Two examples in the Galois theory of free groups

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Abstract

Let F be a free group, and let H be a subgroup of F.

The 'Galois monoid' $\mathrm{End}_H(F)$ consists of all endomorphisms of F which fix every element of H; the 'Galois group' $\mathrm{Aut}_H(F)$ consists of all automorphisms of F which fix every element of H. The $\mathrm{End}(F)$-closure and the $\mathrm{Aut}(F)$-closure of H are the fixed subgroups, $\mathrm{Fix}(\mathrm{End}_H(F))$ and $\mathrm{Fix}(\mathrm{Aut}_H(F))$, respectively.

Martino and Ventura considered examples where

$$\mathrm{Fix}(\mathrm{Aut}_H(F)) \neq \mathrm{Fix}(\mathrm{End}_H(F)) = H.$$

We obtain, for two of their examples, explicit descriptions of $\mathrm{End}_H(F)$, $\mathrm{Aut}_H(F)$, and $\mathrm{Fix}(\mathrm{Aut}_H(F))$, and, hence, give much simpler verifications that $\mathrm{Fix}(\mathrm{Aut}_H(F)) \neq \mathrm{Fix}(\mathrm{End}_H(F))$, in these cases.

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1 Two Galois connections

The Galois theory of free groups consists of two, dual, components. The first component is the study of the fixed subgroup for a given set of endomorphisms of a free group; this has proven to be a very rich area of research, developed by Nielsen, Scott, Dyer, Gersten, Bestvina, Handel, and many others. The dual component is the study of the set of endomorphisms which fix a given subgroup of a free group; currently this component is somewhat less productive. The object of this article is to give more details for some of the examples of this theory that arose in an article of Martino-Ventura [4], and, hence, substantially simplify their proofs in these cases. We refer the reader to [4] for further background, motivation and references concerning the many interesting results which have been obtained about the Galois theory of free groups.

Throughout this section, let F be a free group and let H be a subgroup of F.

1.1 Definitions. Let $\mathrm{End}(F)$ denote the monoid of endomorphisms of F. Let $\mathrm{End}_H(F)$ denote the submonoid consisting of all endomorphisms of F which fix every element of H.

Let $\mathrm{Aut}(F)$ denote the group of invertible elements in $\mathrm{End}(F)$, that is, the group of automorphisms of F. Let $\mathrm{Aut}_H(F)$ denote the subgroup consisting of all automorphisms of F which fix every element of H.

For any subset S of $\mathrm{End}(F)$, let $\mathrm{Fix}(S)$ denote the set of elements of F that are fixed by every element of S.

We think of $\mathrm{Aut}_H(F)$ as a 'Galois group' and $\mathrm{End}_H(F)$ as a 'Galois monoid'.

Now $\mathrm{Aut}_{(-)}(F)$ is a function from the set of subgroups of F to the set of subsets of $\mathrm{Aut}(F)$, and $\mathrm{Fix}(-)$ is a function in the reverse direction. This pair of functions form a Galois connection, and the images of the functions are the sets of closed subsets. We say that H is $\mathrm{Aut}(F)$-*closed* (in F) if $H = \mathrm{Fix}(S)$ for some subset S of $\mathrm{Aut}(F)$. The $\mathrm{Aut}(F)$-*closure* of H (in F) is $\mathrm{Fix}(\mathrm{Aut}_H(F))$, the smallest $\mathrm{Aut}(F)$-closed subgroup of F containing H.

Replacing Aut with End everywhere in the previous paragraph gives another Galois connection.

An F-*retraction* is an idempotent element of $\mathrm{End}(F)$, and an F-*retract* is the image, or set of fixed elements, of an F-retraction. Notice that all F-retracts are $\mathrm{End}(F)$-closed. $\square$

The $\mathrm{End}(F)$-closure of H, $\mathrm{Fix}(\mathrm{End}_H(F))$ is a subgroup of the $\mathrm{Aut}(F)$-closure of H, $\mathrm{Fix}(\mathrm{Aut}_H(F))$. In general, the relation between the two closures is not well understood. A. Martino and E. Ventura [4], [2] gave a family of ingeniously chosen examples where $\mathrm{Fix}(\mathrm{Aut}_H(F)) \neq \mathrm{Fix}(\mathrm{End}_H(F)) = H$; their proof is lengthy and involves many deep results.

In Section 2, using only normal-form methods, we obtain, for two of their examples, an explicit description of the Galois monoid $\mathrm{End}_H(F)$, and, hence, a simple proof that $\mathrm{Fix}(\mathrm{Aut}_H(F)) \neq \mathrm{Fix}(\mathrm{End}_H(F))$, in these cases.

In Section 3, we recall relevant results of Martino and Ventura.

2 Some Galois monoids

Throughout this section we use the following.

2.1 Notation. Let F be a free group of rank three, and let $\{a, b, c\}$ be a basis of F.

We denote an element ρ of $\mathrm{End}(F)$ by the triple $(a\rho, b\rho, c\rho)$. Notice that we write endomorphisms on the right of their arguments.

For $x, y \in F$, $\bar{x}$ denotes x^{-1}, $[x, y]$ denotes $xy\bar{x}\bar{y}$, and we write $x \sim y$ if x and y are conjugate in F; thus, $x \sim \bar{y}xy$.

Let $\phi = (a, b, cb) \in \mathrm{Aut}(F)$.

Let $j \in \mathbb{Z}$, let $d_j = ba[c^j, b]\bar{a} \in F$, let $H_j = \langle a, d_j\rangle = \langle a, ba[c^j, b]\rangle \leq F$, and let $\psi_j = (a, d_j, 1) \in \mathrm{End}(F)$.

Notice that, for each $n \in \mathbb{Z}$, $\phi^n = (a, b, cb^n)$ and $\phi^n\psi_j = (a, d_j, d_j^n)$. $\square$

2.2 Example. Suppose that Notation 2.1 holds, and let $j = 1$.

Thus, we have $H_1 = \langle a, ba[c, b]\rangle$, $\phi = (a, b, cb)$, and $\psi_1 = (a, ba[c, b]\bar{a}, 1)$.

We shall show, in Corollary 2.6(i) below, that

$$\mathrm{End}_{H_1}(F) = \{\phi^n, \phi^n\psi_1 \mid n \in \mathbb{Z}\} = \langle\phi\rangle \cup \langle\phi\rangle\psi_1 \quad \text{and, hence,} \quad \mathrm{Aut}_{H_1}(F) = \langle\phi\rangle.$$

It is then straightforward to verify the following.

Every non-invertible element of $\mathrm{End}_{H_1}(F)$ is an F-retraction with image H_1.

The multiplication in $\mathrm{End}_{H_1}(F)$ is described by the monoid presentation

$$\mathrm{End}_{H_1}(F) = \langle \phi, \bar{\phi}, \psi_1 \mid \psi_1\phi = \psi_1^2 = \psi_1,\ \bar{\phi}\phi = \phi\bar{\phi} = 1\rangle_{\text{monoid}}.$$

Let $K = \langle a, b, cb\bar{c}\rangle$. The two closures of H_1 are

$$\mathrm{Fix}(\mathrm{Aut}_{H_1}(F)) = \mathrm{Fix}(\{\phi\}) = K \quad \text{and} \quad \mathrm{Fix}(\mathrm{End}_{H_1}(F)) = \mathrm{Fix}(\{\psi_1\}) = H_1.$$

Here, $K = H_1 * \langle b\rangle \neq H_1$.

The closure of $\{\psi_1\}$ in $\mathrm{End}(F)$ is $\mathrm{End}_{\mathrm{Fix}(\{\psi_1\})}(F) = \langle\phi\rangle \cup \langle\phi\rangle\psi_1$. $\square$

2.3 Example. Suppose that Notation 2.1 holds, and let $j = 2$.

Thus, we have $H_2 = \langle a, ba[c^2, b]\rangle$, $\phi = (a, b, cb)$, and $\psi_2 = (a, ba[c^2, b]\bar{a}, 1)$.

We shall show, in Corollary 2.6(ii) below, that

$$\mathrm{End}_{H_2}(F) = \{1, \phi^n\psi_2 \mid n \in \mathbb{Z}\} = \{1\} \cup \langle\phi\rangle\psi_2 \quad \text{and, hence,} \quad \mathrm{Aut}_{H_2}(F) = \{1\}.$$

It is then straightforward to verify the following.

Every non-identity element of $\mathrm{End}_{H_2}(F)$ is an F-retraction with image H_2.

The multiplication in $\mathrm{End}_{H_2}(F)$ is described by the monoid presentation

$$\mathrm{End}_{H_2}(F) = \langle \{\phi^n\psi_2 \mid n \in \mathbb{Z}\} \mid \{\phi^n\psi_2 \cdot \phi^m\psi_2 = \phi^n\psi_2 \mid m, n \in \mathbb{Z}\}\rangle_{\text{monoid}}.$$

The two closures of H_2 are

$$\mathrm{Fix}(\mathrm{Aut}_{H_2}(F)) = \mathrm{Fix}(\{1\}) = F \quad \text{and} \quad \mathrm{Fix}(\mathrm{End}_{H_2}(F)) = \mathrm{Fix}(\{\psi_2\}) = H_2.$$

Here, $F \neq H_2$.

The closure of $\{\psi_2\}$ in $\mathrm{End}(F)$ is $\mathrm{End}_{\mathrm{Fix}(\{\psi_2\})}(F) = \{1\} \cup \langle\phi\rangle\psi_2$. $\square$

By adjoining a free-group free factor simultaneously to H_j and to F, one obtains examples where F has arbitrary rank greater than two.

Our argument is concentrated in the next result.

2.4 Lemma. *Suppose that* Notation 2.1 *holds. For all $h \in \langle c\rangle$ and all $x \in F$, the following hold.*

(i). *If $\bar{h}\bar{a}bhx \sim \bar{a}\bar{x}bx$ in F, then $x \in \{1, b\bar{h}\bar{b}ah\bar{a}\}$.*

(ii). *If $x \sim \bar{a}\bar{x}ba[h, b]x$ in F, then $x \in \{b, ba[h, b]\bar{a}\}$.*

Proof. (i). Let $B = \{a, b, c, \bar{a}, \bar{b}, \bar{c}\}$. Each element of F has a unique expression as a reduced monoid word in B, and, where the interpretation is clear, we shall treat the elements of F as reduced monoid words in B. For each $v \in B$ and $z \in F$, we write $|z|_v$ to denote the number of times v occurs in (the reduced monoid expression for) z.

Suppose that

$$\bar{h}\bar{a}bhx \sim \bar{a}\bar{x}bx. \tag{1}$$

Write $x = b^m y a^n$ where $m, n \in \mathbb{Z}$, $y \in F$, y does not begin with b or $\bar{b}$, and y does not end with a or $\bar{a}$.

By abelianizing (1) we see that the derived subgroup of F, denoted F', contains $x = b^m y a^n$; thus

$$(2) \qquad yF' = \bar{a}^n \bar{b}^m F'$$

in F/F', a free abelian group with basis $\{aF',\, bF',\, cF'\}$.

Notice that (1) can be rewritten as

$$(3) \qquad \bar{h}\bar{a}bhb^m y a^n \sim \bar{a}\bar{y}by,$$

and that the right-hand side of (3) is cyclically reduced.

On applying $|-|_a$ to (3) we see that

$$(4) \qquad 0 + |y|_a + \max\{n, 0\} \geq 0 + |y|_{\bar{a}} + 0 + |y|_a\,,$$

and that equality holds in (4) if and only if no a is cancelled in the cyclic reduction of the left-hand side of (3). Now (4) amounts to $\max\{n, 0\} \geq |y|_{\bar{a}}$, and, by (2), it is clear that equality holds. Thus $|y|_{\bar{a}} = \max\{n, 0\}$, and no a is cancelled in the cyclic reduction of the left-hand side of (3).

Similarly, by applying $|-|_{\bar{a}}$ to (3), we find that $|y|_a = \max\{-n, 0\}$.

Also, by applying $|-|_b$ and $|-|_{\bar{b}}$ to (3), we find that $|y|_{\bar{b}} = \max\{m, 0\}$, $|y|_b = \max\{-m, 0\}$, and no b is cancelled in the cyclic reduction of the left-hand side of (3).

We now consider five non-pairwise-disjoint cases.

Case 1: $m = n = 0$.

Here $|y|_a = |y|_{\bar{a}} = |y|_b = |y|_{\bar{b}} = 0$. Thus, each side of (3) has no occurrence of a or $\bar{b}$, has a unique occurrence of $\bar{a}$, and has a unique occurrence of b. In each side, we can equate the cyclic subword between $\bar{a}$ and b and we find that $1 = \bar{y}$. Hence $x = 1$, as desired.

Case 2: $m \leq -1$.

Here $|y|_b = -m$ and $|y|_{\bar{b}} = 0$. Write $y = y'by''$ as a reduced monoid word such that y' has no occurrence of b, and, hence, y'' has $-m-1$ occurrences of b. Notice that $y' \neq 1$ because y does not begin with b. In each side of (3) there are $-m+1$ occurrences of b followed, cyclically, by $-m$ occurrences of $\bar{b}$. In each side, we take the first of the $-m+1$ occurrences of b as the terminating point. For the left-hand side we get $y''a^n\bar{h}\bar{a}bhb^m y'b$, that is, $y''a^n\bar{h}\bar{a}bh\bar{b}^{-m}y'b$. For the right-hand side we get $y\bar{a}\bar{y}b$, that is, $y\bar{a}\bar{y}''\bar{b}\bar{y}'b$. Since there is no cancellation of b or $\bar{b}$, we can equate the cyclic subwords between the last $\bar{b}$ and the first b, and we find that $y' = \bar{y}'$. Hence $y' = 1$, which is a contradiction, as desired.

Case 3: $m \geq 1$.

Here $|y|_{\bar{b}} = m$ and $|y|_b = 0$. In each side of (3), there are $m+1$ occurrences of b followed, cyclically, by m occurrences of $\bar{b}$. In each side, we take the last of the $m+1$ occurrences of b as the terminating point, and find that

$$ya^n\bar{h}\bar{a}bhb^m = y\bar{a}\bar{y}b.$$

Rearranging, we find that $y = \bar{b}^{m-1}\bar{h}\,\bar{b}ah\bar{a}^{n+1}$. Since y does not begin with b or $\bar{b}$, and does not end with a or $\bar{a}$, we see that $m = 1$, $n = -1$, $y = \bar{h}\,\bar{b}ah$, and $x = b\bar{h}\,\bar{b}ah\bar{a}$, as desired.

Case 4: $n \leq -1$.

This is similar to Case 3. Here $|y|_a = -n$ and $|y|_{\bar{a}} = 0$. In each side of (3) there are $-n+1$ occurrences of $\bar{a}$ followed, cyclically, by $-n$ occurrences of a. In each side, we take the first of the $-n+1$ occurrences of $\bar{a}$ as the starting point, and find that $\bar{a}^{-n}\bar{h}\bar{a}bhb^m y = \bar{a}\bar{y}by$. Thus $y = \bar{b}^{m-1}\bar{h}\,\bar{b}ah\bar{a}^{n+1}$, and, as in Case 3, $x = b\bar{h}\,\bar{b}ah\bar{a}$, as desired.

Case 5: $n \geq 1$.

This is similar to Case 2. Here $|y|_a = 0$ and $|y|_{\bar{a}} = n$. Write $y = y'\bar{a}y''$ as a reduced monoid word such that y'' has no occurrence of $\bar{a}$, and, hence, y' has $n-1$ occurrences of $\bar{a}$. Notice that $y'' \neq 1$ since y does not end with $\bar{a}$. In each side of (3) there are $n+1$ occurrences of $\bar{a}$ followed, cyclically, by n occurrences of a. In each side, we take the last of the $n+1$ occurrences of $\bar{a}$ as the starting point, and find $\bar{a}y''a^n\bar{h}\bar{a}bhb^m y' = \bar{a}\bar{y}by = \bar{a}\bar{y}''a\bar{y}'by$. Equating the cyclic subwords between the last $\bar{a}$ and the first a, we find that $y'' = \bar{y}''$. Hence $y'' = 1$, which is a contradiction, as desired.

This completes the proof of (i).

(ii). Suppose that $x \sim \bar{a}\bar{x}ba[h,b]x$. Let $y := \bar{b}x$. Then $x = by$ and

$$by \sim \bar{a}\bar{y}ahb\bar{h}y.$$

Applying $\rho := (a, \bar{h}\bar{a}bh, c) \in \mathrm{Aut}(F)$ and letting $z := y\rho$, we find that

$$\bar{h}\bar{a}bhz \sim \bar{a}\bar{z}bz.$$

By (i), $z \in \{1, b\bar{h}\,\bar{b}ah\bar{a}\}$. Applying $\rho^{-1} = (a, ahb\bar{h}, c)$, we see that

$$y = z\rho^{-1} \in \{1, a[h,b]\bar{a}\}.$$

Left multiplying by b we find that $x = by \in \{b, ba[h,b]\bar{a}\}$, as desired. $\square$

We can now calculate some Galois monoids.

2.5 Theorem. *Suppose that* Notation 2.1 *holds. For each element* (x,y,z) *of* $\mathrm{End}_{H_j}(F)$*, the following hold.*

(i). $y \in \{d_j, b\}$.

(ii). *If* $y = d_j$ *and* $j \geq 1$*, then there exists some* $n \in \mathbb{Z}$ *such that* $z = d_j^n$.

(iii). *If* $y = b$ *and* $j = 1$*, then there exists some* $n \in \mathbb{Z}$ *such that* $z = cb^n$.

(iv). *If* $y = b$ *and* $j \geq 2$*, then* $z = c$.

Proof. (i). Here $x = a$ and $yx[z^j, y] = ba[c^j, b]$. Hence $yaz^jy\bar{z}^j\bar{y} = ba[c^j, b]$ and $z^jy\bar{z}^j = \bar{a}\bar{y}ba[c^j,b]y$. Thus $y \sim \bar{a}\bar{y}ba[c^j,b]y$. By Lemma 2.4(ii),

$$y \in \{b, ba[c^j,b]\bar{a}\} = \{b, d_j\}.$$

This proves (i).

(ii). Suppose that $j \geq 1$ and that $y = d_j$.

Let C denote the centralizer of z^j in F.

Now
$$z^j d_j \bar{z}^j = \bar{a}\bar{d}_j ba[c^j, b]d_j = \bar{a}(a[c^j, b]\bar{a}\bar{b})ba[c^j, b]d_j = d_j.$$
Hence $d_j \in C$.

To show that $z \in \langle d_j \rangle$, we may assume that $z \neq 1$, and, hence, $z^j \neq 1$. Recall that $C = \langle z' \rangle$ for some $z' \in F$; see [1, Proposition I.2.19]. (An easy normal-form argument shows that there exists some $z' \in F$ such that z' is not a proper power and $z^j = z'^n$ for some positive integer n. Another normal-form argument shows that $C = \langle z' \rangle$. Alternatively, C is a free group with a non-trivial centre, and, hence, C is cyclic.)

Thus $d_j = z'^m$ for some integer m. Since $bF' = d_j F' = z'^m F' = (z'F')^m$ in F/F', a free abelian group with basis $\{aF', bF', cF'\}$, we see that $m = \pm 1$. Thus $d_j = z'^{\pm 1}$.

Now $z \in C = \langle z' \rangle = \langle d_j \rangle$. This proves (ii).

(iii) and (iv). Suppose that $j \geq 1$ and that $y = b$.

Then $z^j b \bar{z}^j = \bar{a}\bar{b}ba[c^j, b]b = c^j b \bar{c}^j$. Hence $\bar{c}^j z^j$ commutes with b. A trivial normal-form argument shows that there exists $n \in \mathbb{Z}$ such that $\bar{c}^j z^j = b^n$.

If $j = 1$, then $z = cb^n$. This proves (iii).

If $j \geq 2$, then the equation $c^j b^n = z^j$ clearly implies that $n = 0$ and $z = c$. This proves (iv). □

The cases of Theorem 2.5 where $j = 1$ and $j = 2$ give the key parts of Examples 2.2 and 2.3, respectively.

2.6 Corollary. *Suppose that* Notation 2.1 *holds.*

(i). *For each element (x, y, z) of $\mathrm{End}_{H_1}(F)$, there exists some $n \in \mathbb{Z}$ such that either $(x, y, z) = (a, b, cb^n)$ or $(x, y, z) = (a, d_1, d_1^n)$. Conversely, all these endomorphisms of F fix a and d_1, and, hence, fix H_1.*

(ii). *For each non-identity element (x, y, z) of $\mathrm{End}_{H_2}(F)$, there exists some $n \in \mathbb{Z}$ such that $(x, y, z) = (a, d_2, d_2^n)$. Conversely, all these endomorphisms of F fix a and d_2, and, hence, fix H_2.* □

3 The relevant results of Martino and Ventura

Let F be a finitely generated free group and let H be a subgroup of F.

We say that H is *one-auto fixed* (in F) if $H = \mathrm{Fix}(\{\rho\})$ for some $\rho \in \mathrm{Aut}(F)$. Thus a one-auto-fixed subgroup is $\mathrm{Aut}(F)$-closed.

We define *one-endo fixed* analogously. Thus an F-retract is one-endo fixed, and a one-endo-fixed subgroup is $\mathrm{End}(F)$-closed.

E. Ventura [5, Theorem 3.9] showed that if the rank of F is at most two, then the four concepts, one-endo fixed, $\mathrm{End}(F)$-closed, one-auto fixed, and $\mathrm{Aut}(F)$-closed, all coincide.

A. Martino [2, Corollary 5.3] showed that if the rank of F is three, then the two concepts one-auto fixed and $\mathrm{Aut}(F)$-closed coincide.

If the rank of F is greater than three, it is not known whether or not the $\mathrm{Aut}(F)$-closed subgroups are just the one-auto-fixed subgroups. If the rank of F

is greater than two, it is not known whether or not the End(F)-closed subgroups are just the one-endo-fixed subgroups.

Martino-Ventura [3, Corollary 3.4] showed that if H is Aut(F)-closed, then H is a free factor of some one-auto-fixed subgroup K of F. Also, by [3, Proposition 5.4], there exists an example where H is *not* Aut(F)-closed but H *is* a free factor of some one-auto-fixed subgroup K of F. We find it interesting that the same phenomenon appeared, unsought, in Example 2.2, above.

We now recall the examples of Martino-Ventura [4], together with some observations made to us by Armando Martino. Let Notation 2.1 hold, let i, j, k be integers such that $ijk \neq 0$, and let $H = H_{i,j,k} := \langle a,\, ba^i c^j bc^k \bar{b}\rangle$; thus $H_j = H_{1,j,-j}$. The six-tuple (i, j, k, a, b, c) in our notation corresponds to $(-s, r, t, b, cb^s, a)$ in the notation of [4]. It is straightforward to show that H is an F-retract, and, hence, H is one-endo fixed and End(F)-closed. By [4, Proposition 18], H is not one-auto fixed, and, then, by [2, Corollary 5.3], H is not Aut(F)-closed. Since it strictly contains the rank-two F-retract H, $\mathrm{Fix}(\mathrm{Aut}_H(F))$ has rank at least three, and, hence, is a 'maximal-rank fixed subgroup'. Now [3, Proposition 5.1 and Corollary 5.2] imply the following: $\mathrm{Aut}_H(F)$ is infinite cyclic or trivial, depending as the image of $\mathrm{Fix}(\mathrm{Aut}_H(F))$ in F/F' has rank 2 or 3; every non-identity element of $\mathrm{Aut}_H(F)$ has fixed subgroup exactly $\mathrm{Fix}(\mathrm{Aut}_H(F))$; every non-invertible element of $\mathrm{End}_H(F)$ has fixed subgroup exactly $\mathrm{Fix}(\mathrm{End}_H(F))$, that is, H.

Martino-Ventura use several results, some of them very deep, and their methods do not yield a description of $\mathrm{End}_H(F)$. In Examples 2.2 and 2.3 above, we obtained, directly, a description of $\mathrm{End}_{H_j}(F)$, for $j = 1$ and $j = 2$, respectively. It is straightforward to check that the same techniques apply to $H_{i,j,-j}$ (with $ij \neq 0$), and to verify that no new types of behaviour arise. For $k \neq -j$ (with $ijk \neq 0$), we have been unable to calculate $\mathrm{End}_{H_{i,j,k}}(F)$, or even the Aut($F$)-closure of $H_{i,j,k}$.

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