Lest the Karrass & Solitar proof be forgotten

Notation. For each group G, let $\mathfrak{N}(G)$ denote the set of all normal subgroups of G, and set $\hat{1} := \{1\}$.

Theorem. *Let G be a free group, and H_1 be a finite-rank, infinite-index subgroup of G.*
(i) (M. Hall(1949)) *H_1 is a free factor of some finite-index subgroup H of G.*
(ii) (Karrass & Solitar(1969)) *$H_1 \cap N = \hat{1}$ for some $N \in \mathfrak{N}(G) - \{\hat{1}\}$.*

Proof. *(i)* (Imrich(1977)). Choose a basis E of G, form the Cayley G-tree T with respect to E, take the (finite) base-pointed core of $H_1 \backslash T$, and adjoin all the missing e-edges for each $e \in E$; recall that one *adjoins the missing e-edge* to an e-component which is a line-segment-or-isolated-vertex to form an oriented e-cycle. The resulting base-pointed graph is $H \backslash T$ for a subgroup H with the desired properties.

(ii) (Karrass & Solitar(1969)). By *(i)*, we may write $H = H_1 * H_2 \leq_{f.i.} G$ with $H_2 \neq \hat{1}$. Let H° denote the kernel of the homomorphism from G to the group of permutations of the finite set G/H given by $g \mapsto (xH \mapsto gxH)$. Then $H^\circ \trianglelefteq G$, G/H° is finite, and $H^\circ \leq H$. Let $\widetilde{H}_2$ denote the normal closure of H_2 in H. Since $H^\circ, \widetilde{H}_2 \trianglelefteq H$, $H^\circ \cap \widetilde{H}_2 \trianglelefteq H^\circ$. Since $H^\circ \trianglelefteq_{f.i} G$, $H^\circ \cap \widetilde{H}_2 \trianglelefteq_{f.i} \widetilde{H}_2$. Since $\widetilde{H}_2 \neq \hat{1}$, $H^\circ \cap \widetilde{H}_2 \neq \hat{1}$. Thus, $H^\circ \cap \widetilde{H}_2 \in \mathfrak{N}(H^\circ) - \{\hat{1}\}$. The subset $\{{}^g(H^\circ \cap \widetilde{H}_2) : g \in G\}$ of $\mathfrak{N}(H^\circ) - \{\hat{1}\}$ is finite, since G/H° maps onto it by $gH^\circ \mapsto {}^g(H^\circ \cap \widetilde{H}_2)$. As $\mathfrak{N}(H^\circ) - \{\hat{1}\}$ is closed under finite intersections,[1] $\bigcap_{g \in G} {}^g(H^\circ \cap \widetilde{H}_2) \in \mathfrak{N}(H^\circ) - \{\hat{1}\}$. Thus, $\bigcap_{g \in G} {}^g\widetilde{H}_2 \neq \hat{1}$. As the natural map $H_1 \to H/\widetilde{H}_2$ is bijective, $H_1 \cap \widetilde{H}_2 = \hat{1}$. Thus, we may take $N := \bigcap_{g \in G} {}^g\widetilde{H}_2$. □

Remarks. The above proof of *(ii)*, including the now-standard material in the footnote, consists of part of Karrass & Solitar(1958), page 220, lines 5–16, together with part of Karrass & Solitar(1969), page 211, lines 14–20. Thus, their proof occupies 19 lines, and the relevant part of it occupies 11 lines.

Arzhantseva(2000) generalizes *(ii)* by showing that, in a well-defined sense, almost every $N \in \mathfrak{N}(G)$ satisfies $H_1 \cap N = \hat{1}$.

Lyndon & Schupp(1977), Proposition I.3.17, and Kahrobaei(2005) both offer proofs of *(ii)*. The Lyndon & Schupp proof occupies 8 lines and is not valid; it attributes to subgroups certain properties that their normalizers enjoy, and concludes that $H^\circ \leq \widetilde{H}_2$, which is false unless $H_1 = \hat{1}$. The (valid) Kahrobaei proof occupies 12 lines, is claimed to be 'particularly short and simple', and can be paraphrased as follows.

By *(i)*, we may write $H = H_1 * H_2 \leq_{f.i.} G$ with $H_2 \neq \hat{1}$. Let H° denote the kernel of the homomorphism from G to the group of permutations of the finite set G/H given by $g \mapsto (xH \mapsto gxH)$. Then $H^\circ \trianglelefteq G$, G/H° is finite, and $H^\circ \leq H$. Since H_1 is a free factor of H, it follows from the Kurosh subgroup theorem that $H_1 \cap H^\circ$ is a free factor of H°, say $H^\circ = (H_1 \cap H^\circ) * J$. Since $H^\circ \leq_{f.i.} G$, $J \neq \hat{1}$. Let $\widetilde{J}$ denote the normal closure of J in $H^\circ = (H_1 \cap H^\circ) * J$. The subset $\{\widetilde{J}^{\,g} : g \in G\}$ of $\mathfrak{N}(H^\circ) - \{\hat{1}\}$ is finite, since G/H° maps onto it by $H^\circ g \mapsto \widetilde{J}^{\,g}$. As $\mathfrak{N}(H^\circ) - \{\hat{1}\}$ is closed under finite intersections,[1] $\bigcap_{g \in G} \widetilde{J}^{\,g} \in \mathfrak{N}(H^\circ) - \{\hat{1}\}$. Thus, $\bigcap_{g \in G} \widetilde{J}^{\,g} \neq \hat{1}$. As the natural map $H_1 \cap H^\circ \to \big((H_1 \cap H^\circ) * J\big)/\widetilde{J}$ is bijective, $(H_1 \cap H^\circ) \cap \widetilde{J} = \hat{1}$. Since $\widetilde{J} \leq H^\circ$, $H_1 \cap \widetilde{J} = \hat{1}$. Thus, we may take $N := \bigcap_{g \in G} \widetilde{J}^{\,g}$. □

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[1] By induction, it suffices to show that if $N_1, N_2 \in \mathfrak{N}(H^\circ) - \{\hat{1}\}$, then $N_1 \cap N_2 \neq \hat{1}$. Choose $x_i \in N_i - \hat{1}$, $i = 1, 2$, and recall that $[x_1, x_2] = x_1 \cdot x_2 x_1^{-1} x_2^{-1} = x_1 x_2 x_1^{-1} \cdot x_2^{-1} \in N_1 \cap N_2$. If $[x_1, x_2] \neq 1$, then $N_1 \cap N_2 \neq \hat{1}$, while if $[x_1, x_2] = 1$, then $\langle x_1, x_2 \rangle$ has rank one, and there exist nonzero integers n_1, n_2 such that $x_1^{n_1} = x_2^{n_2} \in (N_1 \cap N_2) - \hat{1}$.