Joel Friedman's proof of the strengthened Hanna Neumann conjecture

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Abstract

For a finite graph Z, $\overline{\mathrm{r}}(Z) \coloneqq e{-}v{+}t$, where e, v, and t denote the number of edges, vertices, and tree components of Z, respectively. Let G be a finite group, Z be a finite G-free G-graph, and X and Y be subgraphs of Z. Using linear algebra and algebraic geometry over a sufficiently large field, Joel Friedman proved that $\sum\limits_{g\in G} \overline{\mathrm{r}}(X \cap gY) \leqslant \overline{\mathrm{r}}(X)\overline{\mathrm{r}}(Y)$. He showed that this inequality implies the strengthened Hanna Neumann conjecture. We simplify Friedman's proof of the foregoing inequality by replacing the sufficiently large field with a field $\mathbb{F}$ on which G acts faithfully and then replacing all the arguments involving algebraic geometry with shorter arguments about the left ideals of the skew group ring $\mathbb{F}G$.

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1 Sheaves on graphs

1.1 Notation. As Bourbaki intended, we let $\mathbb{N}$ denote the set of finite cardinals, $\{0, 1, 2, \ldots\}$.

Throughout this section, let $\mathbb{F}$ be a field. We shall write $\dim(V)$ to denote the $\mathbb{F}$-dimension of an $\mathbb{F}$-module V.

Throughout this section, let $(Z, \mathrm{V}Z, \mathrm{E}Z, \mathrm{E}Z \xrightarrow{\iota,\tau} \mathrm{V}Z)$ be a finite (oriented) graph; here, Z is a finite set, $\mathrm{V}Z \subseteq Z$, $\mathrm{E}Z = Z - \mathrm{V}Z$, and ι and τ are functions. Each $e \in \mathrm{E}Z$ has an associated picture of the form $\overset{\iota e}{\bullet}\xrightarrow{e}\overset{\tau e}{\bullet}$ or $\overset{\iota e = \tau e}{\bullet}{}_{e\circlearrowleft}$. We let the symbol Z also denote the graph.

We shall use the standard concepts of *subgraph*, *connected graph*, *component of a graph*, *tree*, *tree component of a graph*, and *graph map*.

We write $\delta(Z) \coloneqq |\mathrm{E}Z| - |\mathrm{V}Z|$ and $\overline{\mathrm{r}}(Z) \coloneqq \max\{\delta(Y) : Y \text{ is a subgraph of } Z\}$. Each subgraph Y of Z with $\delta(Y) = \overline{\mathrm{r}}(Z)$ is called a *δ-maximizer in Z*. The intersection of all the δ-maximizers in Z is denoted supercore(Z).

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1.2 Lemma. *The following hold for the finite graph Z.*

(i) *If Z is connected, then $\delta(Z) \in \{-1\} \cup \mathbb{N}$ and $\delta(Z) = -1$ if and only if Z is a tree.*

(ii) *If $\mathcal{C}$ is the set of components of Z, then $\delta(Z) = \sum_{X \in \mathcal{C}} \delta(X)$.*

(iii) $\overline{\mathrm{r}}(Z) \geqslant 0$.

(iv) $\delta(\mathrm{supercore}(Z)) = \overline{\mathrm{r}}(Z)$, *and* $\mathrm{supercore}(Z)$ *is the unique $\subseteq$-smallest δ-maximizer in Z.*

(v) $\overline{\mathrm{r}}(Z) - \delta(Z)$ *equals the number of tree components of Z.*

Proof. (i) and (ii) are straightforward, and (iii) holds since $\emptyset$ is a subgraph of Z.

(iv) If X and Y are δ-maximizers in Z, then $\delta(X) \geqslant \delta(X \cap Y)$ and $\delta(Y) \geqslant \delta(X \cup Y)$. Since $\delta(X) + \delta(Y) = \delta(X \cap Y) + \delta(X \cup Y)$, we see that $\delta(X) = \delta(X \cap Y)$. Thus, $X \cap Y$ is a δ-maximizer in Z. Hence, the set of δ-maximizers in Z is closed under finite intersections. Since Z is finite, $\mathrm{supercore}(Z)$ is the intersection of finitely many δ-maximizers in Z. Thus, (iv) holds.

(v) Let $\mathrm{forest}(Z)$ denote the subgraph of Z formed by the tree components. Then $-\delta(\mathrm{forest}(Z))$ equals the number of tree components of Z.

Now $\delta(Z)-\delta(\mathrm{forest}(Z)) = \delta(Z- \mathrm{forest}(Z)) \leqslant \overline{\mathrm{r}}(Z)$.

We shall prove the reverse inequality, $\overline{\mathrm{r}}(Z) \leqslant \delta(Z)-\delta(\mathrm{forest}(Z))$, by induction on $|\mathrm{E}Z|$.

Let $Y \coloneqq \mathrm{supercore}(Z)$. Consider first the case where $\mathrm{E}Z = \mathrm{E}Y$. Then $Z-Y$ consists of isolated vertices, each of which is a tree component of Z. Then

$$\overline{\mathrm{r}}(Z) = \delta(Y) = \delta(Z)-\delta(Z-Y) \leqslant \delta(Z)-\delta(\mathrm{forest}(Z)),$$

as desired. It remains to consider the case where we have some $e \in \mathrm{E}Z-\mathrm{E}Y$. Then,

$$\begin{aligned}\overline{\mathrm{r}}(Z) &= \overline{\mathrm{r}}(Z-\{e\}) \text{ since } e \in \mathrm{E}Z- \mathrm{E}Y \\ &\leqslant \delta(Z-\{e\})-\delta(\mathrm{forest}(Z-\{e\})) \text{ by the implicit induction hypothesis} \\ &= \delta(Z)-1-\delta(\mathrm{forest}(Z-\{e\})) \leqslant \delta(Z)-\delta(\mathrm{forest}(Z))\end{aligned}$$

since adding e to $Z-\{e\}$ reduces the number of tree components by one or zero, depending whether or not e is incident to $\mathrm{forest}(Z-\{e\})$. The result now holds by induction. $\square$

1.3 Definitions. We let $\mathbb{Z}Z$ and $\mathbb{Z}[Z]$ denote the free $\mathbb{Z}$-module with $\mathbb{Z}$-basis the finite set Z. Let V be a finite-dimensional $\mathbb{F}$-module. We shall be interested in the finite-dimensional $\mathbb{F}$-module $V\otimes \mathbb{Z}Z = \bigoplus\limits_{z\in Z} (V\otimes \mathbb{Z}z)$, where we are tensoring over $\mathbb{Z}$. For each $z \in Z$, we have a map $V\otimes \mathbb{Z}Z \to V$, $d \mapsto d_z$, such that, for each $d \in V\otimes \mathbb{Z}Z$, $d = \sum\limits_{z\in Z} (d_z\otimes z)$.

Let D be an $\mathbb{F}$-submodule of $V\otimes \mathbb{Z}Z$. For each $z \in Z$, we let $D_z \coloneqq \{d_z : d \in D\}$; thus, $\bigoplus\limits_{z\in Z} (D_z\otimes \mathbb{Z}z) \supseteq D$. We shall say that D is a *sheaf* in $V\otimes \mathbb{Z}Z$ if, firstly, $\bigoplus\limits_{z\in Z} (D_z\otimes \mathbb{Z}z) = D$, and, secondly, for each $e \in \mathrm{E}X$, $D_e \subseteq D_{\iota e} \cap D_{\tau e}$. Viewed in the lattice of $\mathbb{F}$-submodules of $V\otimes \mathbb{Z}Z$, the set of sheaves in $V\otimes \mathbb{Z}Z$ is closed under sums and intersections.

Let D be a sheaf in $V\otimes\mathbb{Z}Z$. We define $\delta(D) \coloneqq (\sum\limits_{e\in \mathrm{E}Z}\dim(D_e)) - (\sum\limits_{v\in \mathrm{V}Z}\dim(D_v))$. By a *subsheaf C of D*, we mean a sheaf in $V\otimes\mathbb{F}Z$ that is contained in D; in this event, we write $C \preccurlyeq D$. We define $\overline{\mathrm{r}}(D) \coloneqq \max\{\delta(C) : C \preccurlyeq D\}$. Clearly, $\overline{\mathrm{r}}(D) \geqslant 0$, since $\{0\} \preccurlyeq D$. If $C \preccurlyeq D$ and $\delta(C) = \overline{\mathrm{r}}(D)$, then C is called a *δ-maximizer in D*. The intersection of all the δ-maximizers in D is denoted $\mathrm{supercore}(D)$.

1.4 Lemma. *Let V be a finite-dimensional $\mathbb{F}$-module and $D \preccurlyeq V\otimes\mathbb{Z}Z$.*

(i) $\delta(\mathrm{supercore}(D)) = \overline{\mathrm{r}}(D)$*, and* $\mathrm{supercore}(D)$ *is the unique $\subseteq$-smallest δ-maximizer in D.*

(ii) *If $D' \preccurlyeq D$, then* $\mathrm{supercore}(D') \preccurlyeq \mathrm{supercore}(D)$.

Proof. (i) If B and C are δ-maximizers in D, then $\delta(B) \geqslant \delta(B\cap C)$ and $\delta(C) \geqslant \delta(B{+}C)$. By Grassmann's formula, $\delta(B) + \delta(C) = \delta(B\cap C) + \delta(B{+}C)$. Hence, $\delta(B) = \delta(B\cap C)$. Thus, $B\cap C$ is a δ-maximizer in D. Hence, the set of δ-maximizers in D is closed under finite intersections. Since $V\otimes\mathbb{Z}Z$ is finite-dimensional, the descending chain condition holds for $\mathbb{F}$-submodules; hence, $\mathrm{supercore}(D)$ is the intersection of finitely many δ-maximizers in D. It follows that (i) holds.

(ii) Let $C' \coloneqq \mathrm{supercore}(D')$ and $C \coloneqq \mathrm{supercore}(D)$. Then $\delta(C') \geqslant \delta(C'\cap C)$ by (i) for D', and $\delta(C) \geqslant \delta(C'{+}C)$ by (i) for D. Since $\delta(C') + \delta(C) = \delta(C'\cap C) + \delta(C'{+}C)$, we see that $\delta(C') = \delta(C'\cap C)$. Thus, $C'\cap C$ is a δ-maximizer in D'. Since C' is the $\subseteq$-smallest δ-maximizer in D' by (i) for D', $C' \subseteq C'\cap C$. Thus $C' \subseteq C$, as desired. $\square$

1.5 Lemma. *Let V be a finite-dimensional $\mathbb{F}$-module, V' an $\mathbb{F}$-submodule of V, and Z' a subgraph of Z. Then $V'\otimes\mathbb{Z}Z' \preccurlyeq V\otimes\mathbb{Z}Z$.*

Proof. Set $C \coloneqq V'\otimes\mathbb{Z}Z'$. For each $z \in Z$, $C_z = V'$ if $z \in Z'$, while $C_z = \{0\}$ if $z \in Z{-}Z'$. Thus, $C_z\otimes\mathbb{Z}z \subseteq C$. Consider any $e \in \mathrm{E}Z$. If $e \in Z'$, then $C_e = C_{\iota e} = C_{\tau e} = V'$, while if $e \in Z{-}Z'$, then $C_e = \{0\} \subseteq C_{\iota e}\cap C_{\tau e}$. Thus, $C \preccurlyeq V\otimes\mathbb{Z}Z$. $\square$

1.6 Lemma. *Let V be a finite-dimensional $\mathbb{F}$-module, and V' an $\mathbb{F}$-submodule of V. Make the identification $(V/V')\otimes\mathbb{Z}Z = (V\otimes\mathbb{Z}Z)/(V'\otimes\mathbb{Z}Z)$.*

Let $D \preccurlyeq V\otimes\mathbb{Z}Z$. Set $D' \coloneqq D\cap(V'\otimes\mathbb{Z}Z)$ and $D'' \coloneqq (D{+}(V'\otimes\mathbb{Z}Z))/(V'\otimes\mathbb{Z}Z)$. Then $D' \preccurlyeq V'\otimes\mathbb{Z}Z$, $D'' \preccurlyeq (V/V')\otimes\mathbb{Z}Z$, and $\overline{\mathrm{r}}(D) \leqslant \overline{\mathrm{r}}(D') + \overline{\mathrm{r}}(D'')$.

Proof. It is straightforward to show that $D' \preccurlyeq V'\otimes\mathbb{Z}Z$ and $D'' \preccurlyeq (V/V')\otimes\mathbb{Z}Z$. Set $C \coloneqq \mathrm{supercore}(D)$, $C' \coloneqq C\cap(V'\otimes\mathbb{Z}Z)$, and $C'' \coloneqq (C{+}(V'\otimes\mathbb{Z}Z))/(V'\otimes\mathbb{Z}Z) \simeq C/C'$. It is straightforward to show that $C' \preccurlyeq D'$, $C'' \preccurlyeq D''$, and $\delta(C) = \delta(C') + \delta(C'')$. Then we have $\overline{\mathrm{r}}(D) = \delta(C) = \delta(C') + \delta(C'') \leqslant \overline{\mathrm{r}}(D') + \overline{\mathrm{r}}(D'')$. $\square$

We now consider the sheaves that will most interest us.

1.7 Lemma. *Let V be a finite-dimensional $\mathbb{F}$-module, G be an $\mathbb{F}$-basis of V, $(Z_g : g{\in}G)$ be a family of subgraphs of Z, and $D \coloneqq \bigoplus\limits_{g\in G}(\mathbb{F}g\otimes\mathbb{Z}Z_g)$. Then $D \preccurlyeq V\otimes\mathbb{Z}Z$ and $\overline{\mathrm{r}}(D) = \sum\limits_{g\in G}\overline{\mathrm{r}}(Z_g)$. In particular, $\overline{\mathrm{r}}(V\otimes\mathbb{Z}Z) = \dim(V)\times\overline{\mathrm{r}}(Z)$.*

Proof. By Lemma 1.5, $\mathbb{F}g \otimes \mathbb{F}Z_g \preccurlyeq V \otimes \mathbb{F}Z$, for each $g \in G$. Since sums of sheaves are sheaves, it follows that $D \preccurlyeq V \otimes \mathbb{F}Z$. For each $g \in G$, let $X_g := \mathrm{supercore}(Z_g)$. Then

$$\overline{\mathrm{r}}(D) = \overline{\mathrm{r}}(\bigoplus_{g\in G}(\mathbb{F}g \otimes \mathbb{Z}Z_g)) \geqslant \delta(\bigoplus_{g\in G}(\mathbb{F}g \otimes \mathbb{Z}X_g)) = \sum_{g\in G}\delta(X_g) = \sum_{g\in G}\overline{\mathrm{r}}(Z_g).$$

To prove the inequality in the other direction, we argue by induction on $\dim(V)$.

If $\dim(V) = 0$, the inequality is clear. Suppose now that $\dim(V) = 1$, and let $g \in G$. Let $C := \mathrm{supercore}(D)$. Then $C = \bigoplus\limits_{z\in Z_g}(C_z \otimes z)$, and, for each $z \in Z_g$, C_z is an $\mathbb{F}$-submodule of V. Thus, $\dim(C_z) \leqslant 1$. Let $X := \{z \in Z_g : \dim(C_z) = 1\}$. Then $C = \bigoplus\limits_{z\in X}(V \otimes \mathbb{Z}z) = V \otimes \mathbb{Z}X$. Since $C \preccurlyeq V \otimes \mathbb{Z}Z$, it follows that X is a subgraph of Z_g. Now $\overline{\mathrm{r}}(D) = \delta(C) = \delta(X) \leqslant \overline{\mathrm{r}}(Z_g)$.

Thus, we may assume that $\dim(V) \geqslant 2$ and that the inequality holds for all smaller dimensions. Since $|G| \geqslant 2$, we may partition G into two proper subsets G' and G'', and set $V' := \bigoplus\limits_{g\in G'}\mathbb{F}g$, $V'' := \bigoplus\limits_{g\in G''}\mathbb{F}g$, $D' := D \cap (V' \otimes \mathbb{Z}Z)$, and $D'' := (D + (V' \otimes \mathbb{Z}Z))/(V' \otimes \mathbb{Z}Z)$. By Lemma 1.6, $\overline{\mathrm{r}}(D) \leqslant \overline{\mathrm{r}}(D') + \overline{\mathrm{r}}(D'')$. Since $D = \bigoplus\limits_{g\in G}(\mathbb{F}g \otimes \mathbb{Z}Z_g)$, $D' = \bigoplus\limits_{g\in G'}(\mathbb{F}g \otimes \mathbb{Z}Z_g)$ and $D'' = \bigoplus\limits_{g\in G''}(\mathbb{F}g \otimes \mathbb{Z}Z_g)$, the desired inequality follows by induction.

The final assertion is the case where $Z_g = Z$ for each $g \in G$. $\square$

We now impose on Friedman's approach the hypothesis of a faithful group action on $\mathbb{F}$.

1.8 Definition. Let G be a finite multiplicative group given with a faithful left action on $\mathbb{F}$. Let $\mathrm{End}(\mathbb{F})$ denote the ring of all additive-group endomorphisms $r\colon \mathbb{F} \to \mathbb{F}$, $\lambda \mapsto r[\lambda]$. Here, $\mathbb{F}$ is a left $\mathrm{End}(\mathbb{F})$-module. We view G as a subgroup of the group of units of $\mathrm{End}(\mathbb{F})$. We view $\mathbb{F}$ as a subring of $\mathrm{End}(\mathbb{F})$ acting on $\mathbb{F}$ by left multiplication. Let $\mathbb{F}G := \sum\limits_{g\in G}\mathbb{F}g \subseteq \mathrm{End}(\mathbb{F})$. If $g \in G$ and λ, $\mu \in \mathbb{F}$, then $(g\mu)[\lambda] = g[\mu[\lambda]] = g[\mu\lambda] = (g[\mu])(g[\lambda]) = (g[\mu])[g[\lambda]] = (g[\mu]g)[\lambda]$; thus, $g\mu = g[\mu]g$ in $\mathrm{End}(\mathbb{F})$. It follows that $\mathbb{F}G$ is closed under multiplication in $\mathrm{End}(\mathbb{F})$, and, hence, $\mathbb{F}G$ is a subring of $\mathrm{End}(\mathbb{F})$. We call $\mathbb{F}G$ the *skew group ring of G over* $\mathbb{F}$. As is well known, Dedekind showed (publ. 1894) that $\dim(\mathbb{F}G) = |G|$, and Artin gave the following proof (publ. 1938). We shall show that, for each $n \in \mathbb{N}$, each repetition-free sequence $(g_i)_{i=1}^n \in G^n$ is left $\mathbb{F}$-independent; the case $n = |G|$ then gives the desired result. If $n = 0$, the assertion holds. By induction, it remains to consider the case where $n \geqslant 1$ and $(g_i)_{i=1}^{n-1}$ is left $\mathbb{F}$-independent, and to show that, for each $(\lambda_i)_{i=1}^n \in \mathbb{F}^n$, if $\sum_{i=1}^n \lambda_i g_i = 0$, then $(\lambda_i)_{i=1}^n = 0$ in $\mathbb{F}^n$. Let $\mu \in \mathbb{F}$. Notice that

$$\begin{aligned}\sum_{i=1}^{n-1}(\lambda_i(g_i - g_n)[\mu])g_i &= \sum_{i=1}^{n}(\lambda_i(g_i - g_n)[\mu])g_i = \sum_{i=1}^{n}\lambda_i g_i[\mu]g_i - \sum_{i=1}^{n}\lambda_i g_n[\mu]g_i \\ &= \sum_{i=1}^{n}\lambda_i g_i\mu - \sum_{i=1}^{n} g_n[\mu]\lambda_i g_i = 0\mu - g_n[\mu]0 = 0.\end{aligned}$$

Since $(g_i)_{i=1}^{n-1}$ is left $\mathbb{F}$-independent, $(\lambda_i(g_i - g_n)[\mu]))_{i=1}^{n-1} = 0$ in $\mathbb{F}^{n-1}$. Since μ is arbitrary, $(\lambda_i(g_i - g_n))_{i=1}^{n-1} = 0$ in $\mathbb{F}G^{n-1}$. Since each coordinate of $(g_i - g_n)_{i=1}^{n-1}$ is nonzero, $(\lambda_i)_{i=1}^{n-1} = 0$ in $\mathbb{F}^{n-1}$. Finally, $\lambda_n = 0$, since $g_n \neq 0$.

1.9 Setting. Let $\mathbb{F}$ be a field, Z be a finite graph, Y be a subgraph of Z, and G be a finite group. Suppose that G acts freely on Z and acts faithfully on $\mathbb{F}$. Let $\mathbb{F}G$ be the skew group ring. In $\mathbb{F}G\otimes\mathbb{Z}Z$, let $\mathrm{D}(Y) := \bigoplus_{g\in G}(\mathbb{F}g\otimes g\mathbb{Z}Y)$, and, for each left ideal I of $\mathbb{F}G$, let $\mathrm{D}(Y)|_I := \mathrm{D}(Y)\cap(I\otimes\mathbb{Z}Z)$. Here, $\mathrm{D}(Y) \preccurlyeq \mathbb{F}G\otimes\mathbb{Z}Z$ by Lemma 1.7, $I\otimes\mathbb{Z}Z \preccurlyeq \mathbb{F}G\otimes\mathbb{Z}Z$ by Lemma 1.5, and, hence, $\mathrm{D}(Y)|_I \preccurlyeq \mathbb{F}G\otimes\mathbb{Z}Z$.

1.10 Lemma. *In* Setting 1.9*, for each left ideal I of $\mathbb{F}G$, $\overline{\mathrm{r}}(\mathrm{D}(Y)|_I) \in |G|\mathbb{N}$.*

Proof. Set $D := \mathrm{D}(Y)|_I$ and $C := \mathrm{supercore}(D)$. Let G act on $\mathbb{F}G\otimes\mathbb{Z}Z$ with the diagonal action. Let $h \in G$. Then h permutes the sheaves in $\mathbb{F}G\otimes\mathbb{Z}Z$, and stabilizes $\mathrm{D}(Y)$, $I\otimes\mathbb{Z}Z$, and their intersection, D. Thus, $hC \subseteq D$ and $\delta(hC) = \delta(C)$. Hence, hC is a δ-maximizer in D. Since C is the $\subseteq$-smallest δ-maximizer in D, we have $C \subseteq hC$. Now, $\dim(hC) = \dim(C) < \infty$, and, hence, $hC = C$. Thus,

$$\bigoplus_{z\in Z}(C_z\otimes z) = C = hC = \bigoplus_{z\in Z}(h(C_z)\otimes hz) = \bigoplus_{z\in Z}(h(C_{h^{-1}z})\otimes z).$$

Now, for each $z \in Z$, $C_{h^{-1}z} = h^{-1}C_z$, and, hence, $\dim(C_{h^{-1}z}) = \dim(C_z)$. Since G acts freely on Z, it follows that $\delta(C)$ is a multiple of $|G|$, that is, $\overline{\mathrm{r}}(D) \in |G|\mathbb{N}$. □

1.11 Lemma. *In* Setting 1.9*, there exists a left ideal I of $\mathbb{F}G$ such that $\overline{\mathrm{r}}(\mathrm{D}(Y)|_I) = 0$ and $\overline{\mathrm{r}}(Y) \geqslant \dim(\mathbb{F}G/I)$.*

Proof. Let $\mathcal{I}$ denote the set of left ideals I of $\mathbb{F}G$ with $\overline{\mathrm{r}}(\mathrm{D}(Y)|_I) \leqslant |G|(\dim(I)-|G|+\overline{\mathrm{r}}(Y))$.

By Lemma 1.7, $\overline{\mathrm{r}}(\bigoplus_{g\in G}(\mathbb{F}g\otimes g\mathbb{Z}Y)) = \sum_{g\in G}\overline{\mathrm{r}}(gY)$, that is, $\overline{\mathrm{r}}(\mathrm{D}(Y)) = |G|\overline{\mathrm{r}}(Y)$. It follows that $\mathbb{F}G \in \mathcal{I}$. Hence, $\mathcal{I} \neq \emptyset$.

Let I be a $\subseteq$-minimal element of $\mathcal{I}$. Thus, $0 \leqslant \overline{\mathrm{r}}(\mathrm{D}(Y)|_I) \leqslant |G|(\dim(I)-|G|+\overline{\mathrm{r}}(Y))$. In particular, $\overline{\mathrm{r}}(Y) \geqslant |G|-\dim(I) = \dim(\mathbb{F}G/I)$. Set $D := \mathrm{D}(Y)|_I$. If $\overline{\mathrm{r}}(D) = 0$, then we have the desired conclusion. Thus, it suffices to suppose that $\overline{\mathrm{r}}(D) \neq 0$ and obtain a contradiction. Set $C := \mathrm{supercore}(D) \neq \{0\}$. Then there exists some $z \in Z$ and some nonzero $s \in C_z \subseteq D_z \subseteq I \subseteq \mathbb{F}G \subseteq \mathrm{End}(\mathbb{F})$. There exists $\lambda \in \mathbb{F}$ such that $s[\lambda] \neq 0$. Consider the left $\mathbb{F}G$-linear map $\rho_\lambda\colon I \to \mathbb{F}$, $r \mapsto r[\lambda]$. Since $s[\lambda] \neq 0$, ρ_λ is surjective.

Let $I' := \mathrm{Ker}(\rho_\lambda)$. Then I' is a left ideal of $\mathbb{F}G$, $I' \subseteq I$, $\dim(I') = \dim(I)-1$, and $s \notin I'$. Set $D' := \mathrm{D}(Y)|_{I'}$ and $C' := \mathrm{supercore}(D')$. Since $s \notin I' \supseteq D'_z \supseteq C'_z$, we see that $C'_z \neq C_z$. Also, $D' \preccurlyeq D$, and, by Lemma 1.4(i), $C' \preccurlyeq C$. Since $C' \neq C$ and C is the $\subseteq$-smallest δ-maximizer in D, $\delta(C') < \overline{\mathrm{r}}(D)$. Hence, $\overline{\mathrm{r}}(D') < \overline{\mathrm{r}}(D)$. By Lemma 1.10, $\overline{\mathrm{r}}(D') \leqslant \overline{\mathrm{r}}(D)-|G|$. Hence,

$$\overline{\mathrm{r}}(D') \leqslant \overline{\mathrm{r}}(D)-|G| \leqslant |G|(\dim(I)-|G|+\overline{\mathrm{r}}(Y)-1) = |G|(\dim(I')-|G|+\overline{\mathrm{r}}(Y)).$$

It follows that $I' \in \mathcal{I}$. This contradicts the $\subseteq$-minimality of I, as desired. □

1.12 Friedman's theorem. *Let G be a finite group and Z be a finite G-free G-graph. If X and Y are subgraphs of Z, then $\sum_{g\in G}\overline{\mathrm{r}}(X\cap gY) \leqslant \overline{\mathrm{r}}(X)\overline{\mathrm{r}}(Y)$.*

Proof. We may choose $\mathbb{F}$ to be a field with a faithful G-action, for example, $\mathbb{Q}(t_g : g \in G)$ with $h[t_g] = t_{hg}$ for all $h, g \in G$. We may now assume that we are in Setting 1.9. By (Lemma) 1.11, there exists a left ideal I of $\mathbb{F}G$ such that $\overline{\mathrm{r}}(\mathrm{D}(Y)|_I) = 0$ and $\dim(\mathbb{F}G/I) \leqslant \overline{\mathrm{r}}\,(Y)$; hence,

$$(1)\quad \overline{\mathrm{r}}(\mathrm{D}(Y)|_I)+\overline{\mathrm{r}}\Big((\mathbb{F}G/I)\otimes \mathbb{Z}X\Big)=0+\overline{\mathrm{r}}\Big((\mathbb{F}G/I)\otimes \mathbb{Z}X\Big) \overset{1.7}{=} \dim(\mathbb{F}G/I)\overline{\mathrm{r}}(X) \leqslant \overline{\mathrm{r}}(Y)\overline{\mathrm{r}}(X).$$

By 1.5, we now have three sheaves in $\mathbb{F}G\otimes \mathbb{Z}Z$,

$$\mathrm{D}(Y) = \bigoplus_{g\in G}(\mathbb{F}g\otimes g\mathbb{Z}Y), \qquad \mathbb{F}G\otimes \mathbb{Z}X, \qquad I\otimes \mathbb{Z}Z.$$

We then have their pairwise intersections,

$$D \coloneqq \mathrm{D}(Y)\cap (\mathbb{F}G\otimes \mathbb{Z}X), \quad \mathrm{D}(Y)|_I = \mathrm{D}(Y)\cap (I\otimes \mathbb{Z}Z), \quad I\otimes \mathbb{Z}X = (\mathbb{F}G\otimes \mathbb{Z}X)\cap (I\otimes \mathbb{Z}Z),$$

and the intersection of all three, $D' \coloneqq \mathrm{D}(Y)\cap (\mathbb{F}G\otimes \mathbb{Z}X)\cap (I\otimes \mathbb{Z}Z)$.

Notice that $D = \bigoplus_{g\in G}(\mathbb{F}g\otimes \mathbb{Z}[gY\cap X])$, $D' = D\cap (I\otimes \mathbb{Z}X)$, and $D' \preccurlyeq \mathrm{D}(Y)|_I$. By 1.6, $D'' \coloneqq \Big(D+(I\otimes \mathbb{Z}X)\Big)/(I\otimes \mathbb{Z}X) \preccurlyeq (\mathbb{F}G/I)\otimes \mathbb{Z}X$. Now

$$\sum_{g\in G}\overline{\mathrm{r}}(gY\cap X) \overset{1.7}{=} \overline{\mathrm{r}}(D) \overset{1.6}{\leqslant} \overline{\mathrm{r}}(D')+\overline{\mathrm{r}}(D'') \leqslant \overline{\mathrm{r}}(\mathrm{D}(Y)|_I)+\overline{\mathrm{r}}\Big((\mathbb{F}G/I)\otimes \mathbb{Z}X\Big) \overset{(1)}{\leqslant} \overline{\mathrm{r}}(Y)\overline{\mathrm{r}}(X). \qquad \square$$

2 Free groups and graphs

We now quickly review the standard results that Friedman applies in deducing the strengthened Hanna Neumann conjecture. Most of the following can be found in [2, Section I.8], for example.

2.1 Definitions. For a free group F, we define $\overline{\mathrm{r}}(F) \coloneqq \max\{\mathrm{rank}(F)-1, 0\} \in \mathbb{N}\cup\{\infty\}$. Thus, if F is cyclic, then $\overline{\mathrm{r}}(F) = 0$, and otherwise $\overline{\mathrm{r}}(F) > 0$. Similarly, if F is finitely generated, then $\overline{\mathrm{r}}(F) < \infty$, and otherwise $\overline{\mathrm{r}}(F) = \infty$.

2.2 Definitions. Let X and Y be graphs, and let $\alpha\colon X \to Y$ be a graph map. For $x \in \mathrm{V}X$, we write $\mathrm{link}(x, X) \coloneqq \{(e, \nu) \in \mathrm{E}X \times \{\iota, \tau\} : \nu(e) = x\}$, and we see that α induces a map $\mathrm{link}(x, X) \to \mathrm{link}(\alpha(x), Y)$. If the latter map is bijective for each $x \in \mathrm{V}X$, we say that α is *locally bijective*. We define *locally injective* similarly.

2.3 Definitions. Let X be a connected graph and x be a vertex of X that is to serve as a basepoint of X.

Let πX denote the set of all reduced paths in X. Let t denote the element of πX that is the empty path at x. The set πX has a partial binary operation of concatenation-where-defined followed by reduction. For any vertices v, w of X, we let $\pi X[v, w]$ denote the set of elements of πX with initial vertex v and terminal vertex w; we let $\pi X[v, -]$ denote the set of elements of πX with initial vertex v. If X is a tree, then $\pi X[v, w]$ consists of a single element, denoted $X[v, w]$; here, $\pi X[v, -]$ is in bijective correspondence with $\mathrm{V}X$.

The set $F \coloneqq \pi X[x,x]$ inherits a binary operation from πX that makes F into a group, called the *fundamental group of* X *at* x. Let us choose a maximal subtree X_0 of X. It is not difficult to show that the set $\{X_0[x,\iota e]\cdot e\cdot X_0[\tau e,x] : e \in X - X_0\}$ freely generates F. Thus, X is a tree if and only if F is trivial. Also, F is a free group, $\operatorname{rank}(F) = |X - X_0|$, and $|\mathrm{E}X_0| = |\mathrm{V}X| - 1$. If $|\mathrm{V}X| < \infty$, then $\operatorname{rank}(F) - 1 = |\mathrm{E}X| - |\mathrm{V}X|$.

If $|X| < \infty$, we have $\overline{\mathrm{r}}(X)$ as in Notation 1.1, and we shall show that $\overline{\mathrm{r}}(F) = \overline{\mathrm{r}}(X)$. Consider first the case where X is a tree. We have seen that $\overline{\mathrm{r}}(F) = 0$, and, by Lemma 1.2(v), $\overline{\mathrm{r}}(X) = \delta(X) + 1 = 0$. Suppose now that X is not a tree. We have seen that F is nontrivial and $\overline{\mathrm{r}}(F) = |\mathrm{E}X| - |\mathrm{V}X| = \delta(X)$, and, by Lemma 1.2(v), $\overline{\mathrm{r}}(X) = \delta(X)$, as claimed.

The *universal cover of* X *at* x is the graph whose vertex set is $\pi X[x,-]$ with distinguished element t, the empty path at x, and whose edges are given by saying that each element of $(\pi X[x,-]) - \{t\}$ is T-adjacent to the element of $\pi X[x,-]$ obtained by deleting the last edge and the last vertex. Let T denote the universal cover of X at x.

The partial binary operation in πX gives a left action of F on T. It can be seen that F acts freely on T.

There is a natural graph map $\alpha\colon T \to X$ that on vertices is given by assigning to each element of $\pi X[x,-]$ its terminal vertex. Then $\alpha(t) = x$, and α is locally bijective. There is then an induced graph map $F\backslash T \to X$, $Ft \mapsto \alpha(t)$, and it is an isomorphism.

2.4 Definitions. Let B be a set. Let X be the connected graph with one vertex and with edge set B. On applying Definitions 2.3, we obtain a free group F acting freely on a tree T. Here, F is the free group on B, $\mathrm{V}T = F$, and $\mathrm{E}T = F \times B$, with $\iota(f,b) = f$ and $\tau(f,b) = fb$. We call T the *Cayley tree of* F with respect to B.

2.5 Definitions. Let F be a group, T an F-free F-tree, and t_0 a vertex of T that will serve as a basepoint.

Then $X \coloneqq F\backslash T$ is a connected graph with basepoint $x_0 \coloneqq Ft_0$.

There is a natural graph map $T \to X$, $t \mapsto Ft$, and it is locally injective, since F acts freely. To see that it is locally bijective, consider any $Ft \in \mathrm{V}X$ and $(Fe,\nu) \in \operatorname{link}(X,Ft)$. Then there exists a unique $f \in F$ such that $f\cdot\nu e = t$, and then $(f\cdot e,\nu) \in \operatorname{link}(T,t)$.

In summary, Definitions 2.3 associate *to* a basepointed connected graph, a (free) group acting freely on a tree which has a basepoint; and, in the reverse direction, Definitions 2.5 associate a basepointed connected graph *to* a group acting freely on a tree with a basepoint. It can be shown that these two operations are mutually inverse modulo natural identifications. This is an important special case of Bass-Serre theory that was known to Reidemeister and Schreier. We shall need the structure part of the result.

2.6 Theorem. *Let F be a group, T an F-free F-tree, and t_0 a vertex of T. Then $F \simeq \pi(F\backslash T)[Ft_0,Ft_0]$, which is a free group. If $|F\backslash T| < \infty$, then $\overline{\mathrm{r}}(F) = \overline{\mathrm{r}}(F\backslash T)$.*

Proof. Let $X := F\backslash T$ and $x_0 := Ft_0$. We have seen that the natural map $\alpha\colon T \to X$, $t \mapsto Ft$, is locally bijective. It follows that α maps πT to πX. For each $f \in F$, $T[t_0, f{\cdot}t_0]$ is then carried to an element of $\pi X[x_0, x_0]$. Conversely, each element of $\pi X[x_0, x_0]$ lifts to a unique element of $\pi T[t_0, -]$, and then the terminal vertex of this lifted path can be expressed as $f{\cdot}t_0$ for a unique $f \in F$. We then have mutually inverse maps between F and $\pi X[x_0, x_0]$.

The remaining results follow from Definitions 2.3. □

2.7 Reidemeister's theorem. *A group is free if and only if it acts freely on some tree.*

Proof. We saw in Definitions 2.4 that if a group is free, then it acts freely on some tree. Conversely, by Theorem 2.6, if a group acts freely on a tree, then the group is free. □

2.8 The Nielsen-Schreier theorem. *Subgroups of free groups are free.*

Proof. This is clear from Theorem 2.7. □

2.9 The Schreier index theorem. *If F is a free group and H is a finite-index subgroup of F, then*

$$\overline{\mathrm{r}}(H) = (F{:}H) \times \overline{\mathrm{r}}(F). \tag{2}$$

Proof. Let T be the Cayley tree of F with respect to a free generating set B of F. Since $\mathrm{V}T = F$, we see that $|H\backslash \mathrm{V}T| = (F{:}H) < \infty$. By Theorem 2.6, $H \simeq \pi(H\backslash T)[H1, H1]$; hence, by Definitions 2.3, $\mathrm{rank}(H){-}1 = |H\backslash \mathrm{E}T|{-}(F{:}H)$. Since $\mathrm{E}T = F \times B$, we see that $|H\backslash \mathrm{E}T| = (F{:}H) \times \mathrm{rank}(F)$. Thus,

$$\mathrm{rank}(H){-}1 = (F{:}H) \times (\mathrm{rank}(F){-}1). \tag{3}$$

If both sides of (3) are negative, then $\mathrm{rank}(H) = \mathrm{rank}(F) = 0$, and then both sides of (2) are zero. Thus, we may assume that both sides of (3) are non-negative, and here (3) coincides with (2). □

The following is due in steps to M. Hall, M. Tretkoff, L. Babai, and W. Imrich; see [4].

2.10 The geometric Marshall Hall theorem. *Let F be a group, H be a subgroup of F, T be an F-free F-tree, and T_H be an H-subtree of T. If $F\backslash T$ and $H\backslash T_H$ are finite, then there exists a finite-index subgroup L of F containing H such that the induced map $H\backslash T_H \to L\backslash T$ is injective.*

Proof. Notice that $H\backslash T_H \to H\backslash T$ is injective and that $H\backslash T \to F\backslash T$ is locally bijective. Hence, $H\backslash T_H \to F\backslash T$ is locally injective.

To simplify the notation for the next part of the argument, let us write $Z_H := H\backslash T_H$ and $Z_F := F\backslash T$. These are finite graphs by hypothesis, and we have a locally injective graph map $\alpha\colon Z_H \to Z_F$. Let $n := \max\{|\alpha^{-1}\{v\}| : v \in \mathrm{V}Z_F\}$. We shall now add 'missing' vertices and edges to the fibres of α to obtain a finite graph X having Z_H as a subgraph, together with a locally bijective, n to 1, graph map $X \to Z_F$ *extending* α, i.e. the composite $Z_H \hookrightarrow X \to Z_F$ is α.

We first construct a graph map $\beta\colon Z'_H \to Z_F$ extending α by taking Z'_H to be the graph obtained from Z_H by adding, for each $v \in \mathrm{V}Z_F$, $n - |\alpha^{-1}\{v\}|$ isolated vertices, which β then maps to v. Clearly, β is locally injective, extends α, and, for all $v \in \mathrm{V}Z_F$, $|\beta^{-1}\{v\}| = n$.

Consider any $e \in \mathrm{E}Z_F$. Since β is locally injective, the map $\iota\colon \beta^{-1}\{e\} \to \mathrm{V}Z_H$ is injective, and similarly for τ. Thus, $|\iota(\beta^{-1}\{e\})| = |\beta^{-1}\{e\}| = |\tau(\beta^{-1}\{e\})|$. Hence, we may choose a bijective map $\sigma\colon \beta^{-1}\{\iota e\}-\iota(\beta^{-1}\{e\}) \to \beta^{-1}\{\tau e\}-\tau(\beta^{-1}\{e\})$. For each $v \in \beta^{-1}\{\iota e\}-\iota(\beta^{-1}\{e\})$, we add to Z'_H an edge e_v with $\iota e_v = v$ and $\tau e_v = \sigma(v)$, and we map e_v to e. After having done this for each $e \in \mathrm{E}Z_F$, we obtain a graph X containing Z_H and a locally bijective graph map $X \to Z_F$ extending α.

Since Z_H is connected, it lies in a component of X, and we may replace X with this component and still have a locally bijective graph map $\gamma\colon X \to Z_F$ extending α. Returning to the original notation, we have a finite connected graph X containing $H\backslash T_H$ as a subgraph, and a locally bijective graph map $\gamma\colon X \to F\backslash T$ extending $H\backslash T_H \to F\backslash T$.

Choose a vertex v in $H\backslash T_H$. Then v is a vertex of X. By Theorem 2.6, we may make the identifications $H = \pi(H\backslash T_H)[v, v]$ and $F = \pi(F\backslash T)[\gamma(v), \gamma(v)]$. Let $L := \pi X[v, v]$. Since $H\backslash T_H \subseteq X$, $H \leqslant L$. Since $X \to F\backslash T$ is locally bijective, we may identify $\pi X[v, -]$ with $\pi(F\backslash T)[\gamma(v), -]$, and we may identify the latter set with $\mathrm{V}T$. Thus, we may identify the universal cover of X with T, and we may identify L with a subgroup of F. The latter identification respects the copies of H in L and F; thus $H \leqslant L \leqslant F$. The locally bijective graph map $T \to X$ induces a graph map $L\backslash T \to X$ which is bijective. Hence, $(F : L) < \infty$. $\square$

3 The strengthened Hanna Neumann conjecture

For any group F, if $H \leqslant F$ and $f \in F$, then we write ${}^fH := fHf^{-1}$ and $H^f := f^{-1}Hf$. We shall show that if F is a free group and H and K are finitely generated subgroups of F, then $\sum\limits_{HfK \in H\backslash F/K} \bar{\mathrm{r}}(H \cap {}^fK) \leqslant \bar{\mathrm{r}}(H)\bar{\mathrm{r}}(K)$. Notice that $H \cap {}^fK$ does not change if f is multiplied on the right by an element of K. Also $(H \cap {}^fK)^f = H^f \cap K$ and this subgroup does not change if f is multiplied on the left by an element of H. It follows that the conjugacy class of $H \cap {}^fK$ does not change if f is multiplied on the left by an element of H and on the right by an element of K. In particular, $\bar{\mathrm{r}}(H \cap {}^fK)$ is independent of which representative is chosen for the double coset HfK.

3.1 Setting. Let F be a finitely generated, non-cyclic free group, let H and K be finitely generated, non-cyclic subgroups of F, and let

$$\mathrm{SHN}(F,H,K) \coloneqq \sum_{HfK\in H\backslash F/K} \tfrac{\overline{\mathrm{r}}(H\cap {}^fK)}{\overline{\mathrm{r}}(H)\overline{\mathrm{r}}(K)} \in [0,\infty].$$

Let T denote the Cayley tree of F with respect to some free generating set of F. For each subgroup L of F, view L as a subset of $\mathrm{V}T = F$, and let T_L denote the $\subseteq$-smallest subtree of T containing L. Notice that T_L is an L-subtree of T. If there exists some finite generating set $\{f_1,\dots,f_n\}$ for L, then the $\subseteq$-smallest subtree S of T_L containing $\{1,f_1,\dots,f_n\}$ is finite and it can be shown that $LS = T_L$. It follows that $L\backslash T_L$ is finite.

We are now in a position to translate Theorem 1.12 into the desired form.

3.2 Theorem. *In* Setting 3.1*, if there exists a normal, finite-index subgroup N of F such that $N \supseteq H\cup K$ and each of the maps $H\backslash T_H \to N\backslash T$ and $K\backslash T_K \to N\backslash T$ is injective, then* $\mathrm{SHN}(F,H,K) \leqslant 1$.

Proof. Let $Z \coloneqq N\backslash T$, $X \coloneqq H\backslash T_H$, and $Y \coloneqq K\backslash T_K$. By hypothesis, we may view X and Y as subgraphs of Z. Let $G \coloneqq F/N$. Notice that F acts on Z by $f{\cdot}Nt = Nft$, and G then acts freely on Z.

Step 1: $\sum\limits_{HnK\in H\backslash N/K} \overline{\mathrm{r}}(H\cap {}^nK) \leqslant \overline{\mathrm{r}}(X\cap Y)$. Let S be a subset of N such that the map $S\to H\backslash N/K$, $s\mapsto HsN$, is bijective. If s_1, $s_2\in S$, t_1, $t_2\in T$, then we have the following chain of equivalences.

$$\begin{aligned}
&s_1 = s_2 \text{ and } (H\cap {}^{s_1}K)t_1 = (H\cap {}^{s_2}K)t_2\\
&\qquad\Leftrightarrow s_1=s_2 \text{ and } \exists(h,k)\in H\times K \text{ such that } ht_1 = t_2 \text{ and } h = {}^{s_1}k, \text{ i.e., } hs_1 = s_1k\\
&\qquad\Leftrightarrow \exists(h,k)\in H\times K \text{ such that } ht_1=t_2 \text{ and } hs_1 = s_2k, \text{ i.e, } ks_1^{-1}t_1 = s_2^{-1}ht_1 = s_2^{-1}t_2\\
&\qquad\Leftrightarrow Ht_1 = Ht_2 \text{ and } Ks_1^{-1}t_1 = Ks_2^{-1}t_2.
\end{aligned}$$

Thus, we have a well-defined, injective graph map

$$\begin{aligned}
\bigvee_{s\in S}(H\cap {}^sK)\backslash(T_H\cap sT_K) &\to (H\backslash T_H)\times_{N\backslash T}(K\backslash T_K),\\
(H\cap {}^sK)t \qquad &\mapsto \qquad (Ht, Ks^{-1}t) \quad \text{for } s\in S,\ t\in T_H\cap sT_K.
\end{aligned}$$

Now, $(H\backslash T_H)\times_{N\backslash T}(K\backslash T_K) = X\times_Z Y = X\cap Y$. In particular, this codomain is finite. Hence, the domain is finite. The operator $\overline{\mathrm{r}}(-)$ on finite graphs, from Notation 1.1, behaves well with respect to inclusions and disjoint unions. Thus,

$$\overline{\mathrm{r}}(X\cap Y) \geqslant \overline{\mathrm{r}}\Big(\bigvee_{s\in S}(H\cap {}^sK)\backslash(T_H\cap sT_K)\Big) = \sum_{s\in S}\overline{\mathrm{r}}\big((H^s\cap K)\backslash(T_H\cap sT_K)\big). \tag{4}$$

For $s\in S$, we shall now prove that

$$\overline{\mathrm{r}}((H\cap {}^sK)\backslash(T_H\cap sT_K)) = \overline{\mathrm{r}}(H\cap {}^sK). \tag{5}$$

Notice that $H\cap {}^sK$ acts freely on both T_H and sT_K. If $T_H\cap sT_K = \emptyset$, then $H\cap {}^sK$ stabilizes the unique path from T_H to sT_K, and, hence, stabilizes an edge of T, and, hence, is trivial;

here, both sides of (5) are 0. If $T_H \cap sT_K \neq \emptyset$, then $T_H \cap sT_K$ is a tree on which $H \cap {}^sK$ acts freely, and then (5) holds by Theorem 2.6. This proves (5).

By (4) and (5), $\sum\limits_{s\in S} \overline{\mathrm{r}}(H \cap {}^sK) \leqslant \overline{\mathrm{r}}(X \cap Y)$. This completes Step 1.

Step 2. Consider any $f \in F$, and let $g \coloneqq fN \in G$. For each $y \in Y$, there exists some $t \in T_K$ such that $y = Nt$, and then we have $gy = (Nf)(Nt) = Nft$. It is not difficult to see that we have a graph isomorphism $K\backslash T \to {}^fK\backslash T$, $Kt \mapsto ({}^fK)ft$. It follows that, in $N\backslash T$, the image of ${}^fK\backslash fT_K$ is $\{Nft : t \in T_H\}$, that is, gY. On replacing Y, T_K, and K in Step 1 with gY, fT_K, and ${}^fK$, respectively, we find that $\sum\limits_{Hn\,{}^fK\in H\backslash N/{}^fK} \overline{\mathrm{r}}(H \cap {}^n({}^fK)) \leqslant \overline{\mathrm{r}}(X \cap gY)$. It is not difficult to see that there is a bijection $H\backslash N/{}^fK \to H\backslash Nf/K$, $Hn\,{}^fK \mapsto HnfK$. It follows that $\sum\limits_{HnfK\in H\backslash Nf/K} \overline{\mathrm{r}}(H \cap {}^{(nf)}K) \leqslant \overline{\mathrm{r}}(X \cap gY)$.

Step 3. On summing the inequalities obtained in Step 2, one for each $g = Nf \in G$, we find that $\sum\limits_{HfK\in H\backslash F/K} \overline{\mathrm{r}}(H \cap {}^fK) \leqslant \sum\limits_{g\in G} \overline{\mathrm{r}}(X \cap gY)$. By Theorem 1.12, $\sum\limits_{g\in G} \overline{\mathrm{r}}(X \cap gY) \leqslant \overline{\mathrm{r}}(X)\overline{\mathrm{r}}(Y)$. Here, $X = H\backslash T_H$ and $Y = K\backslash T_K$, and, by Theorem 2.6, $\overline{\mathrm{r}}(X) = \overline{\mathrm{r}}(H)$ and $\overline{\mathrm{r}}(Y) = \overline{\mathrm{r}}(K)$. We now see that $\mathrm{SHN}(F, H, K) \leqslant 1$. □

The next result shows that $\mathrm{SHN}(F, H, K)$ is an invariant of the commensurability class of K in F; by symmetry, the same holds for H.

3.3 Lemma. *In* Setting 3.1*, suppose that L is a normal, finite-index subgroup of K. Then* $\mathrm{SHN}(F, H, L) = \mathrm{SHN}(F, H, K)$.

Proof. Consider any $f \in F$. It suffices to show that

$$\sum_{HfkL\in H\backslash HfK/L} \tfrac{\overline{\mathrm{r}}(H\cap {}^{fk}L)}{\overline{\mathrm{r}}(L)} = \tfrac{\overline{\mathrm{r}}(H\cap {}^fK)}{\overline{\mathrm{r}}(K)}.$$

For each $k \in K$, ${}^kL = L$. Thus, it suffices to show that

$$|H\backslash HfK/L| \times \tfrac{\overline{\mathrm{r}}(H^f\cap L)}{\overline{\mathrm{r}}(L)} = \tfrac{\overline{\mathrm{r}}(H^f\cap K)}{\overline{\mathrm{r}}(K)}.$$

Since $(f^{-1})HfkL = H^fkL$, we have a bijective map $H\backslash HfK/L \xrightarrow{\sim} (H^f)\backslash(H^f)K/L$, $HfkL \mapsto H^fkL$. To simplify notation, let us write H in place of H^f. Then it suffices to show that $|H\backslash HK/L| \times \frac{\overline{\mathrm{r}}(H\cap L)}{\overline{\mathrm{r}}(L)} = \frac{\overline{\mathrm{r}}(H\cap K)}{\overline{\mathrm{r}}(K)}$.

By Theorem 2.9, $\overline{\mathrm{r}}(H \cap L) = \overline{\mathrm{r}}(H \cap K) \times (H \cap K : H \cap L)$ and $\overline{\mathrm{r}}(L) = \overline{\mathrm{r}}(K)(K{:}L)$; it then suffices to show that $|H\backslash HK/L| \times (H \cap K : H \cap L) = (K{:}L)$.

The right K-set $H\backslash HK$ is generated by the element $H1$, which has right K-stabilizer $H \cap K$. Hence, $H\backslash HK \simeq (H \cap K)\backslash K$ as right K-sets. Thus, $H\backslash HK/L \simeq (H \cap K)\backslash K/L$ as sets. Hence, $|H\backslash HK/L| = |(H \cap K)\backslash K/L| = |(H \cap K)L\backslash K| = (K : (H \cap K)L)$.

As left $H \cap K$-sets, $((H \cap K)L)/L \simeq (H \cap K)/(H \cap K \cap L) = (H \cap K)/(H \cap L)$. Thus, $(H \cap K : H \cap L) = ((H \cap K)L : L)$.

On multiplying the results of the previous two paragraphs, we find that
$$|H\backslash HK/L| \times (H\cap K : H\cap L) = (K : (H\cap K)L) \times ((H\cap K)L : L) = (K : L),$$
as desired. □

We can now prove the strengthened Hanna Neumann conjecture.

3.4 Theorem. *Let F be a free group, and H, K be finitely generated subgroups of F. Then* $\sum\limits_{HfK\in H\backslash F/K} \overline{\mathrm{r}}(H\cap {}^fK) \leqslant \overline{\mathrm{r}}(H)\overline{\mathrm{r}}(K)$.

Proof. The desired inequality holds if H or K is cyclic; thus, we may assume that H and K are non-cyclic, and, in particular, F is non-cyclic. Choose a free generating set for F and a free product decomposition $F = A{*}B$ such that A is finitely generated and contains generating sets of H and K. The F-graph with vertex set $F/A \vee F/B$ and edge set F, with an edge f joining fA to fB, is a tree, called *the Bass-Serre tree for the free product decomposition*. Consider any $f \in F{-}A$. Then $A \neq fA$ and $H\cap {}^fK$ stabilizes the vertices A and fA, and, hence, stabilizes the path from A to fA. This path contains an edge, and the edges have trivial stabilizers. Thus $H\cap {}^fK = \{1\}$. Hence, $\sum\limits_{HfK\in H\backslash F/K} \overline{\mathrm{r}}(H\cap {}^fK) = \sum\limits_{HaK\in H\backslash A/K} \overline{\mathrm{r}}(H\cap {}^aK)$. Thus, we may replace F with A and assume that F is finitely generated. Now, we may assume that we are in Setting 3.1.

By Theorem 2.10, there exists a finite-index subgroup H_0 of F containing H such that the map $H\backslash T_H \to H_0\backslash T$ is injective. Similarly, there exists a finite-index subgroup K_0 of F containing K such that the map $K\backslash T_K \to K_0\backslash T$ is injective. We have left F-actions on F/H_0 and on F/K_0, and hence an F-action on the finite set $F/H_0 \vee F/K_0$. Let N denote the kernel of this action. Then N is a normal, finite-index subgroup of F. The F-stabilizer of the element $1H_0$ is H_0, and, hence, $N \leqslant H_0$. Similarly, $N \leqslant K_0$.

We shall now apply Theorem 3.2 to $\mathrm{SHN}(F, H\cap N, K\cap N)$. Notice that $H\cap N$ has finite index in H, and, hence, by Theorem 2.9, $H\cap N$ is finitely generated. We claim that the map $(H\cap N)\backslash T_{H\cap N} \to N\backslash T$ is injective. Consider $t_1, t_2 \in T_{H\cap N}$ such that $Nt_1 = Nt_2$. Since $N \leqslant H_0$, $H_0t_1 = H_0t_2$. Since $T_{H\cap N} \subseteq T_H$ and the map $H\backslash T_H \to H_0\backslash T$ is injective, we see that $Ht_1 = Ht_2$. Since H_0 acts freely on T, there is a unique $f \in H_0$ such that $ft_1 = t_2$. We have now seen that $f \in N$ and $f \in H$. Thus, $(H\cap N)t_1 = (H\cap N)t_2$, as desired. Similarly, $K\cap N$ is finitely generated and the map $(K\cap N)\backslash T_{K\cap N} \to N\backslash T$ is injective. By Theorem 3.2, $\mathrm{SHN}(F, H\cap N, K\cap N) \leqslant 1$.

By Lemma 3.3, $\mathrm{SHN}(F, H, K\cap N) = \mathrm{SHN}(F, H\cap N, K\cap N) \leqslant 1$. By the analogue of Lemma 3.3, $\mathrm{SHN}(F, H, K) = \mathrm{SHN}(F, H\cap N, K) \leqslant 1$, as desired. □

Historical note. On May 1, 2011, Joel Friedman posted on the arXiv a proof of the strengthened Hanna Neumann conjecture (SHNC) quite similar to the version presented here; see [3]. Six days later, Igor Mineyev posted on his web page an independent proof of the SHNC; see [5]. (Both [3] and [5] contain other results.) Ten days after that, I emailed Mineyev a one-page proof of the SHNC and encouraged him to add it as an appendix to [5] so that group-theorists would have a proof they could be comfortable with; see [1].

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