\;\, t_{23}{\cdot}e_3^{-1},\, t_{23}\}\big)$$
$$= (t_0 \mapsto t_0^{-1},\;\, e_3 \mapsto e_3{\cdot}t_{23}^{-1}{\cdot}t_0^{-1}{\cdot}t_{23})\mathrm{wrt}\big(E\cup\{t_0,\, t_2,\, t_{23}\}\big).$$

By (11.1.13), $\left[\widehat{\begin{smallmatrix}\mathbf{v}_1\cup\mathbf{v}_4\\ e_1\end{smallmatrix}}\right] = (t_0 \mapsto t_0^{-1},\;\, t_{23} \mapsto t_0{\cdot}t_{23})\mathrm{wrt}(E\cup\{t_0, t_2, t_{23}\})$. Since it is now clear that $\left[\widehat{\begin{smallmatrix}\mathbf{v}_1\cup\mathbf{v}_3\\ e_1\end{smallmatrix}}\right]^{\left[\widehat{\begin{smallmatrix}\mathbf{v}_3\\ e_3\end{smallmatrix}}\right]} \neq \left[\widehat{\begin{smallmatrix}\mathbf{v}_1\cup\mathbf{v}_4\\ e_1\end{smallmatrix}}\right]$, to prove (11.1.P4) we must use the hypothesis that the Γ-vertex $\mathbf{v}_4\cup\{t_3^{-1}\}$ equals $\{e_3^{-1},\;\, t_3^{-1}\}$, and hence has valence equal to two and is not the basepoint. It follows that $\pi\Gamma$ lies in that subgroup $\widehat{\pi\Gamma}$ of $\widehat{F}$ which has basis $(E{-}\{e_3\}^{\pm1})\cup\{t_1,\, t_2,\, e_3{\cdot}t_3^{-1}\}$. Both $\left[\widehat{\begin{smallmatrix}\mathbf{v}_1\cup\mathbf{v}_3\\ e_1\end{smallmatrix}}\right]^{\left[\widehat{\begin{smallmatrix}\mathbf{v}_3\\ e_3\end{smallmatrix}}\right]}$ and $\left[\widehat{\begin{smallmatrix}\mathbf{v}_1\cup\mathbf{v}_4\\ e_1\end{smallmatrix}}\right]$ restrict to the same automorphism of $\widehat{\pi\Gamma}$, namely the one given by $(t_0 \mapsto t_0^{-1},\;\, e_3{\cdot}t_{23}^{-1} \mapsto e_3{\cdot}t_{23}^{-1}{\cdot}t_0^{-1})\mathrm{wrt}\big((E{-}\{e_3\}^{\pm1})\cup\{t_0,\, t_2,\, e_3{\cdot}t_{23}^{-1}\}\big)$. It follows that (11.1.P4) holds. □

By deflating $\{t_1, t_{23}\}$ and setting $t := t_3$, we may see that

$$(11.1.15)\qquad \delta_S(\mathbf{v}_3) = \#(t_3 \in S^{\mathrm{Inflate}}; E\cup\{t_1, t_3, t_{23}\}) = \#(t_2 \in S^{\mathrm{Inflate}}; E\cup\{t_0, t_2, t_{23}\});$$

alternatively, one may check that

$$\#(t_2 \in S^{\text{Inflate}}; E \cup \{t_0, t_2, t_{23}\}) = \#(t_{\{2,3,12\}} \in S^{\text{Inflate}}) \overset{(11.1.8)}{=} \delta_S(\mathbf{v}_3).$$

Proof of (11.1.P5). By (10.1.7), $\#(e_3 \in S^{\left[\begin{smallmatrix}\mathbf{v}_1\\ e_1\end{smallmatrix}\right]}) = \#(e_3 \in S)$. Also,

$$\begin{aligned}
\delta_{S\left[\begin{smallmatrix}\mathbf{v}_1\\ e_1\end{smallmatrix}\right]}(\mathbf{v}_3) &\overset{(11.1.15)}{=} \#(t_2 \in S^{\left[\begin{smallmatrix}\mathbf{v}_1\\ e_1\end{smallmatrix}\right]\circ\,\text{Inflate}}; E \cup \{t_0, t_2, t_{23}\})\\
&\overset{(11.1.10)}{=} \#(t_2 \in S^{\text{Inflate}\,\circ(t_0\mapsto t_0^{-1})\text{wrt}(E\cup\{t_0,t_2,t_{23}\})}; E \cup \{t_0, t_2, t_{23}\})\\
&= \#(t_2 \in S^{\text{Inflate}}; E \cup \{t_0, t_2, t_{23}\})\\
&\overset{(11.1.15)}{=} \delta_S(\mathbf{v}_3).
\end{aligned}$$

It now follows from (10.1.6) that $\Delta_{S\left[\begin{smallmatrix}\mathbf{v}_1\\ e_1\end{smallmatrix}\right]}(\left[\begin{smallmatrix}\mathbf{v}_3\\ e_3\end{smallmatrix}\right]) = \Delta_S(\left[\begin{smallmatrix}\mathbf{v}_3\\ e_3\end{smallmatrix}\right])$, and a simple calculation then shows that (11.1.P5) holds. □

Proof of (11.1.P6). By (10.1.7), $\#(e_3 \in S^{\left[\begin{smallmatrix}\mathbf{v}_1\cup\mathbf{v}_3\\ e_1\end{smallmatrix}\right]}) = \#(e_3 \in S)$. Also,

$$\begin{aligned}
\delta_{S\left[\begin{smallmatrix}\mathbf{v}_1\cup\mathbf{v}_3\\ e_1\end{smallmatrix}\right]}(\mathbf{v}_3) &\overset{(11.1.15)}{=} \#(t_2 \in S^{\left[\begin{smallmatrix}\mathbf{v}_1\cup\mathbf{v}_3\\ e_1\end{smallmatrix}\right]\circ\,\text{Inflate}}; E \cup \{t_0, t_2, t_{23}\})\\
&\overset{(11.1.12)}{=} \#(t_2 \in S^{\text{Inflate}\,\circ(t_0\mapsto t_0^{-1},\ t_2\mapsto t_0\cdot t_2)\text{wrt}(E\cup\{t_0,\, t_2,\, t_{23}\})}; E \cup \{t_0, t_2, t_{23}\})\\
&\leqslant \#(t_2 \in S^{\text{Inflate}}; E \cup \{t_0, t_2, t_{23}\})\\
&\overset{(11.1.15)}{=} \delta_S(\mathbf{v}_3).
\end{aligned}$$

It now follows from (10.1.6) that $\Delta_{S\left[\begin{smallmatrix}\mathbf{v}_1\cup\mathbf{v}_3\\ e_1\end{smallmatrix}\right]}(\left[\begin{smallmatrix}\mathbf{v}_3\\ e_3\end{smallmatrix}\right]) \leqslant \Delta_S(\left[\begin{smallmatrix}\mathbf{v}_3\\ e_3\end{smallmatrix}\right])$; moreover, equality must hold since $\left[\begin{smallmatrix}\mathbf{v}_1\cup\mathbf{v}_3\\ e_1\end{smallmatrix}\right]^2 \overset{(10.1.2)}{\equiv} 1$. A simple calculation then shows that (11.1.P6) holds. □

12. The consequences of five of the six rules

The formulæ verified in this section are based on formulæ given in Higgins & Lyndon(1962) for the case of single cyclic words. They were then modified in McCool(1974) and extended to accommodate the cases of sequences of cyclic words and sequences of straight words, and then modified in Hoare(1979) and extended to accommodate the case of sequences of sets of words. The formulæ are about to be modified again.

12.1. **Theorem** (Higgins & Lyndon)**.** *With the notation of* Section 11*, the following hold.*

(12.1.R0) *The subgroup of* $\operatorname{Aut} F$ *generated by the image of* $\mathcal{E}_0$ *is* $\operatorname{Sym}_{\pm}(E^{\pm1})$.

(12.1.R1) *For all* $\left[\begin{smallmatrix}\mathbf{v}\\ e\end{smallmatrix}\right] \in \mathcal{E}_1$, $\left[\begin{smallmatrix}\mathbf{v}\\ e\end{smallmatrix}\right]^2 \equiv 1$.

(12.1.R2) *For all* $\left[\begin{smallmatrix}\mathbf{v}\\ e\end{smallmatrix}\right] \in \mathcal{E}_1$, $\left[\begin{smallmatrix}\mathbf{v}^*\\ e^{-1}\end{smallmatrix}\right] \equiv \left[\begin{smallmatrix}\mathbf{v}\\ e\end{smallmatrix}\right]$.

(12.1.R3) *For all* $(d{\leftrightarrow}f) \in \mathcal{E}_0$ *and* $\left[\begin{smallmatrix}\mathbf{v}\\ e\end{smallmatrix}\right] \in \mathcal{E}_1$, $\left[\begin{smallmatrix}\mathbf{v}\\ e\end{smallmatrix}\right]^{(d\leftrightarrow f)} \equiv \left[\begin{smallmatrix}\mathbf{v}^{(d\leftrightarrow f)}\\ e^{(d\leftrightarrow f)}\end{smallmatrix}\right]$.

(12.1.R4) *For all* $\left[\begin{smallmatrix}\mathbf{v}\\ e\end{smallmatrix}\right], \left[\begin{smallmatrix}\mathbf{v}\\ e'\end{smallmatrix}\right] \in \mathcal{E}_1$ *with* $e \neq e'$, $\left[\begin{smallmatrix}\mathbf{v}\\ e\end{smallmatrix}\right]^{\left[\begin{smallmatrix}\mathbf{v}\\ e'\end{smallmatrix}\right]} \equiv (e{\leftrightarrow}e')$.

(12.1.R5) *For all* $(\mathbf{u} \overset{d}{\rightsquigarrow}\mathbf{v},\ \mathbf{w}) \in \mathcal{P}_3(E)$, $\left[\begin{smallmatrix}\mathbf{u}\\ d\end{smallmatrix}\right]\cdot(d{\leftrightarrow}d^{-1})\cdot\left[\begin{smallmatrix}\mathbf{w}\cup\{d\}\\ d\end{smallmatrix}\right] \equiv \left[\begin{smallmatrix}\mathbf{v}\\ d^{-1}\end{smallmatrix}\right]$.

(12.1.R6) *For all* $(\mathbf{u} \overset{d}{\rightsquigarrow}\mathbf{v} \overset{e}{\rightsquigarrow}\mathbf{w}) \in \mathcal{P}_3(E)$, $\left[\begin{smallmatrix}\mathbf{v}\\ e\end{smallmatrix}\right]^{\left[\begin{smallmatrix}\mathbf{u}\\ d\end{smallmatrix}\right]} \equiv \left[\begin{smallmatrix}\mathbf{w}\cup\{d^{-1}\}\\ e^{-1}\end{smallmatrix}\right]$.

(12.1.R7) *For all* $(\mathbf{u} \overset{d}{\rightsquigarrow} \mathbf{v} \overset{f}{\leftarrowtail} \mathbf{w}) \in \mathcal{P}_3(E)$, $\big[\begin{smallmatrix}\mathbf{w}\\ f\end{smallmatrix}\big]^{\big[\begin{smallmatrix}\mathbf{u}\\ d\end{smallmatrix}\big]} \equiv \big[\begin{smallmatrix}\mathbf{w}\\ f\end{smallmatrix}\big]$.

(12.1.R8) *For all* $(\mathbf{u} \overset{d}{\rightsquigarrow} \mathbf{v} \overset{e}{\rightsquigarrow} \mathbf{w}) \in \mathcal{P}_3(E)$, $\Delta_S\big(\big[\begin{smallmatrix}\mathbf{v}\\ e\end{smallmatrix}\big]\cdot\big[\begin{smallmatrix}\mathbf{u}\\ d\end{smallmatrix}\big]\big) = \Delta_S\big(\big[\begin{smallmatrix}\mathbf{v}\\ e\end{smallmatrix}\big]\big) + \Delta_S\big(\big[\begin{smallmatrix}\mathbf{u}\\ d\end{smallmatrix}\big]\big)$.

(12.1.R9) *For all* $(\mathbf{u} \overset{d}{\rightsquigarrow} \mathbf{v} \overset{f}{\leftarrowtail} \mathbf{w}) \in \mathcal{P}_3(E)$, $\Delta_S\big(\big[\begin{smallmatrix}\mathbf{w}\\ f\end{smallmatrix}\big]\cdot\big[\begin{smallmatrix}\mathbf{u}\\ d\end{smallmatrix}\big]\big) = \Delta_S\big(\big[\begin{smallmatrix}\mathbf{w}\\ f\end{smallmatrix}\big]\big) + \Delta_S\big(\big[\begin{smallmatrix}\mathbf{u}\\ d\end{smallmatrix}\big]\big)$.

Proof. (12.1.R0) is well-known, (12.1.R1), (12.1.R2), and (12.1.R4) hold by (10.1.2), (10.1.3), and (10.1.4) respectively, and (12.1.R3) is clear.

(12.1.R5). It follows from (8.1.1) that

(12.1.10) for all $e \in E^{\pm 1}$, $\big[\begin{smallmatrix}\{e\}\\ e\end{smallmatrix}\big] \equiv (e \leftrightarrow e^{-1})$.

Set $e_1 := d$, $\mathbf{v}_1 := \{d\}$, $\mathbf{v}_2 := \mathbf{v}$, $\mathbf{v}_3 := \mathbf{u}-\{d\}$, and $\mathbf{v}_4 := \mathbf{w}$. We see that $(\mathbf{v}_1 \overset{e_1}{\rightsquigarrow} \mathbf{v}_2,\ \mathbf{v}_3,\ \mathbf{v}_4)$ is $(\{d\} \overset{d}{\rightsquigarrow} \mathbf{v},\ \mathbf{u}-\{d\},\ \mathbf{w})$. Then

$\big[\begin{smallmatrix}\mathbf{u}\\ d\end{smallmatrix}\big]\cdot(d \leftrightarrow d^{-1})\cdot\big[\begin{smallmatrix}\mathbf{w}\cup\{d\}\\ d\end{smallmatrix}\big] \overset{(12.1.10)}{\equiv} \big[\begin{smallmatrix}\mathbf{u}\\ d\end{smallmatrix}\big]\cdot\big[\begin{smallmatrix}\{d\}\\ d\end{smallmatrix}\big]\cdot\big[\begin{smallmatrix}\mathbf{w}\cup\{d\}\\ d\end{smallmatrix}\big] = \big[\begin{smallmatrix}\mathbf{v}_1\cup\mathbf{v}_3\\ e_1\end{smallmatrix}\big]\cdot\big[\begin{smallmatrix}\mathbf{v}_1\\ e_1\end{smallmatrix}\big]\cdot\big[\begin{smallmatrix}\mathbf{v}_1\cup\mathbf{v}_4\\ e_1\end{smallmatrix}\big] \overset{(11.1.\mathrm{P2})}{\equiv} \big[\begin{smallmatrix}\mathbf{v}_2\\ e_1^{-1}\end{smallmatrix}\big] = \big[\begin{smallmatrix}\mathbf{v}\\ d^{-1}\end{smallmatrix}\big]$.

Hence, (12.1.R5) holds.

(12.1.R6)–(12.1.R9). Set $\mathbf{v}_1 := \mathbf{w},\ \mathbf{v}_2 := \mathbf{v}-\{d^{-1}\},\ \mathbf{v}_3 := \mathbf{u},\ \mathbf{v}_4 := \{d^{-1}\},\ e_1 := f = e^{-1}$, and $e_3 := d$. Then $(\mathbf{v}_1 \overset{e_1}{\rightsquigarrow} \mathbf{v}_2,\ \mathbf{v}_3 \overset{e_3}{\rightsquigarrow} \mathbf{v}_4)$ is $(\mathbf{w} \overset{f=e^{-1}}{\rightsquigarrow} \mathbf{v}-\{d^{-1}\},\ \mathbf{u} \overset{d}{\rightsquigarrow} \{d^{-1}\})$. Now

(12.1.11) $\big[\begin{smallmatrix}\mathbf{v}\\ e\end{smallmatrix}\big] = \big[\begin{smallmatrix}\mathbf{v}_2\cup\mathbf{v}_4\\ e_1^{-1}\end{smallmatrix}\big] \overset{(10.1.3)}{\equiv} \big[\begin{smallmatrix}\mathbf{v}_1\cup\mathbf{v}_3\\ e_1\end{smallmatrix}\big]$,

and we have all of the following.

$\big[\begin{smallmatrix}\mathbf{v}\\ e\end{smallmatrix}\big]^{\big[\begin{smallmatrix}\mathbf{u}\\ d\end{smallmatrix}\big]} \overset{(12.1.11)}{\equiv} \big[\begin{smallmatrix}\mathbf{v}_1\cup\mathbf{v}_3\\ e_1\end{smallmatrix}\big]^{\big[\begin{smallmatrix}\mathbf{v}_3\\ e_3\end{smallmatrix}\big]} \overset{(11.1.\mathrm{P5})}{\equiv} \big[\begin{smallmatrix}\mathbf{v}_1\cup\mathbf{v}_4\\ e_1\end{smallmatrix}\big] = \big[\begin{smallmatrix}\mathbf{w}\cup\{d^{-1}\}\\ e^{-1}\end{smallmatrix}\big]$.

$\big[\begin{smallmatrix}\mathbf{w}\\ f\end{smallmatrix}\big]^{\big[\begin{smallmatrix}\mathbf{u}\\ d\end{smallmatrix}\big]} = \big[\begin{smallmatrix}\mathbf{v}_1\\ e_1\end{smallmatrix}\big]^{\big[\begin{smallmatrix}\mathbf{v}_3\\ e_3\end{smallmatrix}\big]} \overset{(11.1.\mathrm{P4})}{\equiv} \big[\begin{smallmatrix}\mathbf{v}_1\\ e_1\end{smallmatrix}\big] = \big[\begin{smallmatrix}\mathbf{w}\\ f\end{smallmatrix}\big]$.

$\Delta_S\big(\big[\begin{smallmatrix}\mathbf{v}\\ e\end{smallmatrix}\big]\cdot\big[\begin{smallmatrix}\mathbf{u}\\ d\end{smallmatrix}\big]\big) \overset{(12.1.11)}{=} \Delta_S\big(\big[\begin{smallmatrix}\mathbf{v}_1\cup\mathbf{v}_3\\ e_1\end{smallmatrix}\big]\cdot\big[\begin{smallmatrix}\mathbf{v}_3\\ e_3\end{smallmatrix}\big]\big) \overset{(11.1.\mathrm{P6})}{=} \Delta_S\big(\big[\begin{smallmatrix}\mathbf{v}_1\cup\mathbf{v}_3\\ e_1\end{smallmatrix}\big]\big) + \Delta_S\big(\big[\begin{smallmatrix}\mathbf{v}_3\\ e_3\end{smallmatrix}\big]\big) \overset{(12.1.11)}{=} \Delta_S\big(\big[\begin{smallmatrix}\mathbf{v}\\ e\end{smallmatrix}\big]\big) + \Delta_S\big(\big[\begin{smallmatrix}\mathbf{u}\\ d\end{smallmatrix}\big]\big)$.

$\Delta_S\big(\big[\begin{smallmatrix}\mathbf{w}\\ f\end{smallmatrix}\big]\cdot\big[\begin{smallmatrix}\mathbf{u}\\ d\end{smallmatrix}\big]\big) = \Delta_S\big(\big[\begin{smallmatrix}\mathbf{v}_1\\ e_1\end{smallmatrix}\big]\cdot\big[\begin{smallmatrix}\mathbf{v}_3\\ e_3\end{smallmatrix}\big]\big) \overset{(11.1.\mathrm{P5})}{=} \Delta_S\big(\big[\begin{smallmatrix}\mathbf{v}_1\\ e_1\end{smallmatrix}\big]\big) + \Delta_S\big(\big[\begin{smallmatrix}\mathbf{v}_3\\ e_3\end{smallmatrix}\big]\big) = \Delta_S\big(\big[\begin{smallmatrix}\mathbf{w}\\ f\end{smallmatrix}\big]\big) + \Delta_S\big(\big[\begin{smallmatrix}\mathbf{u}\\ d\end{smallmatrix}\big]\big)$.

Thus, (12.1.R6)–(12.1.R9) hold. □

12.2. **Corollary.** *With* Notation 9.1, $\mathcal{R} \subseteq \mathcal{N}$. □

13. Rapaport's decreasable peaks

The following are concepts introduced by Rapaport(1958) to bypass the topological part of the work of Whitehead(1936). They remain some of the main tools for obtaining information about $\operatorname{Aut} F$. The term 'peak' was introduced by Collins & Zieschang(1984), and is now generally accepted. Their phrase 'peak reduction' is also generally accepted, but as peaks are paths and the phrase 'path reduction' is generally accepted for a different concept, we prefer to speak of 'decreasable peaks'.

13.1. **Definitions.** Let E, F, $\mathcal{E}$, and $\mathcal{F}$ be as in Section 8.

By Theorem 5.2, $\mathcal{F}\curvearrowright\mathcal{E}$ is a left-multiplicative $\mathcal{F}$-tree. For all φ, $\varphi' \in \mathcal{F}$, we let $[\![\varphi, \varphi']\!]$ denote the unique reduced $(\mathcal{F}\curvearrowright\mathcal{E})$-path from φ to φ', and we let $\mathrm{V}[\![\varphi, \varphi']\!]$ denote the set of vertices occurring in this path. Then $(\varphi_1, \varphi_2) \mapsto [\![\varphi_1, \varphi_2]\!]$ gives a bijective map from $\mathcal{F} \times \mathcal{F}$ to the set of all reduced $(\mathcal{F}\curvearrowright\mathcal{E})$-paths. There is a natural left-multiplicative $\mathcal{F}$-action on the set of all $(\mathcal{F}\curvearrowright\mathcal{E})$-paths, and hence on the set of all reduced $(\mathcal{F}\curvearrowright\mathcal{E})$-paths. We let $\mathcal{F} \times \mathcal{F}$ act left multiplicatively on the set of all reduced ($\mathcal{F}\frown\mathcal{E}$)-paths by the foregoing bijection, that is, $(\varphi_1,\varphi_2)\cdot[\![\varphi,\varphi']\!] := [\![\varphi_1\cdot\varphi,\varphi_2\cdot\varphi']\!]$, for all $\varphi_1,\varphi_2,\varphi,\varphi' \in \mathcal{F}$. The diagonal $\mathcal{F}$-action agrees with the previously defined $\mathcal{F}$-action.

Let $\mathcal{T}$ be a subtree of $\mathcal{F}\frown\mathcal{E}$. Thus, for all φ_1, $\varphi_2 \in \mathrm{V}\mathcal{T}$, $[\![\varphi_1,\varphi_2]\!]$ is a $\mathcal{T}$-path.

Let S be a finite set of words in F. For each $\varphi \in \mathcal{F}$, set $\hbar_{S,E}(\varphi) := E\text{-length}(S^{\varphi})$. Where E is clear from the context, we shall sometimes write $\hbar_S$ rather than $\hbar_{S,E}$. Where E and S are clear from the context, we shall shall sometimes write $\hbar$ rather than $\hbar_{S,E}$. For all $\varphi \in \mathcal{F}$ and $\mu \in \mathcal{F}_S$, it is clear that $\hbar(\mu\cdot\varphi) = \hbar(\varphi)$, which we express by saying that the map $\hbar : \mathcal{F} \to \mathbb{N}$ is a *left-multiplicative* $\mathcal{F}_S$*-map*; here, we understand that $\mathcal{F}_S$ acts left-multiplicatively on $\mathcal{F}$ naturally and left-multiplicatively on $\mathbb{N}$ trivially.

By a $(\mathcal{T},\hbar)$*-peak*, we mean any reduced, length-two $\mathcal{T}$-path $\varphi_0 \rightarrow\!\!-\varphi_1 \rightarrow\!\!-\varphi_2$ such that $\hbar(\varphi_0) < \hbar(\varphi_1) \geqslant \hbar(\varphi_2)$ or $\hbar(\varphi_0) \leqslant \hbar(\varphi_1) > \hbar(\varphi_2)$. A $\mathcal{T}$-path is said to be an $\hbar$*-peak* if it is a $(\mathcal{T},\hbar)$-peak. A reduced $\mathcal{T}$-path is said to be $\hbar$*-peakfree* if no length-two subpath is an $\hbar$-peak.

Notice that, for any reduced, length-two $\mathcal{T}$-path $\varphi_0 \rightarrow\!\!-\varphi_1 \rightarrow\!\!-\varphi_2$ which is *not* an $\hbar$-peak, if $\hbar(\varphi_0) < \hbar(\varphi_1)$, then $\hbar(\varphi_1) < \hbar(\varphi_2)$, and if $\hbar(\varphi_0) = \hbar(\varphi_1)$, then $\hbar(\varphi_1) \leqslant \hbar(\varphi_2)$. Hence, if a reduced $\mathcal{T}$-path $\varphi_0 \rightarrow\!\!-\varphi_1 \rightarrow\!\!- \cdots \rightarrow\!\!-\varphi_{\ell-1} \rightarrow\!\!-\varphi_\ell$ is $\hbar$-peakfree, then, in the sequence

$$\hbar(\varphi_0)-\hbar(\varphi_1),\ \hbar(\varphi_1)-\hbar(\varphi_2),\ \cdots,\ \hbar(\varphi_{\ell-1})-\hbar(\varphi_\ell), \tag{13.1.1}$$

for each negative term which has a successor, that successor is also negative, while for each zero term which has a successor, that successor is negative or zero. The sequence (13.1.1) then consists of a positive subsequence, followed by a zero subsequence, followed by a negative subsequence, where any of these three subsequences may be empty. Consequently, any $\hbar$-peakfree reduced $\mathcal{T}$-path p has a concatenation factorization $p = p_{\,\mathrm{dec}} \circledcirc p_{\,\mathrm{const}} \circledcirc p_{\,\mathrm{inc}}$ such that $p_{\,\mathrm{dec}}$ is $\hbar$-decreasing, $p_{\,\mathrm{const}}$ is $\hbar$-constant, and $p_{\,\mathrm{inc}}$ is $\hbar$-increasing; any of these three subpaths may have length zero, and all three subpaths are uniquely determined by p and $\hbar$. We then call this concatenation factorization the $\hbar$*-valley factorization* of p. Whitehead(1936) studied $\hbar$-valley factorizations, and Rapaport(1958) studied $\hbar$-peaks.

Let $\mathbf{t}$ be a new symbol, and give the polynomial ring $\mathbb{Z}[\mathbf{t}]$ the structure of an ordered additive group in which the order is denoted $\prec$ and the $\prec$-positive elements are those polynomials with positive leading term. Let $\mathbb{N}[\mathbf{t}]$ denote the ordered additive submonoid of $\mathbb{Z}[\mathbf{t}]$ consisting of those polynomials whose coefficients lie in $\mathbb{N}$. Then $\mathbb{N}[\mathbf{t}]$ is well-ordered under $\prec$.

For φ, $\varphi' \in \mathcal{F}$, we write

$$\widetilde{\hbar}([\![\varphi,\varphi']\!]) := \sum_{\varphi''\in \mathrm{V}[\![\varphi,\varphi']\!]} \mathbf{t}^{\hbar(\varphi'')} \in \mathbb{N}[\mathbf{t}].$$

Any $(\mu,\mu') \in \mathcal{F}\times\mathcal{F}$ is said to $\widetilde{\hbar}$*-decrease* $[\![\varphi,\varphi']\!]$ if $\widetilde{\hbar}\,(\,[\![\mu\cdot\varphi,\mu'\cdot\varphi']\!]) \prec \widetilde{\hbar}([\![\varphi,\varphi']\!])$.

Let $\mathcal{M}$ be any subgroup of $\mathcal{F}_{(S,\mathcal{T})} := \{\varphi \in \mathcal{F} : S^{\varphi} = S, \varphi\cdot\mathcal{T} = \mathcal{T}\}$. Thus, $\mathcal{T}$ is a left-multiplicative $\mathcal{M}$-subtree of $\mathcal{F}\frown\mathcal{E}$, and $\hbar$ is a left-multiplicative $\mathcal{M}$-map.

Consider any φ, $\varphi' \in \mathrm{V}\mathcal{T}$.

We say that $[\![\varphi,\varphi']\!]$ is $(\mathcal{M},\widetilde{\hbar})$*-decreasable* if it is $\widetilde{\hbar}$-decreased by some element of $\mathcal{M}\times\mathcal{M}$. Since $\hbar$ is an $\mathcal{M}$-map, we see that if $[\![\varphi,\varphi']\!]$ is $\widetilde{\hbar}$-decreased by $(\mu_1,\mu_2) \in \mathcal{M}\times\mathcal{M}$, then $[\![\varphi,\varphi']\!]$ is $\widetilde{\hbar}$-decreased by $(\mu\cdot\mu_0,\mu\cdot\mu_1)$ for each $\mu \in \mathcal{M}$; in particular, it is $\widetilde{\hbar}$-decreased by $(1,\mu_0^{-1}\cdot\mu_1)$, and by $(\mu_1^{-1}\cdot\mu_0,1)$. If $\widetilde{\hbar}([\![\mu_1\cdot\varphi,\mu_2\cdot\varphi']\!]) = \widetilde{\hbar}([\![\varphi,\varphi']\!])$, then (μ_1,μ_2) is said to $\widetilde{\hbar}$*-respect* $[\![\varphi,\varphi']\!]$, and we say that $[\![\mu_1\cdot\varphi,\mu_2\cdot\varphi']\!]$ is $(\mathcal{M},\widetilde{\hbar})$*-equivalent* to $([\![\varphi,\varphi']\!])$.

When $\varphi \neq \varphi'$, we let $]\!]\varphi,\varphi']\!]$ and $[\![\varphi,\varphi'[\![$ denote the $\mathcal{T}$-paths obtained from $[\![\varphi,\varphi']\!]$ by omitting the first edge and vertex, and the last edge and vertex, respectively. If $\varphi = \varphi'$, we let $]\!]\varphi,\varphi']\!]$ and $[\![\varphi,\varphi'[\![$ denote the empty set. If the length of $[\![\varphi,\varphi']\!]$ is at least two, we let $]\!]\varphi,\varphi'[\![$ denote the $\mathcal{T}$-subpath of $[\![\varphi,\varphi']\!]$ obtained by omitting the first and last edges and vertices; if the length of $[\![\varphi,\varphi']\!]$ is at most one, we let $]\!]\varphi,\varphi'[\![$ denote the empty set. We define $\widetilde{\hbar}(\emptyset)=0$. If $[\![\varphi,\varphi']\!]$ is $\widetilde{\hbar}$-decreased by (μ_1,μ_2), then $[\![\varphi,\varphi']\!]$ has length at least two, and $\widetilde{h}(\,]\!]\mu_1{\cdot}\varphi,\mu_2{\cdot}\varphi'[\![\,) \prec \widetilde{h}(\,]\!]\varphi,\varphi'[\![\,)$. □

13.2. **Lemma** (Rapaport). *If each $(\mathcal{T},\hbar)$-peak is $(\mathcal{M},\widetilde{\hbar})$-decreasable, then, for all φ, $\varphi'\in \mathrm{V}\mathcal{T}$, there exists some $\mu\in\mathcal{M}$ such that $[\![\varphi,\mu{\cdot}\varphi']\!]$ is $\hbar$-peakfree, and, hence, has an $\hbar$-valley factorization.*

Proof. Since $\mathbb{N}[\mathbf{t}]$ is well-ordered, there exists some $\mu\in\mathcal{M}$ which $\prec$-minimizes $\widetilde{\hbar}([\![\varphi,\mu{\cdot}\varphi']\!])$. It suffices to show that $[\![\varphi,\mu{\cdot}\varphi']\!]$ is $\hbar$-peakfree. Suppose it is not. Then $[\![\varphi,\mu{\cdot}\varphi']\!]$ has some length-two subpath $\varphi_0 \rightarrow\!\!\!-\varphi_1 \rightarrow\!\!\!-\varphi_2$ which is an $\hbar$-peak. By our hypothesis, there exists some $\mu_0\in\mathcal{M}$ such that $\widetilde{\hbar}([\![\varphi_0,\mu_0{\cdot}\varphi_2]\!])\prec\widetilde{\hbar}([\![\varphi_0,\varphi_2]\!])$. Now

$$\begin{aligned}\widetilde{\hbar}([\![\varphi,\mu_0{\cdot}\mu{\cdot}\varphi']\!]) &\preccurlyeq \widetilde{\hbar}([\![\varphi,\varphi_0[\![\,)+\widetilde{\hbar}([\![\varphi_0,\mu_0{\cdot}\varphi_2]\!])+\widetilde{\hbar}(\,]\!]\mu_0{\cdot}\varphi_2,\mu_0{\cdot}\mu{\cdot}\varphi']\!])\\ &= \widetilde{\hbar}([\![\varphi,\varphi_0[\![\,)+\widetilde{\hbar}([\![\varphi_0,\mu_0{\cdot}\varphi_2]\!])+\widetilde{\hbar}(\,]\!]\varphi_2,\mu{\cdot}\varphi']\!])\\ &\prec \widetilde{\hbar}([\![\varphi,\varphi_0[\,)+\widetilde{\hbar}([\![\varphi_0,\varphi_2]\!])+\widetilde{\hbar}(\,]\!]\varphi_2,\mu_0{\cdot}\varphi']\!])=\widetilde{\hbar}([\![\varphi,\mu{\cdot}\varphi']\!]).\end{aligned}$$

This contradicts the $\prec$-minimality of $\widetilde{\hbar}([\![\varphi,\mu{\cdot}\varphi']\!])$, as desired. □

14. All of McCool's peaks are decreasable

In this section we prove a fundamental result about $\operatorname{Aut}F$. It was developed by McCool(1975) using formulæ of Higgins & Lyndon(1962) which simplified algebraic formulæ of Rapaport(1952) which substituted a topological argument of Whitehead(1936). In spite of all the simplifications, the argument remains difficult.

14.1. **Hypotheses.** Let E, F, $\mathcal{E}$, $\mathcal{F}$, $\mathcal{N}$, and $\mathcal{R}$ be as in Section 9. By Corollary 12.2, $\mathcal{R}\subseteq\mathcal{N}$.

For each $i\in\{0,1\}$, let W_i be a finite set of words in F, and, for each $\varphi\in\mathcal{F}$, set $\hbar_i(\varphi):=E\text{-length}(W_i^{\varphi})$. Clearly, $\hbar_i:\mathcal{F}\to\mathbb{N}$ is a left-multiplicative $\mathcal{N}$-map.

Set $\mathcal{F}_{|\hbar_0=\min}:=\{\varphi\in\mathcal{F}:\hbar_0(\varphi)=\min(\hbar_0(\mathcal{F}))\}$.

Let $\mathcal{T}$ be any component of the forest $(\mathcal{F}_{|\hbar_0=\min}\rangle\!\!\curvearrowleft\mathcal{E}$.

Set $\mathcal{M}:=\langle({}^{\mathrm{V}\mathcal{T}}\mathcal{R})_{\mathcal{T}}\rangle$ where $({}^{\mathrm{V}\mathcal{T}}\mathcal{R})_{\mathcal{T}}:=\{{}^{\varphi}\rho\mid\varphi\in\mathrm{V}\mathcal{T},\rho\in\mathcal{R},({}^{\varphi}\rho){\cdot}\mathcal{T}=\mathcal{T}\}$. Clearly, $\mathcal{T}$ is a left-multiplicative $\mathcal{M}$-tree, and $\mathcal{M}\leqslant\mathcal{N}$. □

14.2. **Theorem.** *With* Hypotheses 14.1*, each $(\mathcal{T},\hbar_1)$-peak is $(\mathcal{M},\widetilde{\hbar}_1)$-decreasable.*

Proof. Let p be an arbitrary $(\mathcal{T},\hbar_1)$-peak. Then p has the form $\varphi_0 \rightarrow\!\!\!-\varphi_1 \rightarrow\!\!\!-\varphi_2$. By setting $\varphi:=\varphi_1$, $\psi:=\varphi_1^{-1}{\cdot}\varphi_0$, and $\psi':=\varphi_1^{-1}{\cdot}\varphi_2$, we may express p in the form $\varphi\cdot\psi \rightarrow\!\!\!-\varphi \rightarrow\!\!\!-\varphi\cdot\psi'$ with $\varphi\in\mathcal{F}$, and ψ_1, $\psi_2\in\mathcal{E}^{\pm1}$. Since p is reduced, $\psi\neq\psi'$. Recall (10.1.5), and, for each $i\in\{0,1\}$, set $\Delta_i:=\Delta_{W_i^{\varphi},E}:\mathcal{F}\to\mathbb{N}$.

Status of the peak: $p=\varphi\cdot[\![\psi,\psi']\!]$, $\varphi\in\mathcal{F}$, ψ, $\psi'\in\mathcal{E}^{\pm1}$, $\psi\neq\psi'$. Also, $\Delta_0(\psi)=0=\Delta_0(\psi')$, $\Delta_1(\psi)\leqslant 0\geqslant\Delta_1(\psi')$, and $\Delta_1(\psi)+\Delta_1(\psi')<0$.

The following will be applied frequently. Since Δ_0 and Δ_1 are clearly left-multiplicative $\mathcal{N}$-maps, they are also left-multiplicative $\mathcal{M}$-maps. Notice that $\min(\Delta_0(\mathcal{F}))=0$. For $\psi_1\in\mathcal{E}^{\pm1}$, if $\Delta_0(\psi_1)=0$, then $\varphi\cdot\psi_1\in\mathrm{V}\mathcal{T}$, and, for ψ_1, $\psi_2\in\mathcal{E}^{\pm1}$, if $\Delta_0(\psi_1{\cdot}\psi_2)=\Delta_0(\psi_1)=0$, then $\varphi\cdot\psi_1\cdot\psi_2 \in \mathrm{V}\mathcal{T}$. Notice that $\mathcal{F}_{\hbar_0=\min}\frown\mathcal{E}$ is an $\mathcal{N}$-forest. Since $\mathcal{R} \subseteq \mathcal{N}$, we see that each ${}^{\varphi}\rho \in {}^{\mathrm{V}\mathcal{T}}\mathcal{R}$ permutes the components of $(\mathcal{F}_{\hbar_0=\min})\frown\mathcal{E}$; hence, $({}^{\varphi}\rho)\cdot\mathcal{T} = \mathcal{T}$ if and only if $({}^{\varphi}\rho)\cdot\mathcal{T} \cap \mathcal{T} \neq \emptyset$.

The following 'concatenation result' will be applied twice.

14.3. **Lemma.** *If there exists some $\psi'' \in \mathcal{E}^{\pm1} - \{\psi, \psi'\}$ such that $\Delta_0(\psi'') = 0$, $\Delta_1(\psi'') < 0$, and the resulting $(\mathcal{T}, \hbar_1)$-peaks $\varphi\cdot[\![\psi, \psi''\,]\!]$ and $\varphi\cdot[\![\,\psi'', \psi'\,]\!]$ are $(\mathcal{M}, \widetilde{\hbar}_1)$-decreasable, then the $(\mathcal{T}, \hbar_1)$-peak $\varphi\cdot[\![\,\psi, \psi'\,]\!]$ is $(\mathcal{M}, \widetilde{\hbar}_1)$-decreasable.*

Proof. There exist elements μ_1, $\mu_2 \in \mathcal{M}$ such that $[\![\varphi\cdot\psi, \varphi\cdot\psi''\,]\!]$ and $[\![\,\varphi\cdot\psi'', \varphi\cdot\psi'\,]\!]$ are $\widetilde{\hbar}_1$-decreased by $(\mu_1, 1)$ and $(1, \mu_2)$, respectively. Hence,

$$\begin{aligned} \mathbf{t}^{\hbar_1(\mu_1\cdot\varphi\cdot\psi)} &= \mathbf{t}^{\hbar_1(\varphi\cdot\psi)}, \\ \widetilde{\hbar}_1\big(\,]\!]\mu_1\cdot\varphi\cdot\psi, \varphi\cdot\psi''[\![\,\big) &\prec \widetilde{\hbar}_1(\,]\!]\varphi\cdot\psi, \varphi\cdot\psi''[\![\,) = \mathbf{t}^{\hbar_1(\varphi)}, \\ \mathbf{t}^{\hbar_1(\varphi\cdot\psi'')} &\prec \mathbf{t}^{\hbar_1(\varphi)}, \\ \widetilde{\hbar}_1\big(\,]\!]\varphi\cdot\psi'', \mu_2\cdot\varphi\cdot\psi'[\![\,\big) &\prec \widetilde{\hbar}_1(\,]\!]\varphi\cdot\psi'', \varphi\cdot\psi'[\![\,) = \mathbf{t}^{\hbar_1(\varphi)}, \\ \mathbf{t}^{\hbar_1(\mu_2\cdot\varphi\cdot\psi')} &= \mathbf{t}^{\hbar_1(\varphi\cdot\psi')}. \end{aligned}$$

Let Σ_{left} and Σ_{right} denote the sum of the left-hand and right-hand terms respectively. Now $\widetilde{\hbar}(\,[\![\,\mu_1\cdot\varphi\cdot\psi, \mu_2\cdot\varphi\cdot\psi'\,]\!]) \preccurlyeq \Sigma_{\mathrm{left}}$. Also, $\Sigma_{\mathrm{left}} \prec \Sigma_{\mathrm{right}} - 2\cdot\mathbf{t}^{\hbar_1(\varphi)}$, since $\{f \in \mathbb{N}[\mathbf{t}] : f \prec \mathbf{t}^{\hbar_1(\varphi)}\}$ is closed under summation. Finally, $\Sigma_{\mathrm{right}} - 2\cdot\mathbf{t}^{\hbar_1(\varphi)} = \widetilde{\hbar}(\,[\![\varphi\cdot\psi, \varphi\cdot\psi'\,]\!])$. Hence, $\varphi\cdot[\![\,\psi, \psi'\,]\!]$ is $\widetilde{\hbar}_1$-decreased by $(\mu_1, \mu_2) \in \mathcal{M} \times \mathcal{M}$. □

In outline, our procedure is the following.

Step 1. Clearly, $\widetilde{\hbar}_1(\varphi\cdot[\![\,\psi^{-1}, \psi',]\!]) \preccurlyeq \widetilde{\hbar}_1(\varphi\cdot[\![\,\psi, \psi'\,]\!])$, with equality unless $\psi^{-1} = \psi'$. If $\psi \in \mathcal{E}^{-1}$, we use the fact that $\psi^{-2} \in \mathcal{R}_0 \cup \mathcal{R}_1$ to show that $\varphi\cdot[\![\,\psi^{-1}, \psi'\,]\!]$ lies in the same $\mathcal{M} \times \mathcal{M}$-orbit as $\varphi\cdot[\![\,\psi, \psi'\,]\!]$, and we are free to replace ψ with ψ^{-1}. Similarly for ψ'. Now ψ, $\psi' \in \mathcal{E}$. We are free to interchange ψ and ψ', if desired.

Step 2. We settle the case where ψ or ψ' lies in $\mathcal{E}_0$ by using $\mathcal{R}_3$. Now $\psi = \left[\begin{smallmatrix}\mathbf{u}\\ d\end{smallmatrix}\right]$, $\psi' = \left[\begin{smallmatrix}\mathbf{v}\\ e\end{smallmatrix}\right]$. As in Step 1, we are free to take $\psi' = \left[\begin{smallmatrix}\mathbf{v}^*\\ e^{-1}\end{smallmatrix}\right]$, if desired, by using $\left[\begin{smallmatrix}\mathbf{v}^*\\ e^{-1}\end{smallmatrix}\right]\cdot\left[\begin{smallmatrix}\mathbf{v}\\ e\end{smallmatrix}\right]^{-1} \in \mathcal{R}_2$. Similarly, we are free to take $\psi = \left[\begin{smallmatrix}\mathbf{u}^*\\ d^{-1}\end{smallmatrix}\right]$, if desired. We set $\mathbf{u} \boxplus \mathbf{v} := \{\mathbf{u} \cap \mathbf{v}, \mathbf{u} \cap \mathbf{v}^*, \mathbf{u}^* \cap \mathbf{v}, \mathbf{u}^* \cap \mathbf{v}^*\}$.

Step 3. We settle the case where $\emptyset \in \mathbf{u} \boxplus \mathbf{v}$ as follows. We may assume that $\mathbf{u} \cap \mathbf{v} = \emptyset$ and set $\mathbf{w} := \mathbf{u}^* \cap \mathbf{v}^*$; thus, $(\mathbf{u}, \mathbf{v}, \mathbf{w}) \in \mathcal{P}_3(E)$. There are five possible configurations: $(\mathbf{u} \xrightarrow{d=e^{-1}} \mathbf{v},\ \mathbf{w})$; $(\mathbf{u} \xrightarrow{d\neq e^{-1}} \mathbf{v},\ \mathbf{w})$; $(\mathbf{u} \overset{d}{\rightsquigarrow} \mathbf{v} \overset{e}{\rightsquigarrow} \mathbf{w})$; $(\mathbf{v} \overset{e}{\rightsquigarrow} \mathbf{u} \overset{d}{\rightsquigarrow} \mathbf{w})$; and $(\mathbf{u} \overset{d}{\rightsquigarrow} \mathbf{w} \overset{e}{\leftarrow} \mathbf{v})$. We settle these using elements of $\mathcal{R}_5$, $\mathcal{R}_4$, $\mathcal{R}_6$, $\mathcal{R}_6$, and $\mathcal{R}_7$, respectively. Now $\emptyset \notin \mathbf{u} \boxplus \mathbf{v}$.

Step 4. We prove there exists some $\left[\begin{smallmatrix}\mathbf{x}\\ f\end{smallmatrix}\right] \in \mathcal{E}_1$ such that $\Delta_0(\left[\begin{smallmatrix}\mathbf{x}\\ f\end{smallmatrix}\right]) = 0$, $\Delta_1(\left[\begin{smallmatrix}\mathbf{x}\\ f\end{smallmatrix}\right]) < 0$, and $\mathbf{x} \in \mathbf{u} \boxplus \mathbf{v}$. We apply Step 3 to $\varphi\cdot[\![\,\left[\begin{smallmatrix}\mathbf{u}\\ d\end{smallmatrix}\right], \left[\begin{smallmatrix}\mathbf{x}\\ f\end{smallmatrix}\right]\,]\!]$ and $\varphi\cdot[\![\,\left[\begin{smallmatrix}\mathbf{x}\\ f\end{smallmatrix}\right], \left[\begin{smallmatrix}\mathbf{v}\\ e\end{smallmatrix}\right]\,]\!]$, and then apply Lemma 14.3 to complete the proof.

We now start Step 1, which is a reduction to the case where $\{\psi, \psi'\} \subseteq \mathcal{E}$. If $\psi' \in \mathcal{E}^{-1}$, we set $\rho := \psi'^{-2} \in \mathcal{R}_0 \cup \mathcal{R}_1$ and form the $(\mathcal{F}\frown\mathcal{E})$-path

$$\varphi\cdot\psi \rightarrowtail \varphi \rightarrowtail \varphi\cdot\psi'^{-1} = \varphi\cdot\rho\cdot\psi' = ({}^{\varphi}\rho)\cdot(\varphi\cdot\psi').$$

Since ${}^{\varphi}\rho \in \mathcal{N}$, we have $\Delta_i(\psi'^{-1}) = \Delta_i(\rho\cdot\psi') = \Delta_i(\psi')$, for $i = 0, 1$. In particular, $\varphi\cdot\psi'^{-1} \in \mathrm{V}\mathcal{T}$ and ${}^{\varphi}\rho \in \mathcal{M}$. It may be the case that $\psi'^{-1} = \psi$; in any event, we see that $(1, {}^{\varphi}\rho)$ is an element of $\mathcal{M} \times \mathcal{M}$ which $\widetilde{\hbar}_1$-decreases or $\widetilde{\hbar}_1$-respects p, and in the latter case gives an $(\mathcal{M}, \widetilde{\hbar})$-equivalent

$(\mathcal{T}, \hbar_1)$-peak which may be used in place of p. It therefore suffices to consider the case where $\psi' \in \mathcal{E}$. Similarly, we may assume that $\psi \in \mathcal{E}$.

Status of the peak: ψ, $\psi' \in \mathcal{E}$. We are free to interchange ψ and ψ', if desired.

We now start Step 2, which is a reduction to the case where $\{\psi, \psi'\} \subseteq \mathcal{E}_1$.

14.4. **Lemma.** *If $\{\psi, \psi'\} \cap \mathcal{E}_0 \neq \emptyset$, then p is $(\mathcal{M}, \widetilde{\hbar}_1)$-decreasable.*

Proof. It is not difficult to see that $\Delta_1(\mathcal{E}_0) = \{0\}$; hence, $\{\psi, \psi'\} \cap \mathcal{E}_1 \neq \emptyset$. By interchanging ψ and ψ' if necessary, we may assume that $\psi \in \mathcal{E}_1$ and $\psi' \in \mathcal{E}_0$, say $\psi = \left[\begin{smallmatrix}\mathbf{v}\\ e\end{smallmatrix}\right]$, $\psi' = (d \leftrightarrow f)$. We set $\rho := \left[\begin{smallmatrix}\mathbf{v}\\ e\end{smallmatrix}\right]\cdot(d{\leftrightarrow}f)\cdot\left[\begin{smallmatrix}\mathbf{v}^{(d\leftrightarrow f)}\\ e^{(d\leftrightarrow f)}\end{smallmatrix}\right]\cdot(d{\leftrightarrow}f)^{-1} \in \mathcal{R}_3$, and form the $(\mathcal{F}{\frown}\mathcal{E})$-path

$$\varphi\cdot\left[\begin{smallmatrix}\mathbf{v}\\ e\end{smallmatrix}\right] \rightarrowtail \varphi\cdot\left[\begin{smallmatrix}\mathbf{v}\\ e\end{smallmatrix}\right]\cdot(d \leftrightarrow f) \rightarrowtail \varphi\cdot\left[\begin{smallmatrix}\mathbf{v}\\ e\end{smallmatrix}\right]\cdot(d \leftrightarrow f)\cdot\left[\begin{smallmatrix}\mathbf{v}^{(d\leftrightarrow f)}\\ e^{(d\leftrightarrow f)}\end{smallmatrix}\right] = \varphi\cdot\rho\cdot(d \leftrightarrow f) = ({}^{\varphi}\rho)\cdot(\varphi\cdot(d \leftrightarrow f)).$$

Now $\Delta_0 = 0$ along this path since

$$\Delta_0\big(\left[\begin{smallmatrix}\mathbf{v}\\ e\end{smallmatrix}\right]\cdot(d \leftrightarrow f)\big) = \Delta_0\big(\left[\begin{smallmatrix}\mathbf{v}\\ e\end{smallmatrix}\right]\big) = 0 \text{ and } \Delta_0(\rho\cdot(d \leftrightarrow f)) = \Delta_0((d \leftrightarrow f)) = 0.$$

In particular, ${}^{\varphi}\rho \in \mathcal{M}$. Since $\Delta_1\big(\left[\begin{smallmatrix}\mathbf{v}\\ e\end{smallmatrix}\right]\big) + \Delta_1((d \leftrightarrow f)) < 0$, we also have

$$\Delta_1\big(\left[\begin{smallmatrix}\mathbf{v}\\ e\end{smallmatrix}\right]\cdot(d \leftrightarrow f)\big) = \Delta_1\big(\left[\begin{smallmatrix}\mathbf{v}\\ e\end{smallmatrix}\right]\big) < 0, \quad \Delta_1(\rho\cdot(d \leftrightarrow f)) = \Delta_1((d \leftrightarrow f)) = 0,$$

and

$$\widetilde{\hbar}_1\Big(\varphi\cdot\left[\begin{smallmatrix}\mathbf{v}\\ e\end{smallmatrix}\right] \rightarrowtail \varphi\cdot\left[\begin{smallmatrix}\mathbf{v}\\ e\end{smallmatrix}\right]\cdot(d \leftrightarrow f) \rightarrowtail ({}^{\varphi}\rho)\cdot\big(\varphi\cdot(d \leftrightarrow f)\big)\Big) \prec \widetilde{\hbar}_1\big(\varphi\cdot\left[\begin{smallmatrix}\mathbf{v}\\ e\end{smallmatrix}\right] \leftarrowtail \varphi \rightarrowtail \varphi\cdot(d \leftrightarrow f)\big).$$

Thus, $(1, {}^{\varphi}\rho)$ lies in $\mathcal{M} \times \mathcal{M}$ and $\widetilde{\hbar}_1$-decreases p. □

Status of the peak: $\psi = \left[\begin{smallmatrix}\mathbf{u}\\ d\end{smallmatrix}\right]$ and $\psi' = \left[\begin{smallmatrix}\mathbf{v}\\ e\end{smallmatrix}\right]$. As in Step 1, we are free to take $\psi' = \left[\begin{smallmatrix}\mathbf{v}^*\\ e^{-1}\end{smallmatrix}\right]$, if desired, by using $\rho := \left[\begin{smallmatrix}\mathbf{v}^*\\ e^{-1}\end{smallmatrix}\right]\cdot\left[\begin{smallmatrix}\mathbf{v}\\ e\end{smallmatrix}\right]^{-1} \in \mathcal{R}_2$. Similarly, we are free to take $\psi = \left[\begin{smallmatrix}\mathbf{u}^*\\ d^{-1}\end{smallmatrix}\right]$, if desired.

Step 3 is the following, which is a reduction to the case $\emptyset \notin \mathbf{u} \boxplus \mathbf{v}$.

14.5. **Lemma.** *If $\emptyset \in \mathbf{u} \boxplus \mathbf{v}$, then p is $(\mathcal{M}, \widetilde{\hbar}_1)$-decreasable.*

Proof. Here, by taking $\psi = \left[\begin{smallmatrix}\mathbf{u}^*\\ d^{-1}\end{smallmatrix}\right]$, if necessary, and taking $\psi' = \left[\begin{smallmatrix}\mathbf{v}^*\\ e^{-1}\end{smallmatrix}\right]$, if necessary, we may assume that $\mathbf{u} \cap \mathbf{v} = \emptyset$. Set $\mathbf{w} := \mathbf{u}^* \cap \mathbf{v}^*$. Now $(\mathbf{u}, \mathbf{v}, \mathbf{w}) \in \mathcal{P}_3(E)$, $d \in \mathbf{u}$, $d^{-1} \in \mathbf{v} \cup \mathbf{w}$, $e \in \mathbf{v}$, and $e^{-1} \in \mathbf{u} \cup \mathbf{w}$. We consider five cases.

Case 1: $d^{-1} \in \mathbf{v}$, $e^{-1} \in \mathbf{u}$, and $d = e^{-1}$; thus, $(\mathbf{u} \overset{d=e^{-1}}{\rightsquigarrow} \mathbf{v},\ \mathbf{w}) \in \mathcal{P}_3(E)$.

By interchanging $\left[\begin{smallmatrix}\mathbf{u}\\ d\end{smallmatrix}\right]$ and $\left[\begin{smallmatrix}\mathbf{v}\\ e\end{smallmatrix}\right]$, if necessary, we may assume that $\Delta_1\big(\left[\begin{smallmatrix}\mathbf{u}\\ d\end{smallmatrix}\right]\big) < 0$. We set $\rho := \left[\begin{smallmatrix}\mathbf{u}\\ d\end{smallmatrix}\right]\cdot(d{\leftrightarrow}d^{-1})\cdot\left[\begin{smallmatrix}\mathbf{w}\cup\{d\}\\ d\end{smallmatrix}\right]\cdot\left[\begin{smallmatrix}\mathbf{v}\\ d^{-1}\end{smallmatrix}\right]^{-1} \in \mathcal{R}_5$, and form the $(\mathcal{F}{\frown}\mathcal{E})$-path

$$\varphi\cdot\left[\begin{smallmatrix}\mathbf{u}\\ d\end{smallmatrix}\right] \rightarrowtail \varphi\cdot\left[\begin{smallmatrix}\mathbf{u}\\ d\end{smallmatrix}\right]\cdot(d \leftrightarrow d^{-1}) \rightarrowtail \varphi\cdot\left[\begin{smallmatrix}\mathbf{u}\\ d\end{smallmatrix}\right]\cdot(d \leftrightarrow d^{-1})\cdot\left[\begin{smallmatrix}\mathbf{w}\cup\{d\}\\ d\end{smallmatrix}\right] = \varphi\cdot\rho\cdot\left[\begin{smallmatrix}\mathbf{v}\\ d^{-1}\end{smallmatrix}\right] = {}^{\varphi}\rho\cdot\varphi\cdot\left[\begin{smallmatrix}\mathbf{v}\\ e\end{smallmatrix}\right].$$

It follows as in the proof of Lemma 14.4 that $(1, {}^{\varphi}\rho)$ lies in $\mathcal{M} \times \mathcal{M}$ and $\widetilde{\hbar}_1$-decreases p.

Case 2: $d^{-1} \in \mathbf{v}$, $e^{-1} \in \mathbf{u}$, and $d \neq e^{-1}$; thus, $(\mathbf{u} \overset{d\neq e^{-1}}{\rightsquigarrow} \mathbf{v},\ \mathbf{w}) \in \mathcal{P}_3(E)$.

By four applications of (10.1.6), we have

$$\Delta_1\big(\left[\begin{smallmatrix}\mathbf{u}\\ e^{-1}\end{smallmatrix}\right]\big) + \Delta_1\big(\left[\begin{smallmatrix}\mathbf{v}\\ d^{-1}\end{smallmatrix}\right]\big) = \Delta_1\big(\left[\begin{smallmatrix}\mathbf{u}\\ d\end{smallmatrix}\right]\big) + \Delta_1\big(\left[\begin{smallmatrix}\mathbf{v}\\ e\end{smallmatrix}\right]\big) < 0.$$

By interchanging $\left[\begin{smallmatrix}\mathbf{u}\\ d\end{smallmatrix}\right]$ and $\left[\begin{smallmatrix}\mathbf{v}\\ e\end{smallmatrix}\right]$, if necessary, we may assume that $\Delta_1\big(\left[\begin{smallmatrix}\mathbf{v}\\ d^{-1}\end{smallmatrix}\right]\big) < 0$. Also, $\Delta_0\big(\left[\begin{smallmatrix}\mathbf{v}\\ d^{-1}\end{smallmatrix}\right]\big) = 0$, since

$$0 \leqslant \Delta_0\big(\left[\begin{smallmatrix}\mathbf{v}\\ d^{-1}\end{smallmatrix}\right]\big) \leqslant \Delta_0\big(\left[\begin{smallmatrix}\mathbf{u}\\ e^{-1}\end{smallmatrix}\right]\big) + \Delta_0\big(\left[\begin{smallmatrix}\mathbf{v}\\ d^{-1}\end{smallmatrix}\right]\big) \overset{(10.1.6)}{=} \Delta_0\big(\left[\begin{smallmatrix}\mathbf{u}\\ d\end{smallmatrix}\right]\big) + \Delta_0\big(\left[\begin{smallmatrix}\mathbf{v}\\ e\end{smallmatrix}\right]\big) = 0.$$

By Case 1, the $(\mathcal{T},\hbar_1)$-peak $\varphi\cdot[\![\,\genfrac{[}{]}{0pt}{}{\mathbf{u}}{d},\genfrac{[}{]}{0pt}{}{\mathbf{v}}{d^{-1}}\,]\!]$ is $(\mathcal{M},\widetilde{\hbar}_1)$-decreasable. By Lemma 14.3, it suffices to show that the the $(\mathcal{T},\hbar_1)$-peak $\varphi\cdot[\![\,\genfrac{[}{]}{0pt}{}{\mathbf{v}}{d^{-1}},\genfrac{[}{]}{0pt}{}{\mathbf{v}}{e}\,]\!]$ is also $(\mathcal{M},\widetilde{\hbar}_1)$-decreasable. We set $\rho := \genfrac{[}{]}{0pt}{}{\mathbf{v}}{d^{-1}}\cdot(d^{-1}\leftrightarrow e)\cdot\genfrac{[}{]}{0pt}{}{\mathbf{v}}{d^{-1}}\cdot\genfrac{[}{]}{0pt}{}{\mathbf{v}}{e}^{-1}\in\mathcal{R}_4$, and form the $(\mathcal{F}\frown\mathcal{E})$-path

$$\varphi\cdot\genfrac{[}{]}{0pt}{}{\mathbf{v}}{d^{-1}}\rightarrowtail\varphi\cdot\genfrac{[}{]}{0pt}{}{\mathbf{v}}{d^{-1}}\cdot(d^{-1}\leftrightarrow e)\rightarrowtail\varphi\cdot\genfrac{[}{]}{0pt}{}{\mathbf{v}}{d^{-1}}\cdot(d^{-1}\leftrightarrow e)\cdot\genfrac{[}{]}{0pt}{}{\mathbf{v}}{d^{-1}}=\varphi\cdot\rho\cdot\genfrac{[}{]}{0pt}{}{\mathbf{v}}{e}={}^{\varphi}\rho\cdot\varphi\cdot\genfrac{[}{]}{0pt}{}{\mathbf{v}}{e}.$$

It follows as in the proof of Lemma 14.4 that $(1,{}^{\varphi}\rho)$ lies in $\mathcal{M}\times\mathcal{M}$ and $\widetilde{\hbar}_1$-decreases $\varphi\cdot[\![\,\genfrac{[}{]}{0pt}{}{\mathbf{v}}{d^{-1}},\genfrac{[}{]}{0pt}{}{\mathbf{v}}{e}\,]\!]$, as desired.

Case 3: $d^{-1}\in\mathbf{v}$ and $e^{-1}\in\mathbf{w}$; thus, $(\mathbf{u}\overset{d}{\rightsquigarrow}\mathbf{v}\overset{e}{\rightsquigarrow}\mathbf{w})\in\mathcal{P}_3(E)$.

Here, we set $\rho := \genfrac{[}{]}{0pt}{}{\mathbf{v}}{e}\cdot\genfrac{[}{]}{0pt}{}{\mathbf{u}}{d}\cdot\genfrac{[}{]}{0pt}{}{\mathbf{w}\cup\{d^{-1}\}}{e^{-1}}\cdot\genfrac{[}{]}{0pt}{}{\mathbf{u}}{d}^{-1}\in\mathcal{R}_6$, and form the $(\mathcal{F}\frown\mathcal{E})$-path

$$\varphi\cdot\genfrac{[}{]}{0pt}{}{\mathbf{v}}{e}\rightarrowtail\varphi\cdot\genfrac{[}{]}{0pt}{}{\mathbf{v}}{e}\cdot\genfrac{[}{]}{0pt}{}{\mathbf{u}}{d}\rightarrowtail\varphi\cdot\genfrac{[}{]}{0pt}{}{\mathbf{v}}{e}\cdot\genfrac{[}{]}{0pt}{}{\mathbf{u}}{d}\cdot\genfrac{[}{]}{0pt}{}{\mathbf{w}\cup\{d^{-1}\}}{e^{-1}}=\varphi\cdot\rho\cdot\genfrac{[}{]}{0pt}{}{\mathbf{u}}{d}={}^{\varphi}\rho\cdot\varphi\cdot\genfrac{[}{]}{0pt}{}{\mathbf{u}}{d}.$$

By (12.1.R5),

$$\Delta_0\big(\genfrac{[}{]}{0pt}{}{\mathbf{v}}{e}\cdot\genfrac{[}{]}{0pt}{}{\mathbf{u}}{d}\big)=\Delta_0\big(\genfrac{[}{]}{0pt}{}{\mathbf{v}}{e}\big)+\Delta_0\big(\genfrac{[}{]}{0pt}{}{\mathbf{u}}{d}\big)=0\text{ and }\Delta_1\big(\genfrac{[}{]}{0pt}{}{\mathbf{v}}{e}\cdot\genfrac{[}{]}{0pt}{}{\mathbf{u}}{d}\big)=\Delta_1\big(\genfrac{[}{]}{0pt}{}{\mathbf{v}}{e}\big)+\Delta_1\big(\genfrac{[}{]}{0pt}{}{\mathbf{u}}{d}\big)<0.$$

Thus, $\hbar_0(\varphi\cdot\genfrac{[}{]}{0pt}{}{\mathbf{v}}{e}\cdot\genfrac{[}{]}{0pt}{}{\mathbf{u}}{d})=\hbar_0(\varphi)$ and $\hbar_1(\varphi\cdot\genfrac{[}{]}{0pt}{}{\mathbf{v}}{e}\cdot\genfrac{[}{]}{0pt}{}{\mathbf{u}}{d})<\hbar_1(\varphi)$. As in the proof of Lemma 14.4, it follows that $({}^{\varphi}\rho,1)$ lies in $\mathcal{M}\times\mathcal{M}$ and $\widetilde{\hbar}_1$-decreases p.

Case 4: $d^{-1}\in\mathbf{w}$ and $e^{-1}\in\mathbf{u}$; thus, $(\mathbf{v}\overset{e}{\rightsquigarrow}\mathbf{u}\overset{d}{\rightsquigarrow}\mathbf{w})\in\mathcal{P}_3(E)$.

Here, we may interchange $\genfrac{[}{]}{0pt}{}{\mathbf{u}}{d}$ and $\genfrac{[}{]}{0pt}{}{\mathbf{v}}{e}$, and apply Case 3.

Case 5: $d^{-1}\in\mathbf{w}$ and $e^{-1}\in\mathbf{w}$; thus, $(\mathbf{u}\overset{d}{\rightsquigarrow}\mathbf{w}\overset{e}{\leftarrowtail}\mathbf{v})\in\mathcal{P}_3(E)$.

Here, the argument is similar to that for Case 3. We set $\rho := \genfrac{[}{]}{0pt}{}{\mathbf{v}}{e}\cdot\genfrac{[}{]}{0pt}{}{\mathbf{u}}{d}\cdot\genfrac{[}{]}{0pt}{}{\mathbf{v}}{e}\cdot\genfrac{[}{]}{0pt}{}{\mathbf{u}}{d}^{-1}\in\mathcal{R}_7$, and form the $(\mathcal{F}\frown\mathcal{E})$-path

$$\varphi\cdot\genfrac{[}{]}{0pt}{}{\mathbf{v}}{e}\rightarrowtail\varphi\cdot\genfrac{[}{]}{0pt}{}{\mathbf{v}}{e}\cdot\genfrac{[}{]}{0pt}{}{\mathbf{u}}{d}\rightarrowtail\varphi\cdot\genfrac{[}{]}{0pt}{}{\mathbf{v}}{e}\cdot\genfrac{[}{]}{0pt}{}{\mathbf{u}}{d}\cdot\genfrac{[}{]}{0pt}{}{\mathbf{v}}{e}=\varphi\cdot\rho\cdot\genfrac{[}{]}{0pt}{}{\mathbf{u}}{d}={}^{\varphi}\rho\cdot\varphi\cdot\genfrac{[}{]}{0pt}{}{\mathbf{u}}{d}.$$

By (12.1.R6), $\Delta_0(\genfrac{[}{]}{0pt}{}{\mathbf{v}}{e}\cdot\genfrac{[}{]}{0pt}{}{\mathbf{u}}{d})=0$ and $\Delta_1(\genfrac{[}{]}{0pt}{}{\mathbf{v}}{e}\cdot\genfrac{[}{]}{0pt}{}{\mathbf{u}}{d})<0$. As in the proof of Lemma 14.4, it follows that $({}^{\varphi}\rho,1)$ lies in $\mathcal{M}\times\mathcal{M}$ and $\widetilde{\hbar}_1$-decreases p. □

Status of the peak: $\emptyset\notin\mathbf{u}\boxplus\mathbf{v}$.

14.6. **Lemma.** *There exists* $\genfrac{[}{]}{0pt}{}{\mathbf{x}}{f}\in\mathcal{E}_1$ *such that* $\mathbf{x}\in\mathbf{u}\boxplus\mathbf{v}$, $\Delta_1(\genfrac{[}{]}{0pt}{}{\mathbf{x}}{f})<0$, *and* $\Delta_0(\genfrac{[}{]}{0pt}{}{\mathbf{x}}{f})=0$.

Proof. For each $j\in\{0,1\}$, we set $\delta_j := \delta_{W_j^{\varphi},E}$; as $(\mathbf{u}\cap\mathbf{v},\mathbf{u}^*\cap\mathbf{v}^*,\mathbf{u}\cap\mathbf{v}^*,\mathbf{u}^*\cap\mathbf{v})\in\mathcal{P}_4(E)$, (11.1.P1) gives

$$\delta_j(\mathbf{u}\cap\mathbf{v})+\delta_j(\mathbf{u}^*\cap\mathbf{v}^*)\leqslant\delta_j(\mathbf{u})+\delta_j(\mathbf{v})\geqslant\delta_j(\mathbf{u}\cap\mathbf{v}^*)+\delta_j(\mathbf{u}^*\cap\mathbf{v}).\tag{14.6.1}$$

We consider two cases.

Case 1: For each $\mathbf{y}\in\mathbf{u}\boxplus\mathbf{v}$, $\mathbf{y}\cap\{d,e\}^{\pm1}\neq\emptyset$.

Here, by replacing $\genfrac{[}{]}{0pt}{}{\mathbf{v}}{e}$ with $\genfrac{[}{]}{0pt}{}{\mathbf{v}^*}{e^{-1}}$, if necessary, we may assume that $d\in\mathbf{u}\cap\mathbf{v}$. Then $e\in\mathbf{u}^*\cap\mathbf{v}$, $d^{-1}\in\mathbf{u}^*\cap\mathbf{v}^*$, and $e^{-1}\in\mathbf{u}\cap\mathbf{v}^*$. For each $j\in\{0,1\}$, (14.6.1) and (10.1.6) give

$$\Delta_j(\genfrac{[}{]}{0pt}{}{\mathbf{u}\cap\mathbf{v}}{d}+\Delta_j(\genfrac{[}{]}{0pt}{}{\mathbf{u}^*\cap\mathbf{v}^*}{d^{-1}})+\Delta_j(\genfrac{[}{]}{0pt}{}{\mathbf{u}\cap\mathbf{v}^*}{e^{-1}}+\Delta_j(\genfrac{[}{]}{0pt}{}{\mathbf{u}^*\cap\mathbf{v}}{e}))\leqslant2\cdot\Delta_j(\genfrac{[}{]}{0pt}{}{\mathbf{u}}{d})+2\cdot\Delta_j(\genfrac{[}{]}{0pt}{}{\mathbf{v}}{e}).$$

For $j=1$, since $\Delta_1(\genfrac{[}{]}{0pt}{}{\mathbf{u}}{d})+\Delta_1(\genfrac{[}{]}{0pt}{}{\mathbf{v}}{e})<0$, we see that there exists some $\genfrac{[}{]}{0pt}{}{\mathbf{x}}{f}\in\mathcal{E}_1$ such that $\mathbf{x}\in\mathbf{u}\boxplus\mathbf{v}$ and $\Delta_1(\genfrac{[}{]}{0pt}{}{\mathbf{x}}{f})<0$. For $j=0$, since $\Delta_0(\genfrac{[}{]}{0pt}{}{\mathbf{u}}{d})=0=\Delta_0(\genfrac{[}{]}{0pt}{}{\mathbf{v}}{e})$, we see that $\Delta_0(\genfrac{[}{]}{0pt}{}{\mathbf{x}}{f})=0$.

Case 2: For some $\mathbf{y} \in \mathbf{u} \boxplus \mathbf{v}$, $\mathbf{y} \cap \{d,e\}^{\pm 1} = \emptyset$.

Here, by replacing $\left[\begin{smallmatrix}\mathbf{u}\\ d\end{smallmatrix}\right]$ with $\left[\begin{smallmatrix}\mathbf{u}^*\\ d^{-1}\end{smallmatrix}\right]$, if necessary, and replacing $\left[\begin{smallmatrix}\mathbf{v}\\ e\end{smallmatrix}\right]$ with $\left[\begin{smallmatrix}\mathbf{v}^*\\ e^{-1}\end{smallmatrix}\right]$, if necessary, we may assume that $\mathbf{u} \cap \mathbf{v} \cap \{d,e\} = \emptyset$. Then $d \in \mathbf{u} \cap \mathbf{v}^*$ and $e \in \mathbf{u}^* \cap \mathbf{v}$. For each $j \in \{0,1\}$, (14.6.1) and (10.1.6) give

$$\Delta_j(\left[\begin{smallmatrix}\mathbf{u}\cap\mathbf{v}^*\\ d\end{smallmatrix}\right]) + \Delta_j(\left[\begin{smallmatrix}\mathbf{u}^*\cap\mathbf{v}\\ e\end{smallmatrix}\right]) \leqslant \Delta_j(\left[\begin{smallmatrix}\mathbf{u}\\ d\end{smallmatrix}\right]) + \Delta_j(\left[\begin{smallmatrix}\mathbf{v}\\ e\end{smallmatrix}\right]).$$

As in Case 1, there exists some $\left[\begin{smallmatrix}\mathbf{x}\\ f\end{smallmatrix}\right]$ with the desired properties. □

In Lemma 14.6, the $(\mathcal{T}, \hbar_1)$-peaks $\varphi \cdot [\![\left[\begin{smallmatrix}\mathbf{u}\\ d\end{smallmatrix}\right], \left[\begin{smallmatrix}\mathbf{w}\\ f\end{smallmatrix}\right]]\!]$ and $\varphi \cdot [\![\left[\begin{smallmatrix}\mathbf{w}\\ f\end{smallmatrix}\right], \left[\begin{smallmatrix}\mathbf{v}\\ e\end{smallmatrix}\right]]\!]$ are $(\mathcal{M}, \widetilde{\hbar}_1)$-decreasable by Lemma 14.5. Now Theorem 14.2 follows from Lemma 14.3. □

15. McCool's two-layer results

15.1. **Notation.** Let E, F, $\mathcal{E}$, $\mathcal{F}$, $\mathcal{N}$, and $\mathcal{R}$ be as in Section 9.

By Nielsen's Theorem 7.2, $\mathcal{F}/\mathcal{N} = \mathrm{Aut}(F)$.

By Higgins & Lyndon's Corollary 12.2, $\mathcal{R} \subseteq \mathcal{N}$.

For each $i \in \{0,1\}$, let W_i be a finite set of words in F. Then we have $\mathcal{N} \leqslant \mathcal{F}_{W_i}$ and $\mathcal{F}_{W_i}/\mathcal{N} = \mathrm{Aut}(F, W_i)$. For each $\varphi \in \mathcal{F}$, set $\hbar_i(\varphi) := E\text{-length}(W_i^{\varphi})$. Then $\hbar_i : \mathcal{F} \to \mathbb{N}$ is an $\mathcal{F}_{W_i}$-map. For each nonempty subset $\mathcal{V}$ of $\mathcal{F}$, set $\mathcal{V}_{|\hbar_i=\min} := \{\varphi \in \mathcal{V} : \hbar_i(\varphi) = \min\big(\hbar_i(\mathcal{V})\big)\}$.

Let $\mathfrak{T}_0$ be any component of the $\mathcal{F}_{W_0}$-forest $(\mathcal{F}_{|\hbar_0=\min}) \curvearrowleft \mathcal{E}$. Set $\mathcal{M} := \langle ({}^{\mathrm{V}\mathfrak{T}_0}\mathcal{R})_{\mathfrak{T}_0} \rangle$, where $({}^{\mathrm{V}\mathfrak{T}_0}\mathcal{R})_{\mathfrak{T}_0} := \{{}^{\varphi}\rho \mid \varphi \in \mathrm{V}\mathfrak{T}_0, \rho \in \mathcal{R}, ({}^{\varphi}\rho) \cdot \mathfrak{T}_0 = \mathfrak{T}_0\}$. Since ${}^{\mathrm{V}\mathfrak{T}_0}\mathcal{R} \subseteq \mathcal{N}$, we see $\mathcal{M} \leqslant \mathcal{F}_{(W_0, \mathfrak{T}_0, W_1)}$.

Let $\mathfrak{T}_1$ be any component of the $\mathcal{F}_{(W_0,\mathfrak{T}_0,W_1)}$-forest $\big((\mathrm{V}\mathfrak{T}_0)_{|\hbar_1=\min}\big) \curvearrowleft \mathcal{E}$. □

15.2. **Theorem.** *With* Notation 15.1*, the following hold.*

(*i*) *For all φ, $\varphi' \in \mathrm{V}\mathfrak{T}_0$, there exists some $\mu \in \mathcal{M}$ such that the $\mathfrak{T}_0$-path $p := [\![\varphi, \mu \cdot \varphi']\!]$ has an $\hbar_1$-valley factorization $p = p_{\,dec} \circledcirc p_{\,const} \circledcirc p_{\,inc}$. Moreover, if $\varphi' \in (\mathrm{V}\mathfrak{T}_0)_{|\hbar_1=\min}$, then $\mu \cdot \varphi' \in (\mathrm{V}\mathfrak{T}_0)_{|\hbar_1=\min}$ and, hence, $p_{\,inc}$ has length zero and all the vertices of $p_{\,const}$ lie in $(\mathrm{V}\mathfrak{T}_0)_{|\hbar_1=\min}$. In particular, either $\varphi \in (\mathrm{V}\mathfrak{T}_0)_{|\hbar_1=\min}$ or there exists some length-one $\hbar_1$-decreasing $\mathfrak{T}_0$-path initiating at φ.*

(*ii*) $\mathcal{F}_{(W_0,\mathfrak{T}_0,W_1)} = \mathcal{M} \cdot \mathcal{F}_{(W_0,\mathfrak{T}_0,W_1,\mathfrak{T}_1)}$.

Proof. (*i*). On applying Theorem 14.2 with $\mathcal{T} := \mathfrak{T}_0$, we see that each $(\mathfrak{T}_0, \hbar_1)$-peak is $(\mathcal{M}, \widetilde{\hbar}_1)$-decreasable. By Rapaport's Lemma 13.2, the first sentence of (*i*) holds. For the second sentence, since $\hbar_1$ is an $\mathcal{F}_{W_1}$-map, $\mu \cdot \varphi' \in (\mathrm{V}\mathfrak{T}_0)_{|\hbar_1=\min}$. The third sentence is clear, since $(\mathrm{V}\mathfrak{T}_0)_{|\hbar_1=\min}$ is nonempty.

(*ii*). Consider an arbitrary $\kappa \in \mathcal{F}_{(W_0,\mathfrak{T}_0,W_1)}$. Choose any $\varphi \in \mathrm{V}\mathfrak{T}_1$. Since $\hbar_1$ is an $\mathcal{F}_{W_1}$-map, we see that $\kappa \cdot \varphi \in (\mathrm{V}\mathfrak{T}_0)_{|\hbar_1=\min}$. We now apply the second sentence of (*i*) with $\varphi' := \kappa \cdot \varphi$, and obtain $\mu \in \mathcal{M}$ such that the $\mathfrak{T}_0$-path $p := [\![\varphi, \mu \cdot \kappa \cdot \varphi]\!]$ has an $\hbar_1$-valley factorization $p = p_{\mathrm{dec}} \circledcirc p_{\mathrm{const}} \circledcirc p_{\mathrm{inc}}$ in which p_{dec} and p_{inc} have length zero. Hence, $p = p_{\mathrm{const}}$ is a $\mathfrak{T}_1$-path, and $\mu \cdot \kappa \cdot \varphi \in \mathrm{V}\mathfrak{T}_1$. Since the element $\mu \cdot \kappa$ of $\mathcal{F}_{(W_0,\mathfrak{T}_0,W_1)}$ permutes the components of the $\mathcal{F}_{(W_0,\mathfrak{T}_0,W_1)}$-forest $(\mathrm{V}\mathfrak{T}_0)_{|\hbar_1=\min} \curvearrowleft \mathcal{E}$, we see that $\mu \cdot \kappa \in \mathcal{F}_{(W_0,\mathfrak{T}_0,W_1,\mathfrak{T}_1)}$. Hence, $\kappa \in \mathcal{M} \cdot \mathcal{F}_{(W_0,\mathfrak{T}_0,W_1,\mathfrak{T}_1)}$, as desired. □

16. McCool's presentations

In this section, we give an explicit formula for McCool's presentation for $\mathrm{Aut}(F, S)$.

16.1. **Notation.** Let E, F, $\mathcal{E}$, $\mathcal{F}$, $\mathcal{N}$, $\mathcal{R}$, $\mathbb{W}$, and S be as in Notation 9.2.

By Nielsen's Theorem 7.2, $\mathcal{F}_S/\mathcal{N} = \mathrm{Aut}(F, S)$.

By Higgins & Lyndon's Corollary 12.2, $\mathcal{R} \subseteq \mathcal{N}$.

We have a left-multiplicative $\mathcal{F}_S$-map $\hbar_S : \mathcal{F} \to \mathbb{N}$, $\varphi \mapsto \hbar_S(\varphi) := E\text{-length}(S^{\varphi})$.

For each nonempty subset $\mathcal{V}$ of $\mathcal{F}$, $\mathcal{V}_{|\hbar_S=\min} := \{\varphi \in \mathcal{V} : \hbar_S(\varphi) = \min\big(\hbar_S(\mathcal{V})\big)\}$.

We algorithmically construct an inclusion-maximal E-length-decreasing ($\mathbb{W}\curvearrowleft\mathcal{E}$)-path p_0 initiating at S; for example, we may choose a total order of the finite set $\mathcal{E}_1$, and then, at S and each successive vertex, apply the elements of $\mathcal{E}_1$ in order until either a neighbour with lesser E-length is found, or we know our path is inclusion-maximal and we stop. Let φ_0 denote the Schreier-label of the path p_0. The terminal vertex of p_0 is S^{φ_0}.

Define $\Upsilon := \mathbf{\Gamma}(S^{\varphi_0})$ as in Notation 9.2; thus,

$$\mathbb{W}_{S^{\varphi_0}} := \{W \in \mathbb{W} : E\text{-length}(W) \leqslant E\text{-length}(S^{\varphi_0})\},$$

and Υ is the component of $\mathbb{W}_{S^{\varphi_0}}\curvearrowleft\mathcal{E}$ which contains S^{φ_0}, which is then the basepoint. We choose a maximal subtree Υ_0 of Υ, and then $\overrightarrow{\mathrm{V}\Upsilon}$ and $\overset{\rightarrow\leftarrow}{\mathrm{E}'\Upsilon}$ are subsets of $\mathcal{F}$ given with bijective maps $\mathrm{V}\Upsilon \to \overrightarrow{\mathrm{V}\Upsilon}$ and $\mathrm{E}'\Upsilon \to \overset{\rightarrow\leftarrow}{\mathrm{E}'\Upsilon}$.

Let Γ, resp. Γ_0, denote the subgraph of $\mathbb{W}\curvearrowleft\mathcal{E}$ obtained by adding to Υ, resp. Υ_0, all the vertices and edges of the path p_0. Then Γ is a finite, connected, basepointed subgraph of $\mathbb{W}\curvearrowleft\mathcal{E}$ with basepoint $v_0 := S$, Γ_0 is a maximal subtree of Γ, and $\overrightarrow{\mathrm{V}\Gamma}$ and $\overset{\rightarrow\leftarrow}{\mathrm{E}'\Gamma}$ are subsets of $\mathcal{F}$ given with bijective maps $\mathrm{V}\Gamma \to \overrightarrow{\mathrm{V}\Gamma}$ and $\mathrm{E}'\Gamma \to \overset{\rightarrow\leftarrow}{\mathrm{E}'\Gamma}$. □

Our sole objective is to prove the following.

16.2. **Claim.** $\mathrm{Aut}(F, S) = \big\langle\, \overset{\rightarrow\leftarrow}{\mathrm{E}'\Gamma} \mid {}^{\overrightarrow{\mathrm{V}\Gamma}}\mathcal{R} \cap \langle\, \overset{\rightarrow\leftarrow}{\mathrm{E}'\Gamma}\,\rangle \,\big\rangle$ with the map $\overset{\rightarrow\leftarrow}{\mathrm{E}'\Gamma} \subseteq \mathcal{F} \to \mathrm{Aut}\,F$. □

16.3. **Notation.** The automorphism of $\mathcal{F}$ given by $\varphi \mapsto {}^{\varphi_0}\varphi$ carries $\mathcal{F}_{S^{\varphi_0}}$ to $\mathcal{F}_S$, carries $\mathcal{N}$ to itself, carries $\overset{\rightarrow\leftarrow}{\mathrm{E}'\Upsilon}$ to $\overset{\rightarrow\leftarrow}{\mathrm{E}'\Gamma}$, and carries ${}^{\overrightarrow{\mathrm{V}\Upsilon}}\mathcal{R}$ to ${}^{\overrightarrow{\mathrm{V}\Gamma}}\mathcal{R}$. Thus, it suffices to prove Claim 16.2 for $\mathrm{Aut}(F, S^{\varphi_0})$. Hence, we may replace S with S^{φ_0}, and thereby assume that $\varphi_0 = 1$.

In Notation 15.1, if we take $W_0 := \emptyset$, then we must have $\mathcal{T}_0 = \mathcal{F}\curvearrowleft\mathcal{E}$; if we further take $W_1 := S$, then $\hbar_1 = \hbar_S$, and the third sentence of Theorem 15.2(i) shows that $1 \in \mathcal{F}_{|\hbar_S=\min}$. Let $\mathcal{S}$ denote the component of the $\mathcal{F}_S$-forest $(\mathcal{F}_{|\hbar_S=\min})\curvearrowleft\mathcal{E}$ which contains 1. □

16.4. **Lemma.** *The surjective group morphism* $\mathcal{F} \to \mathrm{Aut}\,F$ *restricts to a surjective group morphism* $\mathcal{F}_{(S,\mathcal{S})} \to \mathrm{Aut}(F, S)$ *which has kernel* $\mathcal{N}_{\mathcal{S}}$.

Proof. In Notation 15.1, if we take $W_0 := \emptyset$, then we must have $\mathcal{T}_0 = \mathcal{F}\curvearrowleft\mathcal{E}$ and $\mathcal{M} = \langle\, {}^{\mathcal{F}}\mathcal{R}\,\rangle$; if we further take $W_1 := S$ and $\mathcal{T}_1 := \mathcal{S}$, then Theorem 15.2($ii$) says that $\mathcal{F}_S = \langle\, {}^{\mathcal{F}}\mathcal{R}\,\rangle \cdot \mathcal{F}_{(S,\mathcal{S})}$.

Since $\langle\, {}^{\mathcal{F}}\mathcal{R}\,\rangle \leqslant \mathcal{N} \leqslant \mathcal{F}_S$, we see that $\mathcal{F}_S = \mathcal{N} \cdot \mathcal{F}_{S,\mathcal{S}}$. Hence,

$$\mathrm{Aut}(F, S) = \mathcal{F}_S/\mathcal{N} = (\mathcal{N} \cdot \mathcal{F}_{(S,\mathcal{S})})/\mathcal{N} \simeq \mathcal{F}_{(S,\mathcal{S})}/(\mathcal{N} \cap \mathcal{F}_{(S,\mathcal{S})}) = \mathcal{F}_{(S,\mathcal{S})}/\mathcal{N}_{\mathcal{S}}.$$ □

To prove Claim 16.2, it now suffices to prove that $\overset{\rightarrow\leftarrow}{\mathrm{E}'\Gamma}$ is an $\mathcal{F}_{(S,\mathcal{S})}$-basis and that $\mathcal{N}_{\mathcal{S}}$ equals the normal closure of ${}^{\overrightarrow{\mathrm{V}\Gamma}}\mathcal{R} \cap \mathcal{F}_{(S,\mathcal{S})}$ in $\mathcal{F}_{(S,\mathcal{S})}$; we shall prove these two assertions in Lemma 16.5 and Corollary 16.8, respectively.

16.5. **Lemma.** $\overset{\rightarrow\leftarrow}{\mathrm{E}'\Gamma}$ *is an* $\mathcal{F}_{(S,\mathcal{S})}$*-basis, and* $\mathcal{F}_{(S,\mathcal{S})}\cdot\overrightarrow{\mathrm{V}\Gamma} = \mathrm{V}\mathcal{S}$.

Proof. Set $\widetilde{\mathrm{V}\Gamma} := \{\varphi \in \mathcal{F} : S^{\varphi} \in \mathrm{V}\Gamma\}$. Since $\mathrm{V}\Gamma \subseteq \mathbb{W}_S$ and $1 \in \mathcal{F}_{|\hbar_S=\min}$, we see that $\widetilde{\mathrm{V}\Gamma} \subseteq \mathcal{F}_{|\hbar_S=\min}$.

To see that $\mathrm{V}\mathcal{S} \subseteq \widetilde{\mathrm{V}\Gamma}$, consider any $\varphi \in \mathrm{V}\mathcal{S}$. Since the reduced $\mathcal{S}$-path from 1 to φ lies in $(\mathcal{F}_{|\hbar_S=\min})\curvearrowleft\mathcal{E}$, we see that $\hbar_S$ is constant along it. Hence, we have a $(\mathbb{W}\curvearrowleft\mathcal{E})$-path from S to S^{φ} along which E-length is constant. Hence, $S^{\varphi} \in \mathrm{V}\Gamma$. Thus, $\varphi \in \widetilde{\mathrm{V}\Gamma}$, as desired.

Now $\mathcal{S} \subseteq \widetilde{\mathrm{V}\Gamma}\curvearrowleft\mathcal{E} \subseteq (\mathcal{F}_{|\hbar_S=\min})\curvearrowleft\mathcal{E}$. Since $\mathcal{S}$ is the component of $(\mathcal{F}_{|\hbar_S=\min})\curvearrowleft\mathcal{E}$ which contains 1, we see that $\mathcal{S}$ is the component of $\widetilde{\mathrm{V}\Gamma}\curvearrowleft\mathcal{E}$ which contains 1.

We have a free group $\mathcal{F}$, an $\mathcal{F}$-basis $\mathcal{E}$, a right-exponential $\mathcal{F}$-set $\mathbb{W}$, a connected, full subgraph Γ of $\mathbb{W}\curvearrowleft\mathcal{E}$, and a basepoint $v_0 := S$ of Γ. We may now apply Definitions 6.1, and find that, as $\mathcal{S}$ is the component of $\widetilde{\mathrm{V}\Gamma}\curvearrowleft\mathcal{E}$ which contains 1, we have $\overline{\pi}(\Gamma, v_0)\cdot\overrightarrow{\mathrm{V}\Gamma} = \mathrm{V}\mathcal{S}$ and $\mathcal{F}_{(v_0,\mathcal{S})} = \overline{\pi}(\Gamma, v_0)$, which has basis $\overset{\to\leftarrow}{\mathrm{E}'\Gamma}$. □

16.6. **Corollary** (Rapaport-McCool). *The group* $\mathrm{Aut}(F, S)$ *is generated by the image in* $\mathrm{Aut}\, F$ *of the finite subset* $\overset{\to\leftarrow}{\mathrm{E}'\Gamma}$ *of* $\mathcal{F}$ *.* □

16.7. **Lemma.** $\mathcal{N}_{\mathcal{S}} = \langle ({}^{\mathrm{V}\mathcal{S}}\mathcal{R})_{\mathcal{S}} \rangle$.

Proof. Let $\mathcal{E}_0^+$ denote the set of those elements of $\mathcal{E}$ whose image in $\mathrm{Aut}\, F$ lies in $\mathrm{Sym}_{\pm}(E^{\pm 1})$. It follows from (8.1.1) that $\mathcal{E}_0^+ = \mathcal{E}_0 \cup \big\{ \big[\begin{smallmatrix}\{e\}\\ e\end{smallmatrix}\big], \big[\begin{smallmatrix}\{e\}^*\\ e^{-1}\end{smallmatrix}\big] : e \in E^{\pm 1} \big\}$.

In Notation 15.1, let us take $W_0 := S$, $\mathcal{T}_0 := \mathcal{S}$, and $W_1 := E^{\pm 1}$. It is clear that $\min\big(\hbar_{E^{\pm 1}}(\mathcal{F})\big) = 2|E|$. Thus, $1 \in \mathrm{V}\mathcal{S}_{|\hbar_{E^{\pm 1}}=\min}$. It is not difficult to see that $\langle \mathcal{E}_0^+ \mid \emptyset \rangle\curvearrowleft\mathcal{E}_0^+$ is a component of the forest $(\mathrm{V}\mathcal{S}_{|\hbar_{E^{\pm 1}}=\min})\curvearrowleft\mathcal{E}$. Thus, we may take $\mathcal{T}_1 := \langle \mathcal{E}_0^+ \mid \emptyset \rangle\curvearrowleft\mathcal{E}_0^+$. Now Theorem 15.2(*ii*) says $\mathcal{F}_{(S,\mathcal{S},E^{\pm 1})} = \langle ({}^{\mathrm{V}\mathcal{S}}\mathcal{R})_{\mathcal{S}} \rangle \cdot \langle \mathcal{E}_0^+ \mid \emptyset \rangle$. Hence, $\mathcal{N}_{\mathcal{S}} = \langle ({}^{\mathrm{V}\mathcal{S}}\mathcal{R})_{\mathcal{S}} \rangle \cdot \big(\mathcal{N} \cap \langle \mathcal{E}_0^+ \mid \emptyset \rangle\big)$.

It remains to show that $\mathcal{N} \cap \langle \mathcal{E}_0^+ \mid \emptyset \rangle \leqslant \langle ({}^{\mathrm{V}\mathcal{S}}\mathcal{R})_{\mathcal{S}} \rangle$. Now $\mathcal{N} \cap \langle \mathcal{E}_0^+ \mid \emptyset \rangle$ is the kernel of the map $\langle \mathcal{E}_0^+ \mid \emptyset \rangle \to \mathrm{Sym}_{\pm}(E^{\pm 1})$. Set $\mathcal{R}_0^+ := \mathcal{R} \cap \langle \mathcal{E}_0^+ \mid \emptyset \rangle$. For $e \in E^{\pm 1}$, $(\{e\} \overset{e}{\rightsquigarrow} \{e\}^*,\ \emptyset) \in \mathcal{P}_3(E)$ and $\big[\begin{smallmatrix}\{e\}\\ e\end{smallmatrix}\big]\cdot(e{\leftrightarrow}e^{-1})\cdot\big[\begin{smallmatrix}\emptyset\cup\{e\}\\ e\end{smallmatrix}\big]\cdot\big[\begin{smallmatrix}\{e\}^*\\ e^{-1}\end{smallmatrix}\big]^{-1} \in \mathcal{R}_5$; thus, $\big[\begin{smallmatrix}\{e\}\\ e\end{smallmatrix}\big]\cdot(e{\leftrightarrow}e^{-1}) \in \langle \mathcal{R}_0^+ \rangle$. It may now be seen that $\langle \mathcal{E}_0^+ \mid \mathcal{R}_0^+ \rangle = \mathrm{Sym}_{\pm}(E^{\pm 1})$. Hence, $\mathcal{N} \cap \langle \mathcal{E}_0^+ \mid \emptyset \rangle = {}^{\langle \mathcal{E}_0^+ \mid \emptyset \rangle}\mathcal{R}_0^+ \leqslant \langle ({}^{\mathrm{V}\mathcal{S}}\mathcal{R})_{\mathcal{S}} \rangle$, as desired. □

16.8. **Corollary.** $\mathcal{N}_{\mathcal{S}}$ *equals the normal closure of* ${}^{\overrightarrow{\mathrm{V}\Gamma}}\mathcal{R} \cap \mathcal{F}_{(S,\mathcal{S})}$ *in* $\mathcal{F}_{(S,\mathcal{S})}$.

Proof. By Lemma 16.7, $\mathcal{N}_{\mathcal{S}} = \langle ({}^{\mathrm{V}\mathcal{S}}\mathcal{R})_{\mathcal{S}} \rangle$; clearly, $({}^{\mathrm{V}\mathcal{S}}\mathcal{R})_{\mathcal{S}} = ({}^{\mathrm{V}\mathcal{S}}\mathcal{R}) \cap \mathcal{N}_{\mathcal{S}} = ({}^{\mathrm{V}\mathcal{S}}\mathcal{R}) \cap \mathcal{F}_{S,\mathcal{S}}$. By Lemma 16.5, $\mathcal{F}_{(S,\mathcal{S})}\cdot\overrightarrow{\mathrm{V}\Gamma} = \mathrm{V}\mathcal{S}$, and the result follows. □

This completes the proof of Claim 16.2, and we have an expression for McCool's finite presentation for $\mathrm{Aut}(F, S)$.

16.9. **Theorem.** $\mathrm{Aut}(F, S) = \big\langle\, \overset{\to\leftarrow}{\mathrm{E}'\Gamma} \mid {}^{\overrightarrow{\mathrm{V}\Gamma}}\mathcal{R} \cap \langle\, \overset{\to\leftarrow}{\mathrm{E}'\Gamma} \,\rangle \,\big\rangle$ *with the map* $\overset{\to\leftarrow}{\mathrm{E}'\Gamma} \subseteq \mathcal{F} \to \mathrm{Aut}\, F$. □

The case $S = \emptyset$ gives McCool's finite presentation for $\mathrm{Aut}\, F$.

16.10. **Corollary.** $\mathrm{Aut}\, F = \langle \mathcal{E} | \mathcal{R} \rangle$ *with the given map* $\mathcal{E} \to \mathrm{Aut}\, F$.

References

Emil Artin: Theorie der Zöpfe. Abh. Math. Sem. Univ. Hamburg **4**, 47–72 (1925).

E. Artin: Aus der Topologie: E. Artins Zöpfe. pp. 346–366 in F. Klein: Vorlesungen über höhere Geometrie (ed. W. Blaschke), Grundlehren Math. Wiss. **22**. Springer, Berlin (1926).

E. Artin: Theory of braids. Ann. of Math. **48**, 101–126 (1947).

Bruce Chandler and Wilhelm Magnus: The history of combinatorial group theory. A case study in the history of ideas. Stud. Hist. Math. and Phys. Sci. **9**. Springer, New York (1982).

Donald J. Collins and Heiner Zieschang: Rescuing the Whitehead method for free products, I: peak reduction. Math. Z. **185**, 487–504 (1984).

Warren Dicks: Groups, trees and projective modules. Lecture Notes Math. **790**. Springer, Berlin (1980).

Walther Dyck: Gruppentheoretische Studien. Math. Ann. **20**, 1–44 (1882).

Ralph H. Fox: Free differential calculus I. Derivation in the free group ring. Ann. Math. **57** (2), 547–560 (1953).

S. M. Gersten: On fixed points of certain automorphisms of free groups. Proc. London Math. Soc. (3) **48**, January issue 72–90 (1984a).

S. M. Gersten: A presentation for the special automorphism group of a free group. J. Pure Appl. Algebra **33**, September issue 269–279 (1984b).

P. Higgins and R. C. Lyndon: Equivalence of elements under automorphisims of a free group. J. London Math. Soc. **8**, 254–258 (1974), reprinting of mimeographed notes from Queen Mary College, London (1962).

A. H. M. Hoare: Coinitial graphs and Whitehead automorphisms. Canadian Jour. Math. **31**, 112–123 (1979).

Wilhelm Magnus: Über Automorphismen von Fundamentalgruppen berandeter Flächen. Math. Ann. **109**, 617–646 (1934).

J. McCool: A presentation for the automorphism group of a free group of finite rank. J. London Math. Soc. (2) **8**, 259–266 (1974).

James McCool: Some finitely presented subgroups of the automorphism group of a free group. J. Algebra **35**, June issue 205–213 (1975a).

J. McCool: On Nielsen's presentation of the automorphism group of a free group. J. London Math. Soc. (2) **10**, July issue 265–270 (1975b).

J. Nielsen: Über die Isomorphismen unendlicher Gruppen ohne Relation. Math. Ann. **79**, 269–272 (1919).

J. Nielsen: Om Regning med ikke-kommutative Faktorer og dens Anvendelse i Gruppeteorien. Mat. Tidsskrift B **107**, 77–94 (1921).

Jakob Nielsen: Die Isomorphismengruppe der freien Gruppen. Math. Ann. **91**, 169–209 (1924).

Elvira Strasser Rapaport: On free groups and their automorphisms. Acta Math. **99**, 139–163 (1958).

Kurt Reidemeister: Knoten und Gruppen. Abh. Math. Univ. Hamburg **5**, 7–23 (1927).

Kurt Reidemeister: Einführung in die Kombinatorische Topologie. Braunschweig (1932). English translation by John Stillwell at `https://arxiv.org/abs/1402.3906`

Otto Schreier: Die Untergruppen der freien Gruppen. Abh. Math. Univ. Hamburg **5**, 161–183 (1927).

Jean-Pierre Serre: Arbres, amalgames, SL_2. Cours au Collège de France rédigé avec la collaboration de Hyman Bass. Astérisque **46**. Soc. Math. de France, Paris (1977).

John R. Stallings: Topology of finite graphs, Invent. Math. **71** 551– 565 (1983).
J. H. C. Whitehead: On equivalent sets of elements in a free group. Ann. of Math. (2) **37**, 782–800 (1936).

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