ON SEMILOCAL RINGS. 2nd edition.

Rosa Camps and Warren Dicks

Departament de Matemàtiques
Universitat Autònoma de Barcelona
08193 - Bellaterra (Barcelona), Spain

Dedicated to the memory of Pere Menal

Abstract

We give several characterizations of semilocal rings and deduce that rationally closed subrings of semisimple artinian rings are semilocal, that artinian modules have semilocal endomorphism rings, and that artinian modules cancel from direct sums.

Throughout, let R be a ring, associative with 1. We write $U(R)$ for the group of units of R, and $J(R)$ for the Jacobson radical of R. The ring R is said to be *semilocal* if $R/J(R)$ is semisimple artinian.

The results in this article were motivated by work of Menal, cf [11], [5], and are briefly summarized as follows.

Theorem 1 below gives various characterizations of semilocal rings, showing, in particular, that R is semilocal if and only if there is a homomorphism from R to a semisimple artinian ring S taking non-units of R to non-units of S. Thus, if R is a subring of a semisimple artinian ring S, then R will be semilocal provided the units of R consist precisely of the units of S which lie in R. This proves a conjecture of Menal.

Corollary 4 shows that if R is a subring of a left artinian ring S such that S/R is artinian as left R-module, then R is semilocal.

Corollary 6 verifies two other conjectures of Menal: if M is an artinian right R-module then $End_R M$ is semilocal, and, for all right R-modules A, B, if $M \oplus A \cong M \oplus B$ then $A \cong B$. In contrast, *indecomposable* artinian modules need not have *local* endomorphism rings; in fact, by Theorem 3.5 of [5], any commutative noetherian semilocal ring can occur as the endomorphism ring of an artinian module, so any indecomposable commutative noetherian semilocal ring can occur as the endomorphism ring of an indecomposable artinian module.

Theorem 7 is a recent result of F. Cedó that classical quotient rings with the maximum condition on left and right annihilators are semilocal.

We have made no attempt to give a detailed history, since a large number of results have been obtained in this area, and the literature is quite extensive, cf [1], [2],[3],[5], [6],[7], [8], [10], [11], [12], [13], [14], [15] and [16].

In order to state our results it will be convenient to have the following terminology and notation.

A homomorphism of rings $S \to R$ is said to be *local* if it carries non-units to non-units, that is, the image of $S \setminus U(S)$ lies in $R \setminus U(R)$. A *rationally closed subring* of R is a subring S such that $U(S) = S \cap U(R)$, which is equivalent to the inclusion map $S \to R$ being a local homomorphism.

Let M be an abelian group and $\mathcal{A}$ a set of subgroups of M. Given X, Y, Z in $\mathcal{A}$ such that $X \oplus Y = Z$, we say that X, Y are *$\mathcal{A}$-summands* of Z. The set $\mathcal{A}$ is said to satisfy the *maximum condition with respect to summands* if every nonempty subset $\mathcal{B}$ of $\mathcal{A}$ contains an element which is an $\mathcal{A}$-summand of no other element of $\mathcal{B}$. This is the maximum condition for the partial order on $\mathcal{A}$ consisting of equality together with the transitive closure of the relation of being a proper $\mathcal{A}$-summand.

Let M be a right R-module. For $r \in R$ we write $l.ann_M(r) = \{ m \in M \mid mr = 0\}$. Let $\mathcal{A} = \{ l.ann_M(r) \mid r \in R\}$. If $\mathcal{A}$ satisfies the maximum condition with respect to summands we say that *in M the left annihilators of elements of R satisfy the maximum condition with respect to summands.* There are many common situations where this occurs. For example, it holds if M satisfies the maximum condition for left annihilators of elements of R. It holds also if there exists a ring S such that M is an S-R-bimodule and $l.u.dim_S M$ is finite, where $l.u.dim_S M$ denotes the *uniform dimension* of M as left S-module, that is, the supremum of the cardinalities of independent sets of nonzero S-submodules of M. The left-right dual notion will be denoted $r.u.dim$. If R is semisimple artinian then $l.u.dim_R R = r.u.dim_R R$, and we denote the common value by $u.dim\, R$.

We can now give our characterizations of semilocal rings.

Theorem 1 *For any ring R the following are equivalent:*
(a) R is semilocal.
(b) There exists a local homomorphism from R to a semisimple artinian ring.
(c) There exists a ring S and an S-R-bimodule M such that $l.u.dim_S M$ is finite and $l.ann_M(r) \neq 0$ for all $r \in R \setminus U(R)$.
(d) There exists a right R-module M such that in M the left annihilators of elements of R satisfy the maximum condition with respect to summands, and $l.ann_M(r) \neq 0$ for all $r \in R \setminus U(R)$.
(e) There exists a non-negative integer n, and a function $d : R \to \{0, \ldots, n\}$ such that for all $a, b \in R$, $d(1 - ab) + d(a) = d(a - aba)$, and if $d(a) = 0$ then $a \in U(R)$.
(f) There exists a partial order $\geq$ on R satisfying the minimum condition, such that for all $a, b \in R$, if $1 - ab \in R \setminus U(R)$ then $a > a - aba$.
(c^*),(d^*),(e^*),(f^*) The left-right duals of (c), (d), (e), (f).

Moreover, if these equivalent conditions hold then, in (c), $u.dim\, R/J(R) \leq l.u.dim_S M$, and, in (e), $u.dim\, R/J(R) \leq n$.

Proof. We shall show *(a) ⇒ (b) ⇒ (c) ⇒ (d) ⇒ (f) ⇒ (a)*, and *(c) ⇒ (e) ⇒ (f)*.

(a) ⇒ (b) If R is semilocal then $R \to R/J(R)$ is a local homomorphism from R to a semisimple artinian ring.

(b) ⇒ (c) Suppose that $f{:}R \to S$ is a local homomorphism and S is semisimple artinian. Let M be S viewed as S-R-bimodule, so $l.u.dim_S M$ is finite. Also, if $r \in R \setminus U(R)$ then $f(r) \in S \setminus U(S)$, so $f(r)$ is a right zerodivisor in S, so $l.ann_M(r) \neq 0$.

(c) ⇒ (d) Suppose that *(c)* holds. Then $\{l.ann_M(r) \mid r \in R\}$ is a set of left S-submodules of M, so satisfies the maximum condition with respect to summands. Thus *(d)* holds.

(d) ⇒ (f) Suppose that *(d)* holds and let $\mathcal{A} = \{l.ann_M(r) \mid r \in R\}$. On $\mathcal{A}$, the transitive closure of the relation is-a-proper-$\mathcal{A}$-summand-of is an irreflexive transitive relation, which we shall denote $<$. For $a, b \in R$, set $a > b$ in R if $l.ann_M(a) < l.ann_M(b)$ in $\mathcal{A}$. On R, the relation $>$ is irreflexive and transitive, so we have a partial order $\geq$ on R. It satisfies the minimum condition because $\mathcal{A}$ satisfies the maximum condition with respect to $\mathcal{A}$-summands. Moreover, if $a, b \in R$ such that $1 - ab \in R \setminus U(R)$, then $l.ann_M(1 - ab) \neq 0$ and $l.ann_M(1 - ab) \oplus l.ann_M(a) = l.ann_M(a - aba)$, so $a > a - aba$.

(f) ⇒ (a) Suppose that *(f)* holds.

Let $\bar{R}$ denote $R/J(R)$, and for $r \in R$ let $\bar{r}$ denote $r + J(R)$.

Added January, 2010. We now give a simplification of the original proof of *(f) ⇒ (a)* incorporating an argument of Camillo-Nielsen [4]. Suppose that we are given a maximal proper right ideal of $\bar{R}$ and let it be expressed in the form $\bar{I}$ where I is a right ideal of R which contains $J(R)$. Let a be an element of $R \setminus I$ that is minimal with respect to the given partial order $\geq$. Consider an arbitrary element of $\bar{a}\bar{R} \cap \bar{I}$ and let it be expressed in the form $\bar{a}\bar{b}$ where $b \in R$. Then $ab \in I$. Consider an arbitrary $c \in R$. Then $abca \in I$ and, hence, $a - abca \in R \setminus I$. It follows from the minimality of a that $1 - abc \in U(R)$. This shows that $ab \in J(R)$, and, hence, $\bar{a}\bar{b} = \bar{0}$. This shows that $\bar{a}\bar{R} \cap \bar{I} = \{\bar{0}\}$. It then follows from the maximality of $\bar{I}$ that $\bar{a}\bar{R}$ is a simple right ideal of $\bar{R}$. The foregoing discussion shows that no proper right ideal of $\bar{R}$ contains every simple right ideal of $\bar{R}$. It follows that $\bar{R}$ is semisimple artinian, and, hence, R is semilocal.

(c) ⇒ (e) Suppose that *(c)* holds. Set $n = l.u.dim_S M$, and define $d : R \to \{0, \dots, n\}$ by $d(r) = l.u.dim_S(l.ann_M(r))$ for all $r \in R$. It is straightforward to verify that for all $a, b \in R$, $d(1 - ab) + d(a) = d(a - aba)$, and if $d(a) = 0$ then $a \in U(R)$.

(e) ⇒ (f) Suppose that *(e)* holds. For $a, b \in R$, set $a > b$ if $d(a) < d(b)$. This defines a partial order $\geq$ on R and *(f)* is readily verified.

Added January, 2010. Now suppose all the equivalent conditions hold. Let $\bar{R}$ denote $R/J(R)$, let $m = u.dim\, R/J(R)$, and for $r \in R$ let $\bar{r}$ denote $r + J(R)$. By *(a)*, there exists some sequence $b_1, \dots, b_m$ in R such that $\bar{b}_1, \dots, \bar{b}_m$ is a sequence of mutually orthogonal nonzero idempotents in $\bar{R}$. Let $a_1 = b_1 + b_2 + \cdots + b_m$, and, for each $i = 1, \dots m$, let $a_{i+1} = a_i - a_i b_i a_i$. It is not difficult to see that for each $i = 1, \dots, m$, $\bar{a}_i = \bar{b}_i + \bar{b}_{i+1} + \cdots + \bar{b}_m$ and, hence, $\bar{1} - \bar{a}_i\bar{b}_i = 1 - \bar{b}_i$, and, hence, $\bar{1} - \bar{a}_i\bar{b}_i$ is idempotent and $\bar{1} - \bar{a}_i\bar{b}_i \neq \bar{1}$; in particular, $1 - a_i b_i \in R \setminus U(R)$, and, hence, $a_i > a_{i+1}$. Thus, with respect to the partial order in *(f)*, there exists a chain in R of length m.

By the proof of *(e) ⇒ (f)*, we may assume that, with respect to the partial order in *(f)*, every chain in R has length at most n. Hence, $m \leq n$.

By the proof of *(c) ⇒ (e)*, we may assume that *(e)* holds with $n = l.u.dim_S M$ and then $l.u.dim_S M = n \geq m$. ■

Corollary 2 *If* $R \to S$ *is a local homomorphism of rings and* S *is semilocal then* R *is semilocal, and* $u.dim\, R/J(R) \leq u.dim\, S/J(S)$.

In particular, rationally closed subrings of semisimple artinian rings are semilocal.

PROOF. Let M be $S/J(S)$ viewed as S-R-bimodule. By Theorem 1 *(c)* $\Rightarrow$ *(a)*, R is semilocal, and by the last part of Theorem 1, $u.dim\, R/J(R) \leq l.u.dim_S M = u.dim\, S/J(S)$. ∎

For example, suppose that R is a subring of a ring S such that R is a direct summand of S as left R-module. Here every element of R which has a right inverse in S already has a right inverse in R, so R is rationally closed in S; thus if S is semilocal then R is semilocal, as was first proved in [13], and moreover $u.dim\, R/J(R) \leq u.dim\, S/J(S)$.

A consequence of the previous paragraph is that if G is a group such that the group ring RG is semilocal then for any subgroup H of G, RH is semilocal, as observed in [13], and moreover $u.dim\, RH/J(RH) \leq u.dim\, RG/J(RG)$.

The following gives another criterion for recognizing semilocal subrings.

Theorem 3 *Let* R *be a ring in which every non-unit is a right zerodivisor, such that in* R *the left annihilators of elements of* R *satisfy the maximum condition with respect to summands. Then* R *is semilocal and* $u.dim\, R/J(R) \leq l.u.dim_R R$.

Moreover, if S *is a subring of* R *such that in* R/S *the left annihilators of elements of* S *satisfy the maximum condition with respect to summands, then* S *is semilocal and* $u.dim\, S/J(S) \leq l.u.dim_R R + l.u.dim_S R/S$.

PROOF. It suffices to prove only the second part, since the first part is then obtained by taking $S = R$. Let M be $R \oplus (R/S)$ viewed as $(R \times S)$-S-bimodule. The hypotheses ensure that in M the left annihilators of elements of S satisfy the maximum condition with respect to summands. Consider any $s \in S \setminus U(S)$. If $s \in S \setminus U(R)$ then s is a right zerodivisor in R, so $l.ann_M(s) \neq 0$; if $s \in U(R)$ then $s^{-1} + S$ is a nonzero element of R/S which is right annihilated by s, so $l.ann_M(s) \neq 0$. Thus in any event $l.ann_M(s) \neq 0$. By Theorem 1 *(d)* $\Rightarrow$ *(a)*, S is semilocal. To see that $u.dim\, S/J(S) \leq l.u.dim_{R\times S}(R \oplus (R/S))$, we can assume that the right-hand side is finite, and apply the inequality concerning *(c)* in the last part of Theorem 1. ∎

We emphasize one particular case of the above result.

Corollary 4 *If S is a subring of a left artinian ring R and $l.u.dim_S R/S$ is finite, then S is semilocal, and $u.dim\, S/J(S) \le l.u.dim_R R + l.u.dim_S R/S$.* ∎

We turn now to consequences for endomorphism rings.

Recall that 1 *is in the stable range of* R if whenever $ax + b = 1$ in R, there exists $c \in R$ such that $a + bc \in U(R)$. Also, a right R-module M *cancels from direct sums* if for all right R-modules A, B, if $M \oplus A \cong M \oplus B$ then $A \cong B$.

Theorem 5 *Let M be a right R-module such that every injective R-linear endomorphism of M is bijective, and such that the set of R-submodules $\{Ker\, f \mid f \in End_R(M)\}$ satisfies the maximum condition with respect to summands. Then $End_R M$ is semilocal, 1 is in the stable range of $End_R M$, and the right R-module M cancels from direct sums. Moreover $u.dim\,((End_R M)/J(End_R M)) \le r.u.dim_R M$.*

Proof. Consider M as an $End_R M$-R-bimodule. In M the right annihilator of an element f of $End_R M$ is precisely $Ker\, f$, and f is a unit if and only if f is bijective. Now Theorem 1 *(d*) ⇒ (a)* shows that $End_R M$ is semilocal. By a result of Bass, 1 is then in the stable range of $End_R M$, cf p.313 of [9]. By a result of Evans, M then cancels from direct sums, cf p.315 of [9].

To see that $u.dim\,((End_R M)/J(End_R M)) \le r.u.dim_R M$, we can assume that the right-hand side is finite, and apply the left-right dual of the last part of Theorem 1. ∎

The following answers Question 16 of [11] in the affirmative.

Corollary 6 *Let M be an artinian right R-module. Then $End_R M$ is semilocal, 1 is in the stable range of $End_R M$, and the right R-module M cancels from direct sums. Moreover $u.dim\,((End_R M)/J(End_R M)) \le r.u.dim_R M$.* ∎

Added October, 1992. Recall that R is said to be *its own classical quotient ring* if every non-unit of R is a zerodivisor, that is, either a right zerodivisor or a left zerodivisor.

F. Cedó has found an elegant application of Theorem 1 to obtain the following generalization of Stafford's result, Corollary 2.7 (i) of [14], that a noetherian ring which is its own classical quotient ring is semilocal. The theorem also generalizes Proposition 2.4 of [5], since it shows that the rings considered in that proposition are actually semilocal; indeed, this was the original motivation for the result.

Theorem 7 (Cedó) *Suppose that R is its own classical quotient ring, and that in R both the set of left annihilators of elements of R, and also the set of right annihilators of elements of R, satisfy the maximum condition with respect to summands. Then R is semilocal.*

In particular, if R is its own classical quotient ring and R satisfies the maximum condition for both left annihilators of elements and for right annihilators of elements then R is semilocal.

Proof. Let $\mathcal{A} = \{l.ann_R(r) \mid r \in R\}$ and $\mathcal{B} = \{r.ann_R(r) \mid r \in R\}$. On $\mathcal{A}$ the transitive closure of the relation is-a-proper-$\mathcal{A}$-summand-of determines a partial order $\leq$ which satisfies the maximum condition. Similarly, on $\mathcal{B}$ the transitive closure of the relation is-a-proper-$\mathcal{B}$-summand-of determines a partial order, which we again denote $\leq$, and which also satisfies the maximum condition. For $a, b \in R$, set $a > b$ if $l.ann_R(a) < l.ann_R(b)$ in $\mathcal{A}$, or $l.ann_R(a) = l.ann_R(b)$ and $r.ann_R(a) < r.ann_R(b)$ in $\mathcal{B}$. This gives a partial order $\geq$ on R satisfying the minimum condition.

Now suppose that $a, b \in R$ such that $1 - ab \in R \setminus U(R)$. By Theorem 1 *(f)* $\Rightarrow$ *(a)*, it suffices to show that $a > a - aba$.

Here R is its own classical quotient ring and $1-ab$ is a non-unit, so $1-ab$ is a zerodivisor, that is, either $l.ann_R(1-ab) \neq 0$ or $r.ann_R(1-ab) \neq 0$. Hence either $l.ann_R(1-ab) \neq 0$ or $r.ann_R(1-ba) \neq 0$, since left multiplication by a determines a right R-linear isomorphism $r.ann_R(1-ba) \to r.ann_R(1-ab)$, with inverse given by left multiplication by b. Now, from the usual equalities $l.ann_R(a - aba) = l.ann_R(a) \oplus l.ann_R(1 - ab)$ and $r.ann_R(a - aba) = r.ann_R(a) \oplus r.ann_R(1 - ba)$, and the definition of the partial order on R, we see that $a > a - aba$. ■

In Example 6.4 of [14], Stafford constructs a right noetherian ring which is its own classical quotient ring, but which is not semilocal; hence the two-sided conditions in Theorem 7 cannot be weakened to one-sided conditions.

As often happens, in the case of rings satisfying polynomial identities, one-sided conditions suffice: by Proposition 2.11 of [14], if R satisfies a polynomial identity, and has the maximum condition on annihilator ideals, and is its own classical quotient ring, then R is semilocal. This result does not seem to follow immediately from any of our characterizations, although there is a connection which can be seen as follows. Stafford first shows that if R satisifies a polynomial identity and has the maximum condition on annihilator ideals, then there is a finite set of prime ideals $P_1, \ldots, P_n$ of R such that the natural map $R \to R/P_1 \times \cdots \times R/P_n$ sends zerodivisors to zerodivisors. As is well-known, $R/P_1 \times \cdots \times R/P_n$ embeds in a semisimple artinian ring Q, so if R is its own classical quotient ring, then we have a local homomorphism from R to Q, and by Corollary 2 above, R is semilocal.

ACKNOWLEDGEMENTS

The authors owe a great debt, both personal and scientific, to the guiding light of the late Pere Menal.

We thank Ferran Cedó for his kindness in allowing us to include Theorem 7.

We gratefully acknowledge partial support by the DGICYT through grants PB89-0296 and PB90-0719.

References

[1] E.P. Armendariz, *Modules with Artinian prime factors,* Proc. Amer. Math. Soc. **78** (1980), 311-314.

[2] E.P. Armendariz, J.W. Fisher and R.L. Snider, *On injective endomorphisms of finitely generated modules*, Comm. in Algebra **6** (1978), 659-672.

[3] E.P. Armendariz and J.K. Park, *Endomorphism Rings of Artinian Modules,* preprint.

[4] V. Camillo and Pace P. Nielsen, *On a theorem of Camps and Dicks*, pp. 83–84 in *Advances in ring theory* (eds. Dinh van Huynh and Sergio R. López-Permouth), Trends Math., Birkhäuser-Verlag, Basel, 2010.

[5] R. Camps and P. Menal, *Power cancellation for Artinian modules*, Comm. in Algebra **19** (1991), 2081-2095.

[6] A. Facchini, *Loewy and Artinian modules over commutative rings,* Annali di Matematica pura ed applicata **128** (1980), 359-374.

[7] J.W. Fisher, *Nil subrings of endomorphism rings of modules,* Proc. Amer. Math. Soc. **34** (1972), 75-78.

[8] J. Guérindon, *Modules de Loewy et modules artiniens,* Journal of Algebra **79** (1982), 1-7.

[9] T.Y. Lam, *A First Course in Noncommutative Rings,* GTM **131**, Springer-Verlag, Berlin, 1991.

[10] P. Menal, *On π-regular rings whose primitive factor rings are artinian,* J. Pure and Applied Algebra **20**(1981), 71-78.

[11] P. Menal, *Cancellation modules over regular rings,* pp. 187-209 in: Proc. Granada Ring Theory Conference, Lecture Notes in Math. **1328**, Springer, Berlin, 1988.

[12] R. Schulz, *The endomorphism ring of an artinian module whose homogeneous length is finite,* Proc. Amer. Math. Soc. **86** (1982), 209-210.

[13] J.B. Srivastava and S.K. Shah, *Semilocal and semiregular group rings,* Indag. Math. **42** (1980), 347-352.

[14] J.T. Stafford, *Noetherian full quotient rings,* Proc. London Math. Soc. **44** (1982), 385-404.

[15] R.B.Warfield, *A Krull-Schmidt Theorem for infinite sums of modules*, Proc. Amer. Math. Soc. **22** (1969), 460-465.

[16] R.B. Warfield, *Cancellation of modules and groups and stable range of endomorphism rings,* Pac. J. Math. **91** (1980), 457-485.