# On hyperbolic once-punctured-torus bundles II: fractal tessellations of the plane.

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*To our wives, Ardyth and Elena*

**Abstract**

We describe fractal tessellations of the complex plane that arise naturally from Cannon-Thurston maps associated to complete, hyperbolic, once-punctured-torus bundles. We determine the symmetry groups of these tessellations.

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## 1 Outline

This section gives an overview of the article. More details, including definitions, will be given in the appropriate sections. To help make the large quantity of notation slightly more manageable, we have added a symbol index at the end of the article.

Let $\langle A, B, C\rangle$ denote the group with presentations

$$\begin{aligned}\langle A, B, C\rangle &= \langle A, B, C, D \mid A^2 = B^2 = C^2 = ABCD = 1\rangle \\ &= \langle A, B, C \mid A^2 = B^2 = C^2 = 1\rangle.\end{aligned}$$

We call the conjugates of $D$ $(= CBA)$ *distinguished* elements of $\langle A, B, C\rangle$.

Let $\mathrm{Aut}\langle A, B, C\rangle$ denote the group of automorphisms of $\langle A, B, C\rangle$. We specify an element $F$ of $\mathrm{Aut}\langle A, B, C\rangle$ by the triple $(F(A), F(B), F(C))$. We will be using the three automorphisms

$$R := (A, BCB, B), \quad L := (B, BAB, C), \quad M := (C, B, A),$$

and this article is mainly about $R$ and $L$. It is known that $\mathrm{Aut}\langle A, B, C\rangle$ stabilizes the set consisting of the distinguished elements and their inverses; see, for example, [5, Notation 2.3]. Let $\mathrm{Aut}^+\langle A, B, C\rangle$ denote the subgroup of $\mathrm{Aut}\langle A, B, C\rangle$ which stabilizes the set of distinguished elements of $\langle A, B, C\rangle$. Then $\mathrm{Aut}\langle A, B, C\rangle = \mathrm{Aut}^+\langle A, B, C\rangle \rtimes \langle M\rangle$.

Let $\hat{\mathbb{C}}$ denote the Riemann sphere $\mathbb{C} \cup \{\infty\}$, and, for any subfield $K$ of $\mathbb{C}$, let $\hat{K}$ denote the subset $K \cup \{\infty\}$ of $\hat{\mathbb{C}}$. Let $\mathrm{Aut}^+(\hat{\mathbb{C}})$ denote the group of all conformal transformations of $\hat{\mathbb{C}}$.

Let $\mathfrak{R}$ denote the set of all discrete, faithful representations

$$\rho \colon \langle A, B, C\rangle \to \mathrm{Aut}^+(\hat{\mathbb{C}})$$

such that $\rho(D)$ is parabolic. Recall that the limit set of $\rho$, denoted $\Lambda(\rho)$, is the closure in $\hat{\mathbb{C}}$ of the $\rho(\langle A, B, C\rangle)$-orbit of the point fixed by $\rho(D)$. Let $\mathfrak{R}_{\hat{\mathbb{C}}}$ and $\mathfrak{R}_{\hat{\mathbb{R}}}$ denote the sets of elements of $\mathfrak{R}$ with limit sets $\hat{\mathbb{C}}$ and $\hat{\mathbb{R}}$, respectively. In a natural way, $\mathfrak{R}$ and $\mathfrak{R}_{\hat{\mathbb{C}}}$ are $\mathrm{Aut}^+\langle A, B, C\rangle \times \mathrm{Aut}^+(\hat{\mathbb{C}})$-sets.

Suppose that we have two representations, $\rho_{\hat{\mathbb{R}}} \in \mathfrak{R}_{\hat{\mathbb{R}}}$ and $\rho_{\hat{\mathbb{C}}} \in \mathfrak{R}_{\hat{\mathbb{C}}}$, fixed for the remainder of this section.

McMullen [8, Theorem 1.1] showed that there exists a (unique) (surjective) continuous $\langle A, B, C\rangle$-map from $\Lambda(\rho_{\hat{\mathbb{R}}})$ to $\Lambda(\rho_{\hat{\mathbb{C}}})$, sometimes denoted

$$\mathrm{CT} = \mathrm{CT}(\rho_{\hat{\mathbb{R}}},\, \rho_{\hat{\mathbb{C}}}) \colon \Lambda(\rho_{\hat{\mathbb{R}}}) \twoheadrightarrow \Lambda(\rho_{\hat{\mathbb{C}}}),$$

and called the *Cannon-Thurston map associated to* $(\rho_{\hat{\mathbb{R}}}, \rho_{\hat{\mathbb{C}}})$. Thus $\mathrm{CT} \colon \hat{\mathbb{R}} \twoheadrightarrow \hat{\mathbb{C}}$ is a sphere-filling curve that inherits a cyclic ordering, or orientation, from $\hat{\mathbb{R}}$.

Results of Thurston [10] and Bonahon [2] give a surjective map

$$\mathfrak{b} \colon \mathfrak{R}_{\hat{\mathbb{C}}} \to \mathrm{IrRLSeqs},$$

where IrRLSeqs denotes the set of all irrational bi-infinite sequences in the two-letter alphabet $\{R, L\}$; see Definitions 3.1 below. Moreover, $\mathfrak{b}$ induces another surjective map

$$\mathrm{Aut}^+(\hat{\mathbb{C}})\backslash\mathfrak{R}_{\hat{\mathbb{C}}}/\mathrm{Aut}^+\langle A, B, C\rangle \twoheadrightarrow \mathrm{IrRLSeqs},$$

and Minsky [9] showed that the latter map is bijective. For example, $\rho_{\hat{\mathbb{C}}}$ arises from (the representation of the fiber of) a complete, hyperbolic, once-punctured-torus bundle if, and only if, there exists a (semigroup) word $F$ in $\{R, L\}$ such that $F$ is not a power of a single letter, and $\mathfrak{b}(\rho_{\hat{\mathbb{C}}}) = \prod_{\mathbb{Z}} F$, the (irrational) bi-infinite $R$-$L$-sequence obtained by concatenating infinitely many copies of $F$ in both directions. In this event, some element of the $\mathrm{Aut}^+\langle A, B, C\rangle$-orbit of $\rho_{\hat{\mathbb{C}}}$ extends (uniquely) to a discrete, faithful representation, in $\mathrm{Aut}^+(\hat{\mathbb{C}})$, of the subgroup $\langle A, B, C, F\rangle$ of $\mathrm{Aut}^+\langle A, B, C\rangle$ .

In Definitions 9.2 below, we recall what it means for $\rho_{\hat{\mathbb{C}}}$ to be 'given by a C-T-model', as in [4]. It is conjectured that this always holds, and Bowditch [3] proved that $\rho_{\hat{\mathbb{C}}}$ is given by a C-T-model if $\rho_{\hat{\mathbb{C}}}$ arises from a complete, hyperbolic, once-punctured-torus bundle, or, more generally, if there is a finite bound on the length of power-of-a-single-letter subwords of $\mathfrak{b}(\rho_{\hat{\mathbb{C}}}) \in \mathrm{IrRLSeqs}$.

Now suppose further that $\rho_{\hat{\mathbb{C}}}$ is given by a C-T-model, and let $q_{\hat{\mathbb{C}}}$ be a point in $\hat{\mathbb{C}}$. The number of elements in the preimage $\mathrm{CT}^{-1}\{q_{\hat{\mathbb{C}}}\}$ is one, two, or countably infinite, and, in this last case, $\mathrm{CT}^{-1}\{q_{\hat{\mathbb{C}}}\}$ consists of the 'head' and 'feet' of a 'double spider'. Let $\mathfrak{I}$ denote the set of connected components of $\hat{\mathbb{R}} - \mathrm{CT}^{-1}\{q_{\hat{\mathbb{C}}}\}$; then $\mathfrak{I}$ consists of one open interval, two open intervals, or a bi-infinite sequence of open intervals. Clearly, CT induces a continuous, surjective map $\hat{\mathbb{R}} - \mathrm{CT}^{-1}\{q_{\hat{\mathbb{C}}}\} \quad \twoheadrightarrow \quad \hat{\mathbb{C}} - \{q_{\hat{\mathbb{C}}}\}$. Let

$$\mathrm{CT}(\mathfrak{I}) := \{\mathrm{CT}(I) \mid I \in \mathfrak{I}\}, \tag{1.0.1}$$

a set consisting of (one, two or a bi-infinite sequence of) subsets of the oriented plane $\hat{\mathbb{C}} - \{q_{\hat{\mathbb{C}}}\}$. We will see that $\mathrm{CT}(\mathfrak{I})$ does not depend on the choice of $\rho_{\hat{\mathbb{R}}}$ within $\mathfrak{R}_{\hat{\mathbb{R}}}$. If $\mathrm{CT}^{-1}\{q_{\hat{\mathbb{C}}}\}$ has exactly one element, then the single element of $\mathrm{CT}(\mathfrak{I})$ is the plane $\hat{\mathbb{C}} - \{q_{\hat{\mathbb{C}}}\}$. If $\mathrm{CT}^{-1}\{q_{\hat{\mathbb{C}}}\}$ has exactly two elements, then the two elements of $\mathrm{CT}(\mathfrak{I})$ arise from complementary Jordan domains in $\hat{\mathbb{C}}$ whose common Jordan-curve boundary contains $q_{\hat{\mathbb{C}}}$. Our interest lies in the third case. Here there is a (unique) distinguished element $D'$ of $\langle A, B, C\rangle$ such that $\rho_{\hat{\mathbb{C}}}(D')$ fixes $q_{\hat{\mathbb{C}}}$. Then $\mathrm{CT}(\mathfrak{I})$ is a $\langle D'\rangle$-set of subsets of the oriented $\langle D'\rangle$-plane $\hat{\mathbb{C}} - \{q_{\hat{\mathbb{C}}}\}$. Here, we call each element of $\mathrm{CT}(\mathfrak{I})$ a *column*, and we arrange the plane so that the columns lie in an up-and-down direction and $D'$ shifts the columns from left to right. By reversing the cyclic ordering of the sphere-filling curve if necessary, we may assume that CT fills in the set of columns from left to right. We then *color* the cut-points and the interior of each column; we use the color 'gray' if CT fills in the column from bottom to top, and 'white' if CT fills in the column from top to bottom. (We like to imagine that the sphere-filling travelling point switches back and forth between its two ink cartridges each time it hits $q_{\hat{\mathbb{C}}}$.) We show, in Definitions 9.4 below, that each column is the concatenation of a bi-infinite sequence of Jordan domains. Together, all the resulting Jordan domains can be used as closed two-cells to endow $\hat{\mathbb{C}} - \{q_{\hat{\mathbb{C}}}\}$ with the structure of an oriented, colored $\mathbb{Z}$-CW-complex, denoted $\mathrm{CW}(\rho_{\hat{\mathbb{C}}}, q_{\hat{\mathbb{C}}})$. The coloring can be used to recover the columns. In Proposition 9.5 below, we will show that each of the following determines the others:

the $\mathrm{Aut}^+(\hat{\mathbb{C}})$-orbit of the oriented, colored $\mathbb{Z}$-CW-complex $\mathrm{CW}(\rho_{\hat{\mathbb{C}}}, q_{\hat{\mathbb{C}}})$;
the isomorphism class of the oriented, colored $\mathbb{Z}$-CW-complex $\mathrm{CW}(\rho_{\hat{\mathbb{C}}}, q_{\hat{\mathbb{C}}})$;
the irrational bi-infinite $R$-$L$-sequence $\mathfrak{b}(\rho_{\hat{\mathbb{C}}})$.

In fact, (the isomorphism class of) the oriented, colored $\mathbb{Z}$-CW-complex

$$\mathrm{CW}(\rho_{\hat{\mathbb{C}}}, q_{\hat{\mathbb{C}}})$$

determines $\mathfrak{b}(\rho_{\hat{\mathbb{C}}})$ visually, giving an alternative definition of $\mathfrak{b}(\rho_{\hat{\mathbb{C}}})$ whenever $\rho_{\hat{\mathbb{C}}}$ is given by a C-T-model.

The final part, Sections 11 to 14, is devoted to the study of the symmetry group of $\mathrm{CW}(\rho_{\hat{\mathbb{C}}}, q_{\hat{\mathbb{C}}})$. We will be particularly interested in the case where $\rho_{\hat{\mathbb{C}}}$ arises from a complete, hyperbolic, once-punctured-torus bundle, or, equivalently, $\mathrm{CW}(\rho_{\hat{\mathbb{C}}}, q_{\hat{\mathbb{C}}})$ has a conformal $\mathbb{Z}^2$-action; see Proposition 13.6 and Section 10.

$\binom{n+m-1}{n}$

**1.1 Examples.** (i) Suppose $\mathfrak{b}(\rho_{\hat{\mathbb{C}}}) = \prod_{\mathbb{Z}} RLLRRRLLLL = \prod_{\mathbb{Z}} R^1L^2R^3L^4$.

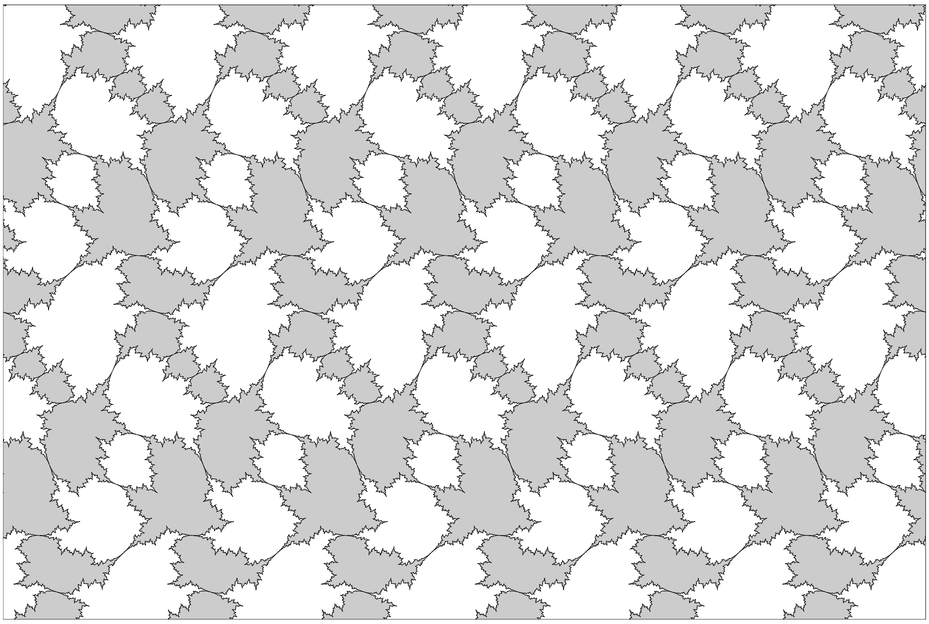

Figure 1.1.1: Part of the tessellation for $\mathfrak{b}(\rho_{\hat{\mathbb{C}}}) = \prod_{\mathbb{Z}} RLLRRRLLLL$.

Figure 1.1.1, which will be repeated as Figure 14.13.1, depicts (1.0.1) and $\mathrm{CW}(\rho_{\hat{\mathbb{C}}}, q_{\hat{\mathbb{C}}})$ for this case. The elements of (1.0.1) appear as colored columns, and

$\rho_{\hat{\mathbb{C}}}(D')$ shifts the columns two steps to the right. The sphere-filling curve CT draws the tessellation (completed with the point $q_{\hat{\mathbb{C}}}$, common to all the columns) from left to right (gradually), alternately proceeding upward (gradually), producing a gray column, and downwards (gradually), producing a white column. We consider such a figure to be normalized. The cut-points of the white (resp. gray) columns are white (resp. gray) zero-cells. By a *gray-white path*, we mean the left border of a white column, or, equivalently, the right border of a gray column; it will be thought of as an upward path given by a bi-infinite sequence of zero-cells. A *white-gray path* is defined dually.

From Figure 1.1.1, we see the following.

The bi-infinite zero-cell-color sequence of any gray-white path is

$$\prod_{\mathbb{Z}} [\text{white, gray, gray, white, white, white, gray, gray, gray, gray,}];$$

on replacing 'white' with $R$ and 'gray' with $L$, we recover the bi-infinite $R$-$L$-sequence $\prod_{\mathbb{Z}} RLLRRRLLLL$.

A white-gray path gives *the same* bi-infinite zero-cell-color sequence.

Each gray zero-cell provides one term in the bi-infinite zero-cell-color sequence of *each* border of a gray column. After the two bi-infinite zero-cell-color sequences are identified in a canonical way, the gray zero-cell provides two terms; these are *consecutive* in the gray subsequence, and the term corresponding to the gray-white path occurs *just after* the term corresponding to the white-gray path.

Similarly, the two borders of a white column intersect in their white zero-cells, and the white-gray path lags one white step behind the gray-white path.

(ii) Suppose $\mathfrak{b}(\rho_{\hat{\mathbb{C}}}) = \prod_{\mathbb{Z}} LLLLRRRLLR$.

A normalized figure for this case can be obtained by rotating Figure 1.1.1 through 180°. Using permutation notation for this example, we say the change from (i) to (ii) is

$$\text{(forwards, backwards)(left, right)(up, down),}$$

by which we mean that, in passing from (i) to (ii), $\mathfrak{b}(\rho_{\hat{\mathbb{C}}})$ is read backwards, while, in the figure, up and down are interchanged, and left and right are interchanged.

(iii) Suppose $\mathfrak{b}(\rho_{\hat{\mathbb{C}}}) = \prod_{\mathbb{Z}} LRRLLLRRRR$.

A normalized figure for this case can be obtained by reflecting Figure 1.1.1 about a vertical line and interchanging the two colors. The change from (i) to (iii) is then

$$(L,\, R)\text{(left, right)(gray, white),}$$

that is, in $\mathfrak{b}(\rho_{\hat{\mathbb{C}}})$, the letters $L$ and $R$ are interchanged, and, in the figure, left and right are interchanged, and gray and white are interchanged.

(iv) Suppose $\mathfrak{b}(\rho_{\hat{\mathbb{C}}}) = \prod_{\mathbb{Z}} RRRRLLLRRL$.

A normalized figure for this case can be obtained by reflecting Figure 1.1.1 about a horizontal line and interchanging the two colors. The change from (i) to (iv) is then

$$\text{(forwards, backwards)}(L,\, R)\text{(up, down)(gray, white).} \qquad \square$$

# 2 General Notation

In this section we record some of our conventions.

**2.1 Notation.** For any metric space $X$, we write $\mathrm{Isom}(X)$ to denote the group of all isometries of $X$. If $X$ is oriented, we write $\mathrm{Isom}^+(X)$ to denote the group of all orientation-preserving isometries of $X$.

Let $\mathbb{N}$ denote $\{0, 1, 2, \ldots\}$, the set of finite cardinals. For any $n \in \mathbb{N}$, we give $\mathbb{R}^n$ the usual Euclidean metric, and give $\mathbb{Z}^n$ the induced metric. There is a natural embedding $\mathrm{Isom}(\mathbb{Z}^n) \to \mathrm{Isom}(\mathbb{R}^n)$, and, at the risk of some confusion, we often view $\mathrm{Isom}(\mathbb{Z}^n)$ as a subgroup of $\mathrm{Isom}(\mathbb{R}^n)$.

Each element of $\mathrm{Isom}(\mathbb{R})$ is of the form $az + b$, where $z$ represents the identity map, while $a \in \{1, -1\}$ and $b \in \mathbb{R}$ represent constant maps. Here $az + b$ lies in $\mathrm{Isom}(\mathbb{Z})$ if and only if $b \in \mathbb{Z}$. Notice that $\mathrm{Isom}(\mathbb{R}) = \mathrm{Isom}^+(\mathbb{R}) \rtimes \langle -z \rangle$.

Each element of $\mathrm{Isom}(\mathbb{R}^2)$ is of the form

$$(x, y) \mapsto (ax + by + e, cx + dy + f), \tag{2.1.1}$$

for some $a, b, c, d, e, f \in \mathbb{R}$ such that $a^2 + b^2 = 1$ and $(c, d) = \pm(-b, a)$. We will denote (2.1.1) by $(ax + by + e, cx + dy + f)$; in this notation, $x$ and $y$ can be thought of as projections onto the first and second coordinates, respectively. The map (2.1.1) lies in $\mathrm{Isom}(\mathbb{Z}^2)$ if and only if all six coefficients are integers. □

**2.2 Notation.** Let $X$ and $Y$ be sets.

Let $Y^X$ denote the set of all maps from $X$ to $Y$.

We let $\mathrm{Sym}(X)$, or $\mathrm{Sym}\,X$, denote the set of all permutations of $X$. Then

$$\mathrm{Sym}(X) \times \mathrm{Sym}(Y)$$

acts on $Y^X$ in the natural way, that is, any $(\alpha, \beta) \in \mathrm{Sym}(X) \times \mathrm{Sym}(Y)$ acting on any $\gamma \in Y^X$ gives $\beta \cdot \gamma \cdot \alpha^{-1}$, where the dot denotes composition.

Any subgroup $G$ of $\mathrm{Sym}(X) \times \mathrm{Sym}(Y)$ acts on $Y^X$, and we may consider the set $G\backslash Y^X$ of the $G$-orbits in $Y^X$. We denote the sets of $\mathrm{Sym}(X)$-orbits, $\mathrm{Sym}(Y)$-orbits, and $\mathrm{Sym}(X) \times \mathrm{Sym}(Y)$-orbits, as $Y^X/\mathrm{Sym}(X)$, $\mathrm{Sym}(Y)\backslash Y^X$, and $\mathrm{Sym}(Y)\backslash Y^X/\mathrm{Sym}(X)$, respectively.

Let us now restrict to the case where $X = \mathbb{Z}$, where we can view $\mathrm{Sym}(\mathbb{Z})$ as containing $\mathrm{Isom}(\mathbb{Z})$. An $\mathrm{Isom}^+(\mathbb{Z})$-orbit in $Y^{\mathbb{Z}}$ will be called a *bi-infinite sequence* in $Y$. □

**2.3 Notation.** Recall the notation at the beginning of Section 1. Let us write

$$\begin{aligned} P &:= LR^{-1}L = R^{-1}LR^{-1} = (C, CBC, CBABC), \\ Q &:= LR^{-1} = (B, C, CBABC). \end{aligned}$$

Then $L = Q^{-1}P$ and $R = Q^{-2}P$. Also $P^2 = Q^3 = D$. The relation $P^2 = Q^3$ presents the subgroup $\langle P, Q \rangle = \langle R, L \rangle$ of $\mathrm{Aut}\langle A, B, C \rangle$. Also, $PRP^{-1} = L^{-1}$ and $PLP^{-1} = R^{-1}$.

For each $W \in \langle A, B, C \rangle$, we write $\langle\!\langle W \rangle\!\rangle$ to denote the smallest *normal* subgroup of $\langle A, B, C \rangle$ containing $W$.

We view $\langle A, B, C\rangle$ as a subgroup of $\mathrm{Aut}\langle A, B, C\rangle$ by identifying each element $W$ of $\langle A, B, C\rangle$ with the automorphism

$$(WAW^{-1}, WBW^{-1}, WBW^{-1})$$

of $\langle A, B, C\rangle$. □

**2.4 Notation.** Consider the complex projective line, or Riemann sphere,

$$\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}.$$

Let $\mathrm{Aut}(\hat{\mathbb{C}})$ denote the group of all the transformations of $\hat{\mathbb{C}}$ which are conformal or anticonformal. We specify an element of $\mathrm{Aut}(\hat{\mathbb{C}})$ by a (non-unique) expression of the form $\frac{az+b}{cz+d}$ or $\frac{a\bar{z}+b}{c\bar{z}+d}$, with $a, b, c, d \in \mathbb{C}$, $ad \neq bc$, were $z$ denotes the identity map on $\hat{\mathbb{C}}$ and $\overline{z}$ denotes complex conjugation on $\hat{\mathbb{C}}$. Alternatively, we can think of $z$ as a generic element of $\hat{\mathbb{C}}$.

Recall that $\mathrm{Aut}^+(\hat{\mathbb{C}})$ is the index-two subgroup of $\mathrm{Aut}(\hat{\mathbb{C}})$ consisting of the conformal transformations of $\hat{\mathbb{C}}$. Thus $\mathrm{Aut}(\hat{\mathbb{C}}) = \mathrm{Aut}^+(\hat{\mathbb{C}}) \rtimes \langle\overline{z}\rangle$, There are natural identifications $\mathrm{Aut}^+(\hat{\mathbb{C}}) = \mathrm{PSL}_2(\mathbb{C}) = \mathrm{PGL}_2(\mathbb{C})$, through the action of $\mathrm{GL}_2(\mathbb{C})$ on $\hat{\mathbb{C}}$ by Möbius transformations.

An element of $\mathrm{Aut}^+(\hat{\mathbb{C}})$ is said to be *parabolic* if it has a unique fixed point in $\hat{\mathbb{C}}$.

A subset of $\mathrm{Aut}^+(\hat{\mathbb{C}})$ ($= \mathrm{PSL}_2(\mathbb{C})$) is *discrete* if its preimage in $\mathrm{SL}_2(\mathbb{C})$ is discrete with respect to the usual metric on $\mathbb{C}^4 \supseteq \mathrm{SL}_2(\mathbb{C})$. □

**2.5 Definitions.** By a *representation of* $\langle A, B, C\rangle$ *in* $\mathrm{Aut}^+(\hat{\mathbb{C}})$, we mean a homomorphism from $\langle A, B, C\rangle$ to $\mathrm{Aut}^+(\hat{\mathbb{C}})$.

There is a natural action of $\mathrm{Aut}(\hat{\mathbb{C}})$ on $\mathrm{Aut}^+(\hat{\mathbb{C}})$ by left conjugation. Thus, by Notation 2.2, the set of all representations of $\langle A, B, C\rangle$ in $\mathrm{Aut}^+(\hat{\mathbb{C}})$ is acted on by the group $\mathrm{Aut}\langle A, B, C\rangle \times \mathrm{Aut}(\hat{\mathbb{C}})$, and hence by all of its subgroups. We will be particularly interested in the orbits under the actions of $\mathrm{Aut}\langle A, B, C\rangle \times \mathrm{Aut}(\hat{\mathbb{C}})$, $\mathrm{Aut}^+\langle A, B, C\rangle \times \mathrm{Aut}^+(\hat{\mathbb{C}})$, $\mathrm{Aut}(\hat{\mathbb{C}})$, and $\mathrm{Aut}^+(\hat{\mathbb{C}})$.

Two representations of $\langle A, B, C\rangle$ in $\mathrm{Aut}^+(\hat{\mathbb{C}})$ are *equivalent* if they are in the same $\mathrm{Aut}^+(\hat{\mathbb{C}})$-orbit.

The $\mathrm{Aut}^+\langle A, B, C\rangle$-orbit of a representation $\rho$ of $\langle A, B, C\rangle$ in $\mathrm{Aut}^+(\hat{\mathbb{C}})$ can be codified as a pair consisting of the subgroup $\rho(\langle A, B, C\rangle)$ of $\mathrm{Aut}^+(\hat{\mathbb{C}})$ together with the image under $\rho$ of the set of distinguished elements of $\langle A, B, C\rangle$.

We say that $\rho$ is *faithful* if it is injective, and that $\rho$ is *discrete* if $\rho(\langle A, B, C\rangle)$ is discrete in $\mathrm{Aut}^+(\hat{\mathbb{C}})$. The *limit set of* $\rho$, $\Lambda(\rho)$, is the smallest nonempty, closed $\rho(\langle A, B, C\rangle)$-invariant subset of $\hat{\mathbb{C}}$. If $\rho(D)$ is parabolic, and $q$ denotes the point fixed by $\rho(D)$, then $\Lambda(\rho)$ is the $\hat{\mathbb{C}}$-closure of the $\rho(\langle A, B, C\rangle)$-orbit of $q$.

Recall that $\mathfrak{R} = \mathfrak{R}(\langle A, B, C\rangle, D)$ denotes the set of all discrete, faithful representations $\rho$ of $\langle A, B, C\rangle$ in $\mathrm{Aut}^+(\hat{\mathbb{C}})$ such that $\rho(D)$ is parabolic. Also, $\mathfrak{R}_{\hat{\mathbb{C}}}$ and $\mathfrak{R}_{\hat{\mathbb{R}}}$ denote the sets of elements of $\mathfrak{R}$ with limit sets $\hat{\mathbb{C}}$ and $\hat{\mathbb{R}}$ ($= \mathbb{R} \cup \{\infty\}$), respectively.

Both $\mathfrak{R}_{\hat{\mathbb{C}}}$ and $\mathfrak{R}$ are closed under the $\mathrm{Aut}\langle A, B, C\rangle \times \mathrm{Aut}(\hat{\mathbb{C}})$-action. □

**2.6 Notation.** We use boldface to denote elements of $\mathbb{R}^2$.

We write $\mathbf{0} := (0,0)$, $\mathbf{x} := (1,0)$, $\mathbf{y} := (0,1)$.

If $(\mathbf{v}_0, \mathbf{v}_1, \dots, \mathbf{v}_n)$ is a sequence in $\mathbb{R}^2$, we denote by $\operatorname{hull}[\mathbf{v}_0, \mathbf{v}_1, \dots, \mathbf{v}_n]$ the convex hull of $\{\mathbf{v}_0, \mathbf{v}_1, \dots, \mathbf{v}_n\}$ in $\mathbb{R}^2$. $\square$

# 3 The principal concepts

Our objective in this article is to derive certain conclusions in the situation where we are given a pair $(\rho_{\hat{\mathbb{C}}}, q_{\hat{\mathbb{C}}}) \in \mathfrak{R}_{\hat{\mathbb{C}}} \times \hat{\mathbb{C}}$ such that $q_{\hat{\mathbb{C}}}$ is a parabolic point of $\rho_{\hat{\mathbb{C}}}(\langle A,B,C\rangle)$, that is, $q_{\hat{\mathbb{C}}}$ is fixed by a (unique) distinguished element of $\rho_{\hat{\mathbb{C}}}(\langle A,B,C\rangle)$. Up to $\mathrm{Aut}^+(\hat{\mathbb{C}})$-orbits, this data can be codified using the following concepts, which provide a language that is useful for expressing our results.

**3.1 Definitions.** An element $\mathfrak{f}$ of $\{R,L\}^{\mathbb{Z}}$ will be called an *R-L-map*; we will write $n \mapsto \mathfrak{f}_n$, rather than $n \mapsto \mathfrak{f}(n)$, and we think of $\mathfrak{f}$ as a formal infinite product, $\prod\limits_{n\in\mathbb{Z}} \mathfrak{f}_n$. We say that $\mathfrak{f}$ is an *irrational* $R$-$L$-map if, as $n$ tends to infinity, neither $\mathfrak{f}_n$ nor $\mathfrak{f}_{-n}$ is eventually constant. We let IrRLMaps denote the set of all irrational $R$-$L$-maps.

Let $\mathfrak{f} \in$ IrRLMaps.

We define the *successor map* of $\mathfrak{f}$ as the (bijective) map

$$\operatorname{succ}_{\mathfrak{f}}\colon \mathbb{Z} \to \mathbb{Z}, \quad n \mapsto \operatorname{succ}_{\mathfrak{f}}(n) := \min\{m \in \mathbb{Z} \mid m > n \text{ and } \mathfrak{f}_m = \mathfrak{f}_n\}.$$

The inverse map is denoted $\operatorname{pred}_{\mathfrak{f}}$.

As in Notation 2.2, $\mathrm{Isom}(\mathbb{Z}) \times \mathrm{Sym}\{R,L\}$ acts on IrRLMaps. The $\mathrm{Isom}^+(\mathbb{Z})$-orbit of $\mathfrak{f}$ will be denoted $[\mathfrak{f}]$.

In Section 1, we denoted IrRLMaps / $\mathrm{Isom}^+(\mathbb{Z})$ by IrRLSeqs, and called the elements *irrational bi-infinite R-L sequences.* $\square$

**3.2 Definitions.** Consider any map $\mathcal{F}\colon \mathbb{Z} \to \mathrm{Aut}^+\langle A,B,C\rangle$, $n \mapsto \mathcal{F}_n$. We define the *quotient map* of $\mathcal{F}$ as

$$\mathfrak{f} = \operatorname{quotient}(\mathcal{F})\colon \mathbb{Z} \to \mathrm{Aut}^+\langle A,B,C\rangle, \qquad n \mapsto \mathfrak{f}_n := \mathcal{F}_{n-1}^{-1}\mathcal{F}_n.$$

We say that $\mathcal{F}$ is an *irrational* $\mathrm{Aut}\langle A,B,C\rangle$*-map* if $\mathfrak{f} \in$ IrRLMaps; in particular, the image of $\mathfrak{f}$ is $\{R,L\}$. We let IrAutABCMaps denote the set of irrational $\mathrm{Aut}\langle A,B,C\rangle$-maps.

Suppose that $\mathcal{F} \in$ IrAutABCMaps, and let $\mathfrak{f} = \operatorname{quotient}(\mathcal{F})$.

We write $D_{\mathcal{F}} := \mathcal{F}_0(D)$, a distinguished element of $\langle A,B,C\rangle$. It can be shown that $\mathcal{F}_n(D) = D_{\mathcal{F}}$, for all $n \in \mathbb{Z}$, since $\mathcal{F}_n(D) = \mathcal{F}_{n-1}\mathfrak{f}_n(D) = \mathcal{F}_{n-1}(D)$.

We define the *residue map* of $\mathcal{F}$ to be the map

$$\mathcal{F}' = \operatorname{residue}(\mathcal{F})\colon \mathbb{Z} \to \mathrm{Aut}^+\langle A,B,C\rangle,$$
$$n \mapsto \mathcal{F}'_n := \mathcal{F}_{n-1}M\mathfrak{f}_nM = \begin{cases} \mathcal{F}_{n-1}L & \text{if } \mathfrak{f}_n = R, \\ \mathcal{F}_{n-1}R & \text{if } \mathfrak{f}_n = L. \end{cases}$$

As in Notation 2.2, $\mathrm{Isom}(\mathbb{Z}) \times \mathrm{Aut}^+\langle A, B, C\rangle$ acts on IrAutABCMaps, where we understand that $\mathrm{Aut}^+\langle A, B, C\rangle$ acts on itself by left multiplication. We can identify the $\mathrm{Aut}^+\langle A, B, C\rangle$-orbit of $\mathcal{F}$ with $\mathfrak{f}$, and get an identification

$$\mathrm{Aut}^+\langle A, B, C\rangle \backslash \,\mathrm{IrAutABCMaps} \xrightarrow{\sim} \mathrm{IrRLMaps}\,. \tag{3.2.1}$$

We can further identify the $\mathrm{Isom}^+(\mathbb{Z}) \times \mathrm{Aut}^+\langle A, B, C\rangle$-orbit of $\mathcal{F}$ with $[\mathfrak{f}]$, the $\mathrm{Isom}^+(\mathbb{Z})$-orbit of $\mathfrak{f}$. □

The main purpose of this article is to investigate IrAutABCMaps, and construct models that correspond to the statements that were given in Section 1. Suppose that $\mathcal{F} \in$ IrAutABCMaps and let $\mathfrak{f} = \mathrm{quotient}(\mathcal{F})$. In Section 4, we associate to $\mathfrak{f}$ simple CW-complexes $\mathrm{CW}'(\mathfrak{f})$ and $\mathrm{CW}(\mathfrak{f})$. Suppose that $\rho_{\hat{\mathbb{R}}} \in \mathfrak{R}_{\hat{\mathbb{R}}}$, and let us use $\rho_{\hat{\mathbb{R}}}$ to endow $\hat{\mathbb{R}}$ with the structure of an $\langle A, B, C\rangle$-space. In Sections 5 and 6, we recall how to construct a model $\mathbb{S}^2 = \mathbb{S}^2(\mathcal{F})$ of $\hat{\mathbb{C}}$, with a distinguished point $q^2_{\mathbb{S}}$, and a continuous surjective $\langle A, B, C\rangle$-map $\mathrm{CT}\colon \hat{\mathbb{R}} \twoheadrightarrow \mathbb{S}^2$. In Section 7, we describe a CW-structure $\mathrm{CW}(\mathcal{F})$, on $\mathbb{S}^2 - \{q^2_{\mathbb{S}}\}$, determined by CT and $q^2_{\mathbb{S}}$ via the method described in Section 1; we then show that $\mathrm{CW}(\mathcal{F})$ is isomorphic to $\mathrm{CW}(\mathfrak{f})$. In Section 8, we recall the deep relationship between IrAutABCMaps and $\mathfrak{R}_{\hat{\mathbb{C}}}$, and, in Section 9, we deduce the statements made in Section 1 about $\mathrm{CW}(\rho_{\hat{\mathbb{C}}}, q_{\hat{\mathbb{C}}})$.

# 4 A model $\mathrm{CW}(\mathfrak{f})$ of a model $\mathrm{CW}(\mathcal{F})$ of $\mathrm{CW}(\rho_{\hat{\mathbb{C}}}, q_{\hat{\mathbb{C}}})$

In this section, we associate, to each $\mathfrak{f} \in$ IrRLMaps, an oriented, colored $\mathbb{Z}$-CW-complex $\mathrm{CW}(\mathfrak{f})$. This will give us a simple model of a tessellated plane that illustrates the more complicated CW-complexes $\mathrm{CW}(\mathcal{F})$ and $\mathrm{CW}(\rho_{\hat{\mathbb{C}}}, q_{\hat{\mathbb{C}}})$ that will be constructed later.

**4.1 Definitions.** By a *colored* CW-*complex* we will mean a two-dimensional CW-complex in which each zero-cell and each open two-cell is assigned a value in the set {white, gray}.

A *color-preserving* CW-*isomorphism* is a homeomorphism between colored CW-complexes which carries cells homeomorphically to cells, and respects the colorings.

A *color-preserving* CW-*automorphism* is defined similarly. (We remark that some authors require that fixed cells be fixed pointwise, but this restriction is not convenient for our purposes.)

For a group $G$, a *colored* $G$-CW-*complex* $X$ is a colored CW-complex together with an action by $G$ such that each element of $G$ acts as a color-preserving CW-automorphism of X. □

**4.2 Definitions.** Let $\mathfrak{f} \in$ IrRLMaps. Let $\mathrm{succ}_{\mathfrak{f}}$ and $\mathrm{pred}_{\mathfrak{f}}$ be as in Definitions 3.1.

We first endow the oriented Euclidean plane $\mathbb{R}^2$ with the structure of an oriented, colored $\mathbb{Z}$-CW-complex, denoted $\mathrm{CW}'(\mathfrak{f})$, as follows.

The set of zero-cells of $\mathrm{CW}'(\mathfrak{f})$ will be $\mathbb{Z}^2 \subseteq \mathbb{R}^2$.

Let $(m, n) \in \mathbb{Z}^2$ be a zero-cell. Let $n_- = \mathrm{pred}_{\mathfrak{f}}(n)$ and $n_+ = \mathrm{succ}_{\mathfrak{f}}(n)$.

We define the *vertical closed one-cell out of* the zero-cell $(m, n)$ to be

$$\mathrm{hull}[m\mathbf{x} + n\mathbf{y}, m\mathbf{x} + (n+1)\mathbf{y}],$$

the straight line segment joining $(m, n)$ to $(m, n+1)$.

We define the *slanted closed one-cell out of* the zero-cell $(2m, n)$ to be the straight line segment joining $(2m, n)$ to $\begin{cases} (2m-1, n_+) & \text{if } \mathfrak{f}_n = L, \\ (2m+1, n_+) & \text{if } \mathfrak{f}_n = R. \end{cases}$

This completes the definition of the one-skeleton of $\mathrm{CW}'(\mathfrak{f})$. The open two-cells will be the (polygonal) components of the complement in $\mathbb{R}^2$ of this one-skeleton. The attaching maps and orientation are induced from $\mathbb{R}^2$.

If $\mathfrak{f}_n = L$, we assign to each zero-cell in $\mathbb{Z} \times \{n\}$ the color 'gray'. The interior of

$$\mathrm{hull}[(2m-1, n), (2m-1, n_+), (2m, n_-), (2m, n)],$$

is an open two-cell in $\mathrm{CW}'(\mathfrak{f})$. We call this the $(m, n)$*th open two-cell of* $\mathrm{CW}'(\mathfrak{f})$, and assign it the color 'gray'. The two slanted one-cells in the boundary join $(2m-1, n)$ to $(2m, n_-)$ and $(2m-1, n_+)$ to $(2m, n)$, and all four zero-cells are gray.

If $\mathfrak{f}_n = R$, we assign the zero-cells in $\mathbb{Z} \times \{n\}$ the color 'white'. The interior of

$$\mathrm{hull}[(2m, n_-), (2m, n), (2m+1, n), (2m+1, n_+)]$$

is an open two-cell in $\mathrm{CW}'(\mathfrak{f})$. We call this the $(m, n)$*th open two-cell of* $\mathrm{CW}'(\mathfrak{f})$, and assign it the color 'white'. The two slanted one-cells in the boundary join $(2m, n_-)$ to $(2m+1, n)$ and $(2m, n)$ to $(2m+1, n_+)$, and all four zero-cells are white.

We take the $\mathbb{Z}$-action on $\mathrm{CW}'(\mathfrak{f})$ corresponding to the natural action of the subgroup $2\mathbb{Z} \times \{0\}$ of $\mathbb{R}^2$. Then $\mathrm{CW}'(\mathfrak{f})$ is an oriented, colored $\mathbb{Z}$-CW-complex.

We now define $\mathrm{CW}(\mathfrak{f})$ to be the oriented, colored $\mathbb{Z}$-CW-complex obtained from $\mathrm{CW}'(\mathfrak{f})$ by collapsing each closed slanted one-cell in $\mathrm{CW}'(\mathfrak{f})$ to a zero-cell. Thus the oriented $\mathbb{Z}$-plane $|\mathrm{CW}(\mathfrak{f})|$ underlying $\mathrm{CW}(\mathfrak{f})$ is obtained from $\mathbb{R}^2$ by collapsing each closed slanted one-cell of $\mathrm{CW}'(\mathfrak{f})$ to a point.

In the one-skeleton of $\mathrm{CW}(\mathfrak{f})$, every zero-cell has valency four.

The $m$th *column* of $\mathrm{CW}'(\mathfrak{f})$ is the CW-subcomplex of $\mathrm{CW}'(\mathfrak{f})$ which has underlying space $[m, m+1] \times \mathbb{R}$. The $m$th *column* of $\mathrm{CW}(\mathfrak{f})$ is the image of the $m$th column of $\mathrm{CW}'(\mathfrak{f})$ under the collapsing map $\mathrm{CW}'(\mathfrak{f}) \to \mathrm{CW}(\mathfrak{f})$. We will say that the $m$th column is $\begin{cases} \text{gray} & \text{if } m \text{ is odd,} \\ \text{white} & \text{if } m \text{ is even.} \end{cases}$

We have an oriented plane with a colored $\mathbb{Z}$-CW-structure, and a bi-infinite sequence of columns; if we divide out by the $\mathbb{Z}$-action, we get an oriented open cylinder (or annulus) with a colored CW-structure, and two columns. □

**4.3 Remarks.** Let $\mathfrak{f} \in \mathrm{IrRLMaps}$.

For $n \geq 2$, a closed two-cell in $\mathrm{CW}(\mathfrak{f})$ with exactly $n$ zero-cells (or one-cells) is called an *n-gon*. Each column of $\mathrm{CW}(\mathfrak{f})$ is an ascending bi-infinite sequence of two-cells, and hence has an associated bi-infinite sequence of polygon sizes.

Consider any map $\mathbb{Z} \to \mathbb{N}$, $n \mapsto s_n$.

If $[\mathfrak{f}] = \prod\limits_{n\in\mathbb{Z}} LR^{s_n}$ then the bi-infinite polygon-size sequence for each gray column of $\mathrm{CW}(\mathfrak{f})$ is $(s_n + s_{n+1} + 2 \mid n \in \mathbb{Z})$.

If $[\mathfrak{f}] = \prod\limits_{n\in\mathbb{Z}} RL^{s_n}$ then the bi-infinite polygon-size sequence for each white column of $\mathrm{CW}(\mathfrak{f})$ is $(s_n + s_{n+1} + 2 \mid n \in \mathbb{Z})$.

Consider the case where $[\mathfrak{f}] = \prod\limits_{\mathbb{Z}} RLLRRRLLLL$. Here

$$[\mathfrak{f}] = \prod_{\mathbb{Z}} LR^0LR^0LR^0LR^1LR^0LR^3 = \prod_{\mathbb{Z}} RL^4RL^2RL^0RL^0,$$

and the foregoing says that, in $\mathrm{CW}(\mathfrak{f})$, each gray, resp. white, column read upward has the form

$$\prod_{\mathbb{Z}} [\text{2-gon}, \text{2-gon}, \text{3-gon}, \text{3-gon}, \text{5-gon}, \text{5-gon}, ],$$

resp.

$$\prod_{\mathbb{Z}} [\text{8-gon}, \text{4-gon}, \text{2-gon}, \text{6-gon}, ].$$

See Figure 1.1.1. □

For $\mathfrak{f} \in$ IrRLMaps, we will now see that the $\mathrm{Isom}^+(\mathbb{Z})$-orbit $[\mathfrak{f}]$, and the isomorphism class of the oriented, colored $\mathbb{Z}$-CW-complex $\mathrm{CW}(\mathfrak{f})$ mutually determine each other.

**4.4 Proposition.** *Let $\mathfrak{f}$ and $\mathfrak{g}$ be elements of* IrRLMaps.

*The following are equivalent.*

(a) $[\mathfrak{g}] = [\mathfrak{f}]$*, that is, there exists $n \in \mathbb{Z}$ such that $\mathfrak{g} = \mathfrak{f} \cdot (z + n)$.*

(b) $\mathrm{CW}(\mathfrak{g}) \simeq \mathrm{CW}(\mathfrak{f})$ *as oriented, colored $\mathbb{Z}$-*CW*-complexes.*

*Proof.* (a) $\Rightarrow$ (b). Suppose there exists $n \in \mathbb{Z}$ such that $\mathfrak{g} = \mathfrak{f} \cdot (z + n)$. Then, in $\mathbb{R}^2$, $\mathrm{CW}'(\mathfrak{g}) = \mathrm{CW}'(\mathfrak{f}) + (0, n)$, and, hence, $\mathrm{CW}(\mathfrak{g}) \simeq \mathrm{CW}(\mathfrak{f})$.

(b) $\Rightarrow$ (a). We will show how to recover $[\mathfrak{f}]$ from the isomorphism class of $\mathrm{CW}(\mathfrak{f})$.

Columns are maximal connected subsets of a single color.

We arrange the oriented plane $|\mathrm{CW}(\mathfrak{f})|$ so that the columns are in an up-down direction, and the action of the distinguished generator of $\mathbb{Z}$ shifts the columns two steps to the right. We can now pick out the upward paths bordering the columns.

An upward path with a gray column on the left and a white column on the right gives a bi-infinite sequence of colored zero-cells, and reading $R$ for 'white' (right) and $L$ for 'gray' (left) recovers the element $[\mathfrak{f}]$ of IrRLMaps / $\mathrm{Isom}^+(\mathbb{Z})$. □

**4.5 Remark.** We will return to these concepts in Remarks 11.4, and we will see that the coloring does not carry much information, since, for $\mathfrak{f}$, $\mathfrak{g}$ in IrRLMaps, the following are equivalent.

(a) There exists a color-interchanging isomorphism $\mathrm{CW}(\mathfrak{f}) \xrightarrow{\sim} \mathrm{CW}(\mathfrak{g})$ of oriented $\mathbb{Z}$-CW-complexes.

(b) $\{[\mathfrak{f}], [\mathfrak{g}]\}$ is one of the three sets $\{\prod\limits_{\mathbb{Z}} RL\}$, $\{\prod\limits_{\mathbb{Z}} RRLL\}$, $\{\prod\limits_{\mathbb{Z}} RRL, \prod\limits_{\mathbb{Z}} RLL\}$.

□

# 5 An $\mathrm{Aut}\langle A, B, C\rangle$-action on $\hat{\mathbb{C}}$

In this section, we recall constructions going back to Nielsen which endow $\hat{\mathbb{C}}$ with an $\mathrm{Aut}\langle A, B, C\rangle$-action.

**5.1 Notation.** Throughout the remainder of the article, we fix $\rho_{\hat{\mathbb{R}}} \in \mathfrak{R}_{\hat{\mathbb{R}}}$, and we let $q_{\hat{\mathbb{R}}} \in \hat{\mathbb{R}}$ denote the point fixed by $\rho_{\hat{\mathbb{R}}}(D)$.

To simplify the exposition, we assume throughout that $\rho_{\hat{\mathbb{R}}}(D)$ shifts the points of $\hat{\mathbb{R}}-\{q_{\hat{\mathbb{R}}}\}$ upward, with respect to the usual induced total order on $\hat{\mathbb{R}}-\{q_{\hat{\mathbb{R}}}\}$. (Otherwise, we compose $\rho_{\hat{\mathbb{R}}}$ with $-z$; this changes the orientation of our sphere-filling curve, but not the resulting oriented, colored $\mathbb{Z}$-CW-complex.) The homomorphism $\rho_{\hat{\mathbb{R}}}\colon \langle A, B, C\rangle \to \mathrm{Aut}^+(\hat{\mathbb{C}})$ gives $\hat{\mathbb{C}}$ the structure of an $\langle A, B, C\rangle$-space.

Let $\mathbb{C}_+$ and $\mathbb{C}_-$ denote the sets of complex numbers with positive and negative imaginary parts, respectively. Then $\mathbb{C}_+$, $\hat{\mathbb{R}}$, and $\mathbb{C}_-$ partition $\hat{\mathbb{C}}$ into three $\langle A, B, C\rangle$-subspaces. We think of them as, respectively, the northern hemisphere, equator, and southern hemisphere of the Riemann sphere.

The homomorphism $\rho_{\hat{\mathbb{R}}}\colon \langle A, B, C\rangle \to \mathrm{Aut}^+(\hat{\mathbb{C}})$ gives $\hat{\mathbb{C}}$ the structure of an $\langle A, B, C\rangle$-space. The group $\langle A, B, C\rangle$ acts faithfully and properly discontinuously

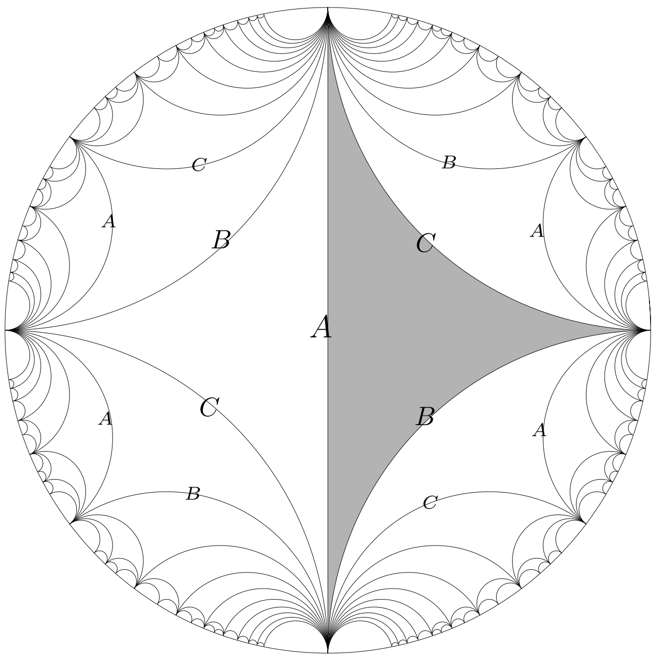


Figure 5.1.1: The $\langle A, B, C\rangle$-tessellation of $\mathbb{C}_+$

on $\mathbb{C}_+$. It can be shown that the orbifold $\langle A,B,C\rangle\backslash\mathbb{C}_+$ is of the form $S_{0,0,3^{(2)}+1}$, a punctured sphere with three double points. (The subscripts indicate that the surface in question has genus zero, and zero boundary components, and $3^{(2)}+1$ distinguished points, that is, three double points and one puncture.) On choosing a fundamental domain, we get a triangulation of $\mathbb{C}_+$; see Figure 5.1.1.

Let $\mathbb{P}$ denote the set of parabolic points of $\rho_{\hat{\mathbb{R}}}(\langle A,B,C\rangle)$; notice that

$$\mathbb{P} \subseteq \Lambda(\rho_{\hat{\mathbb{R}}}) = \hat{\mathbb{R}}.$$

The orbifold $\langle A,B,C\rangle\backslash(\mathbb{C}_+\cup\mathbb{P})$ is of the form $S_{0,0,3^{(2)}+1^{(\infty)}}$, a sphere with three double points and one infinite-cyclic point. □

We want to extend the above $\langle A,B,C\rangle$-action on $\hat{\mathbb{C}}$ to an $\mathrm{Aut}\langle A,B,C\rangle$-action. We deal with $\hat{\mathbb{R}}$, $\mathbb{C}_+$ and $\mathbb{C}_-$ separately. We start with $\hat{\mathbb{R}}$, where it is convenient to use ends.

**5.2 Definitions.** By an *end* of $\langle A,B,C\rangle$, we mean a map $\mathcal{W}\colon \mathbb{N}\to\langle A,B,C\rangle$, $n\mapsto \mathcal{W}_{|n}$, for which there exists a map $\mathfrak{w}\colon \mathbb{N}-\{0\}\to\{A,B,C\}$, $n\mapsto\mathfrak{w}_n$, (called the quotient map of $\mathcal{W}$) such that, for all $n\in\mathbb{N}$, $\mathfrak{w}_1\cdots\mathfrak{w}_n=\mathcal{W}_{|n}$ and $\mathfrak{w}_{n+1}\neq\mathfrak{w}_{n+2}$. We think of $\mathcal{W}$ as a formal infinite reduced product, $\mathcal{W}=\prod_{n=1}^{\infty}\mathfrak{w}_n$.

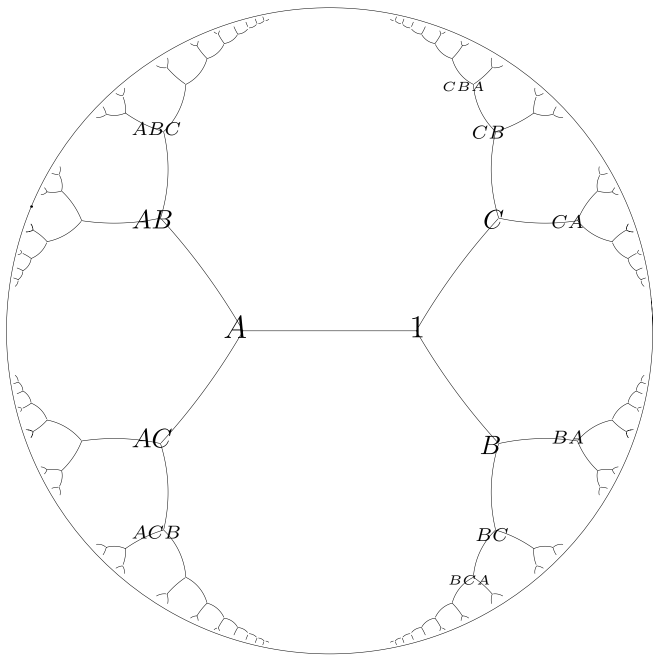


Figure 5.2.1: $\mathfrak{E}/\!\sim\ \simeq\ \hat{\mathbb{R}}$

For example, we write $D^{\infty} := \prod_{n\in\mathbb{N}} (CBA)$ and $D^{-\infty} := \prod_{n\in\mathbb{N}} (ABC)$.

We write $\mathfrak{E} = \mathfrak{E}\langle A,B,C\rangle$ to denote the set of ends of $\langle A,B,C\rangle$.

In a natural way, $\mathfrak{E}$ is a compact, Hausdorff, totally disconnected, topological space, with a continuous $\mathrm{Aut}\langle A,B,C\rangle$-action. McMullen [8, Theorem 1.1] proved that, for each $\rho \in \mathfrak{R}$, there exists a (unique) (surjective) continuous $\langle A,B,C\rangle$-map $\widehat{\rho}\colon \mathfrak{E} \twoheadrightarrow \Lambda(\rho)$. We will make frequent use of this result, and this notation.

If $q_{\hat{\mathbb{C}}}$ denotes the fixed point of $\rho(D)$, then $\widehat{\rho}(D^{\infty}) = q_{\hat{\mathbb{C}}}$, since $\rho(D)$ fixes $\widehat{\rho}(D^{\infty})$. Similarly, $\widehat{\rho}(D^{-\infty}) = q_{\hat{\mathbb{C}}}$. Hence $\widehat{\rho}(D^{-\infty}) = \widehat{\rho}(D^{\infty}) = q_{\hat{\mathbb{C}}}$.

For any $\mathcal{W} \in \mathfrak{E}$, $\lim\limits_{n\to\infty} (\mathcal{W}_{|n} \cdot D^{\infty}) = \mathcal{W}$ in $\mathfrak{E}$, and $\widehat{\rho}(\mathcal{W}) = \lim\limits_{n\to\infty} \rho(\mathcal{W}_{|n})(q_{\hat{\mathbb{C}}})$ in $\hat{\mathbb{C}}$.

Thus, the equivalence relation on $\mathfrak{E}$ determined by $\widehat{\rho}$ contains the equivalence relation $\sim$ on $\mathfrak{E}$ generated by $\{(W\cdot D^{-\infty}, W\cdot D^{\infty}) \mid W \in \langle A,B,C\rangle\}$. It is a classic result that $\Lambda(\rho)$ is a topological circle if and only if the equivalence relation on $\mathfrak{E}$ determined by $\widehat{\rho}$ is equal to $\sim$. Here $\Lambda(\rho)$ becomes an $\mathrm{Aut}\langle A,B,C\rangle$-space, because $\sim$ is invariant under the $\mathrm{Aut}\langle A,B,C\rangle$-action on $\mathfrak{E}^2$.

In particular, $\hat{\mathbb{R}} = \Lambda(\rho_{\hat{\mathbb{R}}}) \simeq \mathfrak{E}/{\sim}$ ; see Figure 5.2.1. Hence $\hat{\mathbb{R}}$ becomes an $\mathrm{Aut}\langle A,B,C\rangle$-space. It is not difficult to see that $\mathbb{P}$ is an $\mathrm{Aut}\langle A,B,C\rangle$-subset, of the form $\langle A,B,C\rangle/\langle D\rangle$. □

We now deal with $\mathbb{C}_+$ and $\mathbb{C}_-$, where it will be convenient to use the Euclidean plane, $\mathbb{R}^2$.

**5.3 Notation.** The natural $\mathrm{Aut}\langle A,B,C\rangle$-action on $\langle BC, BA\rangle/\langle\!\langle D^2 \rangle\!\rangle \simeq \mathbb{Z}^2$ can be used to make $\mathbb{R}^2$ into an affine $\mathrm{Aut}\langle A,B,C\rangle$-space with the action indicated in Table 5.3.1. Notice that this is a rotated version of [5, Definition 3.1].

| $F$ | $A$ | $B$ | $C$ | $D$ |
|---|---|---|---|---|
| $F(x,y)$ | $(1-x,-y)$ | $(1-x,1-y)$ | $(-x,1-y)$ | $(-x,-y)$ |

| $F$ | $R$ | $L$ | $M$ |
|---|---|---|---|
| $F(x,y)$ | $(x+y,y)$ | $(x,x+y)$ | $(y,x)$ |

Table 5.3.1: The action of $\mathrm{Aut}\langle A,B,C\rangle$ on $\mathbb{R}^2$.

By restriction, $\langle A,B,C\rangle$ acts on $\mathbb{R}^2$, and here the action is by multiplication by $\pm1$ followed by translation by an element of $\mathbb{Z}^2$.

The kernel of the $\mathrm{Aut}\langle A,B,C\rangle$-action on $\mathbb{R}^2$ is $\langle\!\langle D^2 \rangle\!\rangle$, and $\langle A,B,C\rangle/\langle\!\langle D^2 \rangle\!\rangle$ acts properly discontinuously on $\mathbb{R}^2$. The quotient orbifold is of the form $S_{0,0,4^{(2)}}$, a sphere with four double points. The action of $\langle A,B,C\rangle/\langle\!\langle D^2 \rangle\!\rangle$ determines a planar symmetry group of type **p 2** of index four in $\mathrm{Isom}(\mathbb{Z}^2)$. We take $\mathrm{hull}[\mathbf{0},\mathbf{x},\mathbf{y}]$ as a fundamental domain, and get a triangulation of $\mathbb{R}^2$; see Figure 5.3.1. (The labelling is discussed in Definitions 5.5.)

The quotient orbifold of $\mathbb{R}^2 - \mathbb{Z}^2$ is of the form $S_{0,0,3^{(2)}+1}$, a once-punctured sphere with three double points.

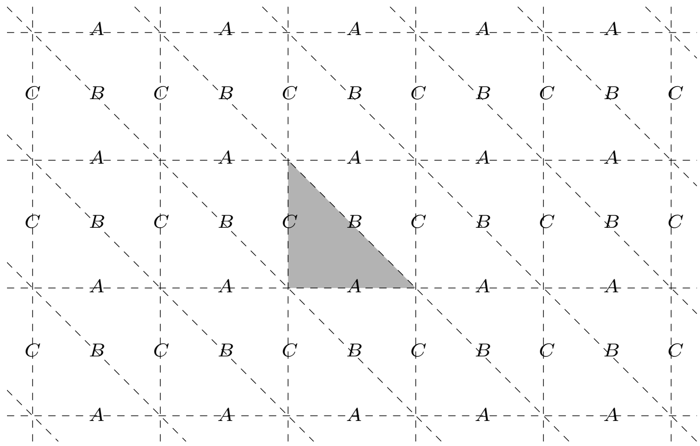


Figure 5.3.1: The $\langle A, B, C\rangle / \langle\!\langle D^2 \rangle\!\rangle$-tessellation of $\mathbb{R}^2$

(We digress to remark that the specified action of $\mathrm{Aut}\langle A, B, C\rangle$ on $\mathbb{R}^2 - \mathbb{Z}^2$ determines an isomorphism from $\mathrm{Aut}\langle A, B, C\rangle / \langle A, B, C\rangle$ to the mapping class group of $S_{0,0,3^{(2)}+1}$, that is, the group of all admissible self-homeomorphisms of $S_{0,0,3^{(2)}+1}$ modulo isotopy. The specified action chooses a self-homeomorphism in each isotopy class.)

Using Notation 5.1, we find that $\langle A, B, C\rangle \backslash \mathbb{C}_+$ and $\langle A, B, C\rangle \backslash (\mathbb{R}^2 - \mathbb{Z}^2)$ are non-canonically homeomorphic orbifolds, of the form $S_{0,0,3^{(2)}+1}$. Lifting to the branched coverings, we find that $\langle\!\langle D^2 \rangle\!\rangle \backslash \mathbb{C}_+$ and $\mathbb{R}^2 - \mathbb{Z}^2$ are $\langle A, B, C\rangle$-homeomorphic spaces. It follows that there exists an $\langle A, B, C\rangle$-homeomorphism

$$\langle\!\langle D^2 \rangle\!\rangle \backslash (\mathbb{C}_+ \cup \mathbb{P}) \xrightarrow{\sim} \mathbb{R}^2, \tag{5.3.1}$$

where $\mathbb{R}^2$ is now viewed as an orbifold in which the elements of $\mathbb{Z}^2$ are infinite-cyclic points.

We want to convert the $\langle A, B, C\rangle$-map (5.3.1) into an $\mathrm{Aut}\langle A, B, C\rangle$-map. It can be shown that $\langle\!\langle D^2 \rangle\!\rangle \backslash \mathbb{P} \xrightarrow{\sim} \mathbb{Z}^2$ is already an $\mathrm{Aut}\langle A, B, C\rangle$-map of the form

$$\begin{aligned}\langle\!\langle D^2 \rangle\!\rangle \backslash \mathbb{P} &\simeq \langle\!\langle D^2 \rangle\!\rangle \backslash \langle A, B, C\rangle / \langle D\rangle \\ &\simeq \langle\!\langle D^2 \rangle\!\rangle \backslash \langle BC, BA\rangle / \langle D^2\rangle \\ &\simeq \langle\!\langle D^2 \rangle\!\rangle \backslash \langle BC, BA\rangle \simeq \mathbb{Z}^2.\end{aligned}$$

It is now straightforward to give $\mathbb{C}_+ \cup\ \mathbb{P}$ the structure of an $\mathrm{Aut}\langle A, B, C\rangle$-space in such a way that (5.3.1) is an $\mathrm{Aut}\langle A, B, C\rangle$-map. It can then be shown that $\mathbb{C}_+ \cup\ \hat{\mathbb{R}}$ is an $\mathrm{Aut}\langle A, B, C\rangle$-space.

Using complex conjugation, we can make all of $\hat{\mathbb{C}}$ into an $\mathrm{Aut}\langle A, B, C\rangle$-space, and we get

$$\langle\!\langle D^2 \rangle\!\rangle \backslash (\mathbb{C}_- \cup \mathbb{P}) \xrightarrow{\sim} \mathbb{R}^2. \tag{5.3.2}$$

We will assume throughout that we have fixed $\mathrm{Aut}\langle A, B, C\rangle$-homeomorphisms (5.3.1) and (5.3.2), and that (5.3.2) is the complex conjugate of (5.3.1). $\square$

**5.4 Definition.** By the *Farey representation* we mean the element $\rho_0 \in \mathfrak{R}_{\hat{\mathbb{R}}}$ given by Table 5.4.1.

| $W$ | $A$ | $B$ | $C$ | $D$ |
|---|---|---|---|---|
| $\rho_0(W)$ | $-\frac{1}{z}$ | $\frac{z-1}{2z-1}$ | $\frac{z-2}{z-1}$ | $z+3$ |

Table 5.4.1: The Farey action of $\langle A, B, C\rangle$ on $\hat{\mathbb{C}}$.

Thus $\rho_0(D)$ is parabolic with fixed point $\infty$. The set of parabolic points of $\rho_0(\langle A, B, C\rangle)$ is then $\hat{\mathbb{Q}}$ $(= \mathbb{Q} \cup \{\infty\})$.

There exists an orientation-preserving self-homeomorphism $\alpha$ of $\hat{\mathbb{C}}$ which transforms the $\langle A, B, C\rangle$-action via $\rho_{\hat{\mathbb{R}}}$ into the $\langle A, B, C\rangle$-action via $\rho_0$, that is, $\rho_0$ and $\rho_{\hat{\mathbb{R}}}$ are 'topologically conjugate'. If we temporarily view $\rho_0$ and $\rho_{\hat{\mathbb{R}}}$ as homomorphisms from $\langle A, B, C\rangle$ to the group of all self-homeomorphisms of $\hat{\mathbb{C}}$, then $\rho_{\hat{\mathbb{R}}}$ followed by conjugation by $\alpha$ gives $\rho_0$. One can construct the restrictions of $\alpha$ to $\mathbb{C}_+$, $\mathbb{C}_-$, and $\hat{\mathbb{R}}$, separately, using the observations in Definitions 5.2 and Notations 5.3.

It follows that, for our purposes, there would be no loss of generality in assuming that $\rho_{\hat{\mathbb{R}}} = \rho_0$ and, hence, $\mathbb{P} = \hat{\mathbb{Q}}$ and $q_{\hat{\mathbb{R}}} = \infty$. □

We now pick out some special points of $\hat{\mathbb{R}}$.

**5.5 Definitions.** Consider the $\langle A, B, C\rangle$-action on $\mathbb{R}^2$.

We label the closed one-cell hull$[\mathbf{0}, \mathbf{x}]$ with the letter $A$; notice that this one-cell is rotated 180° by $A$. We label hull$[\mathbf{x}, \mathbf{y}]$ and hull$[\mathbf{y}, \mathbf{0}]$ with the letters $B$ and $C$, respectively. We now tessellate $\mathbb{R}^2$ with the images under $\langle A, B, C\rangle$ of the labelled triangle hull$[\mathbf{0}, \mathbf{x}, \mathbf{y}]$. Thus the horizontal lines are labelled with $A$'s, the diagonal lines with $B$'s, and the vertical lines with $C$'s, as in Figure 5.3.1.

We think of each zero-cell in $\mathbb{Z}^2$ as an infinitesimally small circle divided into twelve parts, one part in each of the incident one-cells and two-cells. We will be particularly interested in the two parts which subtend angles of 90°; we will call these *quarters* of the point. The *base quarter-point* is the quarter of $\mathbf{0}$ that lies in hull$[\mathbf{0}, \mathbf{x}, \mathbf{y}]$. For any $W \in \langle A, B, C\rangle$, there is a (possibly self-intersecting) path in $\mathbb{R}^2 - \mathbb{Z}^2$ joining the base quarter-point to its image under $W$, such that $W$ is recovered by reading off the labels on the one-cells crossed by the path; we say that the path *reads* $W$. For example, $BCBC$ and $BCA$ carry the base quarter-point to different quarters of $2\mathbf{x}$. Conversely, any path in $\mathbb{R}^2 - \mathbb{Z}^2$ joining the base quarter-point to a quarter-point, and crossing only finitely many one-cells, arises in this way.

Consider any irrational-sloped straight line in $\mathbb{R}^2$, consider its inverse image in $\mathbb{C}_+ \cup \mathbb{P}$ under (5.3.1), and consider a connected component of this inverse image.

If the original straight line is disjoint from $\mathbb{Z}^2$, then the connected component is an open arc in $\mathbb{C}_+$ whose closure in $\hat{\mathbb{C}}$ adds two points of $\hat{\mathbb{R}}$ to the arc. Any such point can be expressed in the form $\widehat{\rho_{\hat{\mathbb{R}}}}(\mathcal{W})$ such that $\mathcal{W}$ $(\in \mathfrak{E})$ is read by a path in $\mathbb{R}^2 - \mathbb{Z}^2$ that starts at the base quarter-point, eventually reaches the straight line, and travels along the straight line to infinity in the appropriate direction.

If the original straight line meets $\mathbb{Z}^2$ (in a single point), then the above connected component is called a *footless spider*, and it consists of a point, called the *head*, in $\mathbb{P}$, and a single infinite-cyclic orbit of open arcs, called the *legs*, in $\mathbb{C}_+$. The closure in $\hat{\mathbb{C}}$ of each leg is a closed arc with one endpoint being the head and the other endpoint lying in $\hat{\mathbb{R}} - \mathbb{P}$, and called a *foot*. The footless spider together with all the feet is called a *spider*.

Similar considerations apply to $\mathbb{C}_- \cup \mathbb{P}$. Notice that the preimages in $\mathbb{C}_- \cup \mathbb{P}$ are obtained by complex conjugation, and the resulting compactification would give the same points.

By a *double spider*, we mean the union, in $\hat{\mathbb{C}}$, of two spiders with a common head (in $\mathbb{P}$). □

# 6 The model $\mathbb{S}^2(\mathcal{F})$ of $\hat{\mathbb{C}}(\rho_{\hat{\mathbb{C}}}, q_{\hat{\mathbb{C}}})$

**6.1 Notation.** Let $\mathcal{F} \in \mathrm{IrAutABCMaps}$, and let $\mathfrak{f} = \mathrm{quotient}(\mathcal{F})$. □

**6.2 Notation.** Suppose that Notation 6.1 holds, and that $\mathcal{F}_0 = 1$. Thus $D_{\mathcal{F}} = D$.

Recall that $\mathbf{0} = (0,0)$, $\mathbf{x} = (1,0)$, $\mathbf{y} = (0,1)$. For $x, y \in \mathbb{R}$, $x\mathbf{x} + y\mathbf{y} = (x,y)$.

For $\mathbf{v} = (x,y) \in \mathbb{R}^2$, we define $|\mathbf{v}| = \sqrt{x^2+y^2}$.

For $\mathbf{v} \in \mathbb{R}^2 - \{\mathbf{0}\}$, we write $h_{\mathbf{v}} := \{t\mathbf{v} \mid t \in [0,\infty[\ \}$. Subsets of $\mathbb{R}^2$ of this form are called *radial lines*. A radial line is said to be *rational*, resp. *irrational*, if its slope is rational, resp. irrational.

If $h_-$, $h_+$ are radial lines, we define $[h_-, h_+] := \bigcup\limits_{\theta \in [\theta_-, \theta_+]} h(\theta)$ for any $\theta_-, \theta_+ \in \mathbb{R}$ such that $0 \le \theta_+ - \theta_- < 2\pi$, $h(\theta_-) = h_-$ and $h_+ = h(\theta_+)$, where we are writing $h(\theta)$ for $h_{\cos(\theta)\mathbf{x}+\sin(\theta)\mathbf{y}}$.

Let $n \in \mathbb{Z}$.

We will study the action of $\mathcal{F}_n$ on $\mathbb{R}^2$. We write $\mathbf{x}_n = \mathcal{F}_n(\mathbf{x})$, $\mathbf{y}_n = \mathcal{F}_n(\mathbf{y})$. Notice that $\mathcal{F}_n$ lies in $\langle R, L\rangle$, hence $\mathcal{F}_n$ acts linearly on $\mathbb{R}^2$, hence $\mathcal{F}_n$ permutes the set of radial lines. Thus, for all $\mathbf{v} \in \mathbb{R}^2 - \{\mathbf{0}\}$, $\mathcal{F}_n(h_{\mathbf{v}}) = h_{\mathcal{F}_n(\mathbf{v})}$, and $\mathcal{F}_n[h_{\mathbf{x}}, h_{\mathbf{y}}] = [h_{\mathbf{x}_n}, h_{\mathbf{y}_n}]$. Since $\mathcal{F}_n$ carries half-planes to half-planes, and the quadrant $[h_{\mathbf{x}}, h_{\mathbf{y}}]$ is strictly contained in a half-plane, we see that $[h_{\mathbf{x}_n}, h_{\mathbf{y}_n}]$ is strictly contained in a half-plane. Thus, for any $\mathbf{v} \in [h_{\mathbf{x}_n}, h_{\mathbf{y}_n}]$, there exist non-negative real numbers $p$, $q$, such that $\mathbf{v} = p\mathbf{x}_n + q\mathbf{y}_n$. □

The following gives standard facts, usually expressed in terms of the Möbius action of $\mathrm{PSL}_2(\mathbb{Z})$ on $\hat{\mathbb{R}}$.

**6.3 Lemma.** *Suppose that* Notation 6.1 *holds, and that* $\mathcal{F}_0 = 1$.

(i) $(\mathcal{F}_n[h_{\mathbf{x}}, h_{\mathbf{y}}] \mid n \in \mathbb{Z})$ *is a strictly decreasing chain of subsets of* $\mathbb{R}^2$, *and its intersection,* $\bigcap\limits_{n\in\mathbb{Z}} \mathcal{F}_n[h_{\mathbf{x}}, h_{\mathbf{y}}]$, *is an irrational radial line.*

(ii) $(\mathcal{F}_n[h_{-\mathbf{y}}, h_{\mathbf{x}}] \mid n \in \mathbb{Z})$ *is a strictly increasing chain of subsets of* $\mathbb{R}^2$, *and its intersection,* $\bigcap\limits_{n\in\mathbb{Z}} \mathcal{F}_n[h_{-\mathbf{y}}, h_{\mathbf{x}}]$, *is an irrational radial line.*

*Proof.* (i). Let $n$ be a variable ranging over $\mathbb{Z}$.

Since $\mathcal{F}_n[h_{\mathbf{x}}, h_{\mathbf{y}}] \in \{\mathcal{F}_{n-1}R[h_{\mathbf{x}}, h_{\mathbf{y}}], \mathcal{F}_{n-1}L[h_{\mathbf{x}}, h_{\mathbf{y}}]\}$, and

$$R[h_{\mathbf{x}}, h_{\mathbf{y}}] = [h_{\mathbf{x}}, h_{\mathbf{x}+\mathbf{y}}] \subset [h_{\mathbf{x}}, h_{\mathbf{y}}], \quad L[h_{\mathbf{x}}, h_{\mathbf{y}}] = [h_{\mathbf{x}+\mathbf{y}}, h_{\mathbf{y}}] \subset [h_{\mathbf{x}}, h_{\mathbf{y}}],$$

it follows that $\mathcal{F}_n[h_{\mathbf{x}}, h_{\mathbf{y}}] \subset \mathcal{F}_{n-1}[h_{\mathbf{x}}, h_{\mathbf{y}}]$. This proves that the sequence is strictly decreasing.

Let $n$ be a variable ranging over $\mathbb{N}$.

Let $\theta_n$ denote the angle between $h_{\mathbf{x}_n}$ and $h_{\mathbf{y}_n}$. By the foregoing, $(\theta_n \mid n \in \mathbb{N})$ is a strictly decreasing sequence. Since $\mathcal{F}_0 = 1$, $\theta_0 = \frac{\pi}{2}$. It follows that $\theta_n \in \left]0, \frac{\pi}{2}\right]$.

The area of hull$[\mathbf{0}, \mathbf{x}_n, \mathbf{y}_n, \mathbf{x}_n + \mathbf{y}_n]$ is $(\sin\theta_n)\,|\mathbf{x}_n|\,|\mathbf{y}_n|$, and the area of

$$\text{hull}[\mathbf{0}, \mathbf{x}, \mathbf{y}, \mathbf{x} + \mathbf{y}]$$

is 1. Since hull$[\mathbf{0}, \mathbf{x}_n, \mathbf{y}_n, \mathbf{x}_n + \mathbf{y}_n] = \mathcal{F}_n(\text{hull}[\mathbf{0}, \mathbf{x}, \mathbf{y}, \mathbf{x}+\mathbf{y}])$, and $\mathcal{F}_n$ preserves area, we see that $(\sin\theta_n)\,|\mathbf{x}_n|\,|\mathbf{y}_n| = 1$. Therefore

$$(\sin\theta_n)^{-2} = |\mathbf{x}_n|^2\,|\mathbf{y}_n|^2 \in \mathbb{N} - \{0\}.$$

Hence the map

$$\mathbb{N} \to \mathbb{N} - \{0\}, \quad n \mapsto (\sin\theta_n)^{-2} = |\mathbf{x}_n|^2\,|\mathbf{y}_n|^2,$$

is a strictly increasing sequence in a discrete set. Therefore,

$$\lim_{n\to\infty}(\sin\theta_n)^{-2} = \lim_{n\to\infty}|\mathbf{x}_n|^2\,|\mathbf{y}_n|^2 = \infty.$$

Thus $\lim\limits_{n\to\infty}\theta_n = 0$. Hence the intersection of the chain is a radial line; let us denote it by $h$.

To see that $h$ is irrational, let us suppose that $h$ is rational. Then there exists some $\mathbf{v} \in h \cap (\mathbb{Z}^2 - \{\mathbf{0}\})$. Now $\mathbf{v} \in [h_{\mathbf{x}_n}, h_{\mathbf{y}_n}]$, and hence there exist non-negative real numbers $p_n$, $q_n$ such that $\mathbf{v} = p_n\mathbf{x}_n + q_n\mathbf{y}_n$. Since we are in the quadrant $[h_{\mathbf{x}}, h_{\mathbf{y}}]$, we see that $|\mathbf{v}| \geq |p_n\mathbf{x}_n|$ and $|\mathbf{v}| \geq |q_n\mathbf{y}_n|$, and thus $|\mathbf{v}|^2 \geq p_nq_n\,|\mathbf{x}_n|\,|\mathbf{y}_n|$. Hence, $\lim\limits_{n\to\infty} p_nq_n = 0$. Also

$$(p_n, q_n) = p_n\mathbf{x} + q_n\mathbf{y} = \mathcal{F}_n^{-1}(p_n\mathbf{x}_n + q_n\mathbf{y}_n) = \mathcal{F}_n^{-1}(\mathbf{v}) \in \mathbb{Z}^2.$$

Thus, for some $N \in \mathbb{N}$, $p_Nq_N = 0$. If $p_N = 0$, then $h = h_{\mathbf{v}} = h_{q_N\mathbf{y}_N} = h_{\mathbf{y}_N}$, and $\mathbf{y}_n = \mathbf{y}_N$ for all $n \geq N$, and $\mathfrak{f}_n = L$ for all $n > N$. This contradicts the fact that $\mathcal{F} \in$ IrAutABCMaps. A similar argument holds if $q_N = 0$. Hence (i) holds.

(ii). Let $\mathcal{H} \in$ IrAutABCMaps be given by $n \mapsto P\mathcal{F}_{-n}P^{-1}$. By applying (i) with $\mathcal{H}$ in place of $\mathcal{F}$, we deduce (ii). □

Let us briefly rework the definitions from [5, Section 3], without giving proofs.

**6.4 Definitions.** Suppose that Notation 6.1 holds, and that $\mathcal{F}_0 = 1$.

Thus $D_{\mathcal{F}} = D$.

Let $h_- = h_-(\mathcal{F}) := \bigcap\limits_{n\in\mathbb{Z}} \mathcal{F}_n[h_{-\mathbf{y}}, h_{\mathbf{x}}]$, and $h_+ = h_+(\mathcal{F}) := \bigcap\limits_{n\in\mathbb{Z}} \mathcal{F}_n[h_{\mathbf{x}}, h_{\mathbf{y}}]$. By Lemma 6.3, $h_-$ and $h_+$ are irrational radial lines. For each $m \in \mathbb{Z}$, we define

$h_{2m-1}(\mathcal{F}) := D^m h_-$ and $h_{2m}(\mathcal{F}) := D^m h_+$. Then $(h_m(\mathcal{F}) \mid m \in \mathbb{Z})$ consists of four radial lines repeating cyclically.

Let $\langle D\rangle h_-$ and $\langle D\rangle h_+$ denote the (whole) lines containing $h_+$ and $h_-$, respectively. The vector-space decomposition $\mathbb{R}^2 = \langle D\rangle h_+ \oplus \langle D\rangle h_-$ gives vector-space retractions

$$\pi_+ \colon \mathbb{R}^2 \twoheadrightarrow \langle D\rangle h_+, \quad \pi_- \colon \mathbb{R}^2 \twoheadrightarrow \langle D\rangle h_-,$$

such that $\pi_+(\langle D\rangle h_-) = \pi_-(\langle D\rangle h_+) = \{\mathbf{0}\}$.

Since $\langle A, B, C\rangle$ permutes the fibers of $\pi_+$, we can view $\langle D\rangle h_+$ as an $\langle A, B, C\rangle$-quotient space of $\mathbb{R}^2$. A similar statement holds for $\langle D\rangle h_-$.

We now apply Notation 5.3. Let $\Pi_- : \mathbb{C}_+ \cup \mathbb{P} \twoheadrightarrow \langle D\rangle h_-$ denote the composition

$$\mathbb{C}_+ \cup \mathbb{P} \twoheadrightarrow \langle\!| D^2 |\!\rangle\backslash(\mathbb{C}_+ \cup \mathbb{P}) \xrightarrow{\sim} \mathbb{R}^2 \overset{\pi_-}{\twoheadrightarrow} \langle D\rangle h_-.$$

Then $\Pi_-$ is a surjective $\langle A, B, C\rangle$-map, and each fiber of $\Pi_-$ has countably many components. Each component of each $\Pi_-$-fiber over $\langle D\rangle h_- - \pi_-(\mathbb{Z}^2)$ is an open arc, and taking its closure in $\hat{\mathbb{C}}$ adds two endpoints, in $\hat{\mathbb{R}} - \mathbb{P}$. Each component of each $\Pi_-$-fiber over $\pi_-(\mathbb{Z}^2)$ is a footless spider, and taking its closure in $\hat{\mathbb{C}}$ adds the feet, in $\hat{\mathbb{R}} - \mathbb{P}$. See Definitions 5.5.

Let $\mathfrak{L}_+$ denote the set consisting of the components of the fibers of $\Pi_-$; thus, $\mathfrak{L}_+$ partitions $\mathbb{C}_+ \cup \mathbb{P}$ into footless spiders and open arcs.

On applying complex conjugation and replacing $\pi_-$ with $\pi_+$, we get a map $\Pi_+$, which is a composition of the following form:

$$\mathbb{C}_- \cup \mathbb{P} \twoheadrightarrow \langle\!| D^2 |\!\rangle\backslash(\mathbb{C}_- \cup \mathbb{P}) \xrightarrow{\sim} \mathbb{R}^2 \overset{\pi_+}{\twoheadrightarrow} \langle D\rangle h_+.$$

As before, we get a partition $\mathfrak{L}_-$ of $\mathbb{C}_- \cup \mathbb{P}$ into footless spiders and open arcs.

Let $\mathfrak{L} = \mathfrak{L}(\mathcal{F})$ denote the set consisting of the following: all the double spiders formed from elements of $\mathfrak{L}_+ \cup \mathfrak{L}_-$, one double spider for each element of $\mathbb{P}$; all the closed arcs which are closures of open arcs belonging to $\mathfrak{L}_+ \cup \mathfrak{L}_-$; all the one-element subsets of $\hat{\mathbb{R}}$ that do not lie in any of the foregoing double spiders and closed arcs. Thus $\mathfrak{L}$ is a partition of $\hat{\mathbb{C}}$, and it is closed under the $\langle A, B, C\rangle$-action.

We now define an oriented $\langle A, B, C\rangle$-space $\mathbb{S}^2 = \mathbb{S}^2(\mathcal{F})$. As a set, $\mathbb{S}^2$ is just $\mathfrak{L}$. As a topological space, $\mathbb{S}^2$ is the quotient space of $\hat{\mathbb{C}}$ obtained by collapsing each element of $\mathfrak{L}$ to a point. By a theorem of R. L. Moore, $\mathbb{S}^2$ is a two-sphere; see, for example, [5, Appendix]. Moreover, $\mathbb{S}^2$ inherits an orientation from $\hat{\mathbb{C}}$ since the collapsing respects the two-sidedness of $\hat{\mathbb{C}}$.

The unique element of $\mathfrak{L}$ fixed by $D$, denoted $\mathfrak{s} = \mathfrak{s}(\mathcal{F})$, is the double spider in $\mathfrak{L}$ with head $q_{\hat{\mathbb{R}}}$. We denote the corresponding element of $\mathbb{S}^2$ by $q_{\mathbb{S}^2}$.

For $m \in \mathbb{Z} \cup \{-\infty, \infty\}$, we define $\mathcal{W}_m = \mathcal{W}_m(\mathcal{F})$ as follows. For each irrational radial line $h$ in $[h_{-\mathbf{y}}, h_{-\mathbf{x}}]$, let $\mathcal{W}(h)$ denote the element of $\mathfrak{E}$ read by $h$, as in Definitions 5.5. For each $m \in \mathbb{Z}$, we define

$$\mathcal{W}_{2m-1} := D^m \mathcal{W}(h_-), \quad \mathcal{W}_{2m} := D^m \mathcal{W}(h_+),$$

and we set $\mathcal{W}_{-\infty} := D^{-\infty}$, $\mathcal{W}_\infty := D^\infty$.

For $m \in \mathbb{Z} \cup \{\pm\infty\}$, we define $w_m = w_m(\mathcal{F})$, as follows. For each $m \in \mathbb{Z}$, we let $w_m := \widehat{\rho_{\hat{\mathbb{R}}}}(\mathcal{W}_m)$. We set $w_{\pm\infty} := \widehat{\rho_{\hat{\mathbb{R}}}}(\mathcal{W}_{-\infty}) = \widehat{\rho_{\hat{\mathbb{R}}}}(\mathcal{W}_{+\infty}) = q_{\hat{\mathbb{R}}}$. Then the head of the double spider $\mathfrak{s}$ is $w_{\pm\infty} = q_{\hat{\mathbb{R}}}$, and the set of feet of $\mathfrak{s}$ is $\{w_m \mid m \in \mathbb{Z}\}$. The cyclic indexing of $\mathfrak{s} \cap \hat{\mathbb{R}} = \{w_m \mid m \in \mathbb{Z} \cup \{\pm\infty\}\}$ is compatible with the cyclic ordering of $\hat{\mathbb{R}}$, since $\rho_{\hat{\mathbb{R}}}(D)$ shifts the points of $\hat{\mathbb{R}} - \{q_{\hat{\mathbb{R}}}\}$ cyclically upward. □

We now remove the restriction $\mathcal{F}_0 = 1$.

**6.5 Definitions.** Suppose that Notation 6.1 holds.

Definitions 6.4 apply to give a partition $\mathfrak{L}(\mathcal{F}_0^{-1}\mathcal{F})$ of $\hat{\mathbb{C}}$, closed under the action of $\langle A, B, C\rangle$. By Notation 5.3, we have an $\mathrm{Aut}\langle A, B, C\rangle$-action on $\hat{\mathbb{C}}$ by continuous maps. On applying $\mathcal{F}_0$ to $\hat{\mathbb{C}}$, we get a partition $\mathfrak{L} = \mathfrak{L}(\mathcal{F}) := \mathcal{F}_0(\mathfrak{L}(\mathcal{F}_0^{-1}\mathcal{F}))$ of $\hat{\mathbb{C}}$, closed under the action of $\langle A, B, C\rangle$. As in Definitions 6.4, $\mathbb{S}^2 = \mathbb{S}^2(\mathcal{F})$ is defined to be the quotient space of $\hat{\mathbb{C}}$ obtained by collapsing each element of $\mathfrak{L}$ to a point. Up to $\langle A, B, C\rangle$-homeomorphism, $\mathbb{S}^2$ will be independent of the choice of $\rho_{\hat{\mathbb{R}}}$ within $\mathfrak{R}_{\hat{\mathbb{R}}}$. We denote the quotient map by $\kappa = \kappa_{\mathcal{F}} \colon \hat{\mathbb{C}} \twoheadrightarrow \mathbb{S}^2$. Thus $\mathbb{S}^2$ is an $\langle A, B, C\rangle$-space and $\kappa$ is an $\langle A, B, C\rangle$-map. Let $\rho_{\mathbb{S}^2}$ denote the implicit homomorphism from $\langle A, B, C\rangle$ to the group of all self-homeomorphisms of $\mathbb{S}^2$.

The action of $\mathcal{F}_0$ on $\hat{\mathbb{C}}$ induces a homeomorphism $\mathcal{F}_0 \colon \mathbb{S}^2(\mathcal{F}_0^{-1}\mathcal{F}) \to \mathbb{S}^2(\mathcal{F})$ such that $\mathcal{F}_0(Wp) = \mathcal{F}_0(W)\mathcal{F}_0(p)$, for all $W \in \langle A, B, C\rangle$ and all $p \in \mathbb{S}^2(\mathcal{F}_0^{-1}\mathcal{F})$. In particular, $\mathbb{S}^2(\mathcal{F})$ is an oriented sphere.

We define $q_{\mathbb{S}^2} = q_{\mathbb{S}^2(\mathcal{F})} := \mathcal{F}_0(q_{\mathbb{S}^2(\mathcal{F}_0^{-1}\mathcal{F})})$, the unique point of $\mathbb{S}^2$ fixed by $D_{\mathcal{F}}$. We define $\mathfrak{s} = \mathfrak{s}(\mathcal{F}) := \mathcal{F}_0(\mathfrak{s}(\mathcal{F}_0^{-1}\mathcal{F})) = \kappa^{-1}\{q_{\mathbb{S}^2}\}$, the unique element of $\mathfrak{L}(\mathcal{F})$ fixed by $D_{\mathcal{F}}$.

For each $m \in \mathbb{Z} \cup \{-\infty, \infty\}$, we let $\mathcal{W}_m(\mathcal{F}) := \mathcal{F}_0(\mathcal{W}_m(\mathcal{F}_0^{-1}\mathcal{F}))$.

For each $m \in \mathbb{Z} \cup \{\pm\infty\}$, we let $w_m(\mathcal{F}) := \mathcal{F}_0(w_m(\mathcal{F}_0^{-1}\mathcal{F}))$. These give the head and feet of the double spider $\mathfrak{s}$, as before.

Let $\mathrm{CT} = \mathrm{CT}(\rho_{\hat{\mathbb{R}}}, \rho_{\mathbb{S}^2})$ denote the composition $\hat{\mathbb{R}} \hookrightarrow \hat{\mathbb{C}} \overset{\kappa}{\twoheadrightarrow} \mathbb{S}^2$. Each element of $\mathfrak{L}$ contains an element of $\hat{\mathbb{R}}$, so CT is a surjective $\langle A, B, C\rangle$-map.

Let $\widehat{\rho_{\mathbb{S}^2}}$ denote the composition $\mathfrak{E} \overset{\widehat{\rho_{\hat{\mathbb{R}}}}}{\twoheadrightarrow} \hat{\mathbb{R}} \hookrightarrow \hat{\mathbb{C}} \overset{\kappa}{\twoheadrightarrow} \mathbb{S}^2$. Again, this is a surjective $\langle A, B, C\rangle$ map: it is the composite $\mathfrak{E} \overset{\widehat{\rho_{\hat{\mathbb{R}}}}}{\twoheadrightarrow} \hat{\mathbb{R}} \overset{\mathrm{CT}}{\twoheadrightarrow} \mathbb{S}^2$. □

# 7 The model $\mathrm{CW}(\mathcal{F})$ of $\mathrm{CW}(\rho_{\hat{\mathbb{C}}}, q_{\hat{\mathbb{C}}})$

**7.1 Notation.** Let $\mathcal{F} \in \mathrm{IrAutABCMaps}$.

Let $\mathfrak{f} = \mathrm{quotient}(\mathcal{F})$ and $\mathcal{F}' = \mathrm{residue}(\mathcal{F})$.

Let $\mathfrak{L} = \mathfrak{L}(\mathcal{F})$, $\mathbb{S}^2 = \mathbb{S}^2(\mathcal{F})$, $\kappa = \kappa_{\mathcal{F}} \colon \hat{\mathbb{C}} \twoheadrightarrow \mathbb{S}^2$, $\mathrm{CT} = \mathrm{CT}(\rho_{\hat{\mathbb{R}}}, \rho_{\mathbb{S}^2}) \colon \hat{\mathbb{R}} \twoheadrightarrow \mathbb{S}^2$, $\widehat{\rho_{\mathbb{S}^2}} = \widehat{\rho_{\mathbb{S}^2(\mathcal{F})}} \colon \mathfrak{E} \twoheadrightarrow \mathbb{S}^2$, $q_{\mathbb{S}^2} = q_{\mathbb{S}^2(\mathcal{F})}$ and $\mathfrak{s} = \mathfrak{s}(\mathcal{F}) = \kappa^{-1}\{q_{\mathbb{S}^2}\}$.

Let $\bar{\mathfrak{s}}$ denote the image of $\mathfrak{s}$ under complex conjugation; thus $\mathfrak{s} \cup \bar{\mathfrak{s}}$ is a quadruple spider.

For each $m \in \mathbb{Z} \cup \{-\infty, +\infty\}$, let $\mathcal{W}_m = \mathcal{W}_m(\mathcal{F})$.

For each $m \in \mathbb{Z} \cup \{\pm\infty\}$, let $w_m = w_m(\mathcal{F})$.

For each $m \in \mathbb{Z}$, let $\ell_m$ denote the closed leg of $\mathfrak{s}$ incident to the foot $w_m$; thus $\ell_m \subseteq \mathbb{C}_{\mathrm{sign}(-1)^m} \cup \hat{\mathbb{R}}$, where $\mathrm{sign}(-1)^m$ is $+$ if $m$ is even, and $-$ if $m$ is odd. Let $\overline{\ell_m}$ denote the image of $\ell_m$ under complex conjugation. It is not difficult to see that $\ell_m \cup \overline{\ell_m}$ is a Jordan curve in $\hat{\mathbb{C}}$. Recall that $\kappa(\ell_m) = \kappa(\mathfrak{s}) = q_{\mathbb{S}^2}$; we write

$\partial_m := \kappa(\ell_m \cup \overline{\ell_m}) = \kappa(\overline{\ell_m})$. In this section, we will see that $\{\partial_m \mid m \in \mathbb{Z}\}$ is a set of arcs which etch a CW-structure, denoted $\mathrm{CW}(\mathcal{F})$, on the plane $\mathbb{S}^2 - \{q_{\mathbb{S}^2}\}$.

When $\mathcal{F}_0 = 1$, we will further understand that $h_- = h_-(\mathcal{F})$, $h_+ = h_+(\mathcal{F})$, and $h_m = h_m(\mathcal{F})$ for each $m \in \mathbb{Z}$, as in Definitions 6.4. Notice that $\ell_m - \{w_m\}$ maps homeomorphically to $h_m$ under the chosen map

$$\mathbb{C}_{\mathrm{sign}(-1)^m} \cup \mathbb{P} \twoheadrightarrow \langle\!\langle D^2 \rangle\!\rangle \backslash (\mathbb{C}_{\mathrm{sign}(-1)^m} \cup \mathbb{P}) \xrightarrow{\sim} \mathbb{R}^2. \qquad \square$$

**7.2 Remarks.** Given elements $u$, $v$, $w$, of $\hat{\mathbb{R}}$, we write $u \prec v \prec w$ if $u \neq v$, $v \neq w$ and $u$, $v$, $w$ are in cyclic order, that is, $v$ is reached before $w$ in travelling cyclically starting from $u$.

Given elements $u$, $w$, of $\hat{\mathbb{R}}$, we write

$$]u,\, w[\ := \{v \in \hat{\mathbb{R}} \mid u \prec v \prec w\}, \quad \text{and} \quad [u,\, w] :=\ ]u,\, w[\ \cup \{u, w\}.$$

The former is called an *open cyclic interval* in $\hat{\mathbb{R}}$, while the latter is a *closed cyclic interval* in $\hat{\mathbb{R}}$.

Notice that $\mathrm{CT}^{-1}\{q_{\mathbb{S}^2}\} = \{w_m \mid m \in \mathbb{Z} \cup \{\pm\infty\}\}$; and

$$\hat{\mathbb{R}} - \mathrm{CT}^{-1}\{q_{\mathbb{S}^2}\} = \bigcup_{m \in \mathbb{Z}} \ ]w_{m-1}, w_m[\ \ ,$$

a disjoint union of open cyclic intervals in $\hat{\mathbb{R}}$. In this section, we study the subsets $\mathrm{CT}(\ ]w_{m-1}, w_m[\ )$ of $\mathbb{S}^2$. $\square$

We will want to speak of cyclic intervals in $\mathfrak{E}$, as suggested by Figure 5.2.1.

**7.3 Definition.** We can carefully pull back the cyclic ordering of $\hat{\mathbb{R}}$ along $\widehat{\rho_{\hat{\mathbb{R}}}} \colon \mathfrak{E} \twoheadrightarrow \hat{\mathbb{R}}$ to induce a cyclic ordering on $\mathfrak{E}$, described algebraically as follows. First we define a total order $<$ on $\mathfrak{E}$. The six two-letter, reduced words in $\langle A, B, C\rangle$ are ordered as:

$$AB < AC < BC < BA < CA < CB.$$

We then endow $\mathfrak{E}$ with the lexicographic order in which two distinct ends $\mathcal{W}$, $\mathcal{V}$, are compared by finding the least $n \in \mathbb{N}$ such that $\mathcal{W}_n^{-1}\mathcal{W}_{n+2} \neq \mathcal{V}_n^{-1}\mathcal{V}_{n+2}$, and then comparing this pair of two-letter words. This gives a total order on $\mathfrak{E}$, and we make it into a cyclic ordering in the obvious way. Given elements $\mathcal{U}$, $\mathcal{V}$, $\mathcal{W}$, we write $\mathcal{U} \prec \mathcal{V} \prec \mathcal{W}$ if $\mathcal{U} \neq \mathcal{V}$, $\mathcal{V} \neq \mathcal{W}$, and $\mathcal{U}$, $\mathcal{V}$, $\mathcal{W}$ are in cyclic order.

For $\mathcal{U}$, $\mathcal{W} \in \mathfrak{E}$, we write

$$]\mathcal{U},\, \mathcal{W}[\ := \{\mathcal{V} \in \mathfrak{E} \mid \mathcal{U} \prec \mathcal{V} \prec \mathcal{W}\}, \quad \text{and} \quad [\mathcal{U},\, \mathcal{W}] :=\ ]\mathcal{U},\, \mathcal{W}[\ \cup \{\mathcal{U}, \mathcal{W}\}.$$

Subsets of $\mathfrak{E}$ of the form $[\mathcal{U},\, \mathcal{W}]$ will be called *closed cyclic intervals* in $\mathfrak{E}$.

We define $[A] := \{\mathcal{W} \in \mathfrak{E} \mid \mathcal{W}_{|1} = A\}$, and similarly for $[B]$ and $[C]$. Thus

$$[A] = [(ABC)^\infty,\, (ACB)^\infty],$$
$$[B] = [(BCA)^\infty,\, (BAC)^\infty],$$
$$[C] = [(CAB)^\infty,\, (CBA)^\infty].$$

It can be shown that the $\mathrm{Aut}\langle A,B,C\rangle$-action on $\mathfrak{E}$ carries closed cyclic intervals to closed cyclic intervals. We recall that

$$A[A] = [B] \cup [C], \quad B[B] = [C] \cup [A], \quad C[C] = [A] \cup [B].$$

Further aspects of this action are recorded in Table 7.3.1 using [5, Lemma 7.1] and the notation $\overline{R} = R^{-1}$, $\overline{L} = L^{-1}$, and $\overline{P} = P^{-1}$; recall the definition of $P$ from Notation 2.3.

| $F$ | $F(A)$ | $F(B)$ | $F(C)$ | $F[A]$ | $F[B]$ | $F[C]$ |
|---|---|---|---|---|---|---|
| $\overline{L}$ | $ABA$ | $A$ | $C$ | $A[B]$ | $A[C]\cup[B]$ | $[C]$ |
| $\overline{R}$ | $A$ | $C$ | $CBC$ | $[A]$ | $[B]\cup C[A]$ | $C[B]$ |
| $R$ | $A$ | $BCB$ | $B$ | $[A]$ | $B[C]$ | $B[A]\cup[C]$ |
| $L$ | $B$ | $BAB$ | $C$ | $[A]\cup B[C]$ | $B[A]$ | $[C]$ |
| $P$ | $C$ | $CBC$ | $CBABC$ | $[A]\cup[B]$ | $C[A]\cup CB[C]$ | $CB[A]$ |
| $\overline{P}$ | $ABCBA$ | $ABA$ | $A$ | $AB[C]$ | $AB[A]\cup A[C]$ | $[B]\cup[C]$ |

Table 7.3.1: The action of $\mathrm{Aut}\langle A,B,C\rangle$ on subsets of $\mathfrak{E}$.

Moreover, $L$ and $R$ respect the total ordering of $\mathfrak{E}$, while $M$ reverses the total ordering of $\mathfrak{E}$. □

**7.4 Proposition.** *Suppose that* Notation 7.1 *holds. Then the following hold.*

(i) $(\mathcal{F}_n[B] \mid n\in\mathbb{Z})$ *is a strictly decreasing chain of closed cyclic intervals in* $\mathfrak{E}$.

(ii) $\bigcap_{n\in\mathbb{Z}} \mathcal{F}_n[B] = \{\mathcal{W}_0\}$.

(iii) $\bigcup_{n\in\mathbb{Z}} \mathcal{F}_n[B] = \;]\mathcal{W}_{-1}, \mathcal{W}_1[\,$.

*Proof.* We may replace $\mathcal{F}$ with $\mathcal{F}_0^{-1}\cdot\mathcal{F}$, and thus assume that $\mathcal{F}_0 = 1$. In particular, $D_{\mathcal{F}} = D$.

(i). From Table 7.3.1, we see that $R[B] = B[C] \subset [B]$ and $L[B] = B[A] \subset [B]$. For each $n\in\mathbb{Z}$, we see that $\mathcal{F}_n[B] = \mathcal{F}_{n-1}\mathfrak{f}_n[B] \subset \mathcal{F}_{n-1}[B]$. Hence (i) holds.

(ii). We first claim that $\bigcap_{n\in\mathbb{Z}} \mathcal{F}_n[B]$ contains at most one point. Let

$$S = \{n\in\mathbb{N} \mid \mathfrak{f}_n = L, \mathfrak{f}_{n+1} = R\}.$$

Then $S$ is infinite, by the definition of IrAutABCMaps. Consider any $n_0\in S$. To prove the claim, we may replace $\mathcal{F}$ with $\mathcal{F}_{n_0}^{-1}\cdot\mathcal{F}\cdot(z+n_0)$, and thus assume that $0\in S$. For any $a,b\in\mathbb{N}-\{0\}$,

$$R^aL^b(BC) = ((BC)^aBA)^bBC,$$
$$R^aL^b(BA) = (BC)^aBA,$$
$$R^aL^b([B]) \subseteq (BC)^a[B].$$

For any $n \in S$, we find that $\mathcal{F}_n([B]) \subseteq W_n[B]$, for some $W_n \in \langle A, B, C\rangle$ ending in $BC$. Moreover, $W_0 = 1$ and the length of $W_n$ increases strictly with $n$. By (i), all the $W_n$, $n \in S$, are initial segments of a single $\mathcal{W} \in \mathfrak{E}$, and $\bigcap\limits_{n\in\mathbb{Z}} \mathcal{F}_n[B] \subseteq \{\mathcal{W}\}$, as desired.

Let $N \in \mathbb{N}$. To complete the proof of (ii), it suffices to show that $\mathcal{W}_0 \in \mathcal{F}_N([B])$.

Let $H$ denote the set of all radial lines of irrational slope in the first quadrant of $\mathbb{R}^2$.

Consider any $h \in H$. Recall from Definitions 6.4 that $\mathcal{W}(h)$ denotes the element of $\mathfrak{E}$ read by $h$ in the labelled triangulation of $\mathbb{R}^2$; here $\mathcal{W}(h) \in [B]$. Now $h$ lifts uniquely to an arc $\widetilde{h} \subseteq \mathbb{C}_+ \cup \mathbb{P}$ such that $\widetilde{h}$ starts at $w_{\pm\infty}$ and the closure of $\widetilde{h}$ ends in $\widehat{\rho_{\hat{\mathbb{R}}}}([B])$; the closure of $\widetilde{h}$ is then $\widetilde{h} \cup \{\widehat{\rho_{\hat{\mathbb{R}}}}(\mathcal{W}(h))\}$.

Let $\mathcal{W}(H) = \{\mathcal{W}(h) \mid h \in H\}$; thus $\mathcal{W}(H) \subseteq [B]$. Let $\widetilde{H} = \{\widetilde{h} \mid h \in H\}$. There are natural bijections between $H$, $\widetilde{H}$, $\widehat{\rho_{\hat{\mathbb{R}}}}(\mathcal{W}(H))$ and $\mathcal{W}(H)$. Since $R$ and $L$ act on $H$, on $\{w_{\pm\infty}\}$, and on $\widehat{\rho_{\hat{\mathbb{R}}}}([B])$, we see that $R$ and $L$ act on $\widetilde{H}$. Since $R$ and $L$ acts continuously on $\hat{\mathbb{C}}$, the natural bijections respect the action of $R$ and $L$.

As in Definitions 6.4, $h_+ = \bigcap\limits_{n\in\mathbb{Z}} \mathcal{F}_n[h_{\mathbf{x}}, h_{\mathbf{y}}] \in H$, and $\mathcal{W}_0 = \mathcal{W}(h_+)$. Let $h = \mathcal{F}_N^{-1}(h_+)$, so $h \in H$. Thus

$$\mathcal{W}_0 = \mathcal{W}(h_+) = \mathcal{W}(\mathcal{F}_N(h)) = \mathcal{F}_N(\mathcal{W}(h)) \in \mathcal{F}_N[B].$$

This completes the proof of (ii).

(iii). Let $\mathcal{H} \in \text{IrAutABCMaps}$ be given by $n \mapsto P\mathcal{F}_{-n}P^{-1}$. Applying (ii) with $\mathcal{H}$ in place of $\mathcal{F}$, we see that

$$\begin{aligned}\{\mathcal{W}_0(\mathcal{H})\} &= \bigcap_{n\in\mathbb{Z}} \mathcal{H}_n[B] = \bigcap_{n\in\mathbb{Z}} \mathcal{H}_{-n}[B] \\ &= \bigcap_{n\in\mathbb{Z}} P\mathcal{F}_n P^{-1}[B] = \bigcap_{n\in\mathbb{Z}} P\mathcal{F}_n(AB[A] \cup A[C]).\end{aligned}$$

Multiplying by $D$, we find that $\bigcap\limits_{n\in\mathbb{Z}} P\mathcal{F}_n(C[A] \cup CB[C]) = \{D\mathcal{W}_0(\mathcal{H})\}$. Now

$$AB[A] \cup A[C], \quad [B], \quad C[A] \cup CB[C],$$

are consecutive closed cyclic intervals in $\mathfrak{E}$; hence, for each $n \in \mathbb{Z}$,

$$\mathcal{F}_n(AB[A] \cup A[C]), \quad \mathcal{F}_n([B]), \quad \mathcal{F}_n(C[A] \cup CB[C]),$$

are also consecutive closed cyclic intervals in $\mathfrak{E}$.

We have just seen that, as $n$ tends to $-\infty$, the first and third intervals shrink to $\{P^{-1}\mathcal{W}_0(\mathcal{H})\}$ and $\{P^{-1}D\mathcal{W}_0(\mathcal{H})\}$, respectively. By (i), the second interval expands as $n$ tends to $-\infty$, and thus $\bigcup\limits_{n\in\mathbb{Z}} \mathcal{F}_n[B]$ is $\left]P^{-1}\mathcal{W}_0(\mathcal{H}), DP^{-1}\mathcal{W}_0(\mathcal{H})\right[$.

Finally, by an argument similar to the second half of the proof of (ii),

$$\begin{aligned}P^{-1}(\mathcal{W}_0(\mathcal{H})) &= P^{-1}(\mathcal{W}(h_+(\mathcal{H}))) = \mathcal{W}(P^{-1}(h_+(\mathcal{H}))) \\ &= \mathcal{W}(h_-(\mathcal{F})) = \mathcal{W}_{-1}(\mathcal{F}) = \mathcal{W}_{-1}.\end{aligned}$$

Thus (iii) holds. ∎

**7.5 Remarks.** Suppose that Notation 7.1 holds, and let $n$ be a variable ranging over $\mathbb{Z}$.

The one-cells of the triangle $\mathcal{F}_n(\mathrm{hull}[\mathbf{x},\mathbf{y},\mathbf{x}+\mathbf{y}])$ are

$$\mathcal{F}_n(\mathrm{hull}[\mathbf{x},\mathbf{y}]), \quad \mathcal{F}_{n+1}(\mathrm{hull}[\mathbf{x},\mathbf{y}]), \quad \text{and} \quad \mathcal{F}'_{n+1}(\mathrm{hull}[\mathbf{x},\mathbf{y}]).$$

The radial line $h_+$ is a subset of

$$\{\mathbf{0}\} \ \cup \bigcup_{n\in\mathbb{Z}} \mathcal{F}_n(\mathrm{hull}[\mathbf{x},\mathbf{y},\mathbf{x}+\mathbf{y}]),$$

and, proceeding outwards from $\mathbf{0}$, $h_+$ enters the triangle $\mathcal{F}_n(\mathrm{hull}[\mathbf{x},\mathbf{y},\mathbf{x}+\mathbf{y}])$ through $\mathcal{F}_n(\mathrm{hull}[\mathbf{x},\mathbf{y}])$ and leaves through $\mathcal{F}_{n+1}(\mathrm{hull}[\mathbf{x},\mathbf{y}])$. It could be said that, for the usual orientation of $\mathbb{R}^2$, $h_+$ exits the triangle on the right (resp. left) if $\mathfrak{f}_n$ is $R$ (resp. $L$). In this way, $h_+$ reads $\mathfrak{b}(\mathcal{F})$; this is closely related to [9, Figure 3].

An interesting feature is that $h_+$ reads both $\mathcal{W}_0(\mathcal{F})$ and $\mathfrak{b}(\mathcal{F})$, from two different sets of triangles. See Figure 7.5.1, where one can read

$$BCBABCBABC\cdots \quad \text{and} \quad \cdots RLLLLRLLR\cdots.$$

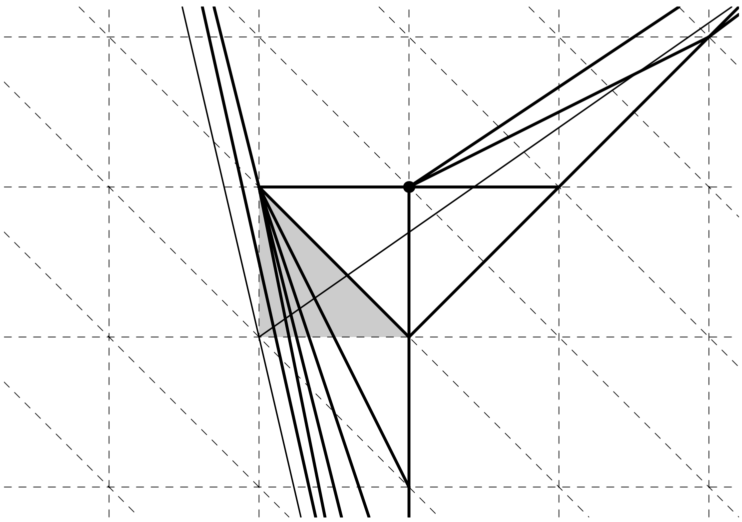

Figure 7.5.1: $\mathcal{F}_{10n} = (RLLRRRLLLL)^n$ for all $n \in \mathbb{Z}$ □

**7.6 Corollary.** *Suppose that* Notation 7.1 *holds. Then the following hold.*

(i) $\{D^m_{\mathcal{F}}\mathcal{F}'_n[B] \mid (m,\, n) \in \mathbb{Z}^2\}$ *is a partition of* $\mathfrak{E} - \{\mathcal{W}_m \mid m \in \mathbb{Z} \cup \{-\infty,\infty\}\}$.

(ii) *For* $m \in \mathbb{Z}$, $\quad ]\mathcal{W}_{2m-1},\mathcal{W}_{2m}[ \ = \bigcup_{n\in\mathbb{Z},\ \mathfrak{f}_n=L} D^m_{\mathcal{F}}\mathcal{F}'_n[B].$

(iii) *For* $m \in \mathbb{Z}$, $\quad ]\mathcal{W}_{2m},\mathcal{W}_{2m+1}[ \ = \bigcup_{n\in\mathbb{Z},\ \mathfrak{f}_n=R} D^m_{\mathcal{F}}\mathcal{F}'_n[B].$

*Proof.* (i) Observe that $\{R[B], L[B]\} = \{B[C], B[A]\}$ partitions $[B]$. For each $n \in \mathbb{Z}$, if we apply $\mathcal{F}_n$, we see that $\{\mathcal{F}_{n+1}[B], \mathcal{F}'_{n+1}[B]\}$ partitions $\mathcal{F}_n[B]$. It follows that $\{\mathcal{F}'_n[B] \mid n \in \mathbb{Z}\}$ partitions

$$\bigcup_{n\in\mathbb{Z}} \mathcal{F}_n[B] \quad - \quad \bigcap_{n\in\mathbb{Z}} \mathcal{F}_n[B].$$

By Proposition 7.4, the latter set is $[\mathcal{W}_{-1}, \mathcal{W}_1] - \{\mathcal{W}_{-1}, \mathcal{W}_0, \mathcal{W}_1\}$. On applying $D^m_{\mathcal{F}}$, for each $m \in \mathbb{Z}$, we obtain (i).

It is then straightforward to prove (ii) and (iii). □

We now have the following.

**7.7 Proposition.** *Suppose that* Notation 7.1 *holds. The following hold for* $\mathrm{CT}\colon \hat{\mathbb{R}} \to \mathbb{S}^2$.

(i) $\mathrm{CT}^{-1}\{q_{\mathbb{S}^2}\} \quad = \quad \{w_m \mid m \in \mathbb{Z} \cup \{\pm\infty\}\}$.

(ii) $\hat{\mathbb{R}} - \mathrm{CT}^{-1}\{q_{\mathbb{S}^2}\} = \bigcup\limits_{m\in\mathbb{Z}} \,]w_{m-1}, w_m[\,$ , *a disjoint union of open cyclic intervals in* $\hat{\mathbb{R}}$.

(iii) *For* $m \in \mathbb{Z}$, $\mathrm{CT}(\,]w_{m-1}, w_m[\,) = \mathrm{CT}(\widehat{\rho_{\hat{\mathbb{R}}}}(\,]\mathcal{W}_{m-1}, \mathcal{W}_m[\,))$
$$= \widehat{\rho_{\mathbb{S}^2}}(\,]\mathcal{W}_{m-1}, \mathcal{W}_m[\,).$$

(iv) *For* $m \in \mathbb{Z}$, $\widehat{\rho_{\mathbb{S}^2}}(\,]\mathcal{W}_{2m-1}, \mathcal{W}_{2m}[\,) = \bigcup\limits_{n\in\mathbb{Z},\ \mathfrak{f}_n = L} \widehat{\rho_{\mathbb{S}^2}}(D^m_{\mathcal{F}}\mathcal{F}'_n[B])$.

(v) *For* $m \in \mathbb{Z}$, $\widehat{\rho_{\mathbb{S}^2}}(\,]\mathcal{W}_{2m}, \mathcal{W}_{2m+1}[\,) = \bigcup\limits_{n\in\mathbb{Z},\ \mathfrak{f}_n = R} \widehat{\rho_{\mathbb{S}^2}}(D^m_{\mathcal{F}}\mathcal{F}'_n[B])$. □

Recall that Notation 7.1 introduced the complex conjugate $\overline{\ell_m}$ of a closed spider leg, and the image $\partial_m$ of $\overline{\ell_m}$ under the collapsing map $\kappa\colon \hat{\mathbb{C}} \twoheadrightarrow \mathbb{S}^2$. In particular, there is a specified surjective map $\overline{\ell_m} \twoheadrightarrow \partial_m$.

**7.8 Proposition.** *Suppose that* Notation 7.1 *holds, and let* $m \in \mathbb{Z}$.

*Then the following hold.*

(i) *If* $\mathcal{F}_0 = 1$, *then* $\overline{\ell_m} - \{w_m\}$ *maps homeomorphically to* $h_m$ *under the map*

$$\mathbb{C}_{\mathrm{sign}(-1)^{m-1}} \cup \mathbb{P} \quad \twoheadrightarrow \quad \langle\!\langle D^2 \rangle\!\rangle \backslash (\mathbb{C}_{\mathrm{sign}(-1)^{m-1}} \cup \mathbb{P}) \quad \xrightarrow{\sim} \quad \mathbb{R}^2.$$

(ii) *The surjective map* $\overline{\ell_m} \twoheadrightarrow \partial_m$ *is the quotient map that identifies the two end points,* $w_{\pm\infty}$ *and* $w_m$, *of* $\overline{\ell_m}$. *Hence the restriction* $\overline{\ell_m} - \{w_m\} \to \partial_m$ *is a bijective continuous map.*

(iii) $\partial_m$ *is a Jordan curve in* $\mathbb{S}^2$.

(iv) $\mathrm{CT}([w_{\pm\infty}, w_m])$ *and* $\mathrm{CT}([w_m, w_{\pm\infty}])$ *are complementary Jordan domains in* $\mathbb{S}^2$ *with common boundary* $\partial_m$.

*Proof.* We may assume that $\mathcal{F}_0 = 1$ by replacing $\mathcal{F}$ with $\mathcal{F}_0^{-1}\mathcal{F}$.

(i). Since $\mathcal{F}_0 = 1$, there is a specified homeomorphism $\ell_m - \{w_m\} \xrightarrow{\sim} h_m$, and complex conjugation gives (i).

(ii). It follows from (i) that no pair of elements of $\overline{\ell_m} - \{w_m\}$ lie in the same element of $\mathfrak{L}$, that is, $\overline{\ell_m} - \{w_m\}$ embeds in $\mathbb{S}^2$. This proves (ii).

(iii) follows from (ii).

(iv). Let $\mathbb{D}_m$, $\mathbb{D}'_m$ denote the complementary Jordan domains in $\hat{\mathbb{C}}$ with common boundary the Jordan curve $\ell_m \cup \overline{\ell_m}$.

Since $(\ell_m \cup \overline{\ell_m}) \cap \hat{\mathbb{R}} = \{w_{\pm\infty}, w_m\}$, we see that $\mathbb{D}_m \cap \hat{\mathbb{R}}$ and $\mathbb{D}'_m \cap \hat{\mathbb{R}}$ are closed cyclic intervals in $\hat{\mathbb{R}}$. By symmetry, we may assume that $\mathbb{D}_m \cap \hat{\mathbb{R}} = [w_{\pm\infty}, w_m]$ and $\mathbb{D}'_m \cap \hat{\mathbb{R}} = [w_m, w_{\pm\infty}]$.

It is not difficult to see that an element of the partition $\mathfrak{L}$ meets $\mathbb{D}_m$ if and only if it meets $\mathbb{D}_m \cap \hat{\mathbb{R}}$. It follows that $\kappa(\mathbb{D}_m) = \mathrm{CT}([w_{\pm\infty}, w_m])$ in $\mathbb{S}^2$.

Notice that an element of $\mathfrak{L}$ meets both $\mathbb{D}_m$ and $\mathbb{D}'_m$ if and only if it meets $\ell_m \cup \overline{\ell_m}$. It follows that $\kappa(\mathbb{D}_m) \cap \kappa(\mathbb{D}'_m) = \kappa(\ell_m \cup \overline{\ell_m}) = \kappa(\mathbb{D}_m \cap \mathbb{D}'_m)$. Of course, $\kappa(\mathbb{D}_m) \cup \kappa(\mathbb{D}'_m) = \mathbb{S}^2$.

Let $\mathfrak{L}_m$ denote the partition of $\mathbb{D}_m$ ($\subseteq \hat{\mathbb{C}}$) induced by the partition $\mathfrak{L}$ of $\hat{\mathbb{C}}$. It can be shown that each element of $\mathfrak{L}_m$ is connected. Let $\mathbb{S}^2_m$ denote the space obtained from $\hat{\mathbb{C}}$ by collapsing each element of $\mathfrak{L}_m$ to a point. Then $\mathbb{S}^2_m$ is also a two-sphere by Moore's Theorem; again, see, for example, [5, Appendix]. Notice that $\mathbb{S}^2$ is a quotient space of $\mathbb{S}^2_m$. The image of $\mathbb{D}_m$ in $\mathbb{S}^2_m$ can be identified with $\kappa(\mathbb{D}_m)$. The image of $\mathbb{D}'_m$ in $\mathbb{S}^2_m$ is the disc obtained from $\mathbb{D}'_m$ by collapsing to a point the interval $\ell_m$ on the boundary, $\ell_m \cup \overline{\ell_m}$, of $\mathbb{D}'_m$. It follows that, in $\mathbb{S}^2_m$, $\kappa(\ell_m \cup \overline{\ell_m})$ is a Jordan curve, and one of the Jordan domains it bounds is $\kappa(\mathbb{D}_m)$. Thus, in $\mathbb{S}^2$, $\kappa(\mathbb{D}_m)$ is a Jordan domain. That is, $\mathrm{CT}([w_{\pm\infty}, w_m])$ is a Jordan domain, and its boundary is $\kappa(\ell_m \cup \overline{\ell_m}\,) = \partial_m$. This completes the proof. □

**7.9 Definitions.** Suppose that Notation 7.1 holds and let $m \in \mathbb{Z}$.

We orient $\ell_m \cup \overline{\ell_m}$ so that it proceeds from $w_{\pm\infty}$ through $\mathbb{C}_-$ to $w_m$ and then through $\mathbb{C}_+$ back to $w_{\pm\infty}$. Then $\partial_m$ has an induced orientation. Now $(\ell_m \cup \overline{\ell_m}) - \{q_{\hat{\mathbb{R}}}\}$ separates the oriented plane $\hat{\mathbb{C}} - \{q_{\hat{\mathbb{R}}}\}$, and $\partial_m - \{q_{\mathbb{S}^2}\}$ separates the oriented plane $\mathbb{S}^2 - \{q_{\mathbb{S}^2}\}$.

We arrange the oriented planes so that $D_{\mathcal{F}}$ moves points from left to right. It follows that $(\ell_m \cup \overline{\ell_m}) - \{q_{\hat{\mathbb{R}}}\}$ and $\partial_m - \{q_{\mathbb{S}^2}\}$, $m \in \mathbb{Z}$, are arranged from left to right as $m$ increases, and are oriented upward.

If $r_1, r_2, \ldots, r_n$ are elements of $\mathbb{S}^2$, we will say that

$$r_1 \le r_2 \le \cdots \le r_n \le r_1 \textit{ in } \partial_m$$

if $r_1, r_2, \ldots, r_n$ are elements of $\partial_m$ and a path proceeding once around $\partial_m$ starting at $r_1$, following the orientation of $\partial_m$, meets the points in the indicated order. We will then speak of the *closed segment of* $\partial_m$ *from* $r_1$ *to* $r_2$. Where, moreover, points are distinct, we will use the usual notation for strict inequality. □

We now begin to study $\partial_{-1} \cap \partial_0$ and $\partial_0 \cap \partial_1$.

**7.10 Lemma.** *Suppose that* Notation 7.1 *holds and that* $\mathcal{F}_0 = 1$.

(i) *If* $\mathfrak{f}_1 = L$ *then the following hold.*

(1) $q_{\mathbb{S}^2} < Aq_{\mathbb{S}^2} < Bq_{\mathbb{S}^2} < q_{\mathbb{S}^2}$ *in both* $\partial_{-1}$ *and* $\partial_0$.

(2) *The union of the closed segments of* $\partial_{-1}$ *and* $\partial_0$ *from* $Aq_{\mathbb{S}^2}$ *to* $Bq_{\mathbb{S}^2}$ *is a Jordan curve in* $\mathbb{S}^2$; *the Jordan domain in* $\mathbb{S}^2 - \{q_{\mathbb{S}^2}\}$ *bounded by this Jordan curve is* $\widehat{\rho_{\mathbb{S}^2}}(B[C])$.

(3) *If* $\mathfrak{f}_0 = R$, *then* $q_{\mathbb{S}^2} < Aq_{\mathbb{S}^2} < Cq_{\mathbb{S}^2} < Bq_{\mathbb{S}^2} < q_{\mathbb{S}^2}$ *in* $\partial_0$.

(4) *If* $\mathfrak{f}_2 = R$, *then* $q_{\mathbb{S}^2} < Aq_{\mathbb{S}^2} < ABq_{\mathbb{S}^2} < Bq_{\mathbb{S}^2} < q_{\mathbb{S}^2}$ *in* $\partial_1$.

(ii) *If* $\mathfrak{f}_1 = R$, *then the following hold.*

(1) $q_{\mathbb{S}^2} < Cq_{\mathbb{S}^2} < Bq_{\mathbb{S}^2} < q_{\mathbb{S}^2}$ *in both* $\partial_0$ *and* $\partial_1$.

(2) *The union of the closed segments of* $\partial_0$ *and* $\partial_1$ *from* $Cq_{\mathbb{S}^2}$ *to* $Bq_{\mathbb{S}^2}$ *is a Jordan curve in* $\mathbb{S}^2$; *the Jordan domain in* $\mathbb{S}^2 - \{q_{\mathbb{S}^2}\}$ *bounded by this Jordan curve is* $\widehat{\rho_{\mathbb{S}^2}}(B[A])$.

(3) *If* $\mathfrak{f}_0 = L$, *then* $q_{\mathbb{S}^2} < Cq_{\mathbb{S}^2} < Aq_{\mathbb{S}^2} < Bq_{\mathbb{S}^2} < q_{\mathbb{S}^2}$ *in* $\partial_0$.

(4) *If* $\mathfrak{f}_2 = L$, *then* $q_{\mathbb{S}^2} < Cq_{\mathbb{S}^2} < CBq_{\mathbb{S}^2} < Bq_{\mathbb{S}^2} < q_{\mathbb{S}^2}$ *in* $\partial_1$. □

*Proof.* We will prove (ii); the proof of (i) is similar. Thus, suppose $\mathcal{F}_1 = R$.

(ii)(1) and (2). Here $\mathcal{F}_1(\mathbf{x}) = \mathbf{x}$ and $\mathcal{F}_1(\mathbf{y}) = \mathbf{x} + \mathbf{y}$. By Lemma 6.3, $h_0$ passes through hull$[\mathbf{x}+\mathbf{y}, \mathbf{x}]$, that is, $h_0$ lies in the first octant.

Also, since $\mathcal{F}_0 = 1$, we see that $h_{-1}$ lies in the fourth quadrant and $h_1$ lies in the second quadrant.

The proof of [5, Theorem 5.6(i)] shows that $\widehat{\rho_{\mathbb{S}^2}}(B[A])$ is a Jordan domain whose boundary is made up of two closed arcs, there denoted $\partial^+[BA]_{\mathbb{S}^2}$, $\partial^-[BA]_{\mathbb{S}^2}$, and these two arcs join $Cq_{\mathbb{S}^2}$ to $Bq_{\mathbb{S}^2}$.

In [5], it was seen that the points on $\partial^+[BA]_{\mathbb{S}^2}$ are given by reading the lines parallel to $h_0$ which pass through hull$[\mathbf{y}, \mathbf{x}+\mathbf{y}]$, provided the path from the base quarter-point to such a line is homotopic, in $\mathbb{R}^2 - \mathbb{Z}^2$, to a path within hull$[\mathbf{0}, \mathbf{x}, \mathbf{y}]$. Thus we can start at the base quarter-point and reach the lines by travelling along $h_1$. Here we can use the continuous bijection from $h_1$ to $\partial_1$ given by Proposition 7.8(i) and (ii). Notice $h_1$ is then oriented towards the origin. Thus $\partial^+[BA]_{\mathbb{S}^2}$ is the segment of $\partial_1$ from $Cq_{\mathbb{S}^2}$ to $Bq_{\mathbb{S}^2}$, and does not contain $q_{\mathbb{S}^2}$.

The points on $\partial^-[BA]_{\mathbb{S}^2}$ are given by reading the lines parallel to $h_1$ which pass through hull$[\mathbf{y}, \mathbf{x}+\mathbf{y}]$, and we can start at the base quarter-point and reach the lines by travelling along $h_0$. Thus we get the segment of $\partial_0$ from $Cq_{\mathbb{S}^2}$ to $Bq_{\mathbb{S}^2}$; in particular the arc does not contain $q_{\mathbb{S}^2}$. Here, $h_0$ is oriented away from the origin.

The boundary of $\widehat{\rho_{\mathbb{S}^2}}(B[A])$ does not contain $q_{\mathbb{S}^2}$, and we now wish to show that $q_{\mathbb{S}^2}$ does not lie in the interior of $\widehat{\rho_{\mathbb{S}^2}}(B[A])$. Since $\mathcal{F}_1 = R$, it is not difficult to show that $q_{\mathbb{S}^2}$ lies in the boundary of $\widehat{\rho_{\mathbb{S}^2}}(B[C])$, and hence, by [5, Theorem 5.6(i)], $q_{\mathbb{S}^2}$ does not lie in the interior of $\widehat{\rho_{\mathbb{S}^2}}(B[A])$, as desired.

(ii)(3). If $\mathfrak{f}_0 = L$, then $\mathcal{F}_{-1} = L^{-1}$. Since $L^{-1}(\mathbf{x}) = -\mathbf{x} - \mathbf{y}$ and $L^{-1}(\mathbf{y}) = \mathbf{y}$, it follows that $h_{-1}$ passes through hull$[-\mathbf{x}-\mathbf{y}, -\mathbf{y}]$. Hence the line parallel to $h_{-1}$ through $\mathbf{x} = A\mathbf{0}$ passes through hull$[\mathbf{y}, \mathbf{x}+\mathbf{y}]$. It follows that $Aq_{\mathbb{S}^2}$ lies on $\partial^-[BA]_{\mathbb{S}^2}$.

(ii)(4). If $\mathfrak{f}_2 = L$, then $\mathcal{F}_2 = RL$. Since $RL(\mathbf{x}) = 2\mathbf{x} + \mathbf{y}$ and $RL(\mathbf{y}) = \mathbf{x} + \mathbf{y}$, it follows that $h_0$ passes through hull$[\mathbf{x}+\mathbf{y}, 2\mathbf{x}+\mathbf{y}]$. Hence the line parallel to $h_0$

through $-\mathbf{x} = CB\mathbf{0}$ passes through hull$[\mathbf{y}, \mathbf{x}+\mathbf{y}]$. It follows that $CBq_{\mathbb{S}^2}$ lies on $\partial^+[BA]_{\mathbb{S}^2}$. □

**7.11 Lemma.** *Suppose that* Notation 7.1 *holds and that* $(m,n) \in \mathbb{Z}^2$.

(i) *If* $\mathfrak{f}_n = L$, *then the following hold*:

(1) $\mathcal{F}_n(AC)D_{\mathcal{F}} = \mathcal{F}_{n-1}(A)$ *and* $D_{\mathcal{F}}^m \mathcal{F}_{\mathrm{succ}_{\mathfrak{f}}(n)}(AC)D_{\mathcal{F}} = D_{\mathcal{F}}^m \mathcal{F}_{n-1}(B)$.

(2) $q_{\mathbb{S}^2} < \mathcal{F}_n(AC)q_{\mathbb{S}^2} < \mathcal{F}_{n+1}(AC)q_{\mathbb{S}^2} \le \mathcal{F}_{n-1}(B)q_{\mathbb{S}^2} < q_{\mathbb{S}^2}$ *in* $\partial_{-1}$.

(3) $q_{\mathbb{S}^2} < \mathcal{F}_n(AC)q_{\mathbb{S}^2} \le \mathcal{F}_{n-2}(B)q_{\mathbb{S}^2} < \mathcal{F}_{n-1}(B)q_{\mathbb{S}^2} < q_{\mathbb{S}^2}$ *in* $\partial_0$.

(4) *The union of the closed segments of* $\partial_{-1}$ *and* $\partial_0$ *from* $\mathcal{F}_n(AC)q_{\mathbb{S}^2}$ *to* $\mathcal{F}_{n-1}(B)q_{\mathbb{S}^2}$ *is a Jordan curve in* $\mathbb{S}^2$; *the Jordan domain in* $\mathbb{S}^2 - \{q_{\mathbb{S}^2}\}$ *bounded by this Jordan curve is* $\widehat{\rho_{\mathbb{S}^2}}(\mathcal{F}'_n[B])$.

(ii) *If* $\mathfrak{f}_n = R$, *then the following hold*:

(1) $D_{\mathcal{F}}\mathcal{F}_n(AC) = \mathcal{F}_{n-1}(C)$ *and* $D_{\mathcal{F}}^{m+1} \mathcal{F}_{\mathrm{succ}_{\mathfrak{f}}(n)}(AC) = D_{\mathcal{F}}^m \mathcal{F}_{n-1}(B)$.

(2) $q_{\mathbb{S}^2} < D_{\mathcal{F}}\mathcal{F}_n(AC)q_{\mathbb{S}^2} \le \mathcal{F}_{n-2}(B)q_{\mathbb{S}^2} < \mathcal{F}_{n-1}(B)q_{\mathbb{S}^2} < q_{\mathbb{S}^2}$ *in* $\partial_0$.

(3) $q_{\mathbb{S}^2} < D_{\mathcal{F}}\mathcal{F}_n(AC)q_{\mathbb{S}^2} < D_{\mathcal{F}}\mathcal{F}_{n+1}(AC)q_{\mathbb{S}^2} \le \mathcal{F}_{n-1}(B)q_{\mathbb{S}^2} < q_{\mathbb{S}^2}$ *in* $\partial_1$.

(4) *The union of the closed segments of* $\partial_0$ *and* $\partial_1$ *from* $D_{\mathcal{F}}\mathcal{F}_n(AC)q_{\mathbb{S}^2}$ *to* $\mathcal{F}_{n-1}(B)q_{\mathbb{S}^2}$ *is a Jordan curve in* $\mathbb{S}^2$; *the Jordan domain in* $\mathbb{S}^2 - \{q_{\mathbb{S}^2}\}$ *bounded by this Jordan curve is* $\widehat{\rho_{\mathbb{S}^2}}(\mathcal{F}'_n[B])$.

*Proof.* We will prove (i); the proof of (ii) is similar. Table 7.3.1 will be useful.

(1). Let $n' = \mathrm{succ}_{\mathfrak{f}}(n)$.
Here $\mathcal{F}_{n'} = \mathcal{F}_{n-1}LR^{n'-n}L$.
Now $\mathcal{F}_n(AC)D_{\mathcal{F}} = \mathcal{F}_n(AC\cdot D) = \mathcal{F}_n(ABA) = \mathcal{F}_nL^{-1}(A) = \mathcal{F}_{n-1}(A)$.
Hence

$$\mathcal{F}_{n'}(AC)D_{\mathcal{F}} = \mathcal{F}_{n'-1}(A) = \mathcal{F}_{n-1}LR^{n'-n}(A) = \mathcal{F}_{n-1}L(A) = \mathcal{F}_{n-1}(B),$$

and (1) follows.

(2)–(4). By replacing $\mathcal{F}$ with $\mathcal{F}\cdot(z+n-1)$, we may assume that $n=1$. Now

$$\widehat{\rho_{\mathbb{S}^2}}(\mathcal{F}'_1[B]) = \widehat{\rho_{\mathbb{S}^2}}(\mathcal{F}_0R[B]) = \widehat{\rho_{\mathbb{S}^2}}(\mathcal{F}_0B[C]),$$

and $\mathcal{F}_1(AC) = \mathcal{F}_0(A)D_{\mathcal{F}}^{-1}$ by (1). Also

$$\mathcal{F}_2(AC) = \mathcal{F}_0L\mathfrak{f}_2(AC) = \begin{cases} \mathcal{F}_0(B)D_{\mathcal{F}}^{-1} & \text{if } \mathfrak{f}_2 = L, \\ \mathcal{F}_0(AB) & \text{if } \mathfrak{f}_2 = R, \end{cases}$$

and $\mathcal{F}_{-1}(B) = \mathcal{F}_0\mathfrak{f}_0^{-1}(B) = \begin{cases} \mathcal{F}_0(A) & \text{if } \mathfrak{f}_0 = L, \\ \mathcal{F}_0(C) & \text{if } \mathfrak{f}_0 = R. \end{cases}$

If $\mathcal{F}_0 = 1$, we get the desired results by Lemma 7.10(i). Hence the results hold for $\mathcal{F}_0^{-1}\mathcal{F}$, and transforming by $\mathcal{F}_0$ gives the general case. □

Recall the construction of CW($\mathfrak{f}$) in Definitions 4.2.

**7.12 Theorem.** *Suppose that* Notation 7.1 *holds.*

*There exists a* $\mathbb{Z}$*-homeomorphism* $|\mathrm{CW}(\mathfrak{f})| \xrightarrow{\sim} \mathbb{S}^2 - \{q_{\mathbb{S}^2}\}$ *which, for each* $(m,n) \in \mathbb{Z}^2$*, carries the* $(m,n)$*th closed two-cell of* $\mathrm{CW}(\mathfrak{f})$ *homeomorphically to* $\widehat{\rho_{\mathbb{S}^2}}(D_{\mathcal{F}}^m \mathcal{F}'_n[B])$.

*Proof.* We first map $\mathbb{R}^2 = |\mathrm{CW}'(\mathfrak{f})|$ to $\mathbb{S}^2 - \{q_{\mathbb{S}^2}\}$ as follows.

Let $(m,n) \in \mathbb{Z}^2$, and let $n_+ = \mathrm{succ}_{\mathfrak{f}}(n)$.

We send the zero-cell $(2m,n)$ to $D_{\mathcal{F}}^m \mathcal{F}_{n-1}(B) q_{\mathbb{S}^2}$.

We send the zero-cell $(2m-1,n)$ to $D_{\mathcal{F}}^m \mathcal{F}_n(AC) q_{\mathbb{S}^2}$.

We map the vertical one-cell out of $(-1,n)$ homeomorphically to the increasing segment of $\partial_{-1}$ from $\mathcal{F}_n(AC)q_{\mathbb{S}^2}$ to $\mathcal{F}_{n+1}(AC)q_{\mathbb{S}^2}$; see Lemma 7.11(i)(2) and (ii)(3). We extend by the $\mathbb{Z}$-action to map the vertical one-cell out of $(2m-1,n)$ homeomorphically to the increasing segment of $\partial_{2m-1}$ from $D_{\mathcal{F}}^m \mathcal{F}_n(AC) q_{\mathbb{S}^2}$ to $D_{\mathcal{F}}^m \mathcal{F}_{n+1}(AC) q_{\mathbb{S}^2}$.

We map the vertical one-cell out of $(0,n)$ homeomorphically to the increasing segment of $\partial_0$ from $\mathcal{F}_{n-1}(B)q_{\mathbb{S}^2}$ to $\mathcal{F}_n(B)q_{\mathbb{S}^2}$; see Lemma 7.11(i)(3) and (ii)(2). We extend by the $\mathbb{Z}$-action to map the vertical one-cell out of $(2m,n)$ homeomorphically to the increasing segment of $\partial_{2m}$ from $D_{\mathcal{F}}^m \mathcal{F}_{n-1}(B) q_{\mathbb{S}^2}$ to $D_{\mathcal{F}}^m \mathcal{F}_n(B) q_{\mathbb{S}^2}$.

If $\mathfrak{f}_n = L$, we collapse the closed slanted one-cell between $(2m,n)$ and $(2m-1,n_+)$ to the point $D_{\mathcal{F}}^m \mathcal{F}_{n-1}(B) q_{\mathbb{S}^2} = D_{\mathcal{F}}^m \mathcal{F}_{n_+}(AC) q_{\mathbb{S}^2}$; see Lemma 7.11(i)(1).

If $\mathfrak{f}_n = R$, we collapse the closed slanted one-cell between $(2m,n)$ and $(2m+1,n_+)$ to the point $D_{\mathcal{F}}^m \mathcal{F}_{n-1}(B) q_{\mathbb{S}^2} = D_{\mathcal{F}}^{m+1} \mathcal{F}_{n_+}(AC) q_{\mathbb{S}^2}$; see Lemma 7.11(ii)(2).

By Lemma 7.11(i), if $\mathfrak{f}_n = L$, then $\widehat{\rho_{\mathbb{S}^2}}(\mathcal{F}'_n[B])$ is a Jordan domain, and its left boundary is the increasing segment of $\partial_{-1}$ from $\mathcal{F}_n(AC)q_{\mathbb{S}^2}$ to $\mathcal{F}_{n-1}(B)q_{\mathbb{S}^2} = \mathcal{F}_{n_+}(AC)q_{\mathbb{S}^2}$, and its right boundary is the increasing segment of $\partial_0$ between the same two points. It follows that $\partial_{-1} \cap \partial_0 = \{\mathcal{F}_n(AC)q_{\mathbb{S}^2} \mid n \in \mathbb{Z}, \mathfrak{f}_n = L\}$, and that

$$\{\widehat{\rho_{\mathbb{S}^2}}(\mathcal{F}'_n[B]) \mid n \in \mathbb{Z},\, \mathfrak{f}_n = L\}$$

is a bi-infinite sequence of Jordan domains with left boundary $\partial_{-1}$ and right boundary $\partial_0$.

Similarly, it can be shown that $\partial_0 \cap \partial_1 = \{D_{\mathcal{F}} \mathcal{F}_n(AC) q_{\mathbb{S}^2} \mid n \in \mathbb{Z}, \mathfrak{f}_n = R\}$ and

$$\{\widehat{\rho_{\mathbb{S}^2}}(\mathcal{F}'_n[B]) \mid n \in \mathbb{Z},\, \mathfrak{f}_n = R\}$$

is a bi-infinite sequence of Jordan domains with left boundary $\partial_0$ and right boundary $\partial_1$.

It is not difficult to show that $\partial_m \cap \partial_k$ is empty if $|m-k| \geq 2$.

We map the $(0,n)$th closed two-cell of $\mathrm{CW}'(\mathfrak{f})$ to the Jordan domain $\widehat{\rho_{\mathbb{S}^2}}(\mathcal{F}'_n[B])$ in a manner that extends the map on the boundary and induces a homeomorphism after collapsing the two closed slanted one-cells. We extend by the $\mathbb{Z}$-action in the natural way.

Since the collapsings in $\mathrm{CW}'(\mathfrak{f})$ give $\mathrm{CW}(\mathfrak{f})$, we have the desired $\mathbb{Z}$-homeomorphism. □

In $\mathrm{CW}(\mathfrak{f})$, the one-cells and zero-cells can be recovered from the closed two-cells by intersection.

**7.13 Definition.** Suppose that Notation 7.1 holds.

In a natural way, Theorem 7.12 gives an oriented, colored $\mathbb{Z}$-CW-complex, which will be denoted $\mathrm{CW}(\mathcal{F})$. The following hold, where $(m,n)$ is a variable ranging over $\mathbb{Z}^2$.

(i) The underlying oriented space is $\mathbb{S}^2 - \{q_{\mathbb{S}^2}\}$.

(ii) The set of zero-cells of $\mathrm{CW}(\mathcal{F})$ is

$$\begin{aligned}\mathrm{CW}_0(\mathcal{F}) &:= \{D_{\mathcal{F}}^m \mathcal{F}_n(B) q_{\mathbb{S}^2} \mid (m,n) \in \mathbb{Z}^2\} \\ &= \{D_{\mathcal{F}}^m \mathcal{F}_n(AC) q_{\mathbb{S}^2} \mid (m,n) \in \mathbb{Z}^2\},\end{aligned}$$

and both of the indicated maps from $\mathbb{Z}^2$ are bijective.

If $\mathfrak{f}_n = L$, then $D_{\mathcal{F}}^m \mathcal{F}_{\mathrm{succ}_{\mathfrak{f}}(n)}(AC) q_{\mathbb{S}^2} = D_{\mathcal{F}}^m \mathcal{F}_{n-1}(B) q_{\mathbb{S}^2}$.

If $\mathfrak{f}_n = R$, then $D_{\mathcal{F}}^m \mathcal{F}_{n-1}(B) q_{\mathbb{S}^2} = D_{\mathcal{F}}^{m+1} \mathcal{F}_{\mathrm{succ}_{\mathfrak{f}}(n)}(AC) q_{\mathbb{S}^2}$.

(iii) The family of zero cells in $\partial_{2m}$, in increasing order, is

$$(D_{\mathcal{F}}^m \mathcal{F}_n(B) q_{\mathbb{S}^2} \mid n \in \mathbb{Z}).$$

The family of zero cells in $\partial_{2m-1}$, in increasing order, is

$$(D_{\mathcal{F}}^m \mathcal{F}_n(AC) q_{\mathbb{S}^2} \mid n \in \mathbb{Z}).$$

Also,

$$\begin{aligned}\partial_{2m-1} \cap \partial_{2m} &= \{D_{\mathcal{F}}^m \mathcal{F}_n(B) q_{\mathbb{S}^2} \mid n \in \mathbb{Z},\, \mathfrak{f}_{n+1} = L\} \\ &= \{D_{\mathcal{F}}^m \mathcal{F}_n(AC) q_{\mathbb{S}^2} \mid n \in \mathbb{Z},\, \mathfrak{f}_n = L\}, \\ \partial_{2m} \cap \partial_{2m+1} &= \{D_{\mathcal{F}}^m \mathcal{F}_n(B) q_{\mathbb{S}^2} \mid n \in \mathbb{Z},\, \mathfrak{f}_{n+1} = R\} \\ &= \{D_{\mathcal{F}}^{m+1} \mathcal{F}_n(AC) q_{\mathbb{S}^2} \mid n \in \mathbb{Z},\, \mathfrak{f}_n = R\}.\end{aligned}$$

(iv) The set of open one-cells of $\mathrm{CW}(\mathcal{F})$ is the set of components of

$$\bigcup_{m \in \mathbb{Z}} \partial_m - \mathrm{CW}_0(\mathcal{F}).$$

Thus, we get the closed one-cells of $\mathrm{CW}(\mathcal{F})$ by taking the closed segment of $\partial_{2m-1}$ between $D_{\mathcal{F}}^m \mathcal{F}_n(AC) q_{\mathbb{S}^2}$ and $D_{\mathcal{F}}^m \mathcal{F}_{n+1}(AC) q_{\mathbb{S}^2}$, and the closed segment of $\partial_{2m}$ between $D_{\mathcal{F}}^m \mathcal{F}_n(B) q_{\mathbb{S}^2}$ and $D_{\mathcal{F}}^m \mathcal{F}_{n+1}(B) q_{\mathbb{S}^2}$.

(v) The set of closed two-cells of $\mathrm{CW}(\mathcal{F})$ is $\{\widehat{\rho_{\mathbb{S}^2}}(D_{\mathcal{F}}^m \mathcal{F}'_n[B]) \mid (m,n) \in \mathbb{Z}^2\}$, and the indicated map from $\mathbb{Z}^2$ is bijective.

(vi) For the $\mathbb{Z}$-action on $\mathrm{CW}(\mathcal{F})$, the distinguished generator acts as $D_{\mathcal{F}}$, that is, $\rho_{\mathbb{S}^2}(D_{\mathcal{F}})$. □

**7.14 Remarks.** Let $(m,n)$ be a variable ranging over $\mathbb{Z}^2$.

(i) In $\hat{\mathbb{R}}$, $([\mathcal{F}_n(A) w_{\pm\infty}, \mathcal{F}_n(C) w_{\pm\infty}] \mid n \in \mathbb{Z})$ is a decreasing sequence of closed cyclic intervals, with intersection $\{w_0\}$ and union $]w_{-1}, w_1[$ . At each stage we divide the interval at $\mathcal{F}_n(B) w_{\pm\infty}$,

$$\begin{aligned}&\{\widehat{\rho_{\hat{\mathbb{R}}}}(\mathcal{F}'_n[B]) \mid n \in \mathbb{Z}, \mathfrak{f}_n = L\} \text{ covers } \ ]w_{-1}, w_0[, \text{ and} \\ &\{\widehat{\rho_{\hat{\mathbb{R}}}}(\mathcal{F}'_{-n}[B]) \mid n \in \mathbb{Z}, \mathfrak{f}_n = R\} \text{ covers } \ ]w_0, w_1[.\end{aligned}$$

By Lemma 7.11,

$$\begin{aligned}\widehat{\rho_{\hat{\mathbb{R}}}}(\mathcal{F}'_n[B]) &= [\mathcal{F}'_n(A)w_{\pm\infty}, \mathcal{F}'_n(C)w_{\pm\infty}] \\ &= \begin{cases} [\mathcal{F}_{\mathrm{pred}_{\mathfrak{f}}(n)-1}(B)w_{\pm\infty}, \mathcal{F}_{n-1}(B)w_{\pm\infty}] & \text{if } \mathfrak{f}_n = L, \\ [\mathcal{F}_{n-1}(B)w_{\pm\infty}, \mathcal{F}_{\mathrm{pred}_{\mathfrak{f}}(n)-1}(B)w_{\pm\infty}] & \text{if } \mathfrak{f}_n = R, \end{cases} \\ &= \begin{cases} [\mathcal{F}_n(AC)w_{\pm\infty}, \mathcal{F}_{\mathrm{succ}_{\mathfrak{f}}(n)}(AC)w_{\pm\infty}] & \text{if } \mathfrak{f}_n = L, \\ [D_{\mathcal{F}}\mathcal{F}_{\mathrm{succ}_{\mathfrak{f}}(n)}(AC)w_{\pm\infty}, D_{\mathcal{F}}\mathcal{F}_n(AC)w_{\pm\infty}] & \text{if } \mathfrak{f}_n = R. \end{cases}\end{aligned}$$

(ii) In $\mathbb{S}^2 - \{q_{\mathbb{S}^2}\}$, $\mathrm{CT}(\,]w_{m-1}, w_m[\,)$ is the $m$th column of $\mathrm{CW}(\mathcal{F})$, and we have seen that this is the union of a bi-infinite sequence of Jordan domains, with each domain overlapping the next in a single point. The two-cells forming the even/gray/up columns correspond to $\mathfrak{f}_n = L$, and the two-cells forming the odd/white/down columns correspond to $\mathfrak{f}_n = R$.

We have also seen that if the oriented plane $\mathbb{S}^2 - \{q_{\mathbb{S}^2}\}$ is arranged so that $D_{\mathcal{F}}$ moves points from left to right, then, in the limit, the points $D^m_{\mathcal{F}}\mathcal{F}_n(B)q_{\mathbb{S}^2}$ move from left to right as $m$ increases, and from bottom to top as $n$ increases. This information plays the role of a frame for $\mathbb{S}^2 - \{q_{\mathbb{S}^2}\}$. □

# 8 The sphere $\hat{\mathbb{C}}(\rho_{\hat{\mathbb{C}}}, q_{\hat{\mathbb{C}}})$

**8.1 Definitions.** Let $\rho_{\hat{\mathbb{C}}} \in \mathfrak{R}_{\hat{\mathbb{C}}}$ and let $q_{\hat{\mathbb{C}}}$ be a parabolic point of $\rho_{\hat{\mathbb{C}}}(\langle A, B, C\rangle)$. We say that $(\rho_{\hat{\mathbb{C}}}, q_{\hat{\mathbb{C}}})$ is a *parabolic pair in* $\mathfrak{R}_{\hat{\mathbb{C}}} \times \hat{\mathbb{C}}$. Let $\hat{\mathbb{C}}(\rho_{\hat{\mathbb{C}}}, q_{\hat{\mathbb{C}}})$ denote $\hat{\mathbb{C}}$ viewed as an $\langle A, B, C\rangle$-space via $\rho_{\hat{\mathbb{C}}}$, and having specified base point $q_{\hat{\mathbb{C}}}$.

Let $\mathcal{F} \in$ IrAutABCMaps, and recall that $\mathbb{S}^2(\mathcal{F})$ has a specified base point, $q_{\mathbb{S}^2(\mathcal{F})}$.

We say that $\mathcal{F}$ *codifies* $(\rho_{\hat{\mathbb{C}}}, q_{\hat{\mathbb{C}}})$ if there exists a (unique) (surjective) continuous, orientation-preserving, base-point-preserving, $\langle A, B, C\rangle$-map

$$\gamma = \gamma(\mathcal{F}, \rho_{\hat{\mathbb{C}}})\colon \mathbb{S}^2(\mathcal{F}) \to \hat{\mathbb{C}}(\rho_{\hat{\mathbb{C}}}, q_{\hat{\mathbb{C}}}).$$

If this holds, then $q_{\hat{\mathbb{C}}}$ is the unique element of $\hat{\mathbb{C}}$ fixed by $\rho_{\hat{\mathbb{C}}}(D_{\mathcal{F}})$. □

**8.2 Definitions.** Codification, as defined above, specifies a subset of

$$\text{IrAutABCMaps} \times \mathfrak{R}_{\hat{\mathbb{C}}} \times \hat{\mathbb{C}},$$

and we call this the *codification pairing relation* between IrAutABCMaps and $\mathfrak{R}_{\hat{\mathbb{C}}} \times \hat{\mathbb{C}}$.

Recall that, as in Notation 2.2, $\mathrm{Isom}(\mathbb{Z}) \times \mathrm{Aut}^+\langle A, B, C\rangle$ acts on

$$\text{IrAutABCMaps},$$

where we understand that $\mathrm{Aut}^+\langle A, B, C\rangle$ acts on itself by left multiplication. We can also let $\mathrm{Aut}^+\langle A, B, C\rangle$ acts on itself by right multiplication by inverses, and we call the resulting action of $\mathrm{Isom}(\mathbb{Z}) \times \mathrm{Aut}^+\langle A, B, C\rangle$ the *right-hand* action. Let us say that two elements $\mathcal{F}$ and $\mathcal{G}$ of IrAutABCMaps are *equivalent* if they are in the same $\langle D\rangle \times \mathrm{Isom}^+(\mathbb{Z})$-orbit for the right-hand action. Suppose the latter condition holds. There exists $(a,b) \in \mathbb{Z}^2$ such that, $\mathcal{G} = \mathcal{F} \cdot (z+b) \cdot (D)^a$, that is,

$$\{D_{\mathcal{F}}^m \mathcal{F}_n \mid (m,n) \in \mathbb{Z}^2\} = \{D_{\mathcal{G}}^m \mathcal{G}_n \mid (m,n) \in \mathbb{Z}^2\}.$$

It is not difficult to see that $\mathbb{S}^2(\mathcal{F}) = \mathbb{S}^2(\mathcal{G})$ and $q_{\mathbb{S}^2(\mathcal{F})} = q_{\mathbb{S}^2(\mathcal{G})}$. Thus $\mathcal{F}$ and $\mathcal{G}$ codify exactly the same elements of $\mathfrak{R}_{\hat{\mathbb{C}}} \times \hat{\mathbb{C}}$.

Recall that, as in Notation 2.2, $\mathrm{Aut}^+\langle A,B,C\rangle \times \mathrm{Aut}^+(\hat{\mathbb{C}})$ acts on $\mathfrak{R}_{\hat{\mathbb{C}}}$; we let $\mathrm{Aut}^+\langle A,B,C\rangle \times \mathrm{Aut}^+(\hat{\mathbb{C}})$ act on $\hat{\mathbb{C}}$, with $\mathrm{Aut}^+\langle A,B,C\rangle$ acting trivially and $\mathrm{Aut}^+(\hat{\mathbb{C}})$ acting in the canonical way. Then $\mathrm{Aut}^+\langle A,B,C\rangle \times \mathrm{Aut}^+(\hat{\mathbb{C}})$ acts on the direct product $\mathfrak{R}_{\hat{\mathbb{C}}} \times \hat{\mathbb{C}}$.

Let $(\rho_1, q_1)$ and $(\rho_2, q_2)$ be parabolic elements of $\mathfrak{R}_{\hat{\mathbb{C}}} \times \hat{\mathbb{C}}$. We say that $(\rho_1, q_1)$ and $(\rho_2, q_2)$ are *equivalent* if they are in the same $\mathrm{Aut}^+(\hat{\mathbb{C}})$-orbit. Suppose that this holds. There exists $\sigma \in \mathrm{Aut}^+(\hat{\mathbb{C}})$ such that $i_\sigma \cdot \rho_1 = \rho_2$ and $\sigma(q_1) = q_2$. Here $\sigma$ induces an orientation-preserving, base-point-preserving, $\langle A,B,C\rangle$-homeomorphism $\hat{\mathbb{C}}(\rho_1, q_1) \xrightarrow{\sim} \hat{\mathbb{C}}(\rho_2, q_2)$. Thus $(\rho_1, q_1)$ and $(\rho_2, q_2)$ are codified by exactly the same elements of IrAutABCMaps.

Hence the codification pairing relation can be collapsed to a pairing relation on the equivalence classes of the foregoing equivalence relations.

It is not difficult to show that the codification pairing relation is closed under the $\mathrm{Aut}^+\langle A,B,C\rangle$-action, that is, if $\mathcal{F}$ codifies $(\rho_{\hat{\mathbb{C}}}, q_{\hat{\mathbb{C}}})$, and $F \in \mathrm{Aut}^+\langle A,B,C\rangle$, then $F \cdot \mathcal{F}$ codifies $(\rho_{\hat{\mathbb{C}}} \cdot F^{-1}, q_{\hat{\mathbb{C}}})$. □

We will now quote various results in the literature to obtain the following; it is stated in a somewhat unusual form, suited to our purposes.

**8.3 Theorem.** (Thurston [10], Bonahon [2], Minsky [9], McMullen [8]) *The codification pairing relation induces a bijective correspondence between*

*the set* IrAutABCMaps $/(\langle D\rangle \times \mathrm{Isom}^+(\mathbb{Z}))$
*of right-hand-action* $(\langle D\rangle \times \mathrm{Isom}^+(\mathbb{Z}))$*-orbits in* IrAutABCMaps

*and*

*the set of* $\mathrm{Aut}^+(\hat{\mathbb{C}})$*-orbits of parabolic pairs in* $\mathfrak{R}_{\hat{\mathbb{C}}} \times \hat{\mathbb{C}}$.

*This correspondence respects the* $\mathrm{Aut}^+\langle A,B,C\rangle$*-action.*

*Proof.* Let $\mathrm{IrAutABCMaps}_D := \{\mathcal{F} \in \mathrm{IrAutABCMaps} \mid D_{\mathcal{F}} = D\}$.

By considering $\langle A,B,C\rangle$-orbits in the codification pairing relation, we see that we may assume that $\mathcal{F} \in \mathrm{IrAutABCMaps}_D$, and that $(\rho_{\hat{\mathbb{C}}}, q_{\hat{\mathbb{C}}}) \in \mathfrak{R}_{\hat{\mathbb{C}}} \times \hat{\mathbb{C}}$ is such that $q_{\hat{\mathbb{C}}}$ is the fixed point of $\rho_{\hat{\mathbb{C}}}(D)$.

For $\mathcal{F} \in \mathrm{IrAutABCMaps}_D$ and $\rho_{\hat{\mathbb{C}}} \in \mathfrak{R}_{\hat{\mathbb{C}}}$, let us say that $\mathcal{F}$ codifies $\rho_{\hat{\mathbb{C}}}$ if there exists a (unique) (surjective) continuous, orientation-preserving $\langle A,B,C\rangle$-map

$$\gamma = \gamma(\mathcal{F}, \rho_{\hat{\mathbb{C}}})\colon \mathbb{S}^2(\mathcal{F}) \to \hat{\mathbb{C}}(\rho_{\hat{\mathbb{C}}}), \tag{8.3.1}$$

where $\hat{\mathbb{C}}(\rho_{\hat{\mathbb{C}}})$ denotes $\hat{\mathbb{C}}$ viewed as an $\langle A,B,C\rangle$-space via $\rho_{\hat{\mathbb{C}}}$.

It now suffices to show that this new codification pairing relation gives a bijective correspondence

$$\langle D\rangle\backslash\,\mathrm{IrAutABCMaps}_D\,/\,\mathrm{Isom}^+(\mathbb{Z}) \quad \xrightarrow{\sim} \quad \mathrm{Aut}^+(\hat{\mathbb{C}})\backslash\mathfrak{R}_{\hat{\mathbb{C}}}. \tag{8.3.2}$$

Thurston and Bonahon [2] associated, to each element of $\mathfrak{R}_{\hat{\mathbb{C}}}$, an ordered pair of distinct irrational numbers, called the *ending invariant*. Thurston [10] showed that any pair of distinct irrational numbers is the ending invariant of some element of $\mathfrak{R}_{\hat{\mathbb{C}}}$. Minsky [9] showed that two elements of $\mathfrak{R}_{\hat{\mathbb{C}}}$ lie in the same $\mathrm{Aut}^+(\hat{\mathbb{C}})$-orbit whenever they have the same ending invariant. (Let us emphasize that we are recording only those aspects of the general theory which we will be using.)

Suppose that $\mathcal{F} \in \mathrm{IrAutABCMaps}_D$.

Let $h_+ = \bigcap\limits_{n\in\mathbb{Z}} \mathcal{F}_n[h_{\mathbf{x}}, h_{\mathbf{y}}]$ and $h_- = \bigcap\limits_{n\in\mathbb{Z}} \mathcal{F}_n[h_{-\mathbf{y}}, h_{\mathbf{x}}]$. It follows from Lemma 6.3, that $h_+$ and $h_-$ are non-collinear irrational radial lines. Reading off these lines in the triangulated plane, we obtain $\mathcal{W}(h_+)$ and $\mathcal{W}(h_-)$ in $\mathfrak{E}$, from which we can construct

$$\{\mathcal{W}_m(\mathcal{F}) \mid m \in \mathbb{Z} \cup \{-\infty, \infty\}\}. \tag{8.3.3}$$

Let $\mu_-$ and $\mu_+$ denote the inverses of the slopes of $h_-$ and $h_+$, respectively; thus, $(\mu_-, \mu_+)$ is a pair of distinct irrational numbers.

Consider any $\rho_{\hat{\mathbb{C}}} \in \mathfrak{R}_{\hat{\mathbb{C}}}$ with ending invariant $(\mu_-, \mu_+)$. McMullen [8] showed that $\widehat{\rho_{\hat{\mathbb{C}}}}\colon \mathfrak{E} \to \hat{\mathbb{C}}$ is constant on (8.3.3). It can be shown that $\mathbb{S}^2(\mathcal{F})$ is the universal Hausdorff $\langle A, B, C\rangle$-space having a continuous $\langle A, B, C\rangle$-map from $\mathfrak{E}$ which collapses (8.3.3) to a point. It follows that there exists a continuous $\langle A, B, C\rangle$-map as in (8.3.1), and it can be shown that this map is orientation-preserving. Hence $\mathcal{F}$ codifies $\rho_{\hat{\mathbb{C}}}$.

Now consider any $\rho_{\hat{\mathbb{C}}} \in \mathfrak{R}_{\hat{\mathbb{C}}}$ that is codified by $\mathcal{F}$. Then $\widehat{\rho_{\hat{\mathbb{C}}}}\colon \mathfrak{E} \to \hat{\mathbb{C}}$ carries (8.3.3) to the point $q_{\hat{\mathbb{C}}} \in \hat{\mathbb{C}}$ fixed by $\rho_{\hat{\mathbb{C}}}(D)$, and McMullen [8] also showed that there are exactly four irrational radial lines (forming two irrational lines) that give rise to elements of $\widehat{\rho_{\hat{\mathbb{C}}}}^{-1}\{q_{\hat{\mathbb{C}}}\}$. Thus the ending invariant of $\rho_{\hat{\mathbb{C}}}$ must be $(\mu_-, \mu_+)$.

Thus we have a well-defined $\mathrm{Aut}^+\langle A, B, C\rangle$-map as in (8.3.2).

We now proceed in the reverse direction. Consider any $\rho_{\hat{\mathbb{C}}} \in \mathfrak{R}_{\hat{\mathbb{C}}}$. Let $(\mu_-, \mu_+)$ be the ending invariant of $\rho_{\hat{\mathbb{C}}}$. There exist two radial lines $h_-$, $h_+$ with slopes $\frac{1}{\mu_-}$, $\frac{1}{\mu_+}$, respectively, such that $h_+ \subseteq [h_-, -h_-]$. The pair $(h_-, h_+)$ is then unique up to multiplying by $-1$. By a standard argument, recalled in Review 8.4 below, there exists $\mathcal{F} \in \mathrm{IrAutABCMaps}_D$ such that

$$\bigcap_{n\in\mathbb{Z}} \mathcal{F}_n[h_{-\mathbf{y}}, h_{\mathbf{x}}] = h_- \text{ and } \bigcap_{n\in\mathbb{Z}} \mathcal{F}_n[h_{\mathbf{x}}, h_{\mathbf{y}}] = h_+, \tag{8.3.4}$$

and the $\langle D^2\rangle \times \mathrm{Isom}^+(\mathbb{Z})$-orbit of $\mathcal{F}$ is unique. It is easy to see that this information gives the inverse of (8.3.2). □

**8.4 Review.** For the convenience of the reader, we recall the standard argument referred to in the last paragraph of the above proof.

(i). We begin by proving the existence of $\mathcal{F}$.

Consider first the case where $h_+ \subseteq [h_{\mathbf{x}}, h_{\mathbf{y}}]$ and $h_- \subseteq [h_{-\mathbf{y}}, h_{\mathbf{x}}]$. We remark that this amounts to $\mu_- < 0 < \mu_+$. Here we construct $\mathcal{F}$ recursively in both directions, as follows.

We set $\mathcal{F}_0 = 1$.

Suppose we are given $n \geq 0$ and $\mathcal{F}_n$ such that $h_+ \subseteq \mathcal{F}_n[h_{\mathbf{x}}, h_{\mathbf{y}}]$. As in the proof of Lemma 6.3,

$$\mathcal{F}_n[h_{\mathbf{x}}, h_{\mathbf{y}}] = \mathcal{F}_n[h_{\mathbf{x}}, h_{\mathbf{x}+\mathbf{y}}] \cup \mathcal{F}_n[h_{\mathbf{x}+\mathbf{y}}, h_{\mathbf{y}}] = \mathcal{F}_n R[h_{\mathbf{x}}, h_{\mathbf{y}}] \cup \mathcal{F}_n L[h_{\mathbf{x}}, h_{\mathbf{y}}]$$

and

$$\mathcal{F}_n R[h_{\mathbf{x}}, h_{\mathbf{y}}] \cap \mathcal{F}_n L[h_{\mathbf{x}}, h_{\mathbf{y}}] = \mathcal{F}_n[h_{\mathbf{x}}, h_{\mathbf{x}+\mathbf{y}}] \cap \mathcal{F}_n[h_{\mathbf{x}+\mathbf{y}}, h_{\mathbf{y}}] = \mathcal{F}_n(h_{\mathbf{x}+\mathbf{y}}).$$

The latter radial line has rational slope and hence is not $h_+$. We choose $\mathcal{F}_{n+1}$ to be the unique element of $\{\mathcal{F}_n R, \mathcal{F}_n L\}$ such that $h_+ \subseteq \mathcal{F}_{n+1}[h_{\mathbf{x}}, h_{\mathbf{y}}]$.

Similarly, if we are given $n \leq 0$ and $\mathcal{F}_n$ such that $h_- \subseteq \mathcal{F}_n[h_{-\mathbf{y}}, h_{\mathbf{x}}]$, we can choose $\mathcal{F}_{n-1}$ to be the unique element of $\{\mathcal{F}_n R^{-1}, \mathcal{F}_n L^{-1}\}$ such that $h_- \subseteq \mathcal{F}_{n-1}[h_{-\mathbf{y}}, h_{\mathbf{x}}]$.

This defines $\mathcal{F} \in \mathrm{IrAutABCMaps}_D$, and (8.3.4) holds by Lemma 6.3.

We next weaken the restraints on $h_+$ and $h_-$ to the assumption that there exists $F \in \mathrm{Aut}^+\langle A, B, C\rangle$ such that

$$F(D) = D, \quad h_+ \subseteq F[h_{\mathbf{x}}, h_{\mathbf{y}}], \quad h_- \not\subseteq F[h_{\mathbf{x}}, h_{\mathbf{y}}], \quad -h_- \not\subseteq F[h_{\mathbf{x}}, h_{\mathbf{y}}].$$

Here

$$F^{-1}(h_+) \subseteq [h_{\mathbf{x}}, h_{\mathbf{y}}], \quad F^{-1}(h_-) \not\subseteq [h_{\mathbf{x}}, h_{\mathbf{y}}], \quad -F^{-1}(h_-) \not\subseteq [h_{\mathbf{x}}, h_{\mathbf{y}}],$$

and $F^{-1}(h_+) \subseteq [F^{-1}(h_-), -F^{-1}(h_-)]$. It follows that $F^{-1}(h_-) \subseteq [-h_{\mathbf{y}}, h_{\mathbf{x}}]$. By the preceding case, there exists $\mathcal{G} \in \mathrm{IrAutABCMaps}_D$ such that

$$\bigcap_{n\in\mathbb{Z}} \mathcal{G}_n[h_{\mathbf{x}}, h_{\mathbf{y}}] = F^{-1}(h_+) \qquad \text{and} \qquad \bigcap_{n\in\mathbb{Z}} \mathcal{G}_n[h_{-\mathbf{y}}, h_{\mathbf{x}}] = F^{-1}(h_-).$$

Now (8.3.4) holds with $\mathcal{F} = F \cdot \mathcal{G}$.

We now remove the restraints on $h_+, h_-$. Since $P$ acts on $\mathbb{R}^2$ as a ninety-degree (counter-clockwise) rotation, there exists $F \in \langle P\rangle$ such that $h_+ \subseteq F[h_{\mathbf{x}}, h_{\mathbf{y}}]$. Choose any irrational radial line $h \subseteq F[-h_{\mathbf{y}}, h_{\mathbf{x}}]$. Then $h \not\subseteq F[h_{\mathbf{x}}, h_{\mathbf{y}}]$, $-h \not\subseteq F[h_{\mathbf{x}}, h_{\mathbf{y}}]$, and $h_+ \subseteq [h, -h]$. By the preceding case, applied to $(h, h_+)$, there exists $\mathcal{G} \in \mathrm{IrAutABCMaps}_D$ such that $\bigcap_{n\in\mathbb{Z}} \mathcal{G}_n[h_{\mathbf{x}}, h_{\mathbf{y}}] = h_+$. Thus, for a sufficiently large $n \in \mathbb{Z}$,

$$h_+ \subseteq \mathcal{G}_n[h_{\mathbf{x}}, h_{\mathbf{y}}], \quad h_- \not\subseteq \mathcal{G}_n[h_{-\mathbf{y}}, h_{\mathbf{x}}], \quad -h_- \not\subseteq \mathcal{G}_n[h_{-\mathbf{y}}, h_{\mathbf{x}}].$$

By the preceding case again, now applied to $(h_-, h_+)$, (8.3.4) holds for some $\mathcal{F}$ in $\mathrm{IrAutABCMaps}_D$.

(ii). We now prove the uniqueness of the $\langle D^2\rangle \times \mathrm{Isom}^+(\mathbb{Z})$-orbit of $\mathcal{F}$.

Thus, suppose that $\mathcal{F}$ and $\mathcal{G}$ are elements of $\mathrm{IrAutABCMaps}_D$ such that

$$\bigcap_{n\in\mathbb{Z}} \mathcal{F}_n[h_{\mathbf{x}}, h_{\mathbf{y}}] = \bigcap_{n\in\mathbb{Z}} \mathcal{G}_n[h_{\mathbf{x}}, h_{\mathbf{y}}] = h_+$$

and

$$\bigcap_{n\in\mathbb{Z}} \mathcal{F}_n[h_{-\mathbf{y}}, h_{\mathbf{x}}] = \bigcap_{n\in\mathbb{Z}} \mathcal{G}_n[h_{-\mathbf{y}}, h_{\mathbf{x}}] = h_-.$$

There exists a sufficiently large $N \in \mathbb{N}$ such that $\mathcal{G}_N[h_{\mathbf{x}}, h_{\mathbf{y}}] \subseteq \mathcal{F}_0[h_{\mathbf{x}}, h_{\mathbf{y}}]$. Let us set $F = \mathcal{F}_0^{-1}\mathcal{G}_N$. Since $F(D) = D$, it is not difficult to use Notation 2.3 to show that there exists $m \in \mathbb{Z}$ such that $P^m F$ is a monoid word in either $\{R, L\}$ or $\{R^{-1}, L^{-1}\}$. Recall that $R$ and $L$ contract $[h_{\mathbf{x}}, h_{\mathbf{y}}]$ onto proper subsets of itself, while $P$ acts as ninety-degree rotation. Since $F[h_{\mathbf{x}}, h_{\mathbf{y}}] \subseteq [h_{\mathbf{x}}, h_{\mathbf{y}}]$, we see that $m \in 4\mathbb{Z}$, and $P^m F$ is a monoid word in $\{R, L\}$. Thus $P^m \in \langle D^2\rangle$, and, by replacing $\mathcal{G}$ with $P^m \cdot \mathcal{G}$, we may assume that $m = 0$. Hence $F$ is a monoid word in $\{R, L\}$; let $d$ denote the length of this word. Since $\mathcal{F}_0 F[h_{\mathbf{x}}, h_{\mathbf{y}}] = \mathcal{G}_n[h_{\mathbf{x}}, h_{\mathbf{y}}] \supseteq h_+$, we see that $F$ is the unique monoid word in $\{R, L\}$ of length $d$ such that $\mathcal{F}_0 F[h_{\mathbf{x}}, h_{\mathbf{y}}] \supseteq h_+$, that is, $\mathcal{F}_0 F = \mathcal{F}_d$. By replacing $\mathcal{G}$ with $\mathcal{G} \cdot (z - d + N)$, we may assume that $N = d$. Now $\mathcal{G}_d = \mathcal{G}_N = \mathcal{F}_0 F = \mathcal{F}_d$. It follows by induction that, for each $n \in \mathbb{N}$, $\mathcal{G}_{d+n} = \mathcal{F}_{d+n}$ and $\mathcal{G}_{d-n} = \mathcal{F}_{d-n}$. Hence $\mathcal{F} = \mathcal{G}$, as desired. $\square$

**8.5 Remark.** It is not difficult to deduce that the ordered pair $(\mathcal{W}_{-1}(\mathcal{F}), \mathcal{W}_0(\mathcal{F}))$ in $\mathfrak{E}^2$ determines the $\mathrm{Isom}^+(\mathbb{Z})$-orbit of $\mathcal{F}$.

Review 8.4 deals with the case where $D_{\mathcal{F}} = D$.

It remains to show that for any irrational radial lines $h$, $h'$ in $[h_{-\mathbf{y}}, h_{\mathbf{y}}]$ and elements $W$, $W'$ of $ABC$, if $W \cdot \mathcal{W}(h) = W' \cdot \mathcal{W}(h')$ in $\mathfrak{E}$ then $W = W'$ and $h = h'$.

By left multiplying the equation $W \cdot \mathcal{W}(h) = W' \cdot \mathcal{W}(h')$ by a suitable element of $\langle A, B, C\rangle$, we may assume that $W$ is completely cancelled in $W \cdot \mathcal{W}(h)$, and that $W'$ is completely cancelled in $W' \cdot \mathcal{W}(h')$. The two sides of the equation can then be used to construct piecewise linear paths in $\mathbb{R}^2$ which must be equal or be separated by an element of $\mathbb{Z}^2$. The result now follows. $\square$

# 9 The complex $\mathrm{CW}(\rho_{\hat{\mathbb{C}}}, q_{\hat{\mathbb{C}}})$

**9.1 Definitions.** Theorem 8.3 gives the bijective $\mathrm{Aut}^+\langle A, B, C\rangle$-map of (8.3.2). On dividing out by the $\mathrm{Aut}^+\langle A, B, C\rangle$-action, and then using (3.2.1), we get a bijective map

$$\mathrm{IrRLMaps} \,/\, \mathrm{Isom}^+(\mathbb{Z}) \ \xrightarrow{\sim}\ \mathrm{Aut}^+(\hat{\mathbb{C}})\backslash \mathfrak{R}_{\hat{\mathbb{C}}} / \,\mathrm{Aut}^+\langle A, B, C\rangle. \qquad (9.1.1)$$

The inverse determines a map

$$\mathfrak{b}\colon \mathfrak{R}_{\hat{\mathbb{C}}} \to \mathrm{IrRLMaps} \,/\, \mathrm{Isom}^+(\mathbb{Z}).$$

in a natural way. The inverse of (9.1.1) was discussed in Section 1, where the domain of (9.1.1) was denoted IrRLSeqs.

Minsky [9] gives (9.1.1) explicitly, in that, by generalizing a construction of Jørgensen [6], he associates a sophisticated model with each element of IrRLMaps / $\mathrm{Isom}^+(\mathbb{Z})$; see [9, Figure 3]. □

We now consider the case where we can identify $\mathbb{S}^2 = \hat{\mathbb{C}}$ and $\widehat{\rho_{\mathbb{S}^2}} = \widehat{\rho_{\hat{\mathbb{C}}}}$.

**9.2 Definitions.** Let $\rho_{\hat{\mathbb{C}}} \in \mathfrak{R}_{\hat{\mathbb{C}}}$. Let $q_{\hat{\mathbb{C}}}$ be a parabolic point for $\rho_{\hat{\mathbb{C}}}(\langle A, B, C\rangle)$.

Choose some $\mathcal{F} \in$ IrAutABCMaps that codifies $(\rho_{\hat{\mathbb{C}}}, q_{\hat{\mathbb{C}}})$, and let $\gamma$ denote the associated map $\gamma(\mathcal{F}, \rho_{\hat{\mathbb{C}}})\colon \mathbb{S}^2(\mathcal{F}) \to \hat{\mathbb{C}}(\rho_{\hat{\mathbb{C}}}, q_{\hat{\mathbb{C}}})$.

It is conjectured that $\gamma$ is a homeomorphism.

If $\gamma$ is a homeomorphism, we say that $\rho_{\hat{\mathbb{C}}}$ is *given by a C-T-model.*

It can be seen that $\rho_{\hat{\mathbb{C}}}$ being given by a C-T-model depends only on the bi-infinite sequence $\mathfrak{b}(\rho_{\hat{\mathbb{C}}})$, that is, the $\mathrm{Isom}^+(\mathbb{Z})$-orbit of quotient$(\mathcal{F})$. □

**9.3 Theorem.** (Bowditch [3]) *If $\rho_{\hat{\mathbb{C}}} \in \mathfrak{R}_{\hat{\mathbb{C}}}$ and there is a finite bound on the length of power-of-a-single-letter subwords of $\mathfrak{b}(\rho_{\hat{\mathbb{C}}})$, then $\rho_{\hat{\mathbb{C}}}$ is given by a C-T-model.* □

**9.4 Definitions.** Let $\rho_{\hat{\mathbb{C}}} \in \mathfrak{R}_{\hat{\mathbb{C}}}$ and suppose that $\rho_{\hat{\mathbb{C}}}$ is given by a C-T-model.

Let $q_{\hat{\mathbb{C}}}$ be a parabolic point for $\rho_{\hat{\mathbb{C}}}(\langle A, B, C\rangle)$.

Choose some $\mathcal{F} \in$ IrAutABCMaps which codifies $(\rho_{\hat{\mathbb{C}}}, q_{\hat{\mathbb{C}}})$.

Let $\mathfrak{f} =$ quotient$(\mathcal{F})$ and let $\gamma = \gamma(\mathcal{F}, \rho_{\hat{\mathbb{C}}})\colon \mathbb{S}^2(\mathcal{F}) \xrightarrow{\sim} \hat{\mathbb{C}}(\rho_{\hat{\mathbb{C}}}, q_{\hat{\mathbb{C}}})$.

Let $(m, n)$ be a variable ranging over $\mathbb{Z}^2$.

The set of all elements of IrAutABCMaps which codify $(\rho_{\hat{\mathbb{C}}}, q_{\hat{\mathbb{C}}})$ is

$$\{D^m_{\mathcal{F}} \cdot \mathcal{F} \cdot (z+n) \mid (m,n) \in \mathbb{Z}^2\},$$

and this set can be reconstructed from the set of values

$$\{D^m_{\mathcal{F}}\mathcal{F}_n \mid (m,n) \in \mathbb{Z}^2\} \subseteq \mathrm{Aut}^+\langle A, B, C\rangle.$$

Thus the latter set captures the $\mathrm{Aut}^+(\hat{\mathbb{C}})$-orbit of $(\rho_{\hat{\mathbb{C}}}, q_{\hat{\mathbb{C}}})$.

Now $\gamma$ carries the CW-structure $\mathrm{CW}(\mathcal{F})$ on $\mathbb{S}^2(\mathcal{F}) - \{q_{\mathbb{S}^2(\mathcal{F})}\}$ to a CW-structure, denoted $\mathrm{CW}(\rho_{\hat{\mathbb{C}}}, q_{\hat{\mathbb{C}}})$, on $\hat{\mathbb{C}} - \{q_{\hat{\mathbb{C}}}\}$.

All the information obtained about $\mathrm{CW}(\mathcal{F})$ in Section 7 passes over to $\mathrm{CW}(\rho_{\hat{\mathbb{C}}}, q_{\hat{\mathbb{C}}})$. Let us mention all of the following.

The set of closed two-cells of $\mathrm{CW}(\rho_{\hat{\mathbb{C}}}, q_{\hat{\mathbb{C}}})$ is $\{\widehat{\rho_{\hat{\mathbb{C}}}}(D^m_{\mathcal{F}}\mathcal{F}'_n[B]) \mid (m,n) \in \mathbb{Z}^2\}$; see Figure 9.6.1.

By Theorem 7.12, there is an isomorphism $\mathrm{CW}(\mathfrak{f}) \xrightarrow{\sim} \mathrm{CW}(\rho_{\hat{\mathbb{C}}}, q_{\hat{\mathbb{C}}})$, of oriented, colored $\mathbb{Z}$-CW-complexes, which carries the $(m,n)$th closed two-cell of $\mathrm{CW}(\mathfrak{f})$ homeomorphically to $\widehat{\rho_{\hat{\mathbb{C}}}}(D^m_{\mathcal{F}}\mathcal{F}'_n[B])$.

Let $\mathrm{CT} = \mathrm{CT}(\rho_{\hat{\mathbb{R}}}, \rho_{\hat{\mathbb{C}}})\colon \hat{\mathbb{R}} \twoheadrightarrow \hat{\mathbb{C}}$. For each $m \in \mathbb{Z} \cup \{\pm\infty\}$, let $w_m = w_m(\mathcal{F})$. Then the following hold.

(i) $\mathrm{CT}^{-1}\{q_{\hat{\mathbb{C}}}\} \quad = \quad \{w_m \mid m \in \mathbb{Z} \cup \{\pm\infty\}\}$.

(ii) $\hat{\mathbb{R}} - \mathrm{CT}^{-1}\{q_{\hat{\mathbb{C}}}\} = \bigcup\limits_{m\in\mathbb{Z}} \,]w_{m-1}, w_m[\,$ . The terms in the union are open cyclic intervals in $\hat{\mathbb{R}}$, and are pairwise disjoint.

(iii) $\hat{\mathbb{C}} - \{q_{\hat{\mathbb{C}}}\} = \bigcup\limits_{m\in\mathbb{Z}} \mathrm{CT}(\,]w_{m-1}, w_m[\,)$. The terms in the union are connected, and are pairwise disjoint with the following exceptions: consecutive terms overlap in an arc that separates $\hat{\mathbb{C}} - \{q_{\hat{\mathbb{C}}}\}$; terms two apart overlap in a discrete countable set.

(iv) $\mathrm{CT}(\,]w_{2m-1}, w_{2m}[\,) = \bigcup\limits_{n\in\mathbb{Z},\ \mathfrak{f}_n = L} \widehat{\rho_{\hat{\mathbb{C}}}}(D^m_{\mathcal{F}}\mathcal{F}'_n[B])$. The terms in the union are Jordan domains, and are pairwise disjoint except that consecutive terms overlap in a single point.

(v) $\mathrm{CT}(\,]w_{2m}, w_{2m+1}[\,) = \bigcup\limits_{n\in\mathbb{Z},\ \mathfrak{f}_n = R} \widehat{\rho_{\hat{\mathbb{C}}}}(D^m_{\mathcal{F}}\mathcal{F}'_n[B])$. The terms in the union are Jordan domains, and are pairwise disjoint except that consecutive terms overlap in a single point.

Now (i)–(v) give the description of $\mathrm{CW}(\rho_{\hat{\mathbb{C}}}, q_{\hat{\mathbb{C}}})$ claimed in Section 1. $\square$

**9.5 Proposition.** *Suppose that $\rho_{\hat{\mathbb{C}}}$ and $\rho'_{\hat{\mathbb{C}}}$ are elements of $\mathfrak{R}_{\hat{\mathbb{C}}}$ that are given by C-T-models. Let $q_{\hat{\mathbb{C}}}$ and $q'_{\hat{\mathbb{C}}}$, be parabolic points of $\rho_{\hat{\mathbb{C}}}(\langle A,B,C\rangle)$ and $\rho'_{\hat{\mathbb{C}}}(\langle A,B,C\rangle)$, respectively. Then the following are equivalent*:

(a) *There exists an element of $\mathrm{Aut}^+(\hat{\mathbb{C}})$ which induces, in a natural way, an isomorphism $\mathrm{CW}(\rho_{\hat{\mathbb{C}}}, q_{\hat{\mathbb{C}}}) \xrightarrow{\sim} \mathrm{CW}(\rho'_{\hat{\mathbb{C}}}, q'_{\hat{\mathbb{C}}})$ of oriented, colored $\mathbb{Z}$-CW-complexes.*

(b) *As oriented, colored $\mathbb{Z}$-CW-complexes, $\mathrm{CW}(\rho_{\hat{\mathbb{C}}}, q_{\hat{\mathbb{C}}})$ and $\mathrm{CW}(\rho'_{\hat{\mathbb{C}}}, q'_{\hat{\mathbb{C}}})$ are isomorphic.*

(c) $\mathfrak{b}(\rho_{\hat{\mathbb{C}}}) = \mathfrak{b}(\rho'_{\hat{\mathbb{C}}})$.

*Proof.* (a) clearly implies (b).

(b) implies (c) by Definitions 9.4 and Proposition 4.4.

(c) $\Rightarrow$ (a). Suppose that (c) holds. By (9.1.1), there exists

$$(F,\sigma) \in \mathrm{Aut}^+\langle A,B,C\rangle \times \mathrm{Aut}^+(\hat{\mathbb{C}})$$

such that $\rho'_{\hat{\mathbb{C}}} = i_\sigma \cdot \rho_{\hat{\mathbb{C}}} \cdot F$. It is not difficult to show that

$$\mathrm{CT}(\rho_{\hat{\mathbb{R}}} \cdot F, i_\sigma \cdot \rho_{\hat{\mathbb{C}}} \cdot F) = \sigma \cdot \mathrm{CT}(\rho_{\hat{\mathbb{R}}}, \rho_{\hat{\mathbb{C}}})\colon \hat{\mathbb{R}} \twoheadrightarrow \hat{\mathbb{C}}.$$

Since $\mathrm{CW}(\rho_{\hat{\mathbb{C}}}, q_{\hat{\mathbb{C}}})$ does not depend on the choice of $\rho_{\hat{\mathbb{R}}} \in \mathfrak{R}_{\hat{\mathbb{R}}}$, it follows that

$$\mathrm{CW}(i_\sigma \cdot \rho_{\hat{\mathbb{C}}} \cdot F, \sigma(q_{\hat{\mathbb{C}}})) = \sigma(\mathrm{CW}(\rho_{\hat{\mathbb{C}}}, q_{\hat{\mathbb{C}}})). \tag{9.5.1}$$

Thus $\mathrm{CW}(\rho'_{\hat{\mathbb{C}}}, \sigma(q_{\hat{\mathbb{C}}})) = \sigma(\mathrm{CW}(\rho_{\hat{\mathbb{C}}}, q_{\hat{\mathbb{C}}}))$; hence, we may assume that $\rho'_{\hat{\mathbb{C}}} = \rho_{\hat{\mathbb{C}}}$.

Now there exists $W \in \langle A,B,C\rangle$ such that $\rho_{\hat{\mathbb{C}}}(W)(q_{\hat{\mathbb{C}}}) = q'_{\hat{\mathbb{C}}}$. We repeat the foregoing argument but taking $F = W^{-1}$ and $\sigma = \rho_{\hat{\mathbb{C}}}(W)$. Here, $i_\sigma \cdot \rho_{\hat{\mathbb{C}}} \cdot F = \rho_{\hat{\mathbb{C}}}$ and $\sigma(q_{\hat{\mathbb{C}}}) = q'_{\hat{\mathbb{C}}}$; by this case of (9.5.1), $\mathrm{CW}(\rho_{\hat{\mathbb{C}}}, q'_{\hat{\mathbb{C}}}) = \sigma(\mathrm{CW}(\rho_{\hat{\mathbb{C}}}, q_{\hat{\mathbb{C}}}))$. We now see that (a) holds. $\square$

**9.6 Example.** There is a unique $\mathcal{F} \in \mathrm{IrAutABCMaps}$ with the property that, for each $n \in \mathbb{Z}$, $\mathcal{F}_{10n} = (RLLRRRLLLL)^n$. Figure 9.6.1 depicts $\widehat{\rho_{\hat{\mathbb{C}}}}(\mathcal{F}'_n[B])$, $-2 \leq n \leq 11$, for this case.

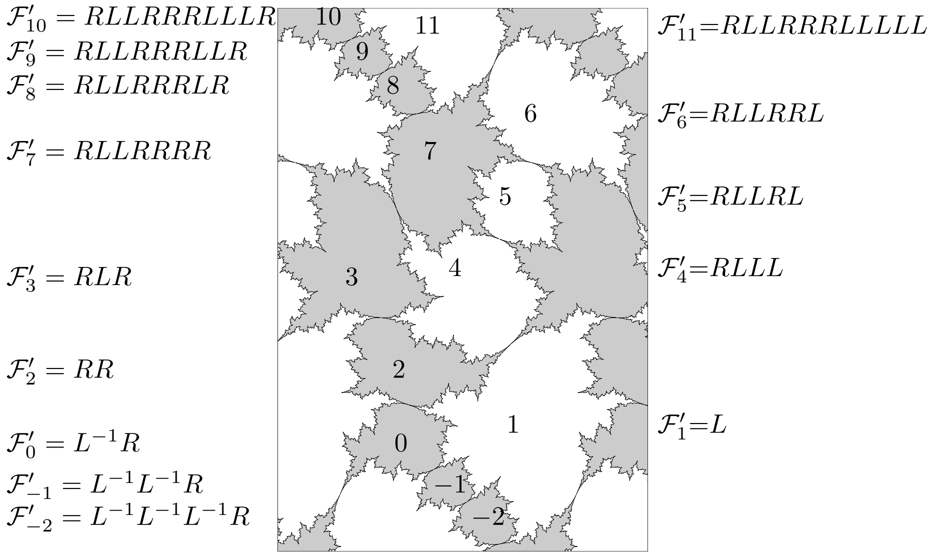


Figure 9.6.1: Some $\widehat{\rho_{\hat{\mathbb{C}}}}(\mathcal{F}'_n[B])$ in the case where $\mathcal{F}_{10n} = (RLLRRRLLLL)^n$. $\square$

# 10 Hyperbolic once-punctured-torus bundles

Let us briefly mention the special case of hyperbolic once-punctured-torus bundles, which is the case we have in mind throughout the article.

Let $F$ be an element of $\mathrm{Aut}\langle A, B, C\rangle$ such that $F$ lies in the semigroup (freely) generated by $\{R, L\}$, and such that $F$ is not a power of $R$ or $L$.

Jørgensen [6] showed that $\langle BC, BA, F\rangle$ ($\leq \mathrm{Aut}\langle A, B, C\rangle$) is the fundamental group of a complete, finite-volume, hyperbolic three-manifold which fibers over a circle with fibers once-punctured tori. It follows that there exists a discrete, faithful representation

$$\rho\colon \langle A, B, C, F\rangle \to \mathrm{Aut}^+(\hat{\mathbb{C}}).$$

For any $\beta \in \mathrm{Aut}^+(\hat{\mathbb{C}})$ such that $\beta \cdot \rho(D) \cdot \beta^{-1} = z+1$, we have $\beta \cdot \rho(F) \cdot \beta^{-1} = z+a$ for some $a \in \mathbb{C} - \mathbb{R}$. By composing $\rho$ with conjugation by $\overline{z}$ if necessary, we may assume that the imaginary part of $a$ is *positive*. By Mostow rigidity, $\rho$ is then unique up to composition with an inner automorphism of $\mathrm{Aut}^+(\hat{\mathbb{C}})$, and $a$ is unique. Restricting $\rho$ to $\langle A, B, C\rangle$ gives an element $\rho_{\hat{\mathbb{C}}} \in \mathfrak{R}_{\hat{\mathbb{C}}}$.

Let $q_{\hat{\mathbb{C}}}$ denote the element of $\hat{\mathbb{C}}$ fixed by $\rho_{\hat{\mathbb{C}}}(D)$.

There exists a unique integer $b \geq 2$, and a unique $\mathfrak{f} \in$ IrRLMaps such that $\mathfrak{f} = \mathfrak{f} \cdot (z + b)$ and $F = \mathfrak{f}_1\mathfrak{f}_2 \cdots \mathfrak{f}_b$. Thus $[\mathfrak{f}] = \prod\limits_{\mathbb{Z}} F$. We say that $\mathfrak{f}$ is *periodic*. There exist a unique $\mathcal{F} \in$ IrAutABCMaps such that $\mathcal{F}_0 = 1$ and quotient$(\mathcal{F}) = \mathfrak{f}$. Then $D_{\mathcal{F}} = D$, and $\mathcal{F}_{nb} = F^n$ for each $n \in \mathbb{Z}$.

Definitions 6.4 give $F$-invariant ('eigen') radial lines, with $h_+ = h_+(\mathcal{F})$ being attracting for $F$ and $h_- = h_-(\mathcal{F})$ being attracting for $F^{-1}$. The funnel indicated in Figure 7.5.1 is invariant under $F$.

Proposition 7.4 gives an attracting point $\mathcal{W}_0 = \mathcal{W}_0(\mathcal{F}) = \mathcal{W}(h_+) = F^{\infty}(B)$ for $F$, and an attracting point $\mathcal{W}_{-1} = \mathcal{W}_{-1}(\mathcal{F}) = \mathcal{W}(h_-) = F^{-\infty}(AC)$ for $F^{-1}$. Moreover,

$$\{\mathcal{W}_m \mid m \in \mathbb{Z} \cup \{-\infty, +\infty\}\}$$

is the set of fixed points for the action of $F$ on $\mathfrak{E}$. See, for example, [5, Section 4].

It follows that $\mathcal{F}$ codifies $(\rho_{\hat{\mathbb{C}}}, q_{\hat{\mathbb{C}}})$, and, hence, $\mathfrak{b}(\rho_{\hat{\mathbb{C}}}) = \prod\limits_{\mathbb{Z}} F$. By Bowditch's Theorem 9.3, $\rho_{\hat{\mathbb{C}}}$ is given by a C-T-model. Thus $\mathrm{CW}(\rho_{\hat{\mathbb{C}}}, q_{\hat{\mathbb{C}}})$ exists by Definitions 9.4.

More generally, we can consider any $\mathcal{G} \in$ IrAutABCMaps with quotient$(\mathcal{G}) = \mathfrak{f}$; that is, $\mathcal{G} = \mathcal{G}_0\mathcal{F}$. Then $D_{\mathcal{G}} = \mathcal{G}_0(D)$; this is $WDW^{-1}$ for some $W \in \langle A, B, C\rangle$. Then $\mathcal{G}$ codifies $(\rho_{\hat{\mathbb{C}}} \cdot \mathcal{G}_0^{-1}, q_{\hat{\mathbb{C}}})$. Notice that $\rho_{\hat{\mathbb{C}}} \cdot \mathcal{G}_0^{-1}$ extends to a discrete faithful representation $\rho \cdot \mathcal{G}_0^{-1}$ of $\langle A, B, C, \mathcal{G}_0 F \mathcal{G}_0^{-1}\rangle$ in $\mathrm{Aut}^+(\hat{\mathbb{C}})$.

# 11 The symmetry group of $\mathrm{CW}(\mathfrak{f})$

In the remainder of the article we will discuss the symmetries of the CW-complexes we have been considering. Our approach was strongly influenced by the analysis of McCullough [7, Section 2.4].

It is convenient to have a description of the symmetries of the model CW-complexes constructed in the first part of Definitions 4.2. We will use Notation 2.1.

**11.1 Definitions.** Let $\mathrm{CW}' := \{\mathrm{CW}'(\mathfrak{f}) \mid \mathfrak{f} \in \mathrm{IrRLMaps}\}$. We think of each element of $\mathrm{CW}'$ as a set of cells, and we think of each cell as a subset of $\mathbb{R}^2$.

Let $\mathrm{Isom}(\mathrm{CW}')$ denote the set of elements of $\mathrm{Isom}(\mathbb{R}^2)$ which act as a CW-isomorphism between some pair of (possibly equal) elements of $\mathrm{CW}'$.

We will find that $\mathrm{Isom}(\mathrm{CW}')$ is a group acting on all of $\mathrm{CW}'$.

Let $\mathrm{Isom}(\mathbb{Z}^2, \mathbb{Z} \times \mathbb{R})$ denote the set of elements of $\mathrm{Isom}(\mathbb{R}^2)$ which permute both $\mathbb{Z}^2$ and $\mathbb{Z} \times \mathbb{R}$. Then

$$\mathrm{Isom}(\mathbb{Z}^2, \mathbb{Z} \times \mathbb{R}) = \{(ax + e, dy + f) \mid a, d, e, f \in \mathbb{Z},\ a^2 = d^2 = 1\},$$

and this has index two in $\mathrm{Isom}(\mathbb{Z}^2)$.

Recall that the slanted one-cells of $\mathrm{CW}'(\mathfrak{f})$ point upward from the zero-cells in $2\mathbb{Z} \times \mathbb{Z}$ to the zero-cells in $(\mathbb{Z} - 2\mathbb{Z}) \times \mathbb{Z}$. It follows that the horizontal reflection $(x, -y)$ carries $\mathrm{CW}'$ to a set disjoint from $\mathrm{CW}'$; in particular, $(x, -y)$ does not lie in $\mathrm{Isom}(\mathbb{R}^2, \mathrm{CW}')$.

It is not difficult to see that $\mathrm{Isom}(\mathrm{CW}')$ is a subset of $\mathrm{Isom}(\mathbb{Z}^2, \mathbb{Z} \times \mathbb{R})$; we will find that it is a subgroup of index two.

Suppose that $\mathfrak{f}, \mathfrak{g} \in$ IrRLMaps.

Let $\mathrm{Isom}(\mathrm{CW}'(\mathfrak{f}), \mathrm{CW}'(\mathfrak{g}))$ denote the set of elements of $\mathrm{Isom}(\mathbb{R}^2)$ which carry $\mathrm{CW}'(\mathfrak{f})$ to $\mathrm{CW}'(\mathfrak{g})$.

Recall that $\mathrm{Isom}(\mathbb{Z}) \times \mathrm{Sym}\{R, L\}$ acts on IrRLMaps, and let $(R, L)$ denote the nontrivial element of $\mathrm{Sym}\{R, L\}$. It is straightforward to check the following.

Horizontal translation: $(x+2, y) \in \mathrm{Isom}(\mathrm{CW}'(\mathfrak{f}), \mathrm{CW}'(\mathfrak{f}))$.
Vertical translation: $(x, y+1) \in \mathrm{Isom}(\mathrm{CW}'(\mathfrak{f}), \mathrm{CW}'(\mathfrak{f} \cdot (z+1)))$.
Half-turn: $(-x-1, -y) \in \mathrm{Isom}(\mathrm{CW}'(\mathfrak{f}), \mathrm{CW}'(\mathfrak{f} \cdot (-z)))$.
Vertical reflection: $(-x, y) \in \mathrm{Isom}(\mathrm{CW}'(\mathfrak{f}), \mathrm{CW}'((R, L) \cdot \mathfrak{f}))$.
Horizontal glide-reflection: $(x-1, -y) \in \mathrm{Isom}(\mathrm{CW}'(\mathfrak{f}), \mathrm{CW}'((R, L) \cdot \mathfrak{f} \cdot (-z)))$.

It follows that $\mathrm{Isom}(\mathrm{CW}')$ is a subgroup of index two in $\mathrm{Isom}(\mathbb{Z}^2, \mathbb{Z} \times \mathbb{R})$, and that

$$\mathrm{Isom}(\mathrm{CW}') = \{(ax+e, dy+f) \mid a, d, e, f \in \mathbb{Z},\ a^2 = d^2 = 1, d = (-1)^e\};$$

notice that either $d = 1$ and $e$ is even, or $d = -1$ and $e$ is odd. Recall that $\mathrm{Isom}(\mathbb{Z}^2, \mathbb{Z} \times \mathbb{R})$ has index two in $\mathrm{Isom}(\mathbb{Z}^2)$.

(For $n \in \mathbb{N}$, let $D_n$ denote a group containing $n$ reflections, generated by reflections, $r$, $s$, with defining relations $r^2 = s^2 = (rs)^n = 1$, and let us mention the following: the planar symmetry group $\mathrm{Isom}(\mathbb{Z}^2)$ is of type **p 4 m** and its quotient orbifold is a right-angled, isosceles triangle with three $D_1$-one-cells ('silvered edges'), one $D_2$-zero-cell and two $D_4$-zero-cells; the planar symmetry group $\mathrm{Isom}(\mathbb{Z}^2, \mathbb{Z} \times \mathbb{R})$ is of type **p m m** and its quotient orbifold is a square with four $D_1$-one-cells, and four $D_2$-zero-cells; the planar symmetry group $\mathrm{Isom}(\mathrm{CW}')$ is of type **p m g** and its quotient orbifold is a closed disc with a $D_1$-boundary and two interior double points.)

It can be seen that

$$\begin{aligned} \mathrm{Isom}(\mathbb{Z}^2, \mathbb{Z} \times \mathbb{R}) &= \langle (x, y+1), (x+1, y), (-x, y), (x, -y) \rangle, \\ \mathrm{Isom}(\mathrm{CW}') &= \langle (x, y+1), (x-1, -y), (-x, y) \rangle. \end{aligned}$$

Notice that we have a homomorphism

$$\begin{array}{lcl} \mathrm{Isom}(\mathrm{CW}') & \to & \mathrm{Isom}(\mathbb{Z}) \times \mathrm{Sym}\{R, L\}, \\ ((-1)^a x + e, (-1)^e y + f) & \mapsto & ((-1)^e z + f, (R, L)^{a+e}). \end{array}$$

It is surjective, and the kernel is $((x+2, y))$. □

**11.2 Definitions.** Let $\mathfrak{f} \in$ IrRLMaps.

The *symmetry group of* $\mathfrak{f}$, denoted $\mathrm{Symm}(\mathfrak{f})$, is defined to be the stabilizer of $\mathfrak{f}$ for the action of $\mathrm{Isom}(\mathbb{Z}) \times \mathrm{Sym}\{R, L\}$ on IrRLMaps.

The *symmetry group of* $\mathrm{CW}'(\mathfrak{f})$, denoted $\mathrm{Isom}(\mathrm{CW}'(\mathfrak{f}))$, is defined to be the stabilizer of $\mathrm{CW}'(\mathfrak{f})$ for the action of $\mathrm{Isom}(\mathrm{CW}')$ on $\mathrm{CW}'$. Recall that $\mathrm{Isom}(\mathrm{CW}')$ has index four in $\mathrm{Isom}(\mathbb{Z}^2)$. □

**11.3 Corollary.** *If* $\mathfrak{f} \in$ IrRLMaps, *then* $\mathrm{Isom}(\mathrm{CW}'(\mathfrak{f}))/((x+2, y)) \simeq \mathrm{Symm}(\mathfrak{f})$. □

We now consider the proof of Proposition 4.4 from a different viewpoint.

**11.4 Remarks.** Suppose that $\mathfrak{f}, \mathfrak{g} \in$ IrRLMaps, and that we are given a self-homeomorphism $\sigma$ of $\mathbb{R}^2$ which acts as a color-permuting CW-isomorphism $\mathrm{CW}'(\mathfrak{f}) \xrightarrow{\sim} \mathrm{CW}'(\mathfrak{g})$. By 'color-permuting' we mean 'either color-preserving or color-interchanging'; it amounts to columns being sent to columns. It follows that the $\mathbb{Z}$-action is preserved or inverted. Obviously, the orientation is preserved or reversed.

We will show that, up to cell-wise homotopy, $\sigma$ may be assumed to be affine on each column. Moreover, with this assumption, we will see that either $\sigma$ lies in $\mathrm{Isom}(\mathrm{CW}'(\mathfrak{f}), \mathrm{CW}'(\mathfrak{g}))$, or $\sigma$ is not globally affine and $\{[\mathfrak{f}], [\mathfrak{g}]\}$ is one of the five sets

$$\{\prod_{\mathbb{Z}} RL\},\ \{\prod_{\mathbb{Z}} RRL\},\ \{\prod_{\mathbb{Z}} RLL\},\ \{\prod_{\mathbb{Z}} RRLL\},\ \{\prod_{\mathbb{Z}} RLL,\ \prod_{\mathbb{Z}} RRL\}.$$

We first show that, by precomposing $\sigma$ with a (unique) element of

$$\mathrm{Isom}(\mathrm{CW}'(\mathfrak{f}), \mathrm{CW}'(\mathfrak{g})),$$

we may assume that $\mathfrak{f} = \mathfrak{g}$ and that, near $\{0\} \times \mathbb{R}$, $\sigma$ acts like either $(x, y)$ or $(-x-1, y)$; here 'acts like' is understood to refer to the behavior on $\{0\} \times \mathbb{Z}$ and on the usual frame on $\mathbb{R}^2$.

Since $\sigma$ permutes columns and one-cells, $\sigma$ permutes the components of $\mathbb{Z} \times \mathbb{R}$.

Let $\sigma((0,0)) = (m, n)$. Then, near $\{0\} \times \mathbb{R}$, $\sigma$ acts like $(x+m, y+n)$, $(-x+m, y+n)$, $(x+m, -y+n)$, or $(-x+m, -y+n)$, By precomposing $\sigma$ with a suitable power of $(x+2, y)$, we may assume that $m$ is $-1$ or $0$.

There are now eight cases to consider.

If $\sigma$ acts like $(x, y+n)$ or $(-x-1, y+n)$ near $\{0\} \times \mathbb{R}$, then $\mathfrak{g} = \mathfrak{f} \cdot (z+n)$, and $\mathrm{Isom}(\mathrm{CW}'(\mathfrak{f}), \mathrm{CW}'(\mathfrak{g}))$ contains $(x, y+n)$, as desired.

If $\sigma$ acts like $(x, -y+n)$ or $(-x-1, -y+n)$ near $\{0\} \times \mathbb{R}$, then $\mathfrak{g} = \mathfrak{f} \cdot (-z+n)$, and $\mathrm{Isom}(\mathrm{CW}'(\mathfrak{f}), \mathrm{CW}'(\mathfrak{g}))$ contains $(-x-1, -y+n)$, as desired.

If $\sigma$ acts like $(-x, y+n)$ or $(x-1, y+n)$ near $\{0\} \times \mathbb{R}$, then $\mathfrak{g} = (R, L) \cdot \mathfrak{f} \cdot (z+n)$, and $\mathrm{Isom}(\mathrm{CW}'(\mathfrak{f}), \mathrm{CW}'(\mathfrak{g}))$ contains $(-x, y+n)$, as desired.

If $\sigma$ acts like $(-x, -y+n)$ or $(x-1, -y+n)$ near $\{0\} \times \mathbb{R}$, then $\mathfrak{g} = (R, L) \cdot \mathfrak{f} \cdot (-z+n)$, and $\mathrm{Isom}(\mathrm{CW}'(\mathfrak{f}), \mathrm{CW}'(\mathfrak{g}))$ contains $(x-1, -y+n)$, as desired.

Thus we may assume that $\mathfrak{f} = \mathfrak{g}$ and that, near $\{0\} \times \mathbb{R}$, $\sigma$ acts like $(x, y)$ or $(-x-1, y)$.

If $\sigma$ acts like $(x, y)$ near $\{0\} \times \mathbb{R}$, then it is easy to see that, up to cell-wise homotopy, $\sigma$ is the identity map.

Finally, suppose that $\sigma$ acts like $(-x-1, y)$ near $\{0\} \times \mathbb{R}$; thus $\sigma$ is a color-preserving, orientation-reversing, $\mathbb{Z}$-action-inverting automorphism of $\mathrm{CW}'(\mathfrak{f})$.

Let $n \in \mathbb{Z}$ and set

$$\mathrm{sign}_{\mathfrak{f}}(n) := \begin{cases} -1 & \text{if } \mathfrak{f}_n = L, \\ +1 & \text{if } \mathfrak{f}_n = R. \end{cases}$$

Since $\sigma$ carries $(0, n)$ to $(-1, n)$, $\sigma$ carries the unique slanted one-cell incident to $(0, n)$ to the unique slanted one-cell incident to $(-1, n)$. Hence $\sigma$ carries $(\mathrm{sign}_{\mathfrak{f}}(n), \mathrm{succ}_{\mathfrak{f}}(n))$ to $(-\mathrm{sign}_{\mathfrak{f}}(n) - 1, \mathrm{pred}_{\mathfrak{f}}(n))$.

Thus $\sigma$ must act like $(-x-1, y-\mathrm{succ}_{\mathfrak{f}}(n)+\mathrm{pred}_{\mathfrak{f}}(n))$ near $\{\mathrm{sign}_{\mathfrak{f}}(n)\}\times\mathbb{R}$. Hence $\mathrm{succ}_{\mathfrak{f}}(n)-\mathrm{pred}_{\mathfrak{f}}(n)$ is constant on both $\mathfrak{f}^{-1}\{R\}$ and $\mathfrak{f}^{-1}\{L\}$; let us denote the constant values by $d_R$ and $d_L$, respectively. Clearly $d_R \geq 2$. Also $d_R$ cannot be 2 since $\mathfrak{f}$ is irrational. If $d_R = 3$, then $[\mathfrak{f}] = \prod_{\mathbb{Z}} RRL$ and $d_L = 6$. If $d_R = 4$ then $[\mathfrak{f}] = \prod_{\mathbb{Z}} RL$ or $[\mathfrak{f}] = \prod_{\mathbb{Z}} RRLL$, and $d_L = 4$. If $d_R \geq 5$ then $d_L \leq 4$. It now follows that

$$[\mathfrak{f}] \in \{\prod_{\mathbb{Z}} RL,\ \prod_{\mathbb{Z}} RLL,\ \prod_{\mathbb{Z}} RRL,\ \prod_{\mathbb{Z}} RRLL\}.$$

In each of these cases, $\sigma$ exists and is (uniquely) realized by a map which is affine on each column, namely, for each $m \in \mathbb{Z}$ and $(x,y)\in\mathbb{R}^2$,

$$\sigma((x,y)) = \begin{cases} (-x-1, \quad d_L x + y - (d_L+d_R)m) & \text{if } x \in [2m-1, 2m], \\ (-x-1, -d_R x + y + (d_L+d_R)m) & \text{if } x \in [2m, 2m+1]. \end{cases}$$

It can be shown that $\sigma^2((x,y)) = (x, y-d_L)$.

Suppose that we are in one of the above cases, and let $(m,n)\in\mathbb{Z}^2$. By using $\sigma$ together with half-turns and translations, we find that there is a CW-automorphism $\nu_{m,n}$ of $\mathrm{CW}'(\mathfrak{f})$ such that $\nu_{m,n}$ maps each column to itself affinely, and $\nu_{m,n}$ carries the $(m,n)$th closed two-cell to itself, interchanging its two slanted one-cells. Then $\nu_{m,n}$ is unique. It is color-preserving, orientation-reversing, $\mathbb{Z}$-action preserving, and is not an isometry.

It is left to the reader to verify Remark 4.5. □

## 12 $\mathrm{Aut}\langle A,B,C\rangle \times \mathrm{Aut}(\hat{\mathbb{C}})$-actions

We now define actions of $\mathrm{Aut}\langle A,B,C\rangle \times \mathrm{Aut}(\hat{\mathbb{C}})$ on IrAutABCMaps and on IrRLMaps.

**12.1 Definitions.** Let $\mathrm{Aut}^-(\hat{\mathbb{C}}) := \mathrm{Aut}(\hat{\mathbb{C}}) - \mathrm{Aut}^+(\hat{\mathbb{C}})$, and

$$\mathrm{Aut}^-\langle A,B,C\rangle := \mathrm{Aut}\langle A,B,C\rangle - \mathrm{Aut}^+\langle A,B,C\rangle.$$

Let $\mathcal{F}\in$ IrAutABCMaps.

The *transpose* of $\mathcal{F}$, denoted transpose($\mathcal{F}$), is the element of IrAutABCMaps given by $n \mapsto \mathcal{F}_{-n-1}P^{-1}$. We let $\mathrm{Aut}(\hat{\mathbb{C}})$ act on IrAutABCMaps with $\mathrm{Aut}^+(\hat{\mathbb{C}})$ acting trivially, and $\mathrm{Aut}^-(\hat{\mathbb{C}})$ acting by transposing. It can be shown that (3.2.1) respects the $\mathrm{Aut}(\hat{\mathbb{C}})$-actions.

We already have $\mathrm{Aut}^+\langle A,B,C\rangle$ acting on IrAutABCMaps.

For any $G$ in $\mathrm{Aut}^-\langle A,B,C\rangle$, define $G\mathcal{F}$ to be the element of IrAutABCMaps given by $n \mapsto (G\mathcal{F})_n := G\mathcal{F}_{-n-1}P^{-1}M$. Again, it can be shown that (3.2.1) respects the $\mathrm{Aut}\langle A,B,C\rangle$-actions.

All these concepts pass to $\mathrm{Isom}^+(\mathbb{Z})$-orbits.

Let $\mathfrak{f}\in$ IrRLMaps.

Define the *transpose* of $\mathfrak{f}$ as transpose($\mathfrak{f}$) $:= (R,L)\cdot\mathfrak{f}\cdot(-z)$; define $\mathfrak{f}$ *read backwards* by backwards($\mathfrak{f}$) $:= \mathfrak{f}\cdot(-z)$; and define the *R-L-interchange* of $\mathfrak{f}$ as interchange($\mathfrak{f}$) $:= (R,L)\cdot\mathfrak{f}$.

We let $\mathrm{Aut}\langle A,B,C\rangle$ act on IrRLMaps with $\mathrm{Aut}^{+}\langle A,B,C\rangle$ acting trivially, and with $\mathrm{Aut}^{-}\langle A,B,C\rangle$ acting by transposing. We let $\mathrm{Aut}\langle A,B,C\rangle$ act on IrRLMaps with $\mathrm{Aut}^{+}\langle A,B,C\rangle$ acting trivially, and $\mathrm{Aut}^{-}\langle A,B,C\rangle$ acting by reading backwards. All these definitions pass to $\mathrm{Isom}^{+}(\mathbb{Z})$-orbits. □

**12.2 Remarks.** Let us record the behavior of the action of

$$\mathrm{Aut}\langle A,B,C\rangle \times \mathrm{Aut}(\hat{\mathbb{C}}).$$

Let $n \in \mathbb{Z}$, $\mathcal{F},\mathcal{G} \in$ IrAutABCMaps. Write $\mathfrak{f} = \mathrm{quotient}(\mathcal{F})$, $\mathfrak{g} = \mathrm{quotient}(\mathcal{G})$, $\mathcal{F}' = \mathrm{residue}(\mathcal{F})$, $\mathcal{G}' = \mathrm{residue}(\mathcal{G})$.

| $\mathcal{G}$ | $F\mathcal{F}$, $F$ in $\mathrm{Aut}^{+}\langle A,B,C\rangle$ | $G\mathcal{F}$, $G$ in $\mathrm{Aut}^{-}\langle A,B,C\rangle$ | $\mathrm{transpose}(\mathcal{F})$ |
|---|---|---|---|
| $\mathcal{G}_n$ | $F\mathcal{F}_n$ | $G\mathcal{F}_{-n-1}P^{-1}M$ | $\mathcal{F}_{-n-1}P^{-1}$ |
| $D_{\mathcal{G}}$ | $F(D_{\mathcal{F}})$ | $G(D_{\mathcal{F}}^{-1})$ | $D_{\mathcal{F}}^{-1}$ |
| $\mathfrak{g}_n$ | $\mathfrak{f}_n$ | $\mathfrak{f}_{-n}$ | $M\mathfrak{f}_{-n}M$ |
| $\mathfrak{g}$ | $\mathfrak{f}$ | $\mathrm{backwards}(\mathfrak{f})$ | $\mathrm{transpose}(\mathfrak{f})$ |
| $\mathcal{G}_n[B]$ | $F(\mathcal{F}_n[B])$ | $G(\mathcal{F}_{-n-1}(AB[A]\cup A[C]))$ | $\mathcal{F}_{-n-1}(AB[A]\cup A[C])$ |
| $\mathcal{W}_m(\mathcal{G})$ | $F\cdot\mathcal{W}_m(\mathcal{F})$ | $G\cdot\mathcal{W}_{-m-1}(\mathcal{F})$ | $\mathcal{W}_{m-1}(\mathcal{F})$ |
| $\mathfrak{s}(\mathcal{G})$ | $F(\mathfrak{s}(\mathcal{F}))$ | $G(\mathfrak{s}(\mathcal{F}))$ | $\mathfrak{s}(\mathcal{F})$ |
| $\mathbb{S}^2(\mathcal{G})$ | $F(\mathbb{S}^2(\mathcal{F}))$ | $G(\mathbb{S}^2(\mathcal{F}))$ | $\mathbb{S}^2(\mathcal{F})$ with the opposite orientation |
| $\mathcal{G}'_n[B]$ | $F(\mathcal{F}'_n[B])$ | $\begin{cases} G(\mathcal{F}'_{-n}[B]) & \text{if } L = \mathfrak{g}_n\ (= \mathfrak{f}_{-n}), \\ G(D_{\mathcal{F}}^{-1}(\mathcal{F}'_{-n}[B])) & \text{if } R = \mathfrak{g}_n\ (= \mathfrak{f}_{-n}) \end{cases}$ | $\begin{cases} \mathcal{F}'_{-n}[B] & \text{if } R = \mathfrak{g}_n\ (\neq \mathfrak{f}_{-n}), \\ D_{\mathcal{F}}^{-1}(\mathcal{F}'_{-n}[B]) & \text{if } L = \mathfrak{g}_n\ (\neq \mathfrak{f}_{-n}) \end{cases}$ |

Table 12.2.1: The action of $\mathrm{Aut}\langle A,B,C\rangle \times \mathrm{Aut}(\hat{\mathbb{C}})$.

Where we write $F(\mathbb{S}^2(\mathcal{F}))$, we are thinking of $\mathbb{S}^2(\mathcal{F})$ as a set of subsets of $\mathfrak{C}$. □

# 13 The symmetry group of $\mathrm{CW}(\mathcal{F})$

**13.1 Notation.** Let $\mathcal{F} \in$ IrAutABCMaps.
Let $\mathfrak{f} = \mathrm{quotient}(\mathcal{F}) \in$ IrRLMaps.
Let $\mathfrak{b} = \mathfrak{b}(\mathcal{F}) = [\mathfrak{f}] \in \mathrm{IrRLMaps} / \mathrm{Isom}^{+}(\mathbb{Z})$. □

**13.2 Remarks.** Suppose that Notation 13.1 holds.

Observe that $\mathrm{CW}'(\mathfrak{f})$ is an $\mathrm{Isom}(\mathrm{CW}'(\mathfrak{f}))$-CW-complex.

In a natural way, $\mathrm{CW}(\mathfrak{f})$ then becomes an $\mathrm{Isom}(\mathrm{CW}'(\mathfrak{f}))$-CW-complex.

By Theorem 7.12 and Definition 7.13, we have an isomorphism

$$\mathrm{CW}(\mathfrak{f}) \xrightarrow{\sim} \mathrm{CW}(\mathcal{F}) \tag{13.2.1}$$

of colored, oriented $\mathbb{Z}$-CW-complexes.

We can use (13.2.1) to make $\mathrm{CW}(\mathcal{F})$ into an $\mathrm{Isom}(\mathrm{CW}'(\mathfrak{f}))$-CW-complex, and $(x+2, y)$ then has the same action as $D_{\mathcal{F}}$.

Later we will adjust (13.2.1) to alter the action of $\mathrm{Isom}(\mathrm{CW}'(\mathfrak{f}))$ on $\mathrm{CW}(\mathcal{F})$. We will not change the action on the set of cells. Notice that the permutation of the cells is determined by the permutation of the two-cells, since every one-cell and every zero-cell is an intersection of a pair of closed two-cells. □

**13.3 Definitions.** Suppose that Notation 13.1 holds.

Recall that, if $G \in \mathrm{Aut}\langle A, B, C\rangle$, then $G(\mathbb{S}^2(\mathcal{F})) = \mathbb{S}^2(G\mathcal{F})$ as partitions of $\hat{\mathbb{C}}$, and as unoriented spaces. Also, $G(q_{\mathbb{S}^2(\mathcal{F})}) = q_{\mathbb{S}^2(G\mathcal{F})}$ as points, and as subsets of $\hat{\mathbb{C}}$.

We define

$$\mathrm{Aut}(\langle A, B, C\rangle, \mathbb{S}^2(\mathcal{F})) \quad := \quad \{G \in \mathrm{Aut}\langle A, B, C\rangle \mid G(\mathbb{S}^2(\mathcal{F})) = \mathbb{S}^2(\mathcal{F})\}.$$

We then define the *symmetry group of* $\mathrm{CW}(\mathcal{F})$, denoted $\mathrm{Aut}(\langle A, B, C\rangle, \mathrm{CW}(\mathcal{F}))$, as the set consisting of those $G$ in $\mathrm{Aut}(\langle A, B, C\rangle, \mathbb{S}^2(\mathcal{F}))$ which act as CW-automorphisms of $\mathrm{CW}(\mathcal{F})$.

Recall that the underlying space of $\mathrm{CW}(\mathcal{F})$ is the oriented $\mathbb{Z}$-plane $\mathbb{S}^2(\mathcal{F}) - \{q_{\mathbb{S}^2(\mathcal{F})}\}$. Hence, if $G \in \mathrm{Aut}(\langle A, B, C\rangle, \mathbb{S}^2(\mathcal{F}))$ then $G \in \mathrm{Aut}\langle A, B, C\rangle$ and $G$ fixes $q_{\mathbb{S}^2(\mathcal{F})}$. □

Over the course of this section we will see the converse. We will then see that $\mathrm{Aut}(\langle A, B, C\rangle, \mathrm{CW}(\mathcal{F}))$ and $\mathrm{Isom}(\mathrm{CW}'(\mathfrak{f}))$ coincide when viewed as groups of permutations of the cells of $\mathrm{CW}(\mathcal{F})$. Also, we will describe the elements of $\mathrm{Aut}(\langle A, B, C\rangle, \mathrm{CW}(\mathcal{F}))$ explicitly in terms of $\mathcal{F}$.

We begin by describing the $G \in \mathrm{Aut}\langle A, B, C\rangle$ which act on the plane

$$\mathbb{S}^2(\mathcal{F}) - \{q_{\mathbb{S}^2(\mathcal{F})}\}.$$

**13.4 Lemma.** *Suppose that* Notation 13.1 *holds. Let* $G \in \mathrm{Aut}\langle A, B, C\rangle$.

*The following are equivalent*:

(a) $G(\mathbb{S}^2(\mathcal{F})) = \mathbb{S}^2(\mathcal{F})$ *and* $G(D_{\mathcal{F}}) = D_{\mathcal{F}}^{\pm 1}$.

(b) $G(\mathbb{S}^2(\mathcal{F})) = \mathbb{S}^2(\mathcal{F})$ *and* $G(q_{\mathbb{S}^2(\mathcal{F})}) = q_{\mathbb{S}^2(\mathcal{F})}$.

(c) $G(q_{\mathbb{S}^2(\mathcal{F})}) = q_{\mathbb{S}^2(\mathcal{F})}$ *as subsets of* $\hat{\mathbb{C}}$.

(d) *There exists* $(m, n) \in \mathbb{Z}^2$ *such that*

$$\textit{either } G \cdot \mathcal{F} = D_{\mathcal{F}}^m \cdot \mathcal{F} \cdot (z+n) \textit{ or } \mathrm{transpose}(G \cdot \mathcal{F}) = D_{\mathcal{F}}^m \cdot \mathcal{F} \cdot (z+n).$$

*Proof.* It is clear that (a) implies (b), and that (b) implies (c).

It follows, from Remark 8.5, that the $\mathrm{Isom}^+(\mathbb{Z})$-orbit of $\mathcal{F}$ is determined by the ordered pair $(\mathcal{W}_{-1}(\mathcal{F}), \mathcal{W}_0(\mathcal{F})) \in \mathfrak{E}^2$. This can be used to show that (c) implies (d).

If (d) holds then $\mathbb{S}^2(G\mathcal{F}) = \mathbb{S}^2(\mathcal{F})$ as unoriented spaces, and (a) holds. $\square$

We have already considered the horizontal-translation symmetries, and we observed the following.

**13.5 Proposition.** *Suppose that* Notation 13.1 *holds.*

*Then* $D_{\mathcal{F}} \in \mathrm{Aut}(\langle A, B, C\rangle, \mathrm{CW}(\mathcal{F}))$*, and the resulting permutation of cells of* $\mathrm{CW}(\mathcal{F})$ *is the same as that determined by the horizontal translation*

$$(x + 2, y) \in \mathrm{Isom}(\mathrm{CW}'(\mathfrak{f})). \qquad \square$$

We now consider the case where $\mathfrak{f}$ is periodic and $\mathrm{CW}'(\mathfrak{f})$, $\mathrm{CW}(\mathcal{F})$ have vertical-translation symmetries. Examples 14.9–14.14 illustrate this.

**13.6 Proposition.** *Suppose that* Notation 13.1 *holds. Let* $d \in \mathbb{Z}$.

*The following are equivalent*:

(a) $(x, y + d) \in \mathrm{Isom}(\mathrm{CW}'(\mathfrak{f}))$.

(b) $\mathfrak{f} \cdot (z + d) = \mathfrak{f}$.

(c) *There exists* $F \in \mathrm{Aut}^+\langle A, B, C\rangle$ *such that* $F \cdot \mathcal{F} = \mathcal{F} \cdot (z + d)$.

*If* $F$ *is as in* (c)*, then* $F = \mathcal{F}_d\mathcal{F}_0^{-1}$, $F \in \mathrm{Aut}(\langle A, B, C\rangle, \mathrm{CW}(\mathcal{F}))$*, and the resulting permutation of cells of* $\mathrm{CW}(\mathcal{F})$ *is the same as that for the vertical translation* $(x, y + d) \in \mathrm{Isom}(\mathrm{CW}'(\mathfrak{f}))$.

*Proof.* The second column of Table 12.2.1 will be useful.

Notice that (a) and (b) are equivalent by Corollary 11.3.

On taking quotients, we see that (c) implies (b).

Now suppose that (b) holds.

Let $F := \mathcal{F}_d\mathcal{F}_0^{-1}$ and $\mathcal{G} := F \cdot \mathcal{F}$.

Now $\mathrm{quotient}(\mathcal{G}) = \mathrm{quotient}(F \cdot \mathcal{F}) = \mathrm{quotient}(\mathcal{F}) = \mathfrak{f} = \mathfrak{f} \cdot (z + d)$.

Also, $\mathcal{G}_0 = F\mathcal{F}_0 = \mathcal{F}_d$.

It follows that $\mathcal{G} = \mathcal{F} \cdot (z + d)$, and (c) holds.

By Lemma 13.4, $F(\mathbb{S}^2(\mathcal{F})) = \mathbb{S}^2(\mathcal{F})$. Hence $F\,\widehat{\rho_{\mathbb{S}^2(\mathcal{F})}}\,F^{-1} = \widehat{\rho_{\mathbb{S}^2(F\mathcal{F})}} = \widehat{\rho_{\mathbb{S}^2(\mathcal{F})}}$.

For all $(m, n) \in \mathbb{Z}^2$, in $\mathfrak{E}$, we have

$$F^{-1}(D^m_{\mathcal{F}}\mathcal{F}'_n[B]) = D^m_{\mathcal{F}}(F^{-1}(\mathcal{F}'_n[B])) = D^m_{\mathcal{F}}(F^{-1}(\mathcal{G}'_{n-d}[B])) = D^m_{\mathcal{F}}\mathcal{F}'_{n-d}[B].$$

A similar equation holds in $\mathbb{S}^2(\mathcal{F})$ since $\widehat{\rho_{\mathbb{S}^2(\mathcal{F})}}$ commutes with the action of $F^{-1}$.

Hence, $F^{-1}$ acts as a self-homeomorphism of $\mathbb{S}^2(\mathcal{F})$ which permutes the two-cells of $\mathrm{CW}(\mathcal{F})$ in the same way that $(x, y - d)$ does. The result now follows. $\square$

We next consider the situation where $\mathrm{backwards}(\mathfrak{b}) = \mathfrak{b}$ and $\mathrm{CW}'(\mathfrak{f})$, $\mathrm{CW}(\mathcal{F})$ have half-turn symmetries. Example 14.9 illustrates the case where $d = 0$, and Example 14.11 illustrates the case where $d = 1$.

**13.7 Proposition.** *Suppose that* Notation 13.1 *holds. Let* $d \in \mathbb{Z}$.

*The following are equivalent*:

(a) $(-x-1, -y+d) \in \mathrm{Isom}(\mathrm{CW}'(\mathfrak{f}))$.

(b) $\mathfrak{f} \cdot (-z+d) = \mathfrak{f}$.

(c) $\mathrm{backwards}(\mathfrak{f}) = \mathfrak{f} \cdot (z+d)$.

(d) *There exists* $G \in \mathrm{Aut}^{-}\langle A, B, C\rangle$ *such that* $G \cdot \mathcal{F} = \mathcal{F} \cdot (z+d)$.

*If* $G$ *is as in* (d), *then* $G = \mathcal{F}_d MP\mathcal{F}_{-1}^{-1}$, $\quad G \in \mathrm{Aut}(\langle A, B, C\rangle, \mathrm{CW}(\mathcal{F}))$, *and the resulting permutation of cells of* $\mathrm{CW}(\mathcal{F})$ *is the same as that for the half-turn* $(-x-1, -y+d) \in \mathrm{Isom}(\mathrm{CW}'(\mathfrak{f}))$.

*Proof.* The third column of Table 12.2.1 will be useful.

Notice that (a) and (b) are equivalent by Corollary 11.3, and clearly (b) and (c) are equivalent.

On taking quotients, we see that (d) implies (c).

Now suppose that (c) holds.

Let $G := \mathcal{F}_d MP\mathcal{F}_{-1}^{-1}$, and $\mathcal{G} := G \cdot \mathcal{F}$.

Now $\mathrm{quotient}(\mathcal{G}) = \mathrm{quotient}(G \cdot \mathcal{F}) = \mathrm{backwards}(\mathfrak{f}) = \mathfrak{f} \cdot (z+d)$.

Also, $\mathcal{G}_0 = G\mathcal{F}_{-1}P^{-1}M = \mathcal{F}_d$.

It follows that $\mathcal{G} = \mathcal{F} \cdot (z+d)$, and (d) holds.

For each $(m,n) \in \mathbb{Z}^2$, in $\mathfrak{C}$,

$$G^{-1}(D_{\mathcal{F}}^{m}\mathcal{F}'_n[B]) = D_{\mathcal{F}}^{-m}(G^{-1}(\mathcal{F}'_n[B])) = D_{\mathcal{F}}^{-m}(G^{-1}(\mathcal{G}'_{n-d}[B]))$$
$$= \begin{cases} D_{\mathcal{F}}^{-m}\mathcal{F}'_{-n+d}[B] & \text{if } L = \mathfrak{f}_{-n+d} \ (= \mathfrak{f}_n), \\ D_{\mathcal{F}}^{-m-1}\mathcal{F}'_{-n+d}[B] & \text{if } R = \mathfrak{f}_{-n+d} \ (= \mathfrak{f}_n). \end{cases}$$

It is now straightforward to use Theorem 7.12 to show that $G^{-1}$ acts as a self-homeomorphism of $\mathbb{S}^2(\mathcal{F})$ which induces the same permutation of the two-cells of $\mathrm{CW}(\mathcal{F})$ that $(-x-1, -y+d)$ does. The result now follows. $\square$

We next consider the situation where $\mathrm{transpose}(\mathfrak{b}) = \mathfrak{b}$ and $\mathrm{CW}'(\mathfrak{f})$, $\mathrm{CW}(\rho_{\hat{\mathbb{C}}}, p)$ have horizontal-glide-reflection symmetries. Example 14.12 illustrates the case $d = 3$.

**13.8 Proposition.** *Suppose that* Notation 13.1 *holds. Let* $d \in \mathbb{Z}$.

*The following are equivalent*:

(a) $(x+1, -y+d) \in \mathrm{Isom}(\mathrm{CW}'(\mathfrak{f}))$.

(b) $(R, L) \cdot \mathfrak{f} \cdot (-z+d) = \mathfrak{f}$.

(c) $\mathrm{transpose}(\mathfrak{f}) = \mathfrak{f} \cdot (z+d)$.

(d) *There exists* $F \in \mathrm{Aut}^{+}\langle A, B, C\rangle$ *such that* $\mathrm{transpose}(F \cdot \mathcal{F}) = \mathcal{F} \cdot (z+d)$.

*If* $F$ *is as in* (d), *then* $d$ *is odd,* $F = \mathcal{F}_d P\mathcal{F}_{-1}^{-1}$, $F \in \mathrm{Aut}(\langle A, B, C\rangle, \mathrm{CW}(\mathcal{F}))$, *and the resulting permutation of cells of* $\mathrm{CW}(\mathcal{F})$ *is the same as that for the horizontal glide-reflection* $(x+1, -y+d) \in \mathrm{Isom}(\mathrm{CW}'(\mathfrak{f}))$.

*Proof.* The second and fourth columns of Table 12.2.1 will be useful.

Notice that (a) and (b) are equivalent by Corollary 11.3, and clearly (b) and (c) are equivalent.

On taking quotients, we see that (d) implies (c).

Now suppose that (c) holds.

Here, for all $n \in \mathbb{Z}$, $M\mathfrak{f}_{-n}M = \mathfrak{f}_{n+d}$, that is, $\mathfrak{f}_n \neq M\mathfrak{f}_n M = \mathfrak{f}_{-n+d}$. Notice that $d$ must be odd.

Let $F := \mathcal{F}_d P \mathcal{F}_{-1}^{-1}$ and $\mathcal{G} := \operatorname{transpose}(F\mathcal{F})$.

Now $\operatorname{quotient}(\mathcal{G}) = \operatorname{quotient}(\operatorname{transpose}(F\mathcal{F})) = \operatorname{transpose}(\mathfrak{f}) = \mathfrak{f} \cdot (z+d)$.

Also $\mathcal{G}_0 = F\mathcal{F}_{-1}P^{-1} = \mathcal{F}_d$.

It follows that $\mathcal{G} = \mathcal{F} \cdot (z+d)$ and (d) holds.

For each $(m,n) \in \mathbb{Z}^2$,

$$\begin{aligned} F^{-1}(D_{\mathcal{F}}^m \mathcal{F}'_n([B])) &= D_{\mathcal{F}}^m(F^{-1}(\mathcal{F}'_n([B]))) = D_{\mathcal{F}}^m(F^{-1}(\mathcal{G}'_{n-d}([B]))) \\ &= \begin{cases} D_{\mathcal{F}}^m \mathcal{F}'_{-n+d}([B])) & \text{if } R = \mathfrak{f}_n \ (\neq \mathfrak{f}_{-n+d}), \\ D_{\mathcal{F}}^{m-1} \mathcal{F}'_{-n+d}([B])) & \text{if } L = \mathfrak{f}_n \ (\neq \mathfrak{f}_{-n+d}). \end{cases} \end{aligned}$$

As in the previous proof, one can show that $F^{-1}$ acts as a self-homeomorphism of $\mathbb{S}^2(\mathcal{F})$ which permutes the two-cells of $\mathrm{CW}(\mathcal{F})$ in the same way that $(x-1, -y+d)$ does. The result now follows. □

We next consider the situation where $\operatorname{interchange}(\mathfrak{b}) = \mathfrak{b}$ and $\mathrm{CW}'(\mathfrak{f})$, $\mathrm{CW}(\mathcal{F})$ have vertical-glide-reflection symmetries. Example 14.14 illustrates the case $d = 6$.

**13.9 Proposition.** *Suppose that* Notation 13.1 *holds. Let* $d \in \mathbb{Z}$.

*The following are equivalent*:

- (a) $(-x,\, y+d) \in \operatorname{Isom}(\mathbb{R}^2, \mathrm{CW}'(\mathfrak{f}))$.
- (b) $(R, L) \cdot \mathfrak{f} \cdot (z+d) = \mathfrak{f}$.
- (c) $\operatorname{interchange}(\mathfrak{f}) = \mathfrak{f} \cdot (z+d)$.
- (d) *There exists* $G \in \operatorname{Aut}^-\langle A, B, C\rangle$ *such that* $\operatorname{transpose}(G \cdot \mathcal{F}) = \mathcal{F} \cdot (z+d)$.

*If* $G$ *is as in* (d), *then* $d \neq 0$, $G = \mathcal{F}_d M \mathcal{F}_0^{-1}$, $G \in \operatorname{Aut}(\langle A, B, C\rangle, \mathrm{CW}(\mathcal{F}))$, *and the resulting permutation of cells of* $\mathrm{CW}(\mathcal{F})$ *is the same as that for the vertical glide-reflection* $(-x, y+d) \in \operatorname{Isom}(\mathrm{CW}'(\mathfrak{f}))$.

*Proof.* The last two columns of Table 12.2.1 will be useful.

Notice that (a) and (b) are equivalent by Corollary 11.3, and clearly (b) and (c) are equivalent.

On taking quotients, we see that (d) implies (c).

Now suppose that (c) holds.

Here, for all $n \in \mathbb{Z}$, $M\mathfrak{f}_{n+d}M = \mathfrak{f}_n$, that is, $\mathfrak{f}_n \neq M\mathfrak{f}_n M = \mathfrak{f}_{n+d}$. Hence $d \neq 0$.

Let $G := \mathcal{F}_d M \mathcal{F}_0^{-1}$ and $\mathcal{G} := \operatorname{transpose}(G\mathcal{F})$.

Then $\operatorname{quotient}(\mathcal{G}) = \operatorname{interchange}(\mathfrak{f}) = \mathfrak{f} \cdot (z+d)$.

Also $\mathcal{G}_0 = G\mathcal{F}_0 P^{-1} M P^{-1} = G\mathcal{F}_0 M = \mathcal{F}_d$.

It follows that $\mathcal{G} = \mathcal{F} \cdot (z+d)$ and (d) holds.

For each $(m,n) \in \mathbb{Z}^2$,

$$\begin{aligned} G^{-1}(D_{\mathcal{F}}^m(\mathcal{F}'_n([B]))) &= D_{\mathcal{F}}^{-m}(G^{-1}(\mathcal{F}'_n([B]))) \\ &= D_{\mathcal{F}}^{-m}(G^{-1}(\mathcal{G}'_{n-d}([B]))) \\ &= D_{\mathcal{F}}^{-m}\mathcal{F}'_{n-d}([B]). \end{aligned}$$

As in the previous proof, one can show that $G^{-1}$ acts as a self-homeomorphism of $\mathbb{S}^2(\mathcal{F})$ which permutes the two-cells of $\mathrm{CW}(\mathcal{F})$ in the same way that $(-x, y-d)$ does. The result now follows. $\square$

**13.10 Remarks.** Let $G \in \mathrm{Aut}\langle A, B, C\rangle$.

We claim that $G \in \mathrm{Aut}(\langle A, B, C\rangle, \mathrm{CW}(\mathcal{F}))$ if and only $G$ satisfies the equivalent conditions of Lemma 13.4.

Clearly if $G \in \mathrm{Aut}(\langle A, B, C\rangle, \mathrm{CW}(\mathcal{F}))$ then $G$ acts on the plane

$$\mathbb{S}^2(\mathcal{F}) - \{q_{\mathbb{S}^2(\mathcal{F})}\},$$

and hence Lemma 13.4(b) holds.

Conversely, if Lemma 13.4(d) holds then Propositions 13.5–13.9 together imply that $G \in \mathrm{Aut}(\langle A, B, C\rangle, \mathrm{CW}(\mathcal{F}))$.

The claim is now proved.

In particular, $\mathrm{Aut}(\langle A, B, C\rangle, \mathbb{S}^2(\mathcal{F})) = \langle A, B, C\rangle \cdot \mathrm{Aut}(\langle A, B, C\rangle, \mathrm{CW}(\mathcal{F}))$ and here $\langle A, B, C\rangle \cap \mathrm{Aut}(\langle A, B, C\rangle, \mathrm{CW}(\mathcal{F})) = \langle D_{\mathcal{F}}\rangle$. $\square$

We now have the main result of this section.

**13.11 Theorem.** *Suppose that* Notation 13.1 *holds.*

*Then* $\mathrm{CW}'(\mathfrak{f})$ *and* $\mathrm{CW}(\mathfrak{f})$ *are* $\mathrm{Isom}(\mathrm{CW}'(\mathfrak{f}))$-CW-*complexes, and* $\mathrm{CW}(\mathcal{F})$ *is an* $\mathrm{Aut}(\langle A, B, C\rangle, \mathrm{CW}(\mathcal{F}))$-CW-*complex.*

*There exists a group isomorphism*

$$\mathrm{Isom}(\mathrm{CW}'(\mathfrak{f})) \xrightarrow{\sim} \mathrm{Aut}(\langle A, B, C\rangle, \mathrm{CW}(\mathcal{F})), \tag{13.11.1}$$

*and it is compatible with some isomorphism*

$$\mathrm{CW}(\mathfrak{f}) \xrightarrow{\sim} \mathrm{CW}(\mathcal{F}) \tag{13.11.2}$$

*of oriented, colored* CW-*complexes, that respects the* $\mathbb{Z}^2$-*labellings of the two-cells.*

*Proof.* We observed that $\mathrm{CW}'(\mathfrak{f})$ and $\mathrm{CW}(\mathfrak{f})$ are $\mathrm{Isom}(\mathrm{CW}'(\mathfrak{f}))$-CW-complexes, in Remarks 13.2.

By definition, $\mathrm{CW}(\mathcal{F})$ is an $\mathrm{Aut}(\langle A, B, C\rangle, \mathrm{CW}(\mathcal{F}))$-CW-complex.

By Theorem 7.12 and Definition 7.13, we have an isomorphism (13.11.2) of oriented, colored $\mathbb{Z}$-CW-complexes, respecting the $\mathbb{Z}^2$-labelling.

By Propositions 13.5–13.9, $\mathrm{Isom}(\mathrm{CW}'(\mathfrak{f}))$ and $\mathrm{Aut}(\langle A, B, C\rangle, \mathrm{CW}(\mathcal{F}))$ determine the same group of permutations of the set of cells of $\mathrm{CW}(\mathcal{F})$. Thus we have an isomorphism (13.11.1) that is compatible with (13.11.2) as a map between sets of cells, but not necessarily as a map of spaces.

It remains to adjust (13.11.2). It is not difficult to see that the quotient orbifolds $\mathrm{Isom}(\mathrm{CW}'(\mathfrak{f}))\backslash \mathrm{CW}(\mathfrak{f})$ and $\mathrm{Aut}(\langle A, B, C\rangle, \mathrm{CW}(\mathcal{F}))\backslash \mathrm{CW}(\mathfrak{f})$ are CW-complexes that are isomorphic. However, it is possible for a half-turn to fix a cell, and then the quotient orbifold has a double point in the interior of a two-cell. We alter the quotient CW-structures by making each such double point into a zero-cell, and then joining it by a new one-cell to an arbitrary zero-cell in the two-cell.

We fix a CW-isomorphism between the two refined structures, and also lift the refined CW-structures back to the original spaces. It is now straightforward to adjust (13.11.2) to make it compatible with (13.11.1), since (13.11.2) is determined as a map of sets of cells, and we lift the quotient homeomorphism of cells. $\square$

**13.12 Remarks.** By Remarks 11.4, the following statements hold.

Let $\mathrm{CW}_0(\mathcal{F})$ denote the set of zero-cells of $\mathrm{CW}(\mathcal{F})$. The group of permutations of $\mathrm{CW}_0(\mathcal{F})$ determined by $\mathrm{Aut}(\langle A,B,C\rangle, \mathrm{CW}(\mathcal{F}))$ is of index at most two in the group of those permutations of $\mathrm{CW}_0(\mathcal{F})$ which can be extended to color-permuting CW-automorphisms of $\mathrm{CW}(\mathcal{F})$. Moreover, the index equals two if and only if

$$\mathfrak{b} \in \{\prod_{\mathbb{Z}} RL,\ \prod_{\mathbb{Z}} RRL,\ \prod_{\mathbb{Z}} RLL,\ \prod_{\mathbb{Z}} RRLL\}.$$

In each of these index-two cases, for each closed two-cell of $\mathrm{CW}(\mathcal{F})$, there is a color-preserving, orientation-reversing, $\mathbb{Z}$-action-preserving, CW-automorphism $\nu$ of $\mathrm{CW}(\mathcal{F})$ that carries the given closed two-cell to itself; see Figures 14.9.1, 14.10.1, and 14.11.1. $\square$

# 14 The symmetry group of $\mathrm{CW}(\rho_{\hat{\mathbb{C}}}, q_{\hat{\mathbb{C}}})$

Using the results of the previous section, we now deduce consequences for $\hat{\mathbb{C}}$.

**14.1 Notation.** Let $\rho_{\hat{\mathbb{C}}} \in \mathfrak{R}_{\hat{\mathbb{C}}}$ and suppose that $\rho_{\hat{\mathbb{C}}}$ is given by a C-T-model.

Choose any parabolic point $q_{\hat{\mathbb{C}}}$ for $\rho_{\hat{\mathbb{C}}}(\langle A,B,C\rangle)$.

Choose any $\mathcal{F} \in \mathrm{IrAutABCMaps}$ that codifies $(\rho_{\hat{\mathbb{C}}}, q_{\hat{\mathbb{C}}})$.

Let $\mathfrak{f} = \mathrm{quotient}(\mathcal{F}) \in \mathrm{IrRLMaps}$.

Let $\mathfrak{b} = \mathfrak{b}(\rho_{\hat{\mathbb{C}}}) = \mathfrak{b}(\mathcal{F}) = [\mathfrak{f}] \in \mathrm{IrRLMaps}\,/\,\mathrm{Isom}^+(\mathbb{Z})$. $\square$

**14.2 Definitions.** Suppose that Notation 14.1 holds.

The *symmetry group of* $\rho_{\hat{\mathbb{C}}}$ is defined to be

$$\mathrm{Symm}(\rho_{\hat{\mathbb{C}}}) := \{(\sigma, G) \in \mathrm{Aut}(\hat{\mathbb{C}}) \times \mathrm{Aut}\langle A,B,C\rangle \mid i_\sigma \cdot \rho_{\hat{\mathbb{C}}} = \rho_{\hat{\mathbb{C}}} \cdot G\},$$

the stabilizer of $\rho_{\hat{\mathbb{C}}}$ under the action of $\mathrm{Aut}(\hat{\mathbb{C}}) \times \mathrm{Aut}\langle A,B,C\rangle$ on $\mathfrak{R}_{\hat{\mathbb{C}}}$.

The *symmetry group of* $(\rho_{\hat{\mathbb{C}}}, q_{\hat{\mathbb{C}}})$ is defined to be

$$\begin{aligned}\mathrm{Symm}(\rho_{\hat{\mathbb{C}}}, q_{\hat{\mathbb{C}}}) : &= \{(\sigma, G) \in \mathrm{Symm}(\rho_{\hat{\mathbb{C}}}) \mid \sigma(q_{\hat{\mathbb{C}}}) = q_{\hat{\mathbb{C}}}\},\\ &= \{(\sigma, G) \in \mathrm{Symm}(\rho_{\hat{\mathbb{C}}}) \mid G(D_{\mathcal{F}}) = D_{\mathcal{F}}^{\pm 1}\}.\end{aligned}$$

Let $\mathrm{Aut}(\hat{\mathbb{C}}, \mathrm{Im}\,\rho_{\hat{\mathbb{C}}})$ denote the normalizer of $\rho_{\hat{\mathbb{C}}}(\langle A,B,C\rangle)$ in $\mathrm{Aut}(\hat{\mathbb{C}})$.

We define the *symmetry group of* $\mathrm{CW}(\rho_{\hat{\mathbb{C}}}, q_{\hat{\mathbb{C}}}))$, denoted

$$\mathrm{Aut}(\hat{\mathbb{C}}, \mathrm{CW}(\rho_{\hat{\mathbb{C}}}, q_{\hat{\mathbb{C}}})),$$

to be the subgroup of $\mathrm{Aut}(\hat{\mathbb{C}})$ consisting of the elements which act as color-permuting CW-automorphisms of $\mathrm{CW}(\rho_{\hat{\mathbb{C}}}, q_{\hat{\mathbb{C}}})$. $\square$

**14.3 Conjectures.** We conjecture that, in the above definition of

$$\mathrm{Aut}(\hat{\mathbb{C}}, \mathrm{CW}(\rho_{\hat{\mathbb{C}}}, q_{\hat{\mathbb{C}}})),$$

the color-permuting restriction is redundant.

Even more, we conjecture that, at each zero-cell of $\mathrm{CW}(\rho_{\hat{\mathbb{C}}}, q_{\hat{\mathbb{C}}})$, the four cyclically-ordered incident one-cells form well-defined angles, two of 180° and two of 0°, and that the two incident open two-cells that fill in the angles of 180° have the same color as the zero-cell. $\square$

We first consider what happens before a distinguished element of $\langle A, B, C\rangle$ is chosen. Here we have rigidity related to Mostow rigidity; see also Theorem 14.8.

**14.4 Lemma.** *Suppose that* Notation 14.1 *holds.*

*Projection on the first and second coordinates give isomorphisms*

$$\mathrm{Aut}(\hat{\mathbb{C}}, \mathrm{Im}\,\rho_{\hat{\mathbb{C}}}) \overset{\sim}{\leftarrow} \mathrm{Symm}(\rho_{\hat{\mathbb{C}}}) \overset{\sim}{\rightarrow} \mathrm{Aut}(\langle A, B, C\rangle, \mathbb{S}^2(\mathcal{F})),$$

*and the induced isomorphism*

$$\mathrm{Aut}(\langle A, B, C\rangle, \mathbb{S}^2(\mathcal{F})) \overset{\sim}{\rightarrow} \mathrm{Aut}(\hat{\mathbb{C}}, \mathrm{Im}\,\rho_{\hat{\mathbb{C}}}) \tag{14.4.1}$$

*is compatible with the homeomorphism* $\gamma(\mathcal{F}, \rho_{\hat{\mathbb{C}}})\colon \mathbb{S}^2(\mathcal{F}) \overset{\sim}{\rightarrow} \hat{\mathbb{C}}(\rho_{\hat{\mathbb{C}}}, q_{\hat{\mathbb{C}}})$.

*Proof.* It is straightforward to check that the first projection is an isomorphism, since $\rho_{\hat{\mathbb{C}}}$ is injective.

We show next that the second projection is injective.

Suppose it is not. Then the centralizer of $\rho_{\hat{\mathbb{C}}}(\langle A, B, C\rangle)$ in $\mathrm{Aut}(\hat{\mathbb{C}})$ contains a non-trivial element.

Lifting back, we obtain some $\gamma \neq \pm 1$ in $\mathrm{SL}_2(\mathbb{C}) \rtimes \langle \overline{z}\rangle$ such that the preimage $S$ of $\rho_{\hat{\mathbb{C}}}(\langle A, B, C\rangle)$ in $\mathrm{SL}_2(\mathbb{C})$ acts by conjugation on $\{\gamma, -\gamma\}$. Hence $S^2$ is a subgroup of $S$, of index at most two, which centralizes $\gamma$.

Now $S^2$ contains a parabolic element; by conjugating and squaring, if necessary, we may assume that $\left(\begin{smallmatrix}1&1\\0&1\end{smallmatrix}\right) \in S^2$. Hence one of $\gamma$, $-\gamma$, $\gamma\overline{z}$, or $-\gamma\overline{z}$ is of the form $\alpha = \left(\begin{smallmatrix}1&a\\0&1\end{smallmatrix}\right)$ for some $a \in \mathbb{C}$. Replacing $\gamma$ with $-\gamma$ if necessary we may assume that $\gamma$ or $\gamma\overline{z}$ is $\alpha$.

Also $S^2$ contains a non-trivial non-parabolic element $\beta$.

The case $\gamma = \alpha$ is impossible since $\gamma$ commutes with $\beta$ and $\gamma \neq 1$.

Thus $\gamma = \alpha\overline{z}$. Now $\beta$ commutes with $\gamma^2 = \left(\begin{smallmatrix}1&a+\overline{a}\\0&1\end{smallmatrix}\right)$. Hence $a + \overline{a} = 0$. It can be shown that $\gamma$ is conjugate to $\overline{z}$. Hence some conjugate of $S^2$ lies in $SL_2(\mathbb{R})$. It follows that the limit set, $\Lambda(\rho_{\hat{\mathbb{C}}})$, is a circle, which contradicts the fact that $\Lambda(\rho_{\hat{\mathbb{C}}}) = \hat{\mathbb{C}}$.

This completes the proof that the second projection is injective.

Let $G$ lie in the image of the second projection. Then there exists $\sigma \in \mathrm{Aut}(\hat{\mathbb{C}})$ such that $\rho_{\hat{\mathbb{C}}} \cdot G = \sigma \cdot \rho_{\hat{\mathbb{C}}}$. Hence $\mathcal{F}$ codifies $\rho_{\hat{\mathbb{C}}}$ and $G \cdot \mathcal{F}$ codifies

$$\rho_{\hat{\mathbb{C}}} \cdot G^{-1} = \sigma^{-1} \cdot \rho_{\hat{\mathbb{C}}}.$$

If $\sigma \in \mathrm{Aut}^+(\hat{\mathbb{C}})$ then, by (8.3.2), there exists $(W, d) \in \langle A, B, C\rangle \times \mathbb{Z}$ such that

$$G \cdot \mathcal{F} = W \cdot \mathcal{F} \cdot (z + d).$$

Similarly, if $\sigma \in \mathrm{Aut}^-(\hat{\mathbb{C}})$ then there exists $(W, d) \in \langle A, B, C\rangle \times \mathbb{Z}$ such that $\mathrm{transpose}(G \cdot \mathcal{F}) = W \cdot \mathcal{F} \cdot (z + d)$. Hence $G \in \mathrm{Aut}(\langle A, B, C\rangle, \mathbb{S}^2(\mathcal{F}))$. Thus the image of the second projection lies in $\mathrm{Aut}(\langle A, B, C\rangle, \mathbb{S}^2(\mathcal{F}))$.

An argument in the reverse direction, using Minsky's uniqueness result, gives the reverse inclusion.

The result now follows. □

We now incorporate the distinguished element of $\langle A, B, C\rangle$.

**14.5 Theorem.** *Suppose that* Notation 14.1 *holds.*

*Projection on the first and second coordinates give isomorphisms*

$$\mathrm{Aut}(\hat{\mathbb{C}}, \mathrm{CW}(\rho_{\hat{\mathbb{C}}}, q_{\hat{\mathbb{C}}})) \xleftarrow{\sim} \mathrm{Symm}(\rho_{\hat{\mathbb{C}}}, q_{\hat{\mathbb{C}}}) \xrightarrow{\sim} \mathrm{Aut}(\langle A, B, C\rangle, \mathrm{CW}(\mathcal{F})),$$

*and the induced isomorphism* $\mathrm{Aut}(\langle A, B, C\rangle, \mathrm{CW}(\mathcal{F})) \xrightarrow{\sim} \mathrm{Aut}(\hat{\mathbb{C}}, \mathrm{CW}(\rho_{\hat{\mathbb{C}}}, q_{\hat{\mathbb{C}}}))$ *is compatible with the* CW*-isomorphism* $\gamma(\mathcal{F}, \rho_{\hat{\mathbb{C}}})\colon \mathrm{CW}(\mathcal{F}) \xrightarrow{\sim} \mathrm{CW}(\rho_{\hat{\mathbb{C}}}, q_{\hat{\mathbb{C}}})$.

*Proof.* Projection on the second coordinate is an isomorphism by Lemma 14.4 and Remarks 13.10.

By Lemma 14.4, projection on the first coordinate is injective, and it remains to describe the image of $\mathrm{Symm}(\rho_{\hat{\mathbb{C}}}, p)$.

Consider any $\sigma$ in this image. By the preceding paragraph, there exists

$$G \in \mathrm{Aut}(\langle A, B, C\rangle, \mathrm{CW}(\mathcal{F}))$$

such that $\rho_{\hat{\mathbb{C}}} \cdot G = i_\sigma \cdot \rho_{\hat{\mathbb{C}}}$. Here $G$ acts on $\mathbb{S}^2(\mathcal{F})$ inducing a color-permuting CW-automorphism of $\mathrm{CW}(\mathcal{F})$. We have a (unique) continuous $\langle A, B, C\rangle$-map

$$\mathbb{S}^2(\mathcal{F}) \xrightarrow{\sim} \hat{\mathbb{C}},$$

inducing a color-permuting CW-isomorphism $\mathrm{CW}(\mathcal{F}) \xrightarrow{\sim} \mathrm{CW}(\rho_{\hat{\mathbb{C}}}, q_{\hat{\mathbb{C}}})$. Now $G$ acts on $\hat{\mathbb{C}}$ inducing a color-permuting CW-automorphism of $\mathrm{CW}(\rho_{\hat{\mathbb{C}}}, q_{\hat{\mathbb{C}}})$. Moreover, on $\mathfrak{E}$,

$$\sigma \cdot \widehat{\rho_{\hat{\mathbb{C}}}} = \widehat{i_\sigma \cdot \rho_{\hat{\mathbb{C}}}} = \widehat{\rho_{\hat{\mathbb{C}}} \cdot G} = \widehat{\rho_{\hat{\mathbb{C}}}} \cdot G = G \cdot \widehat{\rho_{\hat{\mathbb{C}}}}.$$

Thus $G$ and $\sigma$ have the same action on $\hat{\mathbb{C}}$. Hence $\sigma \in \mathrm{Aut}(\hat{\mathbb{C}}, \mathrm{CW}(\rho_{\hat{\mathbb{C}}}, p))$.

This shows that the image of $\mathrm{Symm}(\rho_{\hat{\mathbb{C}}}, q_{\hat{\mathbb{C}}})$ lies in $\mathrm{Aut}(\hat{\mathbb{C}}, \mathrm{CW}(\rho_{\hat{\mathbb{C}}}, q_{\hat{\mathbb{C}}}))$. It follows that we have an embedding

$$\mathrm{Aut}(\langle A, B, C\rangle, \mathrm{CW}(\mathcal{F})) \hookrightarrow \mathrm{Aut}(\hat{\mathbb{C}}, \mathrm{CW}(\rho_{\hat{\mathbb{C}}}, q_{\hat{\mathbb{C}}})) \tag{14.5.1}$$

that is compatible with the isomorphism $\mathrm{CW}(\mathcal{F}) \xrightarrow{\sim} \mathrm{CW}(\rho_{\hat{\mathbb{C}}}, q_{\hat{\mathbb{C}}})$ of oriented, colored $\mathbb{Z}$-CW-complexes.

By Remarks 13.12, if $\mathfrak{b}$ is not in

$$\{\prod_{\mathbb{Z}} RL, \prod_{\mathbb{Z}} RRL, \prod_{\mathbb{Z}} RLL, \prod_{\mathbb{Z}} RRLL\}, \tag{14.5.2}$$

then (14.5.1) must be surjective, and hence an isomorphism.

Also by Remarks 13.12, in the remaining cases, where $\mathfrak{b}$ does lie in (14.5.2), either (14.5.1) is an isomorphism, or, for each closed two-cell of $\mathrm{CW}(\rho_{\hat{\mathbb{C}}}, p)$, an element of $\mathrm{Aut}(\hat{\mathbb{C}})$ acts as a color-preserving, orientation-reversing, $\mathbb{Z}$-action-preserving CW-automorphism $\nu$ of $\mathrm{CW}(\rho_{\hat{\mathbb{C}}}, p)$ that carries the given two-cell to itself. For the purposes of this argument, let us call this a *pathological* symmetry of the closed two-cell.

We will now show that the latter condition does not hold, that is, no closed two-cell has pathological symmetry. One finds that there is a unique $\sigma \in \mathrm{Aut}^-(\hat{\mathbb{C}})$ with the specified behavior on the zero-cells of the closed two-cell, but that $\sigma$ does not permute the one-cells. This can be seen informally by examining Figures 14.9.1, 14.10.1, and 14.11.1, and we now briefly sketch how a formal proof can be constructed.

By an argument similar to the proof of Lemma 7.10(ii)(1)–(2), it can be shown that the curve $\partial_{-1}$ of Notation 7.1 contains the arc that is denoted $\partial^+[A]_{\overline{\mathbb{C}}}$ in [5, Section 8.2]. Thus we can apply the information about $\partial^+[A]_{S^2}$ given in [5, Section 6.3] for the cases where

$$F \in \{RL, RRL, RLL, RRLL\}.$$

For concreteness, let us consider first the case where $\mathfrak{b} = \prod_{\mathbb{Z}} RL$ and $F = RL$. Here the values occurring in [5, Section 6.3] are given by

$$(a_1, a, b, c, d, c + 2a_1c + 2d - 2, \mu) = (1, 2, 1, 1, 1, 3, \frac{1 + \sqrt{5}}{2}),$$

and $\partial^+[A]_{\overline{\mathbb{C}}}$ is mapped into (a proper subset of) itself by Möbius transformations denoted $\overline{E}_{2n}G$, for $0 \le n \le 3$. There is a homeomorphism $[-1, \mu] \xrightarrow{\sim} \partial^+[A]_{\overline{\mathbb{C}}}$ such that, for $0 \le n \le 3$, the resulting action of $\overline{E}_{2n}G$ on $[-1, \mu]$ is an affine action, as

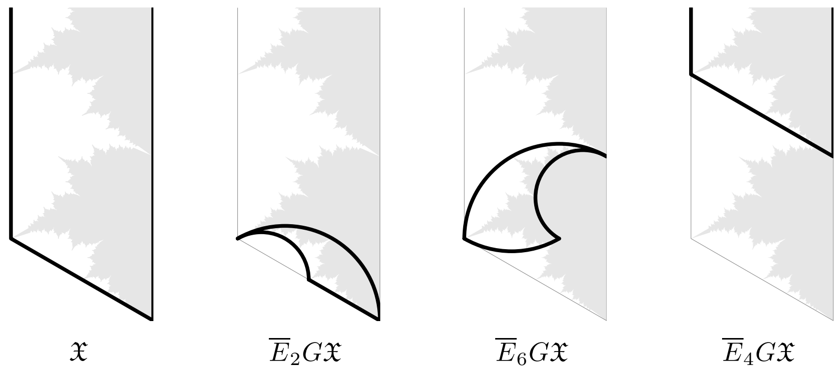


Figure 14.5.1: Regions for $F = RL$.

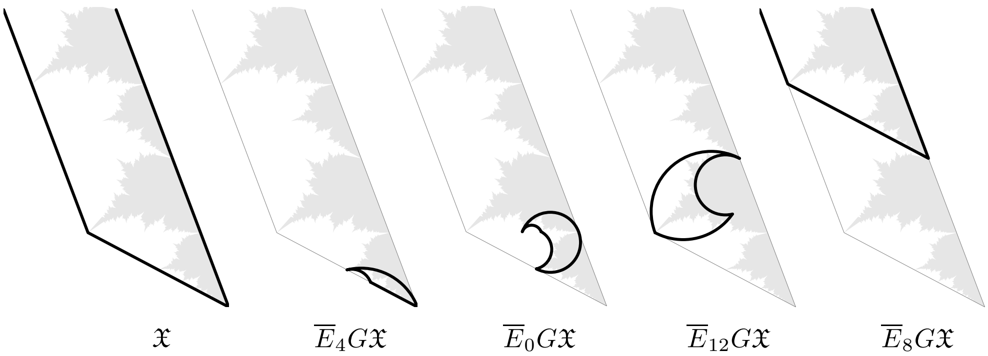


Figure 14.5.2: Regions for $F = RLL$.

described in [5, Section 6.1]. It can be shown that

$$\bigcup_{n\in\{1,3,2\}} \overline{E}_{2n}G([-1,\mu]) = [-1,\mu],$$

and, moreover, that the orbit of an endpoint of $[-1,\mu]$ under the semigroup generated by

$$\{\overline{E}_{2n}G \mid n \in \{1,3,2\}\} \tag{14.5.3}$$

is dense in $[-1,\mu]$. As in [1, Section 8.4], it can be shown that there is a 'triangular' region $\mathfrak{X}$ in $\hat{\mathbb{C}}$ that contains the endpoints of $\partial^+[A]_{\overline{\mathbb{C}}}$ and is mapped to itself by each element of (14.5.3); see Figure 14.5.1. It follows that $\partial^+[A]_{\overline{\mathbb{C}}} \subseteq \mathfrak{X}$ and that, moreover,

$$\partial^+[A]_{\overline{\mathbb{C}}} \;\subseteq \bigcup_{n\in\{1,3,2\}} \overline{E}_{2n}G(\mathfrak{X}).$$

It can then be shown that one of the two-cells in Figure 14.9.1 has no pathological symmetry. Hence no two-cell in Figure 14.9.1 has pathological symmetry.

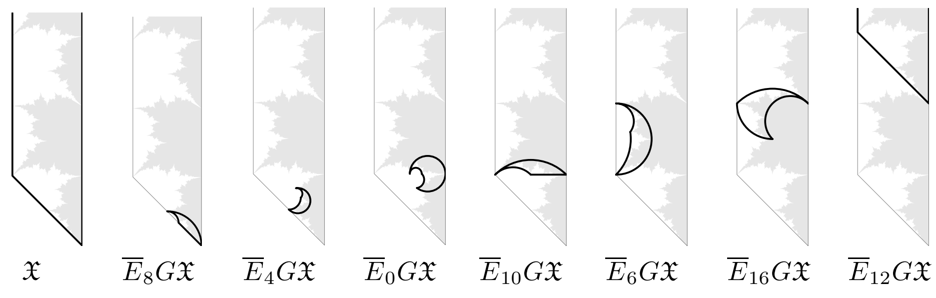


Figure 14.5.3: Regions for $F = RRLL$.

The cases where $\mathfrak{b} = \prod\limits_{\mathbb{Z}} RLL$ and $\mathfrak{b} = \prod\limits_{\mathbb{Z}} RRLL$ can be dealt with as indicated in Figures 14.5.2 and 14.5.3, respectively.

The diagram for $\mathfrak{b} = \prod\limits_{\mathbb{Z}} RRL$ is a color-interchanged vertical reflection of the diagram for $\mathfrak{b} = \prod\limits_{\mathbb{Z}} RLL$; see Example 1.1(iii). $\square$

Theorems 13.11 and 14.5 can be combined.

**14.6 Corollary.** *The natural group isomorphisms*

$$\mathrm{Isom}(\mathrm{CW}'(\mathfrak{f})) \quad \xrightarrow{\sim} \quad \mathrm{Aut}(\langle A, B, C\rangle, \mathrm{CW}(\mathcal{F})) \quad \xrightarrow{\sim} \quad \mathrm{Aut}(\hat{\mathbb{C}}, \mathrm{CW}(\rho_{\hat{\mathbb{C}}}, q_{\hat{\mathbb{C}}}))$$

*are compatible with the previously constructed isomorphisms*

$$\mathrm{CW}(\mathfrak{f}) \quad \xrightarrow{\sim} \quad \mathrm{CW}(\mathcal{F}) \quad \xrightarrow{\sim} \quad \mathrm{CW}(\rho_{\hat{\mathbb{C}}}, q_{\hat{\mathbb{C}}})$$

*of oriented, colored* $\mathbb{Z}$-CW*-complexes.* $\square$

**14.7 Remarks.** Suppose that Notation 14.1 holds.

(i). By conjugating by an element of $\mathrm{Aut}(\hat{\mathbb{C}})$, we may assume that $\rho_{\hat{\mathbb{C}}}(D_{\mathcal{F}})$ is $z+2$. Here $q_{\hat{\mathbb{C}}} = \infty$, and $\mathrm{Aut}(\hat{\mathbb{C}}, \mathrm{CW}(\rho_{\hat{\mathbb{C}}}, q_{\hat{\mathbb{C}}}))$ behaves somewhat like

$$\mathrm{Isom}(\mathrm{CW}'(\mathfrak{f})).$$

By considering all possible conjugation actions on $\langle\, \rho_{\hat{\mathbb{C}}}(D_{\mathcal{F}})\rangle = \langle z{+}2\rangle$, we find that, in $\mathrm{Aut}(\hat{\mathbb{C}}, \mathrm{CW}(\rho_{\hat{\mathbb{C}}}, q_{\hat{\mathbb{C}}}))$, the vertical translations are of the form $z+b$, $b \in \mathbb{C} - \mathbb{R}$; the horizontal glide-reflections are of the form $\overline{z}+b$, $b \in \mathbb{C} - \mathbb{R}\mathbf{i}$; and, the vertical glide-reflections are of the form $-\overline{z}+b$, $b \in \mathbb{C} - \mathbb{R}$.

(ii) By Remarks 13.10, we see that

$$\mathrm{Aut}(\hat{\mathbb{C}}, \mathrm{Im}\,\rho_{\hat{\mathbb{C}}}) = \rho_{\hat{\mathbb{C}}}(\langle A, B, C\rangle) \cdot \mathrm{Aut}(\hat{\mathbb{C}}, \mathrm{CW}(\rho_{\hat{\mathbb{C}}}, q_{\hat{\mathbb{C}}}))$$

and $\rho_{\hat{\mathbb{C}}}(\langle A, B, C\rangle) \;\cap\; \mathrm{Aut}(\hat{\mathbb{C}}, \mathrm{CW}(\rho_{\hat{\mathbb{C}}}, q_{\hat{\mathbb{C}}})) = \rho_{\hat{\mathbb{C}}}(\langle D_{\mathcal{F}}\rangle)$. An analogous statement holds for $\mathrm{Symm}(\rho_{\hat{\mathbb{C}}})$. $\square$

We can reformulate the work of Jørgensen [6] mentioned in Section 10.

**14.8 Theorem.** *Suppose that* Notation 14.1 *holds. Then* (14.4.1) *is a discrete faithful representation extending* $\rho_{\hat{\mathbb{C}}}$.

*Proof.* It remains to show that $\mathrm{Aut}(\hat{\mathbb{C}}, \mathrm{Im}\,\rho_{\hat{\mathbb{C}}})$ is a discrete subgroup of $\mathrm{Aut}(\hat{\mathbb{C}})$.

Consider first the case where $\mathrm{Isom}(\mathrm{CW}'(\mathfrak{f}))$ contains a non-identity vertical translation.

Let $d$ denote the smallest positive integer $d$ such that

$$(x, y+d) \in \mathrm{Isom}(\mathrm{CW}'(\mathfrak{f})).$$

Then $\langle (x{+}2, y), (x, y{+}d)\rangle$ has index at most four in $\mathrm{Isom}(\mathrm{CW}'(\mathfrak{f}))$, since each element of the latter is a translation, or a half-turn, or a horizontal- or a vertical-glide reflection.

By Proposition 13.6, the isomorphism

$$\mathrm{Isom}(\mathrm{CW}'(\mathfrak{f})) \xrightarrow{\sim} \mathrm{Aut}(\langle A, B, C\rangle, \mathrm{CW}(\mathcal{F}))$$

carries $(x, y + d)$ to

$$F := \mathcal{F}_d \mathcal{F}_0^{-1} = \mathcal{F}_0 \cdot \prod_{n=1}^{d} \mathfrak{f}_n \cdot \mathcal{F}_0^{-1}.$$

Hence $\langle F, D_{\mathcal{F}}\rangle$ has index at most four in $\mathrm{Aut}(\langle A, B, C\rangle, \mathrm{CW}(\mathcal{F}))$. Thus,

$$\langle A, B, C, F\rangle$$

has index at most four in $\langle A, B, C\rangle \cdot \mathrm{Aut}(\langle A, B, C\rangle, \mathrm{CW}(\mathcal{F}))$, and the latter equals $\mathrm{Aut}(\langle A, B, C\rangle, \mathbb{S}^2(\mathcal{F}))$, by Remarks 13.10.

The work of Jørgensen[6] mentioned in Section 10 shows that the isomorphism in the statement of the theorem carries $\langle BA, BC, F\rangle$ to a discrete subgroup of $\mathrm{Aut}(\hat{\mathbb{C}}, \mathrm{Im}\,\rho_{\hat{\mathbb{C}}})$; moreover, it is a subgroup of index at most eight, by the preceding discussion. Hence $\mathrm{Aut}(\hat{\mathbb{C}}, \mathrm{Im}\,\rho_{\hat{\mathbb{C}}})$ is itself discrete.

It remains to consider the case where $\mathrm{Isom}(\mathrm{CW}'(\mathfrak{f}))$ contains no non-identity vertical translation.

Then $\mathrm{Isom}(\mathrm{CW}'(\mathfrak{f}))$ contains no vertical-glide-reflection symmetry. It follows that $\mathrm{Isom}(\mathrm{CW}'(\mathfrak{f}))/\langle (x + 2, y)\rangle$ has order at most two, and $\mathrm{Isom}(\mathrm{CW}'(\mathfrak{f}))$ can contain a half-turn symmetry, or a horizontal-glide-reflection symmetry, or neither. (The resulting quotient orbifold is a once-punctured sphere with two double points, or a once-punctured projective plane, or a twice-punctured sphere, respectively. In other words, an open disc with two double points, or an open Möbius band, or an open annulus, respectively.) Hence $\langle D_{\mathcal{F}}\rangle$ has index at most two in $\mathrm{Aut}(\langle A, B, C\rangle, \mathrm{CW}(\mathcal{F}))$. Thus, $\langle A, B, C\rangle$ has index at most two in $\langle A, B, C\rangle \cdot \mathrm{Aut}(\langle A, B, C\rangle, \mathrm{CW}(\mathcal{F}))$, and this is $\mathrm{Aut}(\langle A, B, C\rangle, \mathbb{S}^2(\mathcal{F}))$, by Remarks 13.10. Hence, $\rho_{\hat{\mathbb{C}}}(\langle A, B, C\rangle)$ is a discrete subgroup of $\mathrm{Aut}(\hat{\mathbb{C}}, \mathrm{Im}\,\rho_{\hat{\mathbb{C}}})$ of index at most two, and $\mathrm{Aut}(\hat{\mathbb{C}}, \mathrm{Im}\,\rho_{\hat{\mathbb{C}}})$ is itself discrete. $\square$

**14.9 Example.** Let $\mathfrak{b} = \prod_{\mathbb{Z}} RL$.

Here backwards$(\mathfrak{b}) = \mathfrak{b}$, interchange$(\mathfrak{b}) = \mathfrak{b}$, and transpose$(\mathfrak{b}) = \mathfrak{b}$. We obtain a tessellation of $\mathbb{C}$ with half-turn symmetries, vertical glide-reflection symmetries, and horizontal glide-reflection symmetries. The planar symmetry group is of type **p g g**. The quotient orbifold is a projective plane with two double points.

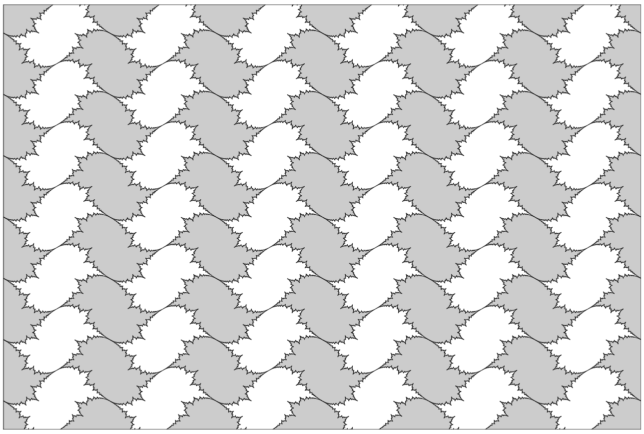

Figure 14.9.1: Part of the tessellation for $\mathfrak{b}(\rho_{\hat{\mathbb{C}}}) = \prod\limits_{\mathbb{Z}} RL$. $\square$

**14.10 Example.** Let $\mathfrak{b} = \prod_{\mathbb{Z}} RLL$.

Here backwards$(\mathfrak{b}) = \mathfrak{b}$, interchange$(\mathfrak{b}) \neq \mathfrak{b}$, and transpose$(\mathfrak{b}) \neq \mathfrak{b}$. We obtain a tessellation of $\mathbb{C}$ with some half-turn symmetries, no vertical glide-reflection symmetries, and no horizontal glide-reflection symmetries. The planar symmetry group is of type **p 2**. The quotient orbifold is a sphere with four double points.

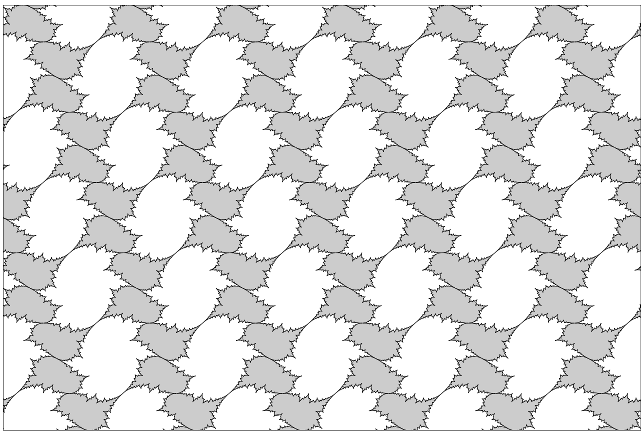

Figure 14.10.1: Part of the tessellation for $\mathfrak{b}(\rho_{\hat{\mathbb{C}}}) = \prod\limits_{\mathbb{Z}} RLL$. $\square$

**14.11 Example.** Let $\mathfrak{b} = \prod_{\mathbb{Z}} RRLL$.

Here backwards$(\mathfrak{b}) = \mathfrak{b}$, interchange$(\mathfrak{b}) = \mathfrak{b}$, and transpose$(\mathfrak{b}) = \mathfrak{b}$. We obtain a tessellation of $\mathbb{C}$ with half-turn symmetries, vertical glide-reflection symmetries, and horizontal glide-reflection symmetries. The planar symmetry group is of type **p g g**. The quotient orbifold is a projective plane with two double points.

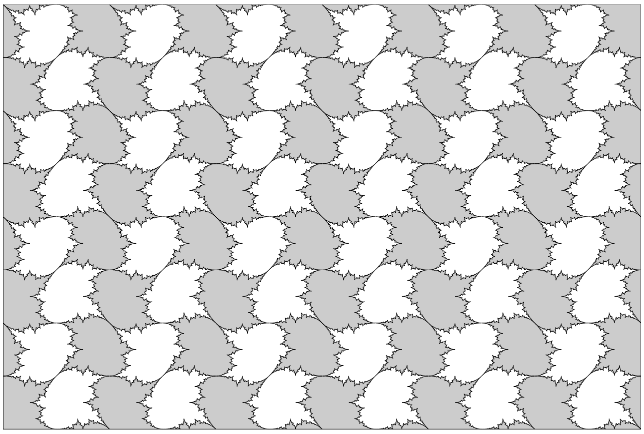

Figure 14.11.1: Part of the tessellation for $\mathfrak{b}(\rho_{\hat{\mathbb{C}}}) = \prod\limits_{\mathbb{Z}} RRLL$. □

**14.12 Example.** Let $\mathfrak{b} = \prod_{\mathbb{Z}} RLRRLL$.

Here backwards$(\mathfrak{b}) \neq \mathfrak{b}$, interchange$(\mathfrak{b}) \neq \mathfrak{b}$, and transpose$(\mathfrak{b}) = \mathfrak{b}$. We obtain a tessellation of $\mathbb{C}$ with no half-turn symmetries, no vertical glide-reflection symmetries, and some horizontal glide-reflection symmetries. The planar symmetry group is of type **p g**. The quotient orbifold is a Klein bottle.

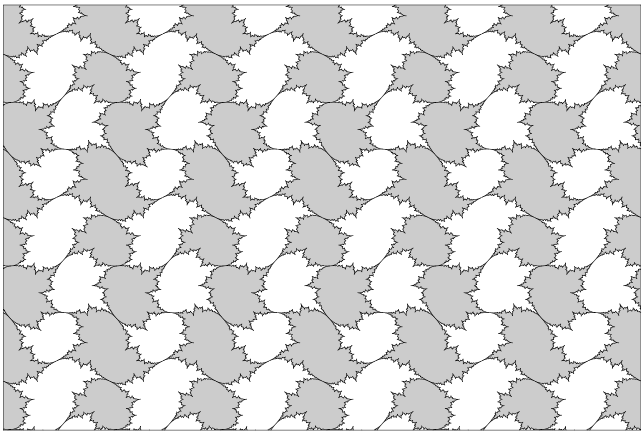

Figure 14.12.1: Part of the tessellation for $\mathfrak{b}(\rho_{\hat{\mathbb{C}}}) = \prod\limits_{\mathbb{Z}} RLRRLL$. □

**14.13 Example.** Let $\mathfrak{b} = \prod_{\mathbb{Z}} RLLRRRLLLL$.

Here backwards$(\mathfrak{b}) \neq \mathfrak{b}$, interchange$(\mathfrak{b}) \neq \mathfrak{b}$, and transpose$(\mathfrak{b}) \neq \mathfrak{b}$. We obtain a tessellation of $\mathbb{C}$ with no half-turn symmetries, no vertical glide-reflection symmetries, and no horizontal glide-reflection symmetries. The planar symmetry group is of type **p 1**. The quotient orbifold is a torus.

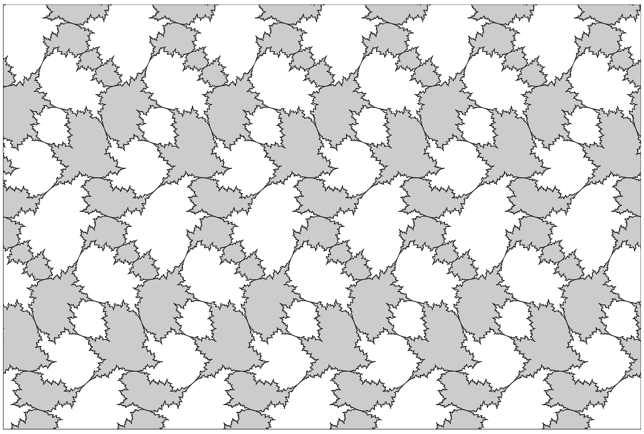

Figure 14.13.1: Part of the tessellation for $\mathfrak{b}(\rho_{\hat{\mathbb{C}}}) = \prod_{\mathbb{Z}} RLLRRRLLLL$. □

**14.14 Example.** Let $\mathfrak{b} = \prod_{\mathbb{Z}} RLLRRRLRRLLL$.

Here backwards$(\mathfrak{b}) \neq \mathfrak{b}$, interchange$(\mathfrak{b}) = \mathfrak{b}$, and transpose$(\mathfrak{b}) \neq \mathfrak{b}$. We obtain a tessellation of $\mathbb{C}$ with no half-turn symmetries, some vertical glide-reflection symmetries, and no horizontal glide-reflection symmetries. The planar symmetry group is of type **p g**. The quotient orbifold is a Klein bottle.

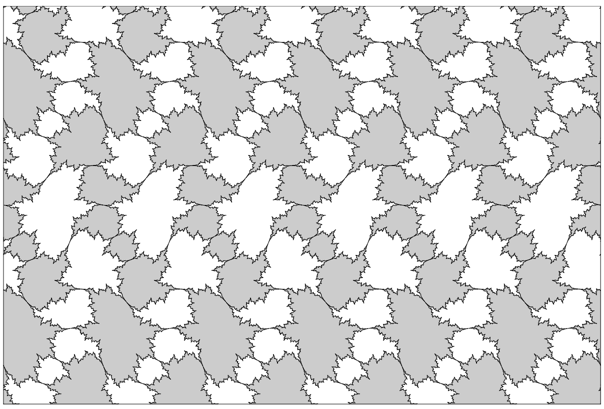

Figure 14.14.1: Part of the tessellation for $\mathfrak{b}(\rho_{\hat{\mathbb{C}}}) = \prod_{\mathbb{Z}} RLLRRRLRRLLL$. □

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Errata and addenda: http://mat.uab.es/∼dicks/Cannon.html

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# Index of symbols