THE ZIESCHANG-McCOOL METHOD FOR GENERATING ALGEBRAIC MAPPING-CLASS GROUPS

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Abstract. Let $g, p \in [0\uparrow\infty[$, the set of non-negative integers. Let $\mathrm{A}_{g,p}$ denote the group consisting of all those automorphisms of the free group on $t_{[1\uparrow p]} \cup x_{[1\uparrow g]} \cup y_{[1\uparrow g]}$ which fix the element $\underset{j\in[p\downarrow 1]}{\Pi} t_j \underset{i\in[1\uparrow g]}{\Pi} [x_i, y_i]$ and permute the set of conjugacy classes $\{\,[t_j] : j \in [1\uparrow p]\}$.

Labruère and Paris, building on work of **A**rtin, Magnus, **D**ehn, Nielsen, **L**ickorish, Zieschang, Birman, **H**umphries, and others, showed that $\mathrm{A}_{g,p}$ is generated by what is called the ADLH set. We use methods of Zieschang and McCool to give a self-contained, algebraic proof of this result.

Labruère and Paris also gave defining relations for the ADLH set in $\mathrm{A}_{g,p}$; we do not know an algebraic proof of this for $g \geqslant 2$.

Consider an orientable surface $\mathbf{S}_{g,p}$ of genus g with p punctures, with $(g,p) \neq (0,0)$, $(0,1)$. The *algebraic* mapping-class group of $\mathbf{S}_{g,p}$, denoted $\mathrm{M}^{\mathrm{alg}}_{g,p}$, is defined as the group of all those outer automorphisms of

$$\langle\, t_{[1\uparrow p]} \cup x_{[1\uparrow g]} \cup y_{[1\uparrow g]} \mid \underset{j\in[p\downarrow 1]}{\Pi} t_j \underset{i\in[1\uparrow g]}{\Pi} [x_i, y_i]\,\rangle$$

which permute the set of conjugacy classes $\{\,[t_j], [\overline{t}_j] : j \in [1\uparrow p]\}$. It now follows from a result of Nielsen that $\mathrm{M}^{\mathrm{alg}}_{g,p}$ is generated by the image of the ADLH set together with a reflection. This gives a new way of seeing that $\mathrm{M}^{\mathrm{alg}}_{g,p}$ equals the (topological) mapping-class group of $\mathbf{S}_{g,p}$, along lines suggested by Magnus, Karrass, and Solitar in 1966.

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1. Introduction

Notation will be explained more fully in Section 2.

1.1. **Definitions.** Let g, $p \in [0\uparrow\infty[$. Let $\mathrm{A}_{g,p}$ denote the group of automorphisms of $\langle\, t_{[1\uparrow p]} \cup x_{[1\uparrow g]} \cup y_{[1\uparrow g]} \mid \quad \rangle$ that fix $\underset{j\in[p\downarrow 1]}{\Pi} t_j \underset{i\in[1\uparrow g]}{\Pi} [x_i, y_i]$ and permute the set of conjugacy classes $\{\,[t_j] : j \in [1\uparrow p]\}$.

We shall usually codify an element $\varphi \in \mathrm{A}_{g,p}$ as a two-row matrix where the first row gives all the elements of $t_{[1\uparrow p]} \cup x_{[1\uparrow g]} \cup y_{[1\uparrow g]}$ that are moved by φ, and the second row equals the φ-image of the first row. We define the following elements of $\mathrm{A}_{g,p}$:

for each $j \in [2\uparrow p]$, $\sigma_j := \left(\begin{smallmatrix} t_j & t_{j-1} \\ t_{j-1} & \overline{t}_{j-1} t_j t_{j-1} \end{smallmatrix}\right)$;

for each $i \in [1\uparrow g]$, $\alpha_i := \left(\begin{smallmatrix} x_i \\ \overline{y}_i x_i \end{smallmatrix}\right)$ and $\beta_i := \left(\begin{smallmatrix} y_i \\ x_i y_i \end{smallmatrix}\right)$;

for each $i \in [2\uparrow g]$, $\gamma_i := \left(\begin{smallmatrix} x_{i-1} & y_{i-1} & x_i \\ \overline{w}_i x_{i-1} & \overline{w}_i y_{i-1} w_i & x_i w_i \end{smallmatrix}\right)$ with $w_i := y_{i-1}\overline{x}_i\overline{y}_i x_i$;

if $\min(1, g, p) = 1$, $\gamma_1 := \left(\begin{smallmatrix} t_1 & x_1 \\ \overline{w}_1 t_1 w_1 & x_1 w_1 \end{smallmatrix}\right)$ with $w_1 := t_1\overline{x}_1\overline{y}_1 x_1$.

We say that $\sigma_{[2\uparrow p]} \cup \alpha_{[1\uparrow g]} \cup \beta_{[1\uparrow g]} \cup \gamma_{[\max(2-p,1)\uparrow g]}$ is the *ADL set*, and that removing $\alpha_{[3\uparrow g]}$ leaves the *ADLH set*, $\sigma_{[2\uparrow p]} \cup \alpha_{[1\uparrow \min(2,g)]} \cup \beta_{[1\uparrow g]} \cup \gamma_{[\max(2-p,1)\uparrow g]}$, named after Artin, Dehn, Lickorish and Humphries. □

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In [13, Proposition 2.10(ii) with $r = 0$], Labruère and Paris showed that $\mathrm{A}_{g,p}$ is generated by the ADLH set. As we shall recall in Section 5, the proof is built on work of Artin, Magnus, Dehn, Nielsen, Lickorish, Zieschang, Birman, Humphries, and others, and some of this work uses topological arguments.

The main purpose of this article is to give a self-contained, *algebraic* proof that $\mathrm{A}_{g,p}$ is generated by the ADL set. Such proofs were given in the case $(g, p) = (1, 0)$ by Nielsen [17], and in the case $g = 0$ by Artin [1], and in the case $p = 0$ by McCool [20]. In the case where $(g, p) = (1, 0)$ or $g = 0$, our proof follows Nielsen's and Artin's. In the case where $p = 0$, McCool proceeds by adding in the free generators two at a time, while, for the general case, we benefit from being able to add in the free generators one at a time.

For completeness, we give a self-contained, algebraic translation of Humphries' proof [12] that the ADLH set, a subset of the ADL set, generates $\mathrm{A}_{g,p}$. We also mention an alternative, recent proof by Labruère and Paris.

1.2**. Remark.** In [13, Theorem 3.1 with $r = 0$], Labruère and Paris use topological and algebraic results of various authors to present the group $\mathrm{A}_{g,p}$ as the quotient of the Artin group on the ADLH graph

$$\begin{array}{ccccccccccccccccccc}
 & & & & & & \alpha_1 & & & & \alpha_2 & & & & & & & & \\
 & & & & & & | & & & & | & & & & & & & & \\
\sigma_p & - & \cdots & - & \sigma_2 & = \gamma_1 - & \beta_1 & - & \gamma_2 & - & \beta_2 & - & \gamma_3 & - & \beta_3 & - \cdots - & \gamma_g & - & \beta_g
\end{array}$$

modulo four-or-less relations, each of which is expressed in terms of centres of Artin groups on subgraphs which agree up to deleting α_1:

'$ZB_4 = Z^2B_3$': if $g \geqslant 1$ and $p \geqslant 2$, then $(\alpha_1\beta_1\gamma_1\sigma_2)^4 = (\beta_1\gamma_1\sigma_2)^6$;
'$ZA_5 = Z^2A_4$': if $g \geqslant 2$, then $(\alpha_1\beta_1\gamma_2\beta_2\alpha_2)^6 = (\beta_1\gamma_2\beta_2\alpha_2)^{10}$;
'$ZD_6 = ZA_5$': if $g \geqslant 2$ and $p \geqslant 1$, then $(\alpha_1\gamma_1\beta_1\gamma_2\beta_2\alpha_2)^5 = (\gamma_1\beta_1\gamma_2\beta_2\alpha_2)^6$;
'$ZE_7 = ZE_6$': if $g \geqslant 3$, then $(\alpha_1\beta_1\gamma_2\beta_2\alpha_2\gamma_3\beta_3)^9 = (\beta_1\gamma_2\beta_2\alpha_2\gamma_3\beta_3)^{12}$.

It would be eminently satisfying to have a direct, algebraic proof of this beautiful, mysterious presentation. Now that we have the ADLH generating set, it would suffice to consider the group with the desired presentation and verify that its action on $\langle\, t_{[1\uparrow p]} \cup x_{[1\uparrow g]} \cup y_{[1\uparrow g]} \mid \quad \rangle$ is faithful. This is precisely the approach carried out by Magnus [15] for both the case $g = 0$, see [3, Section 5], and the case $g = 1$, see [2, Section 6.3]. The algebraic project remains open for $g \geqslant 2$. $\square$

In outline, the article has the following structure.

In Section 2, we fix notation and define the Zieschang groupoid, essentially as in [28, Section 5.2] (developed from [22], [24], [26]), but with modifications taken from work of McCool [8, Lemma 3.2]. We give a simplified proof of a strengthened form of (the orientable, torsion-free case of) Zieschang's result that the Nielsen-automorphism edges and the Artin-automorphism edges together generate the groupoid. Zieschang used group-theoretical techniques of Nielsen [18] and Artin [1], while McCool used group-theoretical techniques of Whitehead [21]. We use all of these.

In Section 3, which is inspired by the proof by McCool [20] of the case $p = 0$, we define the canonical edges in the Zieschang groupoid and use them to find a special generating set for $\mathrm{A}_{g,p}$.

In Section 4, we observe that the results of the previous two sections immediately imply that the ADL set generates $\mathrm{A}_{g,p}$. We then present an algebraic translation of Humphries' proof that the ADLH set also generates $\mathrm{A}_{g,p}$. We also mention an alternative, recent proof by Labruère and Paris.

At this stage, we will have completed our objective. For completeness, we conclude the article with an elementary review of algebraic descriptions of certain mapping-class groups.

In Section 5, we review definitions of some mapping-class groups and mention some of the history of the original proof that the ADLH set generates $\mathrm{A}_{g,p}$.

In Section 6, we recall the definitions of Dehn twists and braid twists, and see that the group $\mathrm{A}_{g,p}$ can be viewed as the mapping-class group of the orientable surface of genus g with p punctures and one boundary component.

In Section 7, we consider an orientable surface $\mathbf{S}_{g,0,p}$ of genus g with p punctures, with $(g,p) \neq (0,0)$, $(0,1)$. The *algebraic* mapping-class group of $\mathbf{S}_{g,0,p}$, denoted $\mathrm{M}^{\mathrm{alg}}_{g,0,p}$, is defined as the group of all those outer automorphisms of

$$\pi_1(\mathbf{S}_{g,0,p}) = \langle\, t_{[1\uparrow p]} \cup x_{[1\uparrow g]} \cup y_{[1\uparrow g]} \mid \prod_{j\in[p\downarrow 1]} t_j \prod_{i\in[1\uparrow g]} [x_i, y_i]\,\rangle$$

which permute the set of conjugacy classes $\{\,[t_j], [\overline{t}_j] : j \in [1\uparrow p]\}$. We review Zieschang's algebraic proof [28, Theorem 5.6.1] of Nielsen's result [19] that $\mathrm{M}^{\mathrm{alg}}_{g,0,p}$ is generated by the natural image of $\mathrm{A}_{g,p}$ together with an outer automorphism $\breve{\zeta}$. Hence, $\mathrm{M}^{\mathrm{alg}}_{g,0,p}$ is generated by the natural image of the ADLH set together with $\breve{\zeta}$. In 1966, Magnus, Karrass and Solitar [16, p.175] remarked that if one could find a generating set of $\mathrm{M}^{\mathrm{alg}}_{g,0,p}$ and self-homeomorphisms of $\mathbf{S}_{g,0,p}$ that induce those generators, then one would be able to prove that $\mathrm{M}^{\mathrm{alg}}_{g,0,p}$ was equal to the (topological) mapping-class group $\mathrm{M}^{\mathrm{top}}_{g,0,p}$, even in the then-unknown case where $g \geqslant 2$ and $p \geqslant 2$. Also in 1966, Zieschang [26, Satz 4] used groupoids to prove equality, and their remark does not seem to have been followed up. The generating set given above fulfills their requirement, since the image of each ADL generator is induced by a braid twist or a Dehn twist of $\mathbf{S}_{g,0,p}$, and $\breve{\zeta}$ is induced by a reflection of $\mathbf{S}_{g,0,p}$. This gives a new way of seeing that $\mathrm{M}^{\mathrm{top}}_{g,0,p} = \mathrm{M}^{\mathrm{alg}}_{g,0,p}$.

2. The Zieschang groupoid and the Nielsen subgraph

In this section, which is based on [28, Section 5.2], we define the Zieschang groupoid $\mathcal{Z}_{g,p}$ and the Nielsen subgraph $\mathcal{N}_{g,p}$, and prove that $\mathcal{N}_{g,p}$ generates $\mathcal{Z}_{g,p}$.

2.1. **Notation.** We will find it useful to have notation for intervals in $\mathbb{Z}$ that is different from the notation for intervals in $\mathbb{R}$. Let i, $j \in \mathbb{Z}$. We define the sequence

$$[\![i\uparrow j]\!] := \begin{cases} (i, i+1, \ldots, j-1, j) \in \mathbb{Z}^{j-i+1} & \text{if } i \leqslant j, \\ () \in \mathbb{Z}^0 & \text{if } i > j. \end{cases}$$

The subset of $\mathbb{Z}$ underlying $[\![i\uparrow j]\!]$ is denoted $[i\uparrow j] := \{i, i+1, \ldots, j-1, j\}$.

Also, $[i\uparrow\infty[\ := \{i, i+1, i+2, \ldots\}$.

We define $[\![j\downarrow i]\!]$ to be the reverse of the sequence $[\![i\uparrow j]\!]$, that is, $(j, j-1, \ldots, i+1, i)$.

Suppose that we have a set X and a map $[i\uparrow j] \to X$, $\ell \mapsto x_\ell$. We define the corresponding sequence in X as

$$x_{[\![i\uparrow j]\!]} := \begin{cases} (x_i, x_{i+1}, \cdots, x_{j-1}, x_j) \in X^{i-j+1} & \text{if } i \leqslant j, \\ () & \text{if } i > j. \end{cases}$$

By abuse of notation, we shall also express this sequence as $(x_\ell \mid \ell \in [\![i\uparrow j]\!])$, although "$\ell \in [\![i\uparrow j]\!]$" on its own will not be assigned a meaning. The set of terms of $x_{[\![i\uparrow j]\!]}$ is denoted $x_{[i\uparrow j]}$. We define $x_{[\![j\downarrow i]\!]}$ to be the reverse of the sequence $x_{[\![i\uparrow j]\!]}$. □

2.2. **Notation.** Let G be a multiplicative group.

For each $u \in G$, we denote the inverse of u by both u^{-1} and $\overline{u}$. For u, $v \in G$, we let $u^v := \overline{v}uv$ and $[u, v] := \overline{u}\,\overline{v}uv$. For $u \in G$, we let $[u] := \{u^v \mid v \in G\}$, called the *$G$-conjugacy class of u*. We let $G/{\sim} := \{[u] : u \in G\}$, the set of all G-conjugacy classes.

Where G is a free group given with a distinguished basis $\mathfrak{B}$, we think of each $u \in G$ as a reduced word in $\mathfrak{B} \cup \mathfrak{B}^{-1}$, and let $|u|$ denote the length of the word. We think of $[u]$ as a cyclically-reduced cyclic word in $\mathfrak{B} \cup \mathfrak{B}^{-1}$.

Suppose that we have $i, j \in \mathbb{Z}$ and a map $[i{\uparrow}j] \to G$, $\ell \mapsto u_\ell$. We write

$$\prod_{\ell \in [\![i{\uparrow}j]\!]} u_\ell := \Pi u_{[\![i{\uparrow}j]\!]} := \begin{cases} u_i u_{i+1} \cdots u_{j-1} u_j \in G & \text{if } i \leqslant j, \\ 1 \in G & \text{if } i > j. \end{cases}$$

$$\prod_{\ell \in [\![j{\downarrow}i]\!]} u_\ell := \Pi u_{[\![j{\downarrow}i]\!]} := \begin{cases} u_j u_{j-1} \cdots u_{i+1} u_i \in G & \text{if } j \geqslant i, \\ 1 \in G & \text{if } j < i. \end{cases}$$

When we have G acting on a set X, then, for each $x \in X$, we let $\mathrm{Stab}(x; G)$ denote the set of elements of G which stabilize, or fix, x.

We let $\mathrm{Aut}\, G$ denote the group of all automorphisms of G, acting on the right, as exponents, $u \mapsto u^\varphi$. In a natural way, $\mathrm{Aut}\, G$ acts on $G/{\sim}$ and on the set of subsets of $G \cup (G/{\sim})$.

We let $\mathrm{Out}\, G$ denote the quotient of $\mathrm{Aut}\, G$ modulo the group of inner automorphisms, we call the elements of $\mathrm{Out}\, G$ *outer automorphisms*, and we denote the quotient map $\mathrm{Aut}\, G \to \mathrm{Out}\, G$ by $\varphi \mapsto \breve{\varphi}$. In a natural way, $\mathrm{Out}\, G$ acts on $G/{\sim}$ and on the set of subsets of $G/{\sim}$. $\square$

2.3. **Notation.** The following will be fixed throughout.

Let $g, p \in [0{\uparrow}\infty[$. Let $F_{g,p} := \langle\, t_{[1\uparrow p]} \cup x_{[1\uparrow g]} \cup y_{[1\uparrow g]} \mid \quad \rangle$, a free group of rank $2g{+}p$ with a distinguished basis. We shall find it convenient to use abbreviations such as

$$[t]_{[1\uparrow p]} := \{[t_j] : j \in [1{\uparrow}p]\},\ \ t^{\pm 1}_{[1\uparrow p]} := \{t_j, \overline{t}_j : j \in [1{\uparrow}p]\},\ \ \Pi[x,y]_{[\![1\uparrow g]\!]} := \prod_{i \in [\![1\uparrow g]\!]} [x_i, y_i].$$

The elements of $t^{\pm1}_{[1\uparrow p]} \cup x^{\pm1}_{[1\uparrow g]} \cup y^{\pm1}_{[1\uparrow g]}$ will be called *letters*. The elements of $t_{[1\uparrow p]}$ will be called *t-letters*. The elements of $\overline{t}_{[1\uparrow p]}$ will be called *inverse t-letters*. The elements of $x^{\pm1}_{[1\uparrow g]} \cup y^{\pm1}_{[1\uparrow g]}$ will be called *x-letters*.

We shall usually codify an element $\varphi \in \mathrm{Aut}\, F_{g,p}$ as a two-row matrix where the first row gives, for some basis consisting of letters, all those elements which are moved by φ, and the second row equals the φ-image of the first row.

We shall be working throughout with the group $\mathrm{Stab}([t]_{[1\uparrow p]}; \mathrm{Aut}\, F_{g,p})$ (which permutes the set of cyclic words $[t]_{[1\uparrow p]}$) and its subgroup

$$\mathrm{A}_{g,p} := \mathrm{Stab}([t]_{[1\uparrow p]} \cup \{\Pi\, t_{[\![p\downarrow 1]\!]}\, \Pi[x,y]_{[\![1\uparrow g]\!]}\}; \mathrm{Aut}\, F_{g,p}). \qquad \square$$

2.4. **Definitions.** Let $g, p \in [0{\uparrow}\infty[$ and let $F_{g,p} := \langle\, t_{[1\uparrow p]} \cup x_{[1\uparrow g]} \cup y_{[1\uparrow g]} \mid \quad \rangle$.

Let $[1{\uparrow}(4g{+}p)] \to t_{[1\uparrow p]} \cup x^{\pm1}_{[1\uparrow g]} \cup y^{\pm1}_{[1\uparrow g]}$, $k \mapsto v_k$, be a bijective map, let $V := \Pi\, v_{[\![1\uparrow(4g+p)]\!]}$, and let Γ denote the graph with

vertex set $t^{\pm1}_{[1\uparrow p]} \cup x^{\pm1}_{[1\uparrow g]} \cup y^{\pm1}_{[1\uparrow g]}$, and

edge set $\{(\overline{t}_j {\rightsquigarrow} t_j) \mid j \in [1{\uparrow}p]\} \ \cup\ \{(v_k {\rightsquigarrow} \overline{v}_{k+1}) \mid k \in [1{\uparrow}(4g{+}p{-}1)]\}$.

If Γ has no cycles (that is, Γ is a forest), then we say that V is a *Zieschang element* of $F_{g,p}$ and that Γ is the *extended Whitehead graph of* V; we note that the condition that Γ has no cycles implies that $\Pi\, v_{[\![1\uparrow(4g+p)]\!]}$ is the reduced expression for V, and, hence, Γ is the usual Whitehead graph of $[\overline{t}]_{[1\uparrow p]} \cup \{V\}$, as in [21]. If $(g,p) \neq (0,0)$ and V is a Zieschang element of $F_{g,p}$, then Γ has the form of an oriented line segment with $4g{+}2p$ vertices and $4g{+}2p{-}1$ edges; here, we define $v_0 := v_{4g+p+1} := 1$, and book-end Γ with the *ghost* edges $(v_0 {\rightsquigarrow} \overline{v}_1)$ and $(v_{4g+p} {\rightsquigarrow} \overline{v}_{4g+p+1})$.

For example, $V_0 := \Pi t_{[\![p\downarrow 1]\!]} \Pi[x,y]_{[\![1\uparrow g]\!]}$ is a Zieschang element of $F_{g,p}$, and its extended Whitehead graph is

$$\overline{t}_p {\rightsquigarrow} t_p {\rightsquigarrow} \overline{t}_{p-1} {\rightsquigarrow} t_{p-1} {\rightsquigarrow} \cdots {\rightsquigarrow} \overline{t}_1 {\rightsquigarrow} t_1 {\rightsquigarrow} x_1 {\rightsquigarrow} \overline{y}_1 {\rightsquigarrow} \overline{x}_1 {\rightsquigarrow} y_1 {\rightsquigarrow} x_2 {\rightsquigarrow} \cdots {\rightsquigarrow} x_g {\rightsquigarrow} \overline{y}_g {\rightsquigarrow} \overline{x}_g {\rightsquigarrow} y_g.$$

The *Zieschang groupoid* for $F_{g,p}$, denoted $\mathcal{Z}_{g,p}$, is defined as follows.

- The set $\mathrm{V}\mathcal{Z}_{g,p}$ of vertices/objects of $\mathcal{Z}_{g,p}$ equals the set of Zieschang elements of $F_{g,p}$.
- The edges/elements/morphisms of $\mathcal{Z}_{g,p}$ are the triples (V, W, φ) such that V, $W \in \mathrm{V}\mathcal{Z}_{g,p}$, and $\varphi \in \mathrm{Stab}([t]_{[1\uparrow p]}; \mathrm{Aut}\, F_{g,p})$, and $V^\varphi = W$. Here, we say that $(V \xrightarrow{\varphi} W)$, or $V \xrightarrow{\varphi} W$, is an edge of $\mathcal{Z}_{g,p}$ *from V to W*, and denote the set of such edges by $\mathcal{Z}_{g,p}(V, W)$.
- The partial multiplication in $\mathcal{Z}_{g,p}$ is defined using the multiplication in $\mathrm{Stab}([t]_{[1\uparrow p]}; \mathrm{Aut}\, F_{g,p})$ in the natural way. (This is an instance of a type of groupoid that arises whenever a group acts on a set.)

If $V \in \mathrm{V}\mathcal{Z}_{g,p}$, then, as a group, $\mathcal{Z}_{g,p}(V, V) = \mathrm{Stab}([t]_{[1\uparrow p]} \cup \{V\}; \mathrm{Aut}\, F_{g,p})$. Thus $\mathcal{Z}_{g,p}(V_0, V_0) = \mathrm{A}_{g,p}$. Throughout, we shall view the elements of $\mathrm{A}_{g,p}$ as edges of $\mathcal{Z}_{g,p}$ from V_0 to V_0. We shall be using V_0 as a basepoint of $\mathcal{Z}_{g,p}$ in Definitions 3.1, where we will verify that $\mathcal{Z}_{g,p}$ is connected. $\square$

2.5. **Definitions.** Let g, $p \in [0\uparrow\infty[$, let $F_{g,p} := \langle\, t_{[1\uparrow p]} \cup x_{[1\uparrow g]} \cup y_{[1\uparrow g]} \mid \quad \rangle$, let V, $W \in F_{g,p}$, and let $\varphi \in \mathrm{Aut}\, F_{g,p}$. Suppose that $V \in \mathrm{V}\mathcal{Z}_{g,p}$, and that $V^\varphi = W$.

If φ permutes the t-letters and permutes the x-letters, then we say that $V \xrightarrow{\varphi} W$ is a *Nielsen*$_1$ *edge in* $\mathcal{Z}_{g,p}$. To see that $(V \xrightarrow{\varphi} W) \in \mathcal{Z}_{g,p}$, notice that $\varphi \in \mathrm{Stab}([t]_{[1\uparrow p]}; \mathrm{Aut}\, F_{g,p})$ and $W \in \mathrm{V}\mathcal{Z}_{g,p}$.

If there exists some $k \in [1\uparrow(4g{+}p{-}1)]$ such that the letter v_k is an x-letter and $\varphi = \left(\begin{smallmatrix} v_k \\ v_k \overline{v}_{k+1} \end{smallmatrix}\right)$, then we say that $V \xrightarrow{\varphi} W$ is a *right Nielsen*$_2$ *edge in* $\mathcal{Z}_{g,p}$. To see that $(V \xrightarrow{\varphi} W) \in \mathcal{Z}_{g,p}$, we note the following. In passing from V to W, we remove the boxed part in $v_k \boxed{v_{k+1}} v_{k+2}$ and add the boxed part in $v_{j-1} \boxed{v_{k+1}} v_j$, where $v_j = \overline{v}_k$. In passing from the extended Whitehead graph of V to the extended Whitehead graph of W, we remove the boxed part in $v_{j-1}{\rightsquigarrow} \boxed{\overline{v}_j{=}v_k{\rightsquigarrow}} \overline{v}_{k+1}$ and add the boxed part in $v_{k+1}{\rightsquigarrow} \boxed{\overline{v}_j{=}v_k{\rightsquigarrow}}\ \overline{v}_{k+2}$, where we have indicated a ghost edge if $j = 1$ or $k = 4g{+}p{-}1$. Hence, $(V \xrightarrow{\varphi} W) \in \mathcal{Z}_{g,p}$.

If there exists some $k \in [2\uparrow(4g{+}p)]$ such that v_k is an x-letter and $\varphi = \left(\begin{smallmatrix} v_k \\ \overline{v}_{k-1} v_k \end{smallmatrix}\right)$, then we say that $V \xrightarrow{\varphi} W$ is a *left Nielsen*$_2$ *edge in* $\mathcal{Z}_{g,p}$. This is an inverse of an edge of the previous type.

By a *Nielsen*$_2$ *edge in* $\mathcal{Z}_{g,p}$, we mean a left or right Nielsen$_2$ edge in $\mathcal{Z}_{g,p}$.

If there exists some $k \in [1\uparrow(4g{+}p{-}1)]$ such that the letter v_k is a t-letter and $\varphi = \left(\begin{smallmatrix} v_k \\ v_{k+1} v_k \overline{v}_{k+1} \end{smallmatrix}\right)$, then we say that $V \xrightarrow{\varphi} W$ is a *right Nielsen*$_3$ *edge in* $\mathcal{Z}_{g,p}$. To see that $(V \xrightarrow{\varphi} W) \in \mathcal{Z}_{g,p}$, we note the following. In passing from V to W, we change $v_{k-1} v_k \boxed{v_{k+1}} v_{k+2}$ to $v_{k-1} \boxed{v_{k+1}} v_k v_{k+2}$. In passing from the extended Whitehead graph of V to the extended Whitehead graph of W, we remove the boxed part in $v_{k-1}{\rightsquigarrow} \boxed{\overline{v}_k{\rightsquigarrow}v_k{\rightsquigarrow}}\ \overline{v}_{k+1}$ and add the boxed part in $v_{k+1}{\rightsquigarrow} \boxed{\overline{v}_k{\rightsquigarrow}v_k{\rightsquigarrow}}\ \overline{v}_{k+2}$, where we have indicated a ghost edge if $k = 1$ or $k = 4g{+}p{-}1$. Hence, $(V \xrightarrow{\varphi} W) \in \mathcal{Z}_{g,p}$.

If there exists some $k \in [2\uparrow(4g{+}p)]$ such that the letter v_k is a t-letter and $\varphi = \left(\begin{smallmatrix} v_k \\ \overline{v}_{k-1} v_k v_{k-1} \end{smallmatrix}\right)$, we say that $V \xrightarrow{\varphi} W$ is a *left Nielsen*$_3$ *edge in* $\mathcal{Z}_{g,p}$. This is an inverse of an edge of the previous type.

By a *Nielsen*$_3$ *edge in* $\mathcal{Z}_{g,p}$, we mean a left or right Nielsen$_3$ edge in $\mathcal{Z}_{g,p}$.

By a *Nielsen edge in* $\mathcal{Z}_{g,p}$, we mean a Nielsen$_i$ edge in $\mathcal{Z}_{g,p}$, for some $i \in \{1, 2, 3\}$.

We define the *Nielsen subgraph of* $\mathcal{Z}_{g,p}$, denoted $\mathcal{N}_{g,p}$, to be the graph with vertex set $\mathrm{V}\mathcal{Z}_{g,p}$ and edges, or elements, the Nielsen edges in $\mathcal{Z}_{g,p}$. $\square$

We now give a simplified proof of a result due to Zieschang and McCool.

2.6. **Theorem.** *Let $g,\ p \in [0{\uparrow}\infty[$, let $F_{g,p} := \langle\, t_{[1\uparrow p]} \cup x_{[1\uparrow g]} \cup y_{[1\uparrow g]} \mid \quad \rangle$, let $V, W \in F_{g,p}$, let H be a free group, let φ be an endomorphism of $H{*}F_{g,p}$, and suppose that the following hold.*
(a). $V \in \mathrm{V}\mathcal{Z}_{g,p}$.
(b). $|W| \leqslant 4g + p$.
(c). $V^{\varphi} = W$.
(d). *There exists some permutation π of $[1{\uparrow}p]$ such that, for each $j \in [1{\uparrow}p]$, t_j^{φ} is $(H{*}F_{g,p})$-conjugate to $t_{j^{\pi}}$.*
(e). $F_{g,p}^{\varphi} \simeq F_{g,p}$.
Then $W \in \mathrm{V}\mathcal{Z}_{g,p}$ and there exists an edge $V \xrightarrow{\varphi'} W$ in the subgroupoid of $\mathcal{Z}_{g,p}$ generated by the Nielsen subgraph $\mathcal{N}_{g,p}$ such that φ acts as φ' on the free factor $F_{g,p}$.

Proof. We may assume that $(g,p) \neq (0,0)$. Extend $t_{[1\uparrow p]} \cup x_{[1\uparrow g]} \cup y_{[1\uparrow g]}$ to a basis $\mathfrak{B}$ of the free group $F_{g,p}{*}H$. For each $A \in F_{g,p}{*}H$, $|A|$ denotes the length of A as a reduced product in $\mathfrak{B}{\cup}\mathfrak{B}^{-1}$. Choose a total order, denoted $\leqslant$, on $\mathfrak{B}{\cup}\mathfrak{B}^{-1}$, and extend $\leqslant$ to a length-lexicographic total order, also denoted $\leqslant$, on $F_{g,p}{*}H$.

Consider the reduced expression $V = \Pi v_{[\![1\uparrow(4g+p)]\!]}$. Let $v_0 := v_{4g+p+1} := 1$.

For each $k \in [0{\uparrow}(4g{+}p)]$, let A_k denote the largest common initial subword of $\overline{v}_k^{\varphi}$ and v_{k+1}^{φ} with respect to $\mathfrak{B}{\cup}\mathfrak{B}^{-1}$. Since $v_0 = v_{4g+p+1} = 1$, we have $A_0 = A_{4g+p} = 1$. For each $k \in [1{\uparrow}(4g{+}p)]$, let $w_k := \overline{A}_{k-1} v_k^{\varphi} A_k \in F_{g,p}{*}H$. Then $v_k^{\varphi} = A_{k-1} w_k \overline{A}_k$, where this expression need not be reduced.

We shall show in Claim 1 that we may assume that $A_k < A_{k-1}w_k$ and that $A_k < A_{k+1}\overline{w}_{k+1}$, and then show in Claim 2 that this ensures that φ permutes the t-letters and permutes the x-letters.

We let $\binom{F_{g,p}*H}{4g+p}$ denote the set of $(4g{+}p)$-element subsets of $F_{g,p}{*}H$, and define a pre-order $\preccurlyeq$ on $\binom{F_{g,p}*H}{4g+p}$ as follows. For each $A \in F_{g,p}{*}H$, there is a unique reduced expression $A = A^{(L)}A^{(R)}$ with the property that $|A^{(L)}| - |A^{(R)}| \in \{0,1\}$. For $A, B \in F_{g,p}{*}H$, we write $A \preccurlyeq B$ if either $|A| < |B|$ or ($|A| = |B|$ and $A^{(L)} \leqslant B^{(L)}$). We can arrange each element of $\binom{F_{g,p}*H}{4g+p}$ as a (not necessarily unique) ascending sequence with respect to $\preccurlyeq$, and assign $\binom{F_{g,p}*H}{4g+p}$ the (unique) lexicographic pre-order, again denoted $\preccurlyeq$. Here, $\mathbb{A} \prec \mathbb{B}$ will mean $\mathbb{A} \preccurlyeq \mathbb{B}$ and $\mathbb{B} \not\preccurlyeq \mathbb{A}$.

Without assigning any meaning to $V \xrightarrow{\varphi} W$, let us write

$$\mu(V \xrightarrow{\varphi} W) := t_{[1\uparrow p]}^{\varphi} \cup (x_{[1\uparrow g]}^{\pm 1})^{\varphi} \cup (y_{[1\uparrow g]}^{\pm 1})^{\varphi} \quad \in \quad (\tbinom{F_{g,p}*H}{4g+p}, \preccurlyeq).$$

It follows from (e) that there are $4g{+}p$ distinct elements in the set $\mu(V \xrightarrow{\varphi} W)$.

Claim 1. *Let $k \in [1{\uparrow}(4g{+}p{-}1)]$. If $A_k \geqslant A_{k-1}w_k$ or $A_k \geqslant A_{k+1}\overline{w}_{k+1}$, then there exists some $(V \xrightarrow{\alpha} U) \in \mathcal{N}_{g,p}$ such that $\mu(U \xrightarrow{\overline{\alpha}\varphi} W) \prec \mu(V \xrightarrow{\varphi} W)$.*

Proof of Claim 1. We have specified reduced expressions $v_k^{\varphi} = B\overline{A}$ and $v_{k+1}^{\varphi} = A\overline{C}$ and $v_k^{\varphi} v_{k+1}^{\varphi} = B\overline{C}$, where $A := A_k$, $B := A_{k-1}w_k$, $C := A_{k+1}\overline{w}_{k+1}$. It follows from (e) that A, B, and C are all different.

By hypothesis, $A \neq \min(\{A,B,C\}, \leqslant)$. We shall consider only the case where $B = \min(\{A,B,C\}, \leqslant)$; the argument where $C = \min(\{A,B,C\}, \leqslant)$ is similar. Thus we have $A > B < C$.

The letter v_{k+1} is either a t-letter or an x-letter.

Case 1. v_{k+1} is an x-letter.

On taking $\alpha := \binom{v_{k+1}}{\overline{v}_k v_{k+1}}$, we have a Nielsen$_2$ edge $(V \xrightarrow{\alpha} U) \in \mathcal{N}_{g,p}$. Here $\overline{\alpha} := \binom{v_{k+1}}{v_k v_{k+1}}$ and $v_{k+1}^{\overline{\alpha}\varphi} = v_k^{\varphi} v_{k+1}^{\varphi} = B\overline{C}$. In this case, the change from $\mu(V \xrightarrow{\varphi} W)$ to $\mu(U \xrightarrow{\overline{\alpha}\varphi} W)$ consists of replacing $\{v_{k+1}^{\varphi}, \overline{v}_{k+1}^{\varphi}\} = \{A\overline{C}, C\overline{A}\}$ with $\{v_{k+1}^{\overline{\alpha}\varphi}, \overline{v}_{k+1}^{\overline{\alpha}\varphi}\} =$

$\{B\overline{C}, C\overline{B}\}$. To show that $\mu(U \xrightarrow{\overline{\alpha}\varphi} W) \prec \mu(V \xrightarrow{\varphi} W)$, it now suffices to show that $B\overline{C} \prec A\overline{C}$ and $C\overline{B} \preccurlyeq C\overline{A}$.

If $|A| > |B|$, then $|B\overline{C}| = |C\overline{B}| = |B| + |C| < |A| + |C| = |A\overline{C}| = |C\overline{A}|$, and, hence, $B\overline{C} \prec A\overline{C}$ and $C\overline{B} \prec C\overline{A}$.

If $|A| = |B|$, then, since $A > B < C$, we have $|A| = |B| \leqslant |C|$ and $B < A$. Hence $B\overline{C} \prec A\overline{C}$ and $C\overline{B} \preccurlyeq C\overline{A}$.

Case 2. v_{k+1} is a t-letter.

On taking $\alpha \coloneqq \binom{v_{k+1}}{\overline{v}_k v_{k+1} w_k}$, we have a Nielsen$_3$ edge $(V \xrightarrow{\alpha} U) \in \mathcal{N}_{g,p}$. Here $\overline{\alpha} \coloneqq \binom{v_{k+1}}{v_k v_{k+1} \overline{v}_k}$. In this case, the change from $\mu(V \xrightarrow{\varphi} W)$ to $\mu(U \xrightarrow{\overline{\alpha}\varphi} W)$, consists of replacing $v_{k+1}^{\varphi} = A\overline{C}$ with $v_{k+1}^{\overline{\alpha}\varphi} = v_k^{\varphi} v_{k+1}^{\varphi} \overline{v}_k^{\varphi} = (B\overline{A})(A\overline{C})(A\overline{B}) = B\overline{C}A\overline{B}$. To show that $\mu(U \xrightarrow{\overline{\alpha}\varphi} W) \prec \mu(V \xrightarrow{\varphi} W)$, it suffices to show that $B\overline{C}A\overline{B} \prec A\overline{C}$.

Let $D \coloneqq \min(\{A, C\}, \leqslant)$. Since v_{k+1} is a t-letter, there exists some $j \in [1{\uparrow}p]$ such that v_{k+1}^{φ} is a conjugate of t_j, that is, $A\overline{C}$ is a conjugate of t_j. Thus, both $A\overline{C}$ and $C\overline{A}$ begin with D, and we can write $A\overline{C} = DEt_j\overline{E}\,\overline{D}$ with no cancellation. Now $Et_j\overline{E} = \overline{D}A\overline{C}D = \overline{C}A$. Hence $B\overline{C}A\overline{B} = BEt_j\overline{E}\,\overline{B}$ where this expression may have cancellation. Recall that $B < D$. Thus $BEt_j\overline{E}\,\overline{B} \prec DEt_j\overline{E}\,\overline{D}$, that is, $B\overline{C}A\overline{B} \prec A\overline{C}$.

This completes the proof of Claim 1. □

Claim 1 gives a procedure for reducing $\mu(V \xrightarrow{\varphi} W)$. Once φ is specified, only a finite subset of $\mathfrak{B} \cup \mathfrak{B}^{-1}$ is ever involved, and, moreover, there is an upper bound for the lengths of the elements of $F_{g,p} * H$ which will appear. It follows that we can repeat the procedure only a finite number of times. Hence, we may now assume that, for each $k \in [1{\uparrow}(4g{+}p{-}1)]$, $A_k < A_{k-1} w_k$ and $A_k < A_{k+1} \overline{w}_{k+1}$.

Claim 2. *Under the latter assumption, φ permutes the t-letters and permutes the x-letters, and the desired conclusion holds.*

Proof of Claim 2. For each $k \in [1{\uparrow}(4g{+}p)]$, $A_k < A_{k-1} w_k$ and $A_{k-1} < A_k \overline{w}_k$ (even for $k = 1$ and $k = 4g{+}p$). It follows that $w_k \neq 1$ and also that the expression $v_k^{\varphi} = A_{k-1} w_k \overline{A}_k$ is reduced. It then follows that, for each $k \in [1{\uparrow}(4g{+}p{-}1)]$, $A_{k-1} w_k w_{k+1} \overline{A}_{k+1}$ is a reduced expression for $v_k^{\varphi} v_{k+1}^{\varphi}$. Now

$$W = V^{\varphi} = (\Pi v_{[\![1\uparrow(4g+p)]\!]})^{\varphi} = \underset{k \in [\![1\uparrow(4g+p)]\!]}{\Pi} (A_{k-1} w_k \overline{A}_k) = \Pi w_{[\![1\uparrow(4g+p)]\!]},$$

and we have just seen that the expression $\Pi w_{[\![1\uparrow(4g+p)]\!]}$ is reduced. By (b),

$$4g{+}p \;\geqslant\; |W| = |\,\Pi\, w_{[\![1\uparrow(4g+p)]\!]}| = \sum_{k=1}^{4g+p} |w_k| \;\geqslant\; 4g{+}p.$$

Hence, equality holds throughout, and, for each $k \in [1{\uparrow}(4g{+}p)]$, $|w_k| = 1$ and w_k is a letter.

Let $s_{[\![1\uparrow(4g+2p)]\!]}$ be the vertex sequence in the extended Whitehead graph of V, that is, $s_{[1\uparrow(4g+2p)]} = t^{\pm 1}_{[1\uparrow p]} \cup x^{\pm 1}_{[1\uparrow g]} \cup y^{\pm 1}_{[1\uparrow g]}$ and $\{(s_\ell \rightsquigarrow s_{\ell+1}) \mid \ell \in [1{\uparrow}(4g{+}2p{-}1)]\}$ equals $\{(\overline{t}_j \rightsquigarrow t_j) \mid j \in [1{\uparrow}p]\} \,\cup\, \{(v_k \rightsquigarrow \overline{v}_{k+1}) \mid k \in [1{\uparrow}(4g{+}p{-}1)]\}$.

We assume that there exists some $\ell \in [1{\uparrow}(4g{+}2p)]$ such that $|s_\ell^{\varphi}| > 1$, and we shall obtain a contradiction. Let s_ℓ^{φ} end in $b \in \mathfrak{B} \cup \mathfrak{B}^{-1}$. Assume further that ℓ has been chosen to minimize $\overline{b}$ in $(\mathfrak{B} \cup \mathfrak{B}^{-1}, \leqslant)$. Assume further that ℓ has been chosen maximal. In particular, if $|s_{\ell+1}^{\varphi}| > 1$, then $s_{\ell+1}^{\varphi}$ does not end in b.

Recall that $(s_\ell \rightsquigarrow s_{\ell+1})$ can be expressed either as $(\overline{t}_j \rightsquigarrow t_j)$ or as $(v_k \rightsquigarrow \overline{v}_{k+1})$, possibly a ghost edge. If $(s_\ell \rightsquigarrow s_{\ell+1}) = (\overline{t}_j \rightsquigarrow t_j)$, then $|s_{\ell+1}^{\varphi}| = |s_\ell^{\varphi}| > 1$, and, also, $s_{\ell+1}^{\varphi}$ ends in b. This is a contradiction. Thus, we may assume that

$(s_\ell \rightsquigarrow s_{\ell+1}) = (v_k \rightsquigarrow \overline{v}_{k+1})$, possibly with $k = 4g{+}p$. Then $s_\ell^\varphi = v_k^\varphi = A_{k-1}w_k\overline{A}_k$ and $A_{k-1} < A_k\overline{w}_k$.

We claim that $A_k = 1$. Suppose not. Then $k < 4g{+}p$ and, also, $\overline{A}_k$ ends in b. Now $\overline{s}_{\ell+1}^\varphi = v_{k+1}^\varphi = A_k w_{k+1}\overline{A}_{k+1}$. Thus $|s_{\ell+1}^\varphi| > 1$ and $s_{\ell+1}^\varphi$ ends in b. This is a contradiction. Hence $A_k = 1$.

Now, $s_\ell^\varphi = A_{k-1}w_k\overline{A}_k = A_{k-1}w_k$. Here, $w_k = b$ and, also, $A_{k-1} \neq 1$. Now, $A_{k-1} < A_k\overline{w}_k = \overline{w}_k = \overline{b}$. Thus, $A_{k-1} \in \mathfrak{B}\cup\mathfrak{B}^{-1}$. Write $a := \overline{A}_{k-1} \in \mathfrak{B}\cup\mathfrak{B}^{-1}$. Then $s_\ell^\varphi = \overline{a}b$ and $\overline{a} < \overline{b}$. There exists some $\ell' \in [1{\uparrow}(4g{+}2p)]$ such that $s_{\ell'} = \overline{s}_\ell$. Then $s_{\ell'}^\varphi = \overline{b}a$ and $\overline{a} < \overline{b}$. This contradicts the minimality of $\overline{b}$.

We have now shown that φ permutes the t-letters and maps the x-letters to letters. It follows from (e) that φ permutes the x-letters. Hence, φ gives a Nielsen$_1$ edge in $\mathcal{N}_{g,p}$.

This completes the proof of Claim 2 and the proof of the theorem. □ □

Theorem 2.6 combines Zieschang's approach [28, Section 5.2] and McCool's approach [8, Lemma 3.2]. Zieschang does not use Whitehead graphs explicitly and McCool does not use Nielsen$_3$ edges explicitly. For Claim 1, the ingenious pre-order and the proof of Case 1 go back to Nielsen [18], and the proof of Case 2 goes back to Artin [1]. The proof of Claim 2 goes back to Whitehead [21]. Zieschang refers to Nielsen [18] for the proof of his version of Claim 1 and gives a long proof of his version of Claim 2. McCool uses results of Whitehead [21] for the proof of his version of Theorem 2.6.

We shall be interested in five special cases.

In Theorem 2.6, we can take $H = 1$ and take $(V \xrightarrow{\varphi} W) \in \mathcal{Z}_{g,p}$, and get a decomposition of this latter edge as a path in $\mathcal{N}_{g,p}$. Thus we have the following.

2.7. **Consequence.** $\mathcal{Z}_{g,p}$ is generated by $\mathcal{N}_{g,p}$. □

In Theorem 2.6, we can take $H = 1$ and take φ to be an automorphism to obtain the following weak form of results of Whitehead.

2.8. **Consequence.** For $V \in \mathrm{V}\mathcal{Z}_{g,p}$ and $\varphi \in \mathrm{Stab}([t]_{[1\uparrow p]}; \mathrm{Aut}\, F_{g,p})$, if $|V^\varphi| \leqslant 4g{+}p$, then $V^\varphi \in \mathrm{V}\mathcal{Z}_{g,p}$. □

It is a classic result of Nielsen [18] that every surjective endomorphism of a finite-rank free group is an automorphism, and his proof is the basis of the above proof of Claim 1. A special case of this classic result will be used later in reviewing a proof of another result of Nielsen, Theorem 7.2, and to make our exposition self-contained, we now note that we have proved the desired special case. We have also proved one of Zieschang's results concerning injective endomorphisms being automorphisms.

In Theorem 2.6, we can take $H = 1$ to obtain the following.

2.9. **Consequence.** Suppose that φ is an endomorphism of $F_{g,p}$ such that φ is surjective or injective, and such that φ fixes $\Pi t_{[\![p\downarrow 1]\!]}\Pi[x,y]_{[\![1\uparrow g]\!]}$ and such that there exists some permutation π of $[1{\uparrow}p]$ such that, for each $j \in [1{\uparrow}p]$, t_j^φ is $F_{g,p}$-conjugate to t_{j^π}. Then φ is an automorphism. □

2.10. **Consequence.** Suppose that $p \geqslant 1$.

Let us identify $F_{g,p} = H{*}F_{g,p-1}$ where $H := \langle\, t_p \mid \quad \rangle$.

Let $V := \Pi t_{[\![(p-1)\downarrow 1]\!]}\Pi[x,y]_{[\![1\uparrow g]\!]} \in F_{g,p-1}$ and $\varphi \in \mathrm{Stab}(V; \mathrm{A}_{g,p}) = \mathrm{Stab}(t_p; \mathrm{A}_{g,p})$. By Theorem 2.6, φ acts as an automorphism φ' on $F_{g,p-1}$ and φ' lies in $\mathrm{A}_{g,p-1}$.

Thus, we have a natural isomorphism $\mathrm{Stab}(t_p; \mathrm{A}_{g,p}) \xrightarrow{\sim} \mathrm{A}_{g,p-1}$, $\varphi \mapsto \varphi'$. □

2.11. Consequence. Suppose that $p = 0$ and $g \geqslant 1$.

Let us identify $F_{g,0} = H{*}K$ where $H \coloneqq \langle\, x_1 \mid \quad \rangle$ and $K \coloneqq \langle\, y_{[1\uparrow g]} \cup x_{[2\uparrow g]} \mid \quad \rangle$. We have an isomorphism $K \xrightarrow{\sim} F_{g-1,1}$ with $y_1 \mapsto t_1$, and, for each $i \in [2\uparrow g]$, $x_i \mapsto x_{i-1}$, $y_i \mapsto y_{i-1}$.

Let $V \coloneqq y_1\Pi[x,y]_{[\![2\uparrow g]\!]}$ and $\varphi \in \operatorname{Stab}(V; \mathrm{A}_{g,0}) = \operatorname{Stab}(\overline{x}_1\overline{y}_1x_1; \mathrm{A}_{g,0})$. Then φ stabilizes the $F_{g,0}$-conjugacy class $[y_1]$. By Theorem 2.6, φ acts as an automorphism on K such that the induced action on $F_{g-1,1}$ is an element φ' of $\mathrm{A}_{g-1,1}$.

Then we have a homomorphism $\operatorname{Stab}(\overline{x}_1\overline{y}_1x_1; \mathrm{A}_{g,0}) \to \mathrm{A}_{g-1,1}$, $\varphi \mapsto \varphi'$. It is straightforward to see that this map is surjective and that the kernel is generated by a central element, $\alpha_1 \coloneqq \left(\begin{smallmatrix} x_1 \\ \overline{y}_1 x_1 \end{smallmatrix}\right)$. In particular, $\operatorname{Stab}(\overline{x}_1\overline{y}_1x_1; \mathrm{A}_{g,0})/\langle\, \alpha_1 \,\rangle \simeq \mathrm{A}_{g-1,1}$. □

3. The canonical edges in the Zieschang groupoid

In this section, we develop methods introduced by McCool in [20]. We define the canonical edges in $\mathcal{Z}_{g,p}$ and use them to find a special generating set for $\mathrm{A}_{g,p}$.

Throughout this section, all products AB are understood to be without cancellation; any product where cancellation might be possible will be written as $A{\circ}B$. Upper-case letters will be used to denote elements of $F_{g,p}$, and lower-case letters will be used to denote t-letters and x-letters.

3.1. Definitions. Let $g,\ p \in [0\uparrow\infty[$, let $F_{g,p} \coloneqq \langle\, t_{[1\uparrow p]} \cup x_{[1\uparrow g]} \cup y_{[1\uparrow g]} \mid \quad \rangle$, and let $V \in \mathrm{V}\mathcal{Z}_{g,p}$. We shall now recursively construct a path in $\mathcal{Z}_{g,p}$ from V to $\Pi t_{[\![p\downarrow 1]\!]}\Pi[x,y]_{[\![1\uparrow g]\!]}$. In particular, $\mathcal{Z}_{g,p}$ is connected. At each step, we specify an automorphism and tacitly apply Consequence 2.8 to see that we have an edge in $\mathcal{Z}_{g,p}$.

(i). If $p \geqslant 1$ and $V = Pt_jQ$ where t_j is the first t-letter which occurs in V and $P \neq 1$, then we travel along the edge
$$Pt_jQ \xrightarrow{\left(\begin{smallmatrix} t_j \\ t_j^P \end{smallmatrix}\right)} t_jPQ.$$

(ii). If $p \geqslant 1$ and $V = t_jP$ and $j \neq p$, then we travel along the edge
$$t_jP \xrightarrow{\left(\begin{smallmatrix} t_j & t_p \\ t_p & t_j \end{smallmatrix}\right)} t_pP'.$$

(iii). If $j \in [2\uparrow p]$ and V begins with $\Pi t_{[\![p\downarrow(j+1)]\!]}$ but not with $\Pi t_{[\![p\downarrow j]\!]}$, then we proceed analogously to steps (i) and (ii).

(iv). If $g \geqslant 1$ and $V = \Pi t_{[\![p\downarrow 1]\!]}aP\overline{a}Q$ where a is an x-letter and $a \neq \overline{x}_1$, then we travel along the edge
$$\Pi t_{[\![p\downarrow 1]\!]}aP\overline{a}Q \xrightarrow{\left(\begin{smallmatrix} a & x_1 \\ \overline{x}_1 & \overline{a} \end{smallmatrix}\right)} \Pi t_{[\![p\downarrow 1]\!]}\overline{x}_1P'x_1Q'.$$

(v). Suppose that $g \geqslant 1$ and $V = \Pi t_{[\![p\downarrow 1]\!]}\overline{x}_1Px_1Q$ and $|P| \geqslant 2$. If the set of letters which occur in P were closed under taking inverses, then the extended Whitehead graph of V would have a cycle $\overline{P_{\text{first}}} \rightsquigarrow \cdots \rightsquigarrow P_{\text{last}} \rightsquigarrow \overline{x}_1 \rightsquigarrow \overline{P_{\text{first}}}$, which is a contradiction. Let b denote the first letter that occurs in P such that $\overline{b}$ occurs in Q. We write $P = P_1bP_2$ and $Q = Q_1\overline{b}Q_2$, and we travel along the edge
$$\Pi t_{[\![p\downarrow 1]\!]}\overline{x}_1P_1bP_2x_1Q_1\overline{b}Q_2 \xrightarrow{\left(\begin{smallmatrix} b \\ \overline{P}_1 b \overline{P}_2 \end{smallmatrix}\right)} \Pi t_{[\![p\downarrow 1]\!]}\overline{x}_1bx_1Q_1P_2\overline{b}P_1Q_2.$$

(vi). If $g \geqslant 1$ and $V = \Pi t_{[\![p\downarrow 1]\!]}\overline{x}_1bx_1P\overline{b}Q$ where b is an x-letter and $b \neq \overline{y}_1$, then we travel along the edge
$$\Pi t_{[\![p\downarrow 1]\!]}\overline{x}_1bx_1P\overline{b}Q \xrightarrow{\left(\begin{smallmatrix} b & y_1 \\ \overline{y}_1 & \overline{b} \end{smallmatrix}\right)} \Pi t_{[\![p\downarrow 1]\!]}\overline{x}_1\overline{y}_1x_1P'y_1Q'.$$

(vii). Suppose that $g \geqslant 1$ and $V = \Pi t_{[\![p\downarrow 1]\!]}\overline{x}_1\overline{y}_1x_1Py_1Q$ and $P \neq 1$. Here the extended Whitehead graph of V has the form
$$\overline{t}_p \rightsquigarrow t_p \rightsquigarrow \cdots \rightsquigarrow \overline{t}_1 \rightsquigarrow t_1 \rightsquigarrow x_1 \rightsquigarrow \overline{P_{\text{first}}} \rightsquigarrow \cdots \rightsquigarrow P_{\text{last}} \rightsquigarrow \overline{y}_1 \rightsquigarrow \overline{x}_1 \rightsquigarrow y_1 \rightsquigarrow \overline{Q_{\text{first}}} \rightsquigarrow \cdots \rightsquigarrow Q_{\text{last}}.$$

Let φ denote the (Whitehead) automorphism of $F_{g,p}$ such that, for each letter u,
$u^\varphi := y_1^{\text{Truth}(\overline{u} \text{ appears in } \overline{P_{\text{first}}}\rightsquigarrow\cdots\rightsquigarrow P_{\text{last}})} \circ u \circ \overline{y}_1^{\text{Truth}(u \text{ appears in } \overline{P_{\text{first}}}\rightsquigarrow\cdots\rightsquigarrow P_{\text{last}})}$
where $\text{Truth}(-)$ assigns the value 1 to true statements and the value 0 to false statements. Then φ stabilizes each t-letter and x_1 and y_1. For all but two edges $(v_k\rightsquigarrow\overline{v}_{k+1})$, the right multiplier for v_k equals the right multiplier for $\overline{v}_{k+1}$, that is, the inverse of the left multiplier for v_{k+1}. The two exceptional edges are $x_1\rightsquigarrow\overline{P_{\text{first}}}$ and $P_{\text{last}}\rightsquigarrow\overline{y}_1$. It follows that $Q^\varphi = Q$ and $P^\varphi = y_1 P\overline{y}_1$.
We travel along the edge
$\Pi t_{[\![p\downarrow 1]\!]}\overline{x}_1\overline{y}_1 x_1 P y_1 Q \xrightarrow{\varphi} \Pi t_{[\![p\downarrow 1]\!]}[x_1, y_1]PQ.$

(viii). If $i \in [2\uparrow g]$ and V begins with $\Pi t_{[\![p\downarrow 1]\!]}\Pi[x,y]_{[\![1\uparrow(i-1)]\!]}$ but V does not begin with $\Pi t_{[\![p\downarrow 1]\!]}\Pi[x,y]_{[\![1\uparrow i]\!]}$, then we proceed analogously to steps (iv)–(vii).

The foregoing procedure specifies a path in $\mathcal{Z}_{g,p}$ from V to $\Pi t_{[\![p\downarrow 1]\!]}\Pi[x,y]_{[\![1\uparrow g]\!]}$, and, hence, a *canonical edge* in $\mathcal{Z}_{g,p}$, denoted

$$V \xrightarrow{\Phi_V} \Pi t_{[\![p\downarrow 1]\!]}\Pi[x,y]_{[\![1\uparrow g]\!]}.$$

We understand that $\Phi_{\Pi t_{[\![p\downarrow 1]\!]}\Pi[x,y]_{[\![1\uparrow g]\!]}}$ is the identity map. The only information about Φ_V that we shall need is that the following hold; all of these assertions can be seen from the construction.

(3.1.1) If $p = 0$ and $g \geqslant 1$ and $V = aP\overline{a}Q$, then Φ_V sends a to $\overline{x}_1$, and P to $\overline{y}_1$.
(3.1.2) If $p = 1$ and $V = Pt_1Q = (t_1^{\overline{P}}) \circ (PQ)$, then Φ_V sends $t_1^{\overline{P}}$ to t_1.
(3.1.3) If $p = 1$ and $g \geqslant 1$ and $V = t_1 aP\overline{a}Q$, then Φ_V sends t_1 to t_1, a to $\overline{x}_1$, and P to $\overline{y}_1$.
(3.1.4) If $p \geqslant 2$ and $V = Pt_{j_1}Qt_{j_2}R = (t_{j_1}^{\overline{P}})\circ(t_{j_2}^{\overline{Q}\,\overline{P}})\circ(PQR)$, and no t-letters occur in P or Q, then Φ_V sends $t_{j_1}^{\overline{P}}$ to t_p, and $t_{j_2}^{\overline{Q}\,\overline{P}}$ to t_{p-1}. □

3.2**. Remark.** We shall be given a special subset A' of $\mathrm{A}_{g,p}$ that we wish to show generates $\mathrm{A}_{g,p}$. We view $\mathrm{A}_{g,p}$ as the set of edges of $\mathcal{Z}_{g,p}$ from $\Pi t_{[\![p\downarrow 1]\!]}\Pi[x,y]_{[\![1\uparrow g]\!]}$ to itself, and we let $\mathcal{Z}'_{g,p}$ denote the subgroupoid of $\mathcal{Z}_{g,p}$ generated by the edges in A' together with all the canonical edges of $\mathcal{Z}_{g,p}$. Using methods introduced by McCool [20], we shall prove that $\mathcal{Z}'_{g,p}$ contains the Nielsen subgraph $\mathcal{N}_{g,p}$ of $\mathcal{Z}_{g,p}$. By Consequence 2.7, $\mathcal{Z}'_{g,p} = \mathcal{Z}_{g,p}$. Now when any edge in $\mathrm{A}_{g,p}$ is expressed as a product of canonical edges and edges in A' and their inverses, then the nontrivial canonical edges and their inverses must pair off and cancel out, and we are left with an expression that involves no nontrivial canonical edges. Here, $\mathrm{A}_{g,p}$ is generated by A'. □

3.3**. Theorem.** *Let $g \in [1\uparrow\infty[$, $p = 0$. Then the group $\mathrm{A}_{g,0}$ is generated by $\mathrm{Stab}(\overline{x}_1\overline{y}_1x_1; \mathrm{A}_{g,0}) \cup \{\beta_1\}$, where $\beta_1 := \left(\begin{smallmatrix} y_1 \\ x_1y_1 \end{smallmatrix}\right)$.*

Proof. Let $\mathcal{Z}'_{g,0}$ denote the subgroupoid of $\mathcal{Z}_{g,0}$ generated by the given set together with all the canonical edges. By Remark 3.2, it suffices to show that $\mathcal{N}_{g,0} \subseteq \mathcal{Z}'_{g,0}$.

Recall that $\alpha_1 := \left(\begin{smallmatrix} x_1 \\ \overline{y}_1x_1 \end{smallmatrix}\right) \in \mathrm{Stab}(\overline{x}_1\overline{y}_1x_1; \mathrm{A}_{g,0}) \subseteq \mathcal{Z}'_{g,0}$. In $F_{g,0}$, $(\overline{x}_1\overline{y}_1x_1)^{\beta_1\alpha_1} = (\overline{x}_1\overline{y}_1)^{\alpha_1} = \overline{x}_1$. Hence $\mathrm{Stab}(\overline{x}_1; \mathrm{A}_{g,0}) \subseteq \mathcal{Z}'_{g,0}$. Thus $\mathcal{Z}'_{g,0}$ contains all the edges of the forms

$$(\text{I.1})\colon V \in \mathrm{V}\mathcal{Z}_{g,0} \xrightarrow{\Phi_V} \Pi[x,y]_{[\![1\uparrow g]\!]},$$

$$(\text{I.2})\colon \Pi[x,y]_{[\![1\uparrow g]\!]} \xrightarrow{\text{map in } \mathrm{A}_{g,0} \text{ that stabilizes } \overline{x}_1 \text{ or } \overline{x}_1\overline{y}_1x_1} \Pi[x,y]_{[\![1\uparrow g]\!]}.$$

We next describe two more families of edges in $\mathcal{Z}'_{g,0}$, expressed as products of edges of types (I.1) and (I.2) and their inverses.

$$\begin{tikzcd}[column sep=huge]
(\mathrm{I.3})\colon a_1P_1 \in \mathrm{V}\mathcal{Z}_{g,0} \arrow[r, "\text{map in } \mathrm{Aut}\,F_{g,0} \text{ with } a_1\mapsto a_2,\, P_1\mapsto P_2"] \arrow[d, "(\mathrm{I.1})\overset{(3.1.1)}{\Rightarrow}(a_1\mapsto\overline{x}_1)"] & a_2P_2 \in \mathrm{V}\mathcal{Z}_{g,0} \arrow[d, "(\mathrm{I.1})\overset{(3.1.1)}{\Rightarrow}(a_2\mapsto\overline{x}_1)"] \\
\Pi[x,y]_{[\![1\uparrow g]\!]} \arrow[r, "\text{map making square commute} \Rightarrow(\overline{x}_1\mapsto\overline{x}_1)\Rightarrow(\mathrm{I.2})"'] & \Pi[x,y]_{[\![1\uparrow g]\!]}
\end{tikzcd}$$

$$\begin{tikzcd}[column sep=huge]
(\mathrm{I.4})\colon ab P\overline{a}Q \in \mathrm{V}\mathcal{Z}_{g,0} \arrow[r, "\left(\begin{smallmatrix} a \\ a\overline{b}\end{smallmatrix}\right)\Rightarrow(abP\overline{a}\,\mapsto\, aPb\overline{a})"] \arrow[d, "(\mathrm{I.1})\overset{(3.1.1)}{\Rightarrow}(abP\overline{a}\mapsto\overline{x}_1\overline{y}_1x_1)"] & aPb\overline{a}Q \in \mathrm{V}\mathcal{Z}_{g,0} \arrow[d, "(\mathrm{I.1})\overset{(3.1.1)}{\Rightarrow}(aPb\overline{a}\mapsto\overline{x}_1\overline{y}_1x_1)"] \\
\Pi[x,y]_{[\![1\uparrow g]\!]} \arrow[r, "\text{make square commute} \Rightarrow(\overline{x}_1\overline{y}_1x_1\mapsto\overline{x}_1\overline{y}_1x_1)\Rightarrow(\mathrm{I.2})"'] & \Pi[x,y]_{[\![1\uparrow g]\!]}
\end{tikzcd}$$

We then have the family

$$(\mathrm{I.5})\colon abP\bar{b}Q \in \mathrm{V}\mathcal{Z}_{g,0} \xrightarrow{\left(\begin{smallmatrix} b \\ \overline{a}b\end{smallmatrix}\right)} bP\bar{b}aQ \in \mathrm{V}\mathcal{Z}_{g,0},$$

since, here, we have the factorization

$$abP\bar{b}Q \xrightarrow{\left(\begin{smallmatrix} a \\ a\bar{b}\end{smallmatrix}\right)\Rightarrow(\mathrm{I.4})} aP'\bar{b}Q' \xrightarrow{\text{makes triangle commute} \Rightarrow(a\mapsto b)\Rightarrow(\mathrm{I.3})} bP\bar{b}aQ.$$

It can be seen that the edges of type (I.3) include all the Nielsen$_1$ edges in $\mathcal{Z}_{g,0}$, and also all the Nielsen$_2$ edges in $\mathcal{Z}_{g,0}$ that do not involve a_1. The remaining Nielsen$_2$ edges in $\mathcal{Z}_{g,0}$ are of type (I.4) or (I.5) or their inverses. Since $p = 0$, there are no Nielsen$_3$ edges. We have now shown that $\mathcal{N}_{g,0} \subseteq \mathcal{Z}'_{g,0}$, as desired. □

3.4. **Theorem.** *Let $g \in [1\uparrow\infty[$, $p = 1$. Then the group $\mathrm{A}_{g,1}$ is generated by $\mathrm{Stab}(t_1;\mathrm{A}_{g,1}) \cup \{\gamma_1\}$ where $\gamma_1 := \left(\begin{smallmatrix} t_1 & x_1 \\ t_1^{w_1} & x_1w_1\end{smallmatrix}\right)$ with $w_1 := t_1\overline{y}_1^{x_1}$.*

Proof. Let $\mathcal{Z}'_{g,1}$ denote the subgroupoid of $\mathcal{Z}_{g,1}$ generated by the given set together with all the canonical edges. By Remark 3.2, it suffices to show that $\mathcal{N}_{g,1} \subseteq \mathcal{Z}'_{g,1}$.

Now $\mathcal{Z}'_{g,1}$ contains $\mathrm{Stab}(t_1;\mathrm{A}_{g,1})\gamma_1$, which consists of the maps in $\mathrm{A}_{g,1}$ with $t_1 \mapsto t_1^{\gamma_1} = t_1^{w_1} = t_1^{\overline{x}_1\overline{y}_1x_1}$. Thus, $\mathcal{Z}'_{g,1}$ contains all the edges of the forms

$$(\mathrm{II.1})\colon V \in \mathrm{V}\mathcal{Z}_{g,1} \xrightarrow{\Phi_V} t_1\Pi[x,y]_{[\![1\uparrow g]\!]},$$

$$(\mathrm{II.2})\colon t_1\Pi[x,y]_{[\![1\uparrow g]\!]} \xrightarrow{\text{map in } \mathrm{A}_{g,1} \text{ with } t_1\mapsto t_1 \text{ or } t_1\mapsto t_1^{\overline{x}_1\overline{y}_1x_1}} t_1\Pi[x,y]_{[\![1\uparrow g]\!]}.$$

We next describe another family of edges in $\mathcal{Z}'_{g,1}$.

$$\begin{tikzcd}[column sep=huge]
(\mathrm{II.3})\colon P_1t_1Q_1 \in \mathrm{V}\mathcal{Z}_{g,1} \arrow[r, "\text{map in } \mathrm{Aut}\,F_{g,1} \text{ with } t_1^{\overline{P}_1}\mapsto t_1^{\overline{P}_2},\, P_1Q_1\mapsto P_2Q_2"] \arrow[d, "(\mathrm{II.1})\overset{(3.1.2)}{\Rightarrow}(t_1^{\overline{P}_1}\mapsto t_1)"] & P_2t_1Q_2 \in \mathrm{V}\mathcal{Z}_{g,1} \arrow[d, "(\mathrm{II.1})\overset{(3.1.2)}{\Rightarrow}(t_1^{\overline{P}_2}\mapsto t_1)"] \\
t_1\Pi[x,y]_{[\![1\uparrow g]\!]} \arrow[r, "\text{map makes the square commute} \Rightarrow(t_1\mapsto t_1)\Rightarrow(\mathrm{II.2})"'] & t_1\Pi[x,y]_{[\![1\uparrow g]\!]}
\end{tikzcd}$$

Edges of type (II.3) include all the Nielsen$_1$ edges, and all the Nielsen$_2$ edges which do not involve t_1, and all the Nielsen$_3$ edges, since these have the form

$$Pat_1Q \xrightarrow{\left(\begin{smallmatrix} t_1 \\ t_1^{a}\end{smallmatrix}\right)\Rightarrow(t_1^{\overline{a}\,\overline{P}}\mapsto t_1^{\overline{P}},\, PaQ\mapsto PaQ)} Pt_1aQ, \text{ or its inverse.}$$

It remains to consider the Nielsen$_2$ edges which involve t_1; these are of the forms

$Pt_1aQ\overline{a}R \xrightarrow{\binom{a}{\overline{t}_1 a}} PaQ\overline{a}t_1R$, $\quad Pat_1Q\overline{a}R \xrightarrow{\binom{a}{a\overline{t}_1}} PaQt_1\overline{a}R$, and their inverses. To construct a commuting hexagon, we define the following edges.

$$\text{(II.4)}\colon PaQ\overline{a}t_1R \in \mathrm{V}\mathcal{Z}_{g,1} \xrightarrow{\binom{t_1}{t_1^{PaQ\overline{a}}} \Rightarrow \text{(II.3)}, (t_1^{\overline{P}} \mapsto t_1^{PaQ\overline{a}\overline{P}})} t_1PaQ\overline{a}R \in \mathrm{V}\mathcal{Z}_{g,1},$$

$$\text{(II.5)}\colon t_1PaQ\overline{a}R \in \mathrm{V}\mathcal{Z}_{g,1} \xrightarrow{\binom{a}{\overline{P}a} \Rightarrow \text{(II.3)}, (t_1^{PaQ\overline{a}\overline{P}} \mapsto t_1^{aQ\overline{a}})} t_1aQ\overline{a}PR \in \mathrm{V}\mathcal{Z}_{g,1},$$

$$\text{(II.6)}\colon t_1aQ\overline{a}PR \in \mathrm{V}\mathcal{Z}_{g,1} \xrightarrow{\text{(II.1)}\overset{(3.1.3)}{\Rightarrow}(t_1^{aQ\overline{a}} \mapsto t_1^{\overline{x}_1\overline{y}_1x_1})} t_1\Pi[x,y]_{[\![1\uparrow g]\!]}.$$

Then we have the factorization

$$\begin{tikzcd}
\text{(II.7)}\colon Pt_1aQ\overline{a}R \in \mathrm{V}\mathcal{Z}_{g,1} \arrow[r, "\binom{a}{\overline{t}_1 a}\Rightarrow(t_1^{\overline{P}}\mapsto t_1^{\overline{P}})"] \arrow[d, "\text{(II.1)}\overset{(3.1.2)}{\Rightarrow}(t_1^{\overline{P}}\mapsto t_1)"] & PaQ\overline{a}t_1R \in \mathrm{V}\mathcal{Z}_{g,1} \arrow[d, "\substack{\text{(II.4)}-\text{(II.6)} \\ \Rightarrow(t_1^{\overline{P}}\mapsto t_1^{\overline{x}_1\overline{y}_1x_1})}"] \\
t_1\Pi[x,y]_{[\![1\uparrow g]\!]} \arrow[r, "\text{map makes square commute }\Rightarrow(t_1\mapsto t_1^{\overline{x}_1\overline{y}_1x_1})\Rightarrow\text{(II.2)}"'] & t_1\Pi[x,y]_{[\![1\uparrow g]\!]}
\end{tikzcd}$$

We also have the factorization

$$\begin{tikzcd}
\text{(II.8)}\colon Pat_1Q\overline{a}R \in \mathrm{V}\mathcal{Z}_{g,1} \arrow[r, "\binom{a}{a\overline{t}_1}"] \arrow[d, "\binom{t_1}{t_1^a}\Rightarrow\text{(II.3)}"] & PaQt_1\overline{a}R \in \mathrm{V}\mathcal{Z}_{g,1} \arrow[d, "\binom{t_1}{t_1^a}\Rightarrow\text{(II.3)}"] \\
Pt_1aQ\overline{a}R \in \mathrm{V}\mathcal{Z}_{g,1} \arrow[r, "\text{map makes square commute }\Rightarrow\binom{a}{\overline{t}_1 a}\Rightarrow\text{(II.7)}"'] & PaQ\overline{a}t_1R \in \mathrm{V}\mathcal{Z}_{g,1}
\end{tikzcd}$$

We have now shown that $\mathcal{N}_{g,1} \subseteq \mathcal{Z}'_{g,1}$, as desired. □

3.5. **Theorem.** *Let $g \in [0\uparrow\infty[$, $p \in [2\uparrow\infty[$. Then the group $\mathrm{A}_{g,p}$ is generated by $\mathrm{Stab}(t_p; \mathrm{A}_{g,p}) \cup \{\sigma_p\}$ where $\sigma_p \coloneqq \left(\begin{smallmatrix} t_p & t_{p-1} \\ t_{p-1} & t_p^{t_{p-1}} \end{smallmatrix}\right)$.*

Proof. Let $\mathcal{Z}'_{g,p}$ denote the subgroupoid of $\mathcal{Z}_{g,p}$ generated by the given set together with all the canonical edges. By Remark 3.2, it suffices to show that $\mathcal{N}_{g,p} \subseteq \mathcal{Z}'_{g,p}$.

Now $\mathcal{Z}'_{g,p}$ contains $\mathrm{Stab}(t_p; \mathrm{A}_{g,p})\sigma_p$, which consists of the maps in $\mathrm{A}_{g,p}$ with $t_p \mapsto t_p^{\sigma_p} = t_{p-1}$. Thus $\mathcal{Z}'_{g,p}$ contains all the edges of the forms

$$\text{(III.1)}\colon V \in \mathrm{V}\mathcal{Z}_{g,p} \xrightarrow{\Phi_V} \Pi t_{[\![p\downarrow 1]\!]}\Pi[x,y]_{[\![1\uparrow g]\!]},$$

$$\text{(III.2)}\colon \Pi t_{[\![p\downarrow 1]\!]}\Pi[x,y]_{[\![1\uparrow g]\!]} \xrightarrow{\text{map in } \mathrm{A}_{g,p} \text{ with } t_p\mapsto t_p \text{ or } t_p\mapsto t_{p-1}} \Pi t_{[\![p\downarrow 1]\!]}\Pi[x,y]_{[\![1\uparrow g]\!]}.$$

We now describe some more families of edges in $\mathcal{Z}'_{g,p}$.

In the following, we assume that no t-letters occur in P_1 or P_2.

$$\begin{tikzcd}
\text{(III.3)}\colon P_1t_{j_1}Q_1 \in \mathrm{V}\mathcal{Z}_{g,p} \arrow[r, "\text{in Stab}([t]_{[1\uparrow p]}; F_{g,p}),\ P_1\mapsto P_2, t_{j_1}\mapsto t_{j_2}, Q_1\mapsto Q_2"] \arrow[d, "\text{(III.1)}\overset{(3.1.4)}{\Rightarrow}(t_{j_1}^{\overline{P}_1}\mapsto t_p)"] & P_2t_{j_2}Q_2 \in \mathrm{V}\mathcal{Z}_{g,p} \arrow[d, "\text{(III.1)}\overset{(3.1.4)}{\Rightarrow}(t_{j_2}^{\overline{P}_2}\mapsto t_p)"] \\
\Pi t_{[\![p\downarrow 1]\!]}\Pi[x,y]_{[\![1\uparrow g]\!]} \arrow[r, "\text{makes square commute }\Rightarrow(t_p\mapsto t_p)\Rightarrow\text{(III.2)}"'] & \Pi t_{[\![p\downarrow 1]\!]}\Pi[x,y]_{[\![1\uparrow g]\!]}
\end{tikzcd}$$

The edges of type (III.3) include all the Nielsen$_1$ edges.

In the following, we assume that no t-letters occur in P or Q.

$$
\text{(III.4)}\colon\
\begin{tikzcd}[column sep=huge]
PQt_{j_1}t_{j_2}R \in \mathrm{V}\mathcal{Z}_{g,p} \arrow[r, "\begin{pmatrix} t_{j_2} \\ t_{j_2}^{Qt_{j_1}} \end{pmatrix} \Rightarrow (t_{j_1}^{\overline{Q}\,\overline{P}} \mapsto t_{j_1}^{\overline{Q}\,\overline{P}})"] \arrow[d, "\text{(III.1)} \overset{(3.1.4)}{\Rightarrow} (t_{j_1}^{\overline{Q}\,\overline{P}} \mapsto t_p)"'] & Pt_{j_2}Qt_{j_1}R \in \mathrm{V}\mathcal{Z}_{g,p} \arrow[d, "\text{(III.1)} \overset{(3.1.4)}{\Rightarrow} (t_{j_1}^{\overline{Q}\,\overline{P}} \mapsto t_{p-1})"] \\
\Pi t_{\llbracket p\downarrow 1\rrbracket}\Pi[x,y]_{\llbracket 1\uparrow g\rrbracket} \arrow[r, "\text{square commutes} \Rightarrow (t_p \mapsto t_{p-1}) \Rightarrow \text{(III.2)}"'] & \Pi t_{\llbracket p\downarrow 1\rrbracket}\Pi[x,y]_{\llbracket 1\uparrow g\rrbracket}
\end{tikzcd}
$$

In the following, we assume that no t-letters occur in P.

$$\text{(III.5}'\text{)}\colon\ Pt_{j_1}Qt_{j_2}R \in \mathrm{V}\mathcal{Z}_{g,p} \xrightarrow{\begin{pmatrix} t_{j_2} \\ t_{j_2}^{Pt_{j_1}Q} \end{pmatrix}} t_{j_2}Pt_{j_1}QR \in \mathrm{V}\mathcal{Z}_{g,p}$$

has the factorization

$$Pt_{j_1}Qt_{j_2}R \xrightarrow{\begin{pmatrix} t_{j_2} \\ t_{j_2}^{Q} \end{pmatrix} \Rightarrow \text{(III.3)}} Pt_{j_1}t_{j_2}QR \xrightarrow{\begin{pmatrix} t_{j_2} \\ t_{j_2}^{Pt_{j_1}} \end{pmatrix} \Rightarrow \text{(III.4)}} t_{j_2}Pt_{j_1}QR.$$

In the following, we do allow t-letters to occur in P, and rewrite (III.5′) as

$$\text{(III.5)}\colon\ Pt_jQ \in \mathrm{V}\mathcal{Z}_{g,p} \xrightarrow{\begin{pmatrix} t_{j} \\ t_{j}^{P} \end{pmatrix}} t_jPQ \in \mathrm{V}\mathcal{Z}_{g,p}.$$

In the following, we do allow t-letters to occur in P_1, P_2.

$$
\text{(III.6)}\colon\
\begin{tikzcd}[column sep=huge]
P_1t_jQ_1 \in \mathrm{V}\mathcal{Z}_{g,p} \arrow[r, "\text{in } \mathrm{Stab}([t]_{[1\uparrow p]};F_{g,p}) \text{ with } P_1\mapsto P_2,\, t_j\mapsto t_j,\, Q_1\mapsto Q_2"] \arrow[d, "\begin{pmatrix} t_{j} \\ t_{j}^{P_1} \end{pmatrix} \Rightarrow \text{(III.5)}"'] & P_2t_jQ_2 \in \mathrm{V}\mathcal{Z}_{g,p} \arrow[d, "\begin{pmatrix} t_{j} \\ t_{j}^{P_2} \end{pmatrix} \Rightarrow \text{(III.5)}"] \\
t_jP_1Q_1 \in \mathrm{V}\mathcal{Z}_{g,p} \arrow[r, "\text{map makes square commute} \Rightarrow (t_j \mapsto t_j) \Rightarrow \text{(III.3)}"'] & t_jP_2Q_2 \in \mathrm{V}\mathcal{Z}_{g,p}
\end{tikzcd}
$$

Since $p \geqslant 2$, any Nielsen$_2$ edge of $\mathcal{Z}_{g,p}$ will be of type (III.6) for some j, as will any Nielsen$_3$ edge except where $p = 2$ and we have an edge of the form $Pt_{j_1}t_{j_2}Q \in \mathrm{V}\mathcal{Z}_{g,p} \xrightarrow{\begin{pmatrix} t_{j_2} \\ t_{j_2}^{t_{j_1}} \end{pmatrix}} Pt_{j_2}t_{j_1}Q \in \mathrm{V}\mathcal{Z}_{g,p}$, and or its inverse, and, since $p = 2$, these are of type (III.4).

We have now shown that $\mathcal{N}_{g,p} \subseteq \mathcal{Z}'_{g,p}$, as desired. $\square$

4. The ADLH generating set

The results of the preceding two sections combine to give an algebraic proof of the algebraic form of [13, Proposition 2.10(ii) with $r = 0$]. We start with the ADL set.

4.1. Theorem. *Let $g,\ p \in [0\uparrow\infty[$. Let $\mathrm{A}_{g,p}$ denote the group of automorphisms of $\langle\, t_{[1\uparrow p]} \cup x_{[1\uparrow g]} \cup y_{[1\uparrow g]} \mid \quad \rangle$ that fix $\Pi\, t_{\llbracket p\downarrow 1\rrbracket}\, \Pi[x,y]_{\llbracket 1\uparrow g\rrbracket}$ and permute the set of conjugacy classes $[t]_{[1\uparrow p]}$. Then $\mathrm{A}_{g,p}$ is generated by*

$$\sigma_{[2\uparrow p]} \cup \alpha_{[1\uparrow g]} \cup \beta_{[1\uparrow g]} \cup \gamma_{[\max(2-p,1)\uparrow g]},$$

where, for $j \in [2\uparrow p]$, $\sigma_j \coloneqq \begin{pmatrix} t_j & t_{j-1} \\ t_{j-1} & t_j^{t_{j-1}} \end{pmatrix}$, for $i \in [1\uparrow g]$, $\alpha_i \coloneqq \begin{pmatrix} x_i \\ \overline{y}_i x_i \end{pmatrix}$ and $\beta_i \coloneqq \begin{pmatrix} y_i \\ x_i y_i \end{pmatrix}$, for $i \in [2\uparrow g]$, $\gamma_i \coloneqq \begin{pmatrix} x_{i-1} & y_{i-1} & x_i \\ \overline{w}_i x_{i-1} & y_{i-1}^{w_i} & x_i w_i \end{pmatrix}$ with $w_i \coloneqq y_{i-1}\overline{y}_i^{x_i}$, and if $\min(1,g,p) = 1$, $\gamma_1 \coloneqq \begin{pmatrix} t_1 & x_1 \\ t_1^{w_1} & x_1 w_1 \end{pmatrix}$ with $w_1 \coloneqq t_1\overline{y}_1^{x_1}$.

Proof. We use induction on $2g + p$. If $2g + p \leqslant 1$, then $\mathrm{A}_{g,p}$ is trivial and the proposed generating set is empty. Thus we may assume that $2g + p \geqslant 2$, and that the conclusion holds for smaller pairs (g,p).

Case 1. $p = 0$.

Here $g \geqslant 1$.

By Consequence 2.11, we have a homomorphism $\mathrm{Stab}(\overline{x}_1\overline{y}_1x_1; \mathrm{A}_{g,0}) \to \mathrm{A}_{g-1,1}$ such that the kernel is $\langle\, \alpha_1 \,\rangle$, and such that $\alpha_{[2\uparrow g]} \cup \beta_{[2\uparrow g]} \cup \gamma_{[2\uparrow g]}$ is mapped bijectively to $\alpha_{[1\uparrow(g-1)]} \cup \beta_{[1\uparrow(g-1)]} \cup \gamma_{[1\uparrow(g-1)]}$. The latter is a generating set of $\mathrm{A}_{g-1,1}$ by the induction hypothesis. It follows that $\mathrm{Stab}(\overline{x}_1\overline{y}_1x_1; \mathrm{A}_{g,0})$ is generated by $\alpha_{[1\uparrow g]} \cup \beta_{[2\uparrow g]} \cup \gamma_{[2\uparrow g]}$.

By Theorem 3.3, $\mathrm{A}_{g,0}$ is generated by $\mathrm{Stab}(\overline{x}_1\overline{y}_1x_1; \mathrm{A}_{g,0}) \cup \{\beta_1\}$.

Hence $\mathrm{A}_{g,0}$ is generated by $\alpha_{[1\uparrow g]} \cup \beta_{[1\uparrow g]} \cup \gamma_{[2\uparrow g]}$, as desired.

Case 2. $p \geqslant 1$.

It follows from Consequence 2.10 that we can identify $\mathrm{Stab}(t_p; \mathrm{A}_{g,p})$ with $\mathrm{A}_{g,p-1}$ in a natural way. By the induction hypothesis, $\mathrm{Stab}(t_p; \mathrm{A}_{g,p})$ is generated by $\sigma_{[2\uparrow(p-1)]} \cup \alpha_{[1\uparrow g]} \cup \beta_{[1\uparrow g]} \cup \gamma_{[\max(3-p,1)\uparrow g]}$. We consider two cases.

Case 2.1. $p = 1$.

Here $g \geqslant 1$. By Theorem 3.4, $\mathrm{A}_{g,1}$ is generated by $\mathrm{Stab}(t_1; \mathrm{A}_{g,1}) \cup \{\gamma_1\}$.

Hence, $\mathrm{A}_{g,1}$ is generated by $\alpha_{[1\uparrow g]} \cup \beta_{[1\uparrow g]} \cup \gamma_{[1\uparrow g]}$, as desired.

Case 2.2. $p \geqslant 2$.

By Theorem 3.5, $\mathrm{A}_{g,p}$ is generated by $\mathrm{Stab}(t_p; \mathrm{A}_{g,p}) \cup \{\sigma_p\}$.

Hence, $\mathrm{A}_{g,p}$ is generated by $\sigma_{[2\uparrow p]} \cup \alpha_{[1\uparrow g]} \cup \beta_{[1\uparrow g]} \cup \gamma_{[1\uparrow g]}$, as desired. □

We next recall Humphries' result [12] that the $\alpha_{[3\uparrow g]}$ part is not needed, and, hence, the ADHL set suffices.

4.2. **Corollary.** $\mathrm{A}_{g,p}$ *is generated by* $\sigma_{[2\uparrow p]} \cup \alpha_{[1\uparrow \min(2,g)]} \cup \beta_{[1\uparrow g]} \cup \gamma_{[\max(2-p,1)\uparrow g]}$.

Proof. It is not difficult to check that there exists an element of $\mathrm{A}_{3,0}$ given by

$$\eta := \left(\begin{smallmatrix} x_1 & y_1 & x_2 & y_2 & x_3 & y_3 \\ \overline{y}_3\overline{x}_3\overline{y}_3 & \overline{y}_3^{x_3y_3} & x_2^{[x_3,y_3]} & y_2^{[x_3,y_3]} & x_1^{[x_2,y_2][x_3,y_3]} & y_1^{[x_2,y_2][x_3,y_3]} \end{smallmatrix}\right),$$

and that $(\overline{x}_1\overline{y}_1x_1)^\eta = y_3$, and that both $\alpha_1\eta$ and $\eta\alpha_3$ equal

$$\left(\begin{smallmatrix} x_1 & y_1 & x_2 & y_2 & x_3 & y_3 \\ \overline{y}_3\overline{x}_3 & \overline{y}_3^{x_3y_3} & x_2^{[x_3,y_3]} & y_2^{[x_3,y_3]} & x_1^{[x_2,y_2][x_3,y_3]} & y_1^{[x_2,y_2][x_3,y_3]} \end{smallmatrix}\right).$$

By Consequence 2.11, each element of $\mathrm{Stab}(\overline{x}_1\overline{y}_1x_1; \mathrm{A}_{3,0})$ centralizes α_1, and, hence, each element of $\mathrm{Stab}(\overline{x}_1\overline{y}_1x_1; \mathrm{A}_{3,0})\eta$ conjugates α_1 into α_3. Notice that $\mathrm{Stab}(\overline{x}_1\overline{y}_1x_1; \mathrm{A}_{3,0})\eta$ is the set of elements of $\mathrm{A}_{3,0}$ with $\overline{x}_1\overline{y}_1x_1 \mapsto y_3$.

One can compute

$$\begin{array}{lllllllll}
\overline{x}_1\overline{y}_1x_1 & \overset{\beta_1}{\mapsto} & \overline{x}_1\overline{y}_1 & \overset{\gamma_2}{\mapsto} & \overline{x}_1\overline{x}_2\overline{y}_2x_2 & \overset{\beta_2}{\mapsto} & \overline{x}_1\overline{x}_2\overline{y}_2 & \overset{\alpha_2}{\mapsto} & \overline{x}_1\overline{x}_2 = \\
\overline{x}_1\overline{x}_2 & \overset{\gamma_3}{\mapsto} & \overline{x}_1\overline{x}_2y_2\overline{x}_3\overline{y}_3x_3 & \overset{\beta_3}{\mapsto} & \overline{x}_1\overline{x}_2y_2\overline{x}_3\overline{y}_3 & \overset{\beta_2}{\mapsto} & \overline{x}_1y_2\overline{x}_3\overline{y}_3 & \overset{\gamma_3}{\mapsto} & \overline{x}_1\overline{x}_3 = \\
\overline{x}_1\overline{x}_3 & \overset{\gamma_2}{\mapsto} & \overline{x}_1y_1\overline{x}_2\overline{y}_2x_2\overline{x}_3 & \overset{\beta_2}{\mapsto} & \overline{x}_1y_1\overline{x}_2\overline{y}_2\overline{x}_3 & \overset{\beta_1}{\mapsto} & y_1\overline{x}_2\overline{y}_2\overline{x}_3 & \overset{\gamma_2}{\mapsto} & \overline{x}_2\overline{x}_3 = \\
\overline{x}_2\overline{x}_3 & \overset{\alpha_2}{\mapsto} & \overline{x}_2y_2\overline{x}_3 & \overset{\beta_2}{\mapsto} & y_2\overline{x}_3 & \overset{\gamma_3}{\mapsto} & \overline{x}_3y_3 & \overset{\beta_3}{\mapsto} & y_3;
\end{array}$$

this is the algebraic translation of [12, Figure 2]. We then see that, as in [12], $\alpha_1^{\beta_1\gamma_2\beta_2\alpha_2\gamma_3\beta_3\beta_2\gamma_3\gamma_2\beta_2\beta_1\gamma_2\alpha_2\beta_2\gamma_3\beta_3} = \alpha_3$. By shifting the indices upward, we see that $\alpha_{[3\uparrow g]}$ can be removed from the ADL set and still leave a generating set.

Alternatively, it would also suffice to prove the relation (R5) of [13, Theorem 3.1] which says that $\alpha_3 = (\alpha_1\beta_1\gamma_2\beta_2\gamma_3\alpha_2)^5(\alpha_1\beta_1\gamma_2\beta_2\gamma_3)^{-6}$; notice that this expression is longer and does not involve β_3. □

We have now completed our objective. For completeness, we conclude the article with an elementary review of some classic results.

5. Some background on mapping-class groups

5.1. **Notation.** Let us define $F_{g,p-1} \coloneqq \langle\, t_{[1\uparrow p]} \cup x_{[1\uparrow g]} \cup y_{[1\uparrow g]} \mid \Pi t_{[\![p\downarrow 1]\!]}\Pi[x,y]_{[\![1\uparrow g]\!]} \,\rangle$.

Then $F_{g,p} = \langle\, t_{[1\uparrow (p+1)]} \cup x_{[1\uparrow g]} \cup y_{[1\uparrow g]} \mid \Pi t_{[\![(p+1)\downarrow 1]\!]}\Pi[x,y]_{[\![1\uparrow g]\!]} \,\rangle$, and we still have $F_{g,p} = \langle\, t_{[1\uparrow p]} \cup x_{[1\uparrow g]} \cup y_{[1\uparrow g]} \mid \quad \rangle$ and here $\overline{t}_{p+1} = \Pi t_{[\![p\downarrow 1]\!]}\Pi[x,y]_{[\![1\uparrow g]\!]}$. □

5.2. **Definitions.** We construct an orientable surface $\mathbf{S}_{g,1,p}$, of genus g with p punctures and one boundary component, as follows. We start with a vertex which will be the basepoint. We attach a set of $2g+p+1$ oriented edges $t_{[1\uparrow(p+1)]} \cup x_{[1\uparrow g]} \cup y_{[1\uparrow g]}$. We attach a $(4g+p+1)$-gon with counter-clockwise boundary label $\Pi t_{[\![(p+1)\downarrow 1]\!]}\Pi[x,y]_{[\![1\uparrow g]\!]} \in \langle\, t_{[1\uparrow(p+1)]} \cup x_{[1\uparrow g]} \cup y_{[1\uparrow g]} \mid \quad \rangle$. For each $j \in [1\uparrow p]$, we attach a punctured disk with counterclockwise boundary label $\overline{t}_j$. This completes the definition of $\mathbf{S}_{g,1,p}$. Notice that the boundary of $\mathbf{S}_{g,1,p}$ is the edge labelled t_{p+1}.

We may identify $\pi_1(\mathbf{S}_{g,1,p}) = F_{g,p}$. We call $\mathrm{Stab}([\overline{t}]_{[1\uparrow p]} \cup \{\overline{t}_{p+1}\}; \mathrm{Aut}\, F_{g,p})$ the *algebraic mapping-class group of* $\mathbf{S}_{g,1,p}$. This is our group $\mathrm{A}_{g,p}$. (In [9], $\mathrm{A}_{g,p}$ is denoted $\mathrm{Aut}^{+}_{g,0,p\perp\hat{1}}$, and, in [9, Proposition 7.1(v)], the latter group is shown to be isomorphic to what is there called the orientation-preserving algebraic mapping-class group of $\mathbf{S}_{g,1,p}$, denoted $\mathrm{Out}^{+}_{g,1,p}$.)

Let $\mathrm{Aut}\,\mathbf{S}_{g,1,p}$ denote the group of self-homeomorphisms of $\mathbf{S}_{g,1,p}$ which stabilize each point on the boundary. The quotient of $\mathrm{Aut}\,\mathbf{S}_{g,1,p}$ modulo the group of elements of $\mathrm{Aut}\,\mathbf{S}_{g,1,p}$ which are isotopic to the identity map through a boundary-fixing isotopy is called the (topological) *mapping-class group of* $\mathbf{S}_{g,1,p}$, denoted $\mathrm{M}^{\mathrm{top}}_{g,1,p}$.

Then $\mathrm{Aut}\,\mathbf{S}_{g,1,p}$ acts on $F_{g,p}$ stabilizing $[\overline{t}]_{[1\uparrow p]} \cup \{\overline{t}_{p+1}\}$, and we have a homomorphism $\mathrm{M}^{\mathrm{top}}_{g,1,p} \to \mathrm{A}_{g,p}$. □

5.3. **Definitions.** Let $\mathbf{S}_{g,0,p}$ denote the quotient space obtained from $\mathbf{S}_{g,1,p}$ by collapsing the boundary to a point. Then $\mathbf{S}_{g,0,p}$ is an orientable surface of genus g with p punctures.

We may identify $\pi_1(\mathbf{S}_{g,0,p}) = F_{g,p-1}$. We define the *algebraic mapping-class group of* $\mathbf{S}_{g,0,p}$ as $\mathrm{M}^{\mathrm{alg}}_{g,0,p} \coloneqq \mathrm{Stab}([t]_{[1\uparrow p]} \cup [\overline{t}]_{[1\uparrow p]}; \mathrm{Out}\, F_{g,p-1})$. (In [9], if $(g,p) \neq (0,0)$, $(0,1)$, then $\mathrm{M}^{\mathrm{alg}}_{g,0,p}$ is denoted $\mathrm{Out}_{g,0,p}$.)

Let $\mathrm{Aut}\,\mathbf{S}_{g,0,p}$ denote the group of self-homeomorphisms of $\mathbf{S}_{g,0,p}$. The quotient of $\mathrm{Aut}\,\mathbf{S}_{g,0,p}$ modulo the group of elements which are isotopic to the identity map is called the (topological) *mapping-class group of* $\mathbf{S}_{g,0,p}$, denoted $\mathrm{M}^{\mathrm{top}}_{g,0,p}$.

Then $\mathrm{Aut}\,\mathbf{S}_{g,0,p}$ acts on $F_{g,p-1}/\!\sim$ stabilizing $[t]_{[1\uparrow p]} \cup [\overline{t}]_{[1\uparrow p]}$. This action factors through a natural homomorphism $\mathrm{Aut}\,\mathbf{S}_{g,0,p} \to \mathrm{Out}\, F_{g,p-1}$, and we have a homomorphism $\mathrm{M}^{\mathrm{top}}_{g,0,p} \to \mathrm{M}^{\mathrm{alg}}_{g,0,p}$.

Consider the simply-connected case, that is, $F_{g,p-1} = 1$. Then, (g,p) is either $(0,0)$ or $(0,1)$, corresponding to the sphere $\mathbf{S}_{0,0,0}$ and the open disk $\mathbf{S}_{0,0,1}$. Here, $\mathrm{M}^{\mathrm{alg}}_{g,0,p}$ is trivial, while $\mathrm{M}^{\mathrm{top}}_{g,0,p}$ has order two, with one mapping class consisting of the reflections. □

It has been the work of many years to show that $\mathrm{M}^{\mathrm{top}}_{g,1,p} = \mathrm{A}_{g,p}$ and to show that both are generated by the ADLH set. Also, if $(g,p) \neq (0,0), (0,1)$, then $\mathrm{M}^{\mathrm{top}}_{g,0,p} = \mathrm{M}^{\mathrm{alg}}_{g,0,p}$, and their orientation-preserving subgroups are generated by the ADLH set. The proofs developed in stages, roughly as follows, although we are omitting many important results.

- In 1917, Nielsen [17] proved that if $(g,p) = (1,0)$ then the ADL set generates $\mathrm{A}_{g,p}$.
- In 1925, Artin [1] introduced *braid twists*, and proved that if $g = 0$ then the ADL set generates $\mathrm{A}_{g,p}$ and $\mathrm{M}^{\mathrm{top}}_{g,1,p} = \mathrm{A}_{g,p}$.

- In 1927, Nielsen [19] presented unpublished results of Dehn and proved that if $p = 0$ then $\mathrm{M}^{\mathrm{top}}_{g,1,p}$ maps onto $\mathrm{A}_{g,p}$, and that if $p \leqslant 1$ then $\mathrm{M}^{\mathrm{top}}_{g,0,p}$ maps onto $\mathrm{M}^{\mathrm{alg}}_{g,0,p}$.
- In 1928, Baer [4] proved that if $p = 0$ then $\mathrm{M}^{\mathrm{top}}_{g,0,p}$ embeds in $\mathrm{M}^{\mathrm{alg}}_{g,0,p}$ for all $g \geqslant 2$.
- In 1934, Magnus [15] proved that if $g = 1$ then $\mathrm{M}^{\mathrm{top}}_{g,1,p} = \mathrm{A}_{g,p}$ and $\mathrm{M}^{\mathrm{top}}_{g,0,p} = \mathrm{M}^{\mathrm{alg}}_{g,0,p}$.
- In 1939, Dehn [6] introduced what are now called *Dehn twists*, and proved, among other results, that a finite number of Dehn twists generate the orientation-preserving subgroup of $\mathrm{M}^{\mathrm{top}}_{g,0,p}$.; see [6, Section 10.3.c].
- In 1964, Lickorish [14] rediscovered and refined Dehn's 1939 methods and proved that if $p = 0$ then the ADL set generates the orientation-preserving subgroup of $\mathrm{M}^{\mathrm{top}}_{g,0,p}$.
- In 1966, Epstein [11] refined Baer's 1928 methods and proved that $\mathrm{M}^{\mathrm{top}}_{g,1,p}$ embeds in $\mathrm{A}_{g,p}$ and that, if $(g, p) \neq (0, 0)$, $(0, 1)$, then $\mathrm{M}^{\mathrm{top}}_{g,0,p}$ embeds in $\mathrm{M}^{\mathrm{alg}}_{g,0,p}$.
- In 1966, Zieschang [26, Satz 4], [28, Theorem 5.7.1] proved that $\mathrm{M}^{\mathrm{top}}_{g,1,p}{=}\mathrm{A}_{g,p}$ and that, for $(g, p) \neq (0, 0)$, $(0, 1)$, $\mathrm{M}^{\mathrm{top}}_{g,0,p} = \mathrm{M}^{\mathrm{alg}}_{g,0,p}$, and called these results the Baer-Dehn-Nielsen Theorem.
- In 1979, Humphries[12] showed that the ADHL set generates the same group as the ADL set.
- In 2001, Labruère and Paris [13, Proposition 2.10(ii) with $r = 0$] used some of the foregoing results and a theorem of Birman [5] to prove that the ADLH set generates $\mathrm{M}^{\mathrm{top}}_{g,1,p}$.

6. The topological source of the ADL set

In this section, we shall recall the definitions of Dehn twists and braid twists and see that the ADL set lies in $\mathrm{M}^{\mathrm{top}}_{g,1,p}$. The diagram [13, Figure 12] illustrates the elements of the ADLH set acting on $\mathbf{S}_{g,1,p}$.

6.1. **Definitions.** Let $\mathbf{A} \coloneqq [0, 1] \times (\mathbb{R}/\mathbb{Z})$, a closed annulus. Let z denote the oriented boundary component $\{1\} \times (\mathbb{R}/\mathbb{Z})$ with basepoint $(1, \mathbb{Z})$. Let z' denote the oriented boundary component $\{0\} \times (\mathbb{R}/\mathbb{Z})$ with basepoint $(0, \mathbb{Z})$. Let e denote the edge $[0, 1] \times \{\mathbb{Z}\}$ oriented from $(1, \mathbb{Z})$ to $(0, \mathbb{Z})$.

The *model Dehn twist* is the self-homeomorphism τ of $\mathbf{A} = [0, 1] \times (\mathbb{R}/\mathbb{Z})$ given by $(x, y + \mathbb{Z}) \mapsto (x, -x + y + \mathbb{Z})$. Notice that τ fixes every point of $z' \cup z$, and τ acts on e as $(x, \mathbb{Z}) \mapsto (x, 1 - x + \mathbb{Z})$. Thus $e^{\tau}\,\overline{e}\,\overline{z}$ bounds a triangle; hence e^{τ} is homotopic to ze.

Suppose now that we have an embedding of $\mathbf{A}$ in a surface $\mathbf{S}$. Then the image of z is an oriented simple closed curve $\mathbf{c}$, and τ induces a self-homeomorphism of $\mathbf{S}$ which is the identity outside the copy of $\mathbf{A}$. We call the resulting map of $\mathbf{S}$ a (left) *Dehn twist about* $\mathbf{c}$; see [6]. □

Recall the construction of $\mathbf{S}_{g,1,p}$ in Definitions 5.2.

6.2. **Examples.** Let $i \in [1{\uparrow}g]$.

Recall that $\overline{x}_i\overline{y}_i x_i$ is a subword of the boundary label of the $(4g{+}p{+}1)$-gon used in the construction of $\mathbf{S}_{g,1,p}$. We place the annulus $\mathbf{A}$ on $\mathbf{S}_{g,1,p}$ with the image of z along the boundary edge labelled $\overline{y}_i$. The image of z' enters the $(4g{+}p{+}1)$-gon near the end of the boundary edge labelled $\overline{x}_i$, travels near $z = \overline{y}_i$, and exits near the beginning of x_i, completing the cycle. The only oriented edge of the one-skeleton of $\mathbf{S}_{g,1,p}$ that crosses $\mathbf{A}$ from right to left is x_i, near its beginning. Incident to the basepoint of z are, in clockwise order, the end of z, the beginning of x_i, and the beginning of z. The Dehn twist about $\overline{y}_i$ induces $\left(\begin{smallmatrix} x_i \\ \overline{y}_i x_i \end{smallmatrix}\right)$ on $\pi_1(\mathbf{S}_{g,1,p}) = F_{g,p}$. Hence $\alpha_i \in \mathrm{M}^{\mathrm{top}}_{g,1,p}$.

Recall that $\overline{y}_i x_i y_i$ is a subword of the boundary label of the $(4g+p+1)$-gon used in the construction of $\mathbf{S}_{g,1,p}$. We place the annulus $\mathbf{A}$ on $\mathbf{S}_{g,1,p}$ with the image of z along the boundary edge labelled x_i. The image of z' enters the $(4g+p+1)$-gon near the end of the boundary edge labelled $\overline{y}_i$, travels near $z = x_i$, and exits near the beginning of y_i, completing the cycle. The only oriented edge of the one-skeleton of $\mathbf{S}_{g,1,p}$ that crosses $\mathbf{A}$ from right to left is y_i, near its beginning. Incident to the basepoint of z are, in clockwise order, the end of z, the beginning of y_i, and the beginning of z. The Dehn twist about x_i induces $\left(\begin{smallmatrix} y_i \\ x_i y_i \end{smallmatrix}\right)$ on $\pi_1(\mathbf{S}_{g,1,p}) = F_{g,p}$. Hence $\beta_i \in \mathrm{M}^{\mathrm{top}}_{g,1,p}$. $\square$

6.3. **Example.** Let $i \in [2{\uparrow}g]$. Recall that $\overline{x}_{i-1}\overline{y}_{i-1}x_{i-1}y_{i-1}\overline{x}_i\overline{y}_i x_i$ is a subword of the boundary label of the $(4g+p+1)$-gon used in the construction of $\mathbf{S}_{g,1,p}$. We place the annulus $\mathbf{A}$ on $\mathbf{S}_{g,1,p}$ with the image of z marking out, in the $(4g+p+1)$-gon, a pentagon with boundary label $y_{i-1}\overline{x}_i\overline{y}_i x_i z$. The image of z'

- enters (the $(4g+p+1)$-gon) near the end of (the boundary edge labelled) $\overline{y}_{i-1}$, travels counter-clockwise near the basepoint, exits near the beginning of x_{i-1},
- enters near the end of $\overline{x}_{i-1}$, travels counter-clockwise near the basepoint, exits near the beginning of $\overline{y}_{i-1}$,
- enters near the end of y_{i-1}, travels counter-clockwise near the basepoint, exits near the beginning of $\overline{x}_i$,
- enters near the end of x_i, travels near z, passing $\overline{y}_i$, $\overline{x}_i$, exits near the beginning of y_{i-1},

completing the cycle. The entrances correspond to $\overline{y}_{i-1} \rightsquigarrow \overline{x}_{i-1} \rightsquigarrow y_{i-1} \rightsquigarrow x_i$ in the extended Whitehead graph. The oriented edges of the one-skeleton of $\mathbf{S}_{g,1,p}$ that cross $\mathbf{A}$ from right to left are the exits: x_{i-1} near its beginning, $\overline{y}_{i-1}$ near its beginning, $\overline{x}_i$ near its beginning, and y_{i-1} near its beginning. Incident to the basepoint of z are, in clockwise order, the end of z, and the beginnings of y_{i-1}, x_{i-1}, $\overline{y}_{i-1}$, $\overline{x}_i$, and z. Let $w_i \coloneqq y_{i-1}\overline{x}_i\overline{y}_i x_i$. The Dehn twist about $\overline{w}_i$ induces $\left(\begin{smallmatrix} x_{i-1} & y_{i-1} & x_i \\ \overline{w}_i x_{i-1} & y_{i-1}^{w_i} & x_i w_i \end{smallmatrix}\right)$ on $\pi_1(\mathbf{S}_{g,1,p}) = F_{g,p}$. Hence $\gamma_i \in \mathrm{M}^{\mathrm{top}}_{g,1,p}$. $\square$

6.4. **Example.** Suppose that $\min(g,p,1) = 1$. Recall that $t_1\overline{x}_1\overline{y}_1 x_1$ is a subword of the boundary label of the $(4g+p+1)$-gon used in the construction of $\mathbf{S}_{g,1,p}$ and that $\bar{t}_1$ is the boundary label of the $\bar{t}_1$-disk. We place the annulus $\mathbf{A}$ on $\mathbf{S}_{g,1,p}$ with z marking out, in the $(4g+p+1)$-gon, a pentagon with boundary label $t_1\overline{x}_1\overline{y}_1 x_1 z$. The image of z'

- enters the $\bar{t}_1$-disk near the end of $\bar{t}_1$, travels counter-clockwise near the basepoint, exits near the beginning of $\bar{t}_1$,
- enters the $(4g+p+1)$-gon near the end of t_1, travels counter-clockwise near the basepoint, exits near the beginning of $\overline{x}_1$,
- enters the $(4g+p+1)$-gon near the end of x_1, travels near z passing $\overline{y}_1$, $\overline{x}_1$, exits near the beginning of t_1,

completing the cycle. The entrances correspond to $\bar{t}_1 \rightsquigarrow t_1 \rightsquigarrow x_1$ in the extended Whitehead graph. The oriented edges of the one-skeleton of $\mathbf{S}_{g,1,p}$ that cross $\mathbf{A}$ from right to left are the exits: $\bar{t}_1$ near its beginning, $\overline{x}_1$ near its beginning, and t_1 near its beginning. Incident to the basepoint of z are, in clockwise order, the end of z, and the beginnings of t_1, $\bar{t}_1$, $\overline{x}_1$, and z. Let $w_1 \coloneqq t_1\overline{x}_1\overline{y}_1 x_1$. The Dehn twist about $\overline{w}_1$ induces $\left(\begin{smallmatrix} t_1 & x_1 \\ t_1^{w_1} & x_1 w_1 \end{smallmatrix}\right)$ on $\pi_1(\mathbf{S}_{g,1,p}) = F_{g,p}$. Hence $\gamma_1 \in \mathrm{M}^{\mathrm{top}}_{g,1,p}$. $\square$

6.5. **Definitions.** Recall the annulus $\mathbf{A} = [0,1] \times (\mathbb{R}/\mathbb{Z})$ of Definitions 6.1. Let $\mathbf{D}$ denote the space that is obtained from $\mathbf{A}$ by deleting the two points $p_2 \coloneqq (\frac{1}{2}, \mathbb{Z})$ and $p_1 \coloneqq (\frac{1}{2}, \frac{1}{2}+\mathbb{Z})$ and collapsing to a point the boundary component $z' = \{0\} \times (\mathbb{R}/\mathbb{Z})$. We take $p_0 \coloneqq (1, \mathbb{Z})$ as the basepoint of $\mathbf{D}$.

Thus $\mathbf{D}$ is a closed disk with two punctures, and the model Dehn twist τ has an induced action on $\mathbf{D}$, called the *model braid twist*. We now determine the induced action on $\pi_1(\mathbf{D})$.

Let z_2 denote an infinitesimal clockwise circle around p_2, and let $z_1 \coloneqq z_2^{\tau}$, an infinitesimal clockwise circle around p_1. Then τ interchanges z_2 and z_1. Let e_2 denote the oriented subedge of $\overline{e}$ from z_2 to p_0 starting at a point p_2' on z_2. Let $e_1 \coloneqq e_2^{\tau}$, an oriented subedge of $\overline{e}^{\tau}$ from z_1 to p_0 starting at $p_1' \coloneqq p_2'^{\tau}$ on z_1. Then τ interchanges p_1' and p_2', and acts on e_1 as $(x, 1-x+\mathbb{Z}) \mapsto (x, 2-2x+\mathbb{Z})$. Here, e_1^{τ} is an oriented edge from p_2' to p_0 such that $e_1^{\tau} z \overline{e}_2$ bounds a triangle; hence, e_1^{τ} is homotopic to $e_2 \overline{z}$.

We view $z_2^{e_2}$ and $z_1^{e_1}$ as closed paths, and then $z_2^{e_2} z_1^{e_1} z$ bounds a disk in $\mathbf{D}$. Now $\pi_1(\mathbf{D}) = \langle\, z_2^{e_2}, z_1^{e_1}, z \mid z_2^{e_2} z_1^{e_1} z \,\rangle = \langle\, z_2^{e_2}, z_1^{e_1} \mid \quad \rangle$, and the induced action of τ on $\pi_1(\mathbf{D})$ is given by $z_2^{e_2} \mapsto z_1^{e_1}$ and $z_1^{e_1} \mapsto z_2^{e_2\,\overline{z}} = z_2^{e_2\, z_2^{e_2} z_1^{e_1}} = (z_2^{e_2})^{z_1^{e_1}}$.

Suppose that we have an embedding of $\mathbf{D}$ in a surface $\mathbf{S}$ which carries punctures to punctures. Then τ induces a self-homeomorphism of $\mathbf{S}$ which is the identity outside the copy of $\mathbf{D}$. The resulting map of $\mathbf{S}$ is called a *braid twist*; see [1]. $\square$

6.6. **Example.** Let $j \in [2{\uparrow}p]$. We place the twice-punctured disk $\mathbf{D}$ on $\mathbf{S}_{g,1,p}$ with the image of z marking out, in the $(4g{+}p{+}1)$-gon, a triangle with boundary label $t_j t_{j-1} z$. This is possible since z now bounds a twice-punctured disk in $\mathbf{S}_{g,1,p}$. Here t_j is homotopic to $z_2^{e_2}$ and t_{j-1} is homotopic to $z_1^{e_1}$. The resulting braid twist of $\mathbf{S}_{g,1,p}$ induces $\left(\begin{smallmatrix} t_j & t_{j-1} \\ t_{j-1} & t_j^{t_{j-1}} \end{smallmatrix}\right)$. Hence $\sigma_j \in \mathrm{M}_{g,1,p}^{\mathrm{top}}$. $\square$

We now see that the ADL set lies in $\mathrm{M}_{g,1,p}^{\mathrm{top}}$. By Theorem 4.1, the homomorphism $\mathrm{M}_{g,1,p}^{\mathrm{top}} \to \mathrm{A}_{g,p}$ is surjective; that is, by using Zieschang's proof, we have recovered Zieschang's result [26, Satz 4], [28, Theorem 5.7.1]. Assuming Epstein's result [11], we now have $\mathrm{M}_{g,1,p}^{\mathrm{top}} = \mathrm{A}_{g,p}$, and both are generated by the ADLH set.

7. Collapsing the boundary

In this section we review Zieschang's algebraic proof of a result of Nielsen. We then describe a generating set for $\mathrm{M}_{g,0,p}^{\mathrm{alg}}$ which lies in the image of $\mathrm{M}_{g,0,p}^{\mathrm{top}}$.

7.1. **Definitions.** Recall $F_{g,p-1} = \langle\, t_{[1\uparrow p]} \cup x_{[1\uparrow g]} \cup y_{[1\uparrow g]} \mid \Pi t_{[\![p\downarrow 1]\!]} \Pi [x,y]_{[\![1\uparrow g]\!]} \,\rangle$.
Let $\zeta \in \operatorname{Aut} F_{g,p-1}$ be defined by

$$\forall i \in [1{\uparrow}g] \;\; x_i^{\zeta} \coloneqq y_{g+1-i}, \;\; y_i^{\zeta} \coloneqq x_{g+1-i}, \;\; \forall j \in [1{\uparrow}p] \;\; t_j^{\zeta} \coloneqq \overline{t}_{p+1-j}.$$

We then have the outer automorphism $\breve{\zeta} \in \mathrm{M}_{g,0,p}^{\mathrm{alg}}$. $\square$

7.2. **Theorem.** *For $g, p \in [0{\uparrow}\infty[$, $\mathrm{M}_{g,0,p}^{\mathrm{alg}}$ is generated by the natural image of $\mathrm{A}_{g,p}$ together with $\breve{\zeta}$. Hence, $\mathrm{M}_{g,0,p}^{\mathrm{alg}}$ is generated by the image of the ADLH set together with $\breve{\zeta}$.*

Steve Humphries has pointed out to us that there are some cases where it is known that an element can be omitted from the resulting generating set of $\mathrm{M}_{g,0,p}^{\mathrm{alg}}$. For $g \geqslant 1$, the relations (R9a), (R9b) in [13, Theorem 3.2] show that if $p = 1$, then $\breve{\gamma}_1$ can be omitted, while if $p \geqslant 2$ and $p = 2g{-}2{\pm}1$, then $\breve{\alpha}_1$ can be omitted.

Sketched proof of Theorem 7.2. For $p \geqslant 1$, this is a straightforward exercise which we leave to the reader. Thus we may assume that $p = 0$. We may further assume that $g \geqslant 1$. The remaining case is now a result of Nielsen [19] for which Zieschang has given an algebraic proof [28, Theorem 5.6.1] developed from [23, 24, 25] along the following lines.

Let $\varphi \in \operatorname{Aut} F_{g,-1}$. We wish to show that the element $\breve{\varphi} \in \operatorname{Out} F_{g,-1} = \mathrm{M}^{\mathrm{alg}}_{g,0,0}$ lies in the subgroup generated by the image of $\mathrm{A}_{g,0} = \operatorname{Stab}(t_1, \operatorname{Aut} F_{g,0})$ together with $\breve{\zeta}$. It is clear that φ lifts back to an endomorphism $\tilde{\varphi}$ of $F_{g,0}$ such that $t_1^{\tilde{\varphi}}$ lies in the normal closure of t_1.

Now $\mathrm{H}^2(F_{g,-1}, \mathbb{Z}) \simeq \mathbb{Z}$; see, for example, [7, Theorem V.4.9]. The image of φ under the natural map $\operatorname{Aut} F_{g,-1} \to \operatorname{Aut} \mathrm{H}^2(F_{g,-1}, \mathbb{Z}) \simeq \{1, -1\}$ is denoted $\deg(\varphi)$. By a cohomology calculation, if we express $t_1^{\tilde{\varphi}}$ as a product of n_+ conjugates of t_1 and n_- conjugates of $\bar{t}_1$, then $n_+ - n_- = \deg(\varphi) = \pm 1$. By using van Kampen diagrams on a surface, one can alter $\tilde{\varphi}$ and arrange that $n_- = 0$ or $n_+ = 0$; this was also done in [10, Theorem 4.9]. Thus $t_1^{\tilde{\varphi}}$ is now a conjugate of t_1 or $\bar{t}_1$. By composing $\tilde{\varphi}$ with an inner automorphism of $F_{g,0}$, we may assume that $t_1^{\tilde{\varphi}}$ is t_1 or $\bar{t}_1$.

Notice that ζ lifts back to $\tilde{\zeta} \in \operatorname{Aut} F_{g,0}$ where, for each $i \in [1{\uparrow}g]$, $x_i^{\tilde{\zeta}} \coloneqq y_{g+1-i}$ and $y_i^{\tilde{\zeta}} \coloneqq x_{g+1-i}$. Then $\bar{t}_1^{\tilde{\zeta}} = (\Pi[x,y]_{[\![1\uparrow g]\!]})^{\tilde{\zeta}} = \Pi[y,x]_{[\![g\downarrow 1]\!]} = t_1$. By replacing φ with $\varphi\zeta$ if necessary, we may now assume that $t_1^{\tilde{\varphi}} = t_1$.

We next prove a result, due to Nielsen [17] for $g = 1$, and Zieschang [22] for $g \geqslant 1$, that $t_1^{\tilde{\varphi}} = t_1$ implies that $\tilde{\varphi}$ is an automorphism of $F_{g,0}$.

We shall show first that $\tilde{\varphi}$ is surjective, by an argument of Formanek [7, Theorem V.4.11]. Let w be an element of the basis $x_{[1\uparrow g]} \cup y_{[1\uparrow g]}$ of $F_{g,0}$. The map of sets $x_{[1\uparrow g]} \cup y_{[1\uparrow g]} \to \mathrm{GL}_2(\mathbb{Z}F_{g,0})$, $v \mapsto \left(\begin{smallmatrix} v & 0 \\ \delta_{v,w} & 1 \end{smallmatrix}\right)$ (where $\delta_{v,w}$ equals 1 if $v = w$ and equals 0 if $v \neq w$) extends uniquely to a group homomorphism

$$F_{g,0} \to \mathrm{GL}_2(\mathbb{Z}F_{g,0}), \quad v \mapsto \left(\begin{smallmatrix} v & 0 \\ v^{\partial_w} & 1 \end{smallmatrix}\right).$$

The map $\partial_w \colon F_{g,0} \to \mathbb{Z}F_{g,0}$, called the *Fox derivative with respect to* w, satisfies, for all u, $v \in F_{g,0}$, $(uv)^{\partial_w} = (u^{\partial_w})v + v^{\partial_w}$. On applying ∂_w to $u\bar{u} = 1$, we see that $\bar{u}^{\partial_w} = -u^{\partial_w}\bar{u}$. For each $i \in [1{\uparrow}g]$, let $X_i \coloneqq x_i^{\tilde{\varphi}}$ and $Y_i \coloneqq y_i^{\tilde{\varphi}}$. Since $\tilde{\varphi}$ fixes $\bar{t}_1 = \Pi[x,y]_{[\![1\uparrow g]\!]}$, we have $\Pi[X,Y]_{[\![1\uparrow g]\!]} = \Pi[x,y]_{[\![1\uparrow g]\!]}$. On applying ∂_w, we obtain

$$\begin{aligned}
&\sum_{i=1}^{g} \Bigg(\Big(X_i^{\partial_w} \cdot Y_i \cdot (1 - \overline{Y}_i^{X_i Y_i}) + Y_i^{\partial_w} \cdot (1 - X_i^{Y_i}) \Big) \cdot \Pi[X,Y]_{[\![(i+1)\uparrow g]\!]} \Bigg) \\
&= \sum_{i=1}^{g} \Bigg(\Big(x_i^{\partial_w} \cdot y_i \cdot (1 - \overline{y}_i^{x_i y_i}) + y_i^{\partial_w} \cdot (1 - x_i^{y_i}) \Big) \cdot \Pi[x,y]_{[\![(i+1)\uparrow g]\!]} \Bigg).
\end{aligned}$$

On applying the natural left $\mathbb{Z}F_{g,0}$-linear map $\mathbb{Z}F_{g,0} \to \mathbb{Z}[F_{g,0}/F_{g,0}^{\tilde{\varphi}}]$, denoted $f \mapsto fF_{g,0}^{\tilde{\varphi}}$, we obtain

$$0 = \sum_{i=1}^{g} \Bigg(\Big(x_i^{\partial_w} \cdot y_i \cdot (1 - \overline{y}_i^{x_i y_i}) + y_i^{\partial_w} \cdot (1 - x_i^{y_i}) \Big) \cdot \Pi[x,y]_{[\![(i+1)\uparrow g]\!]} \Bigg) F_{g,0}^{\tilde{\varphi}}. \tag{1}$$

Consider any $i \in [1{\uparrow}g]$ such that $x_{[(i+1)\uparrow g]} \cup y_{[(i+1)\uparrow g]} \subseteq F_{g,0}^{\tilde{\varphi}}$. By taking $w = y_i$ in (1), we obtain

$$0 = (1 - x_i^{y_i}) \cdot \Pi[x,y]_{[\![(i+1)\uparrow g]\!]} F_{g,0}^{\tilde{\varphi}} = (1 - x_i^{y_i}) F_{g,0}^{\tilde{\varphi}}.$$

Hence $x_i^{y_i} \in F_{g,0}^{\varphi}$, that is, $x_i^{x_i y_i} \in F_{g,0}^{\varphi}$. By taking $w = x_i$ in (1) and left multiplying by $\bar{y}_i$, we obtain

$$0 = (1 - \overline{y}_i^{x_i y_i}) \cdot \Pi[x,y]_{[\![(i+1)\uparrow g]\!]} F_{g,0}^{\tilde{\varphi}} = (1 - \overline{y}_i^{x_i y_i}) F_{g,0}^{\tilde{\varphi}}.$$

Hence, $\overline{y}_i^{x_i y_i} \in F_{g,p}^{\varphi}$. It follows that x_i, $y_i \in F_{g,0}^{\tilde{\varphi}}$.

By induction, $x_{[1\uparrow g]} \cup y_{[1\uparrow g]} \subseteq F_{g,0}^{\tilde{\varphi}}$. Thus $\tilde{\varphi}$ is surjective.

By Consequence 2.9, $\tilde{\varphi}$ is an automorphism, as desired. □

Recall that $\mathbf{S}_{g,0,p}$ was constructed in Definitions 5.3 as the quotient space obtained from $\mathbf{S}_{g,1,p}$ by collapsing the boundary component to a point. We then have a natural embedding of $\mathrm{Aut}\,\mathbf{S}_{g,1,p}$ in $\mathrm{Aut}\,\mathbf{S}_{g,0,p}$. Thus the Dehn twists and braid twists of $\mathbf{S}_{g,1,p}$ constructed in Section 6 induce Dehn twists and braid twists of $\mathbf{S}_{g,0,p}$. It follows that the image of the ADL set in $\mathrm{M}^{\mathrm{alg}}_{g,0,p}$ lies in $\mathrm{M}^{\mathrm{top}}_{g,0,p}$. Also, $\breve{\zeta}$ lies in $\mathrm{M}^{\mathrm{top}}_{g,0,p}$, since $\breve{\zeta}$ is easily seen to arise from a reflection of $\mathbf{S}_{g,0,p}$. We now see, in the manner proposed by Magnus, Karrass and Solitar [16, p.175], that the homomorphism $\mathrm{M}^{\mathrm{top}}_{g,0,p} \to \mathrm{M}^{\mathrm{alg}}_{g,0,p}$ is surjective, by Theorem 7.2. Assuming Epstein's result [11], if $(g,p) \neq (0,0),(0,1)$, then $\mathrm{M}^{\mathrm{top}}_{g,0,p}$ equals $\mathrm{M}^{\mathrm{alg}}_{g,0,p}$, and both are generated by the image of the ADLH set together with $\breve{\zeta}$; see [13, Corollary 2.11(ii)].

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