On hyperbolic once-punctured-torus bundles IV: automata for lightning curves.

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Abstract. Let $\langle A, B, C \rangle \coloneqq \langle A, B, C, D \mid A^2 = B^2 = C^2 = ABCD = 1 \rangle$. Let R and L denote the automorphisms of $\langle A, B, C \rangle$ determined by ${}^R(A, B, C) = (A, BCB, B)$, ${}^L(A, B, C) = (B, BAB, C)$. Let $(a_1, b_1, a_2, b_2, \ldots, a_p, b_p)$ be a non-empty, even-length, positive-integer sequence, let F denote $R^{a_1}L^{b_1}R^{a_2}L^{b_2} \cdots R^{a_p}L^{b_p}$, and let $\langle A, B, C, F \rangle$ denote the semidirect product $\langle F \mid \; \rangle \ltimes \langle A, B, C \rangle$. In an influential unfinished work, Jørgensen constructed a discrete faithful representation $\rho_F \colon \langle A, B, C, F \rangle \to \mathrm{PSL}_2(\mathbb{C})$. The group $\langle A, B, C, F \rangle$ then acts conformally on the Riemann sphere $\hat{\mathbb{C}}$ via ρ_F. Using results of Thurston, Minsky, McMullen, Bowditch, and others, Cannon-Dicks showed that $\hat{\mathbb{C}}$ has a CW-structure formed from three closed two-cells, denoted $[A]$, $[B]$ and $[C]$, that are Jordan disks satisfying the ping-pong conditions $A[A]=[B] \cup [C]$, $B[B]=[C] \cup [A]$, and $C[C]=[A] \cup [B]$. Further, Cannon-Dicks expressed the resulting theta-shaped one-skeleton as the union of two arcs, denoted $\partial^- A$ and $\partial^+ B$, and expressed each of these lightning curves as limit sets of finitely generated subsemigroups of $\langle A, B, C, F \rangle$. The foregoing results had previously been obtained by Alperin-Dicks-Porti for $F = RL$ by elementary methods. Independently, Mumford, Scorza, Series, Wright, and others studied more general lightning curves that arise as limits of sequences of finite chains of round disks in $\hat{\mathbb{C}}$. Later, Cannon-Dicks showed that the set of $\langle D, F \rangle$-translates of $\partial^- A \cup \partial^+ B$ gives a tessellation $\mathrm{CW}(F)$ of $\mathbb{C}$ with tiles that are Jordan disks.

In this article, we find that classic Adler-Weiss automata codify $\partial^- A$ and $\partial^+ B$ in terms of ends of trees. The $\partial^+ B$-automaton distinguishes a tree of words in a certain finite alphabet S that is a subset of $\langle A, B, C, F \rangle$. The $\partial^- A$-automaton distinguishes a tree of words in the finite alphabet S^{-1}. The automata allow depth-first searches which give drawings of $\partial^- A$ and $\partial^+ B$ that, while requiring less computer time and memory, are more detailed than those that have hitherto been obtained.

We show that the limit set of the semigroup generated by S is $\partial^+ B$ and the limit set of the semigroup generated by S^{-1} is $\partial^- A$. We use this to show that the Hausdorff dimensions of $\partial^- A$ and $\partial^+ B$ are equal. We raise the problem of whether or not the common Hausdorff dimension can be calculated by applying a famous technique of McMullen to the $\partial^- A$-automaton.

We note that the improved drawings of $\partial^- A$ and $\partial^+ B$ give improved drawings of the planar tessellation $\mathrm{CW}(F)$. We review Riley's sufficient condition for the columns of $\mathrm{CW}(F)$ to be vertical. We review Helling's description of Jørgensen's ρ_{RL^n} and Hodgson-Meyerhoff-Weeks' ρ_{RL^∞}, and we draw $\mathrm{CW}(RL^{100})$ together with something we call $\mathrm{CW}(RL^\infty)$.

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1 Background and outline

Let φ be an orientation-preserving, pseudo-Anosov self-homeomorphism of the once-punctured torus $\mathbf{T}^* := (\mathbb{R}^2 - \mathbb{Z}^2)/\mathbb{Z}^2$. The mapping torus $\mathbf{T}^*_\varphi := (\mathbf{T}^* \times \mathbb{R})/((x,t) \sim (\varphi(x), t+1))$ is a bundle over the circle $\mathbb{R}/\mathbb{Z}$ with fiber $\mathbf{T}^*$ and monodromy φ. By Thurston's uniformization theorem for surface bundles, $\mathbf{T}^*_\varphi$ admits a complete, finite-volume, hyperbolic structure; see, for example, [39], [32]. Since $\mathbf{T}^*_\varphi$ has a single torus cusp, $\mathbf{T}^*_\varphi$ admits a *canonical decomposition* into ideal tetrahedra; see, for example, [19], [41]. The universal cover of $\mathbf{T}^*_\varphi$ can then be identified with hyperbolic three-space, $\mathbf{H}^3$, and the fundamental group $\pi_1(\mathbf{T}^*_\varphi)$ can be identified with a Kleinian subgroup of $\mathrm{PSL}_2(\mathbb{C})$. The canonical decomposition of $\mathbf{T}^*_\varphi$ then lifts to a $\pi_1(\mathbf{T}^*_\varphi)$-invariant tessellation of $\mathbf{H}^3$ by ideal tetrahedra. The hyperbolic structure and the canonical decomposition of $\mathbf{T}^*_\varphi$ and the tessellation of $\mathbf{H}^3$ were constructed by Jørgensen in his famous unfinished work [24]; rigorous treatments of part of his results were given in [5], [6], [20], [21], [25], and [33].

The punctured torus $\mathbf{T}^*$ $(= (\mathbb{R}^2 - \mathbb{Z}^2)/\mathbb{Z}^2)$ admits the *hyper-elliptic involution* ι induced by the multiplication-by-(-1) involution of $\mathbb{R}^2$. The quotient orbifold $\mathbf{O}^* := \mathbf{T}^*/\langle\, \iota \,\rangle$ is a once-punctured sphere with three index-two cone points. Since the mapping class of ι lies in the center of the mapping-class group of $\mathbf{T}^*$, ι extends to a fiber-preserving, isometric involution ι_φ of $\mathbf{T}^*_\varphi$. The quotient orbifold $\mathbf{O}^*_\varphi := \mathbf{T}^*_\varphi/\langle\, \iota_\varphi \,\rangle$ is a bundle over $\mathbb{R}/\mathbb{Z}$ with fiber $\mathbf{O}^*$, and $\mathbf{O}^*_\varphi$ inherits a complete, finite-volume, hyperbolic structure. The fundamental orbifold group $\pi_1(\mathbf{O}^*_\varphi)$ can be identified with a Kleinian subgroup of $\mathrm{PSL}_2(\mathbb{C})$ that respects the tessellation of $\mathbf{H}^3$ and contains $\pi_1(\mathbf{T}^*_\varphi)$ as an index-two subgroup.

By specifying a complete, finite-area, hyperbolic structure on $\mathbf{O}^*$, we may view the universal orbifold cover of $\mathbf{O}^*$ as the hyperbolic plane, $\mathbf{H}^2$, and we may view the fundamental orbifold group $\pi_1(\mathbf{O}^*)$ as a Kleinian subgroup of $\mathrm{PSL}_2(\mathbb{R})$ given with an embedding in $\pi_1(\mathbf{O}^*_\varphi)$. The fibration of $\mathbf{O}^*_\varphi$ over $\mathbb{R}/\mathbb{Z}$ with fibre $\mathbf{O}^*$ lifts to the universal orbifold covers giving a fibration of $\mathbf{H}^3$ over $\mathbb{R}$ with (very twisted) fiber $\mathbf{H}^2$. The boundary of $\mathbf{H}^2$ can be identified with the real projective line $\hat{\mathbb{R}} := \mathbb{R} \cup \{\infty\}$, and the boundary of $\mathbf{H}^3$ can be identified with the Riemann sphere $\hat{\mathbb{C}} := \mathbb{C} \cup \{\infty\}$. It was conjectured by Cannon-Thurston [16], and proved by McMullen [29], using results of Minsky [30] and others, that the $\pi_1(\mathbf{O}^*)$-map $\mathbf{H}^2 \to \mathbf{H}^3$ can be extended continuously to the boundaries. The resulting map on the boundaries is a continuous, surjective $\pi_1(\mathbf{O}^*)$-map $\mathrm{CT}\colon \hat{\mathbb{R}} \twoheadrightarrow \hat{\mathbb{C}}$ called the *Cannon-Thurston map* (or path) for $\mathbf{O}^*_\varphi$ and $\mathbf{T}^*_\varphi$. The case of these results where $\mathbf{T}^*_\varphi$ is the complement of the figure-eight knot had been obtained by Alperin-Dicks-Porti [7] by elementary methods.

Information is easily transferred between the orbifolds $\mathbf{O}^*_\varphi$ and $\mathbf{T}^*_\varphi$. In this article, we find that our results are easier to express in terms of $\mathbf{O}^*_\varphi$ and that many of our proofs are easier to express in terms of $\mathbf{T}^*_\varphi$.

The orbifold group $\pi_1(\mathbf{O}^*)$ is isomorphic to the group

$$\langle\, A, B, C \,\rangle := \langle\, A, B, C, D \mid A^2 = B^2 = C^2 = ABCD = 1 \,\rangle,$$

and the orbifold group $\pi_1(\mathbf{O}^*_\varphi)$ is isomorphic to a semi-direct product

$$\langle\, A, B, C, F\,\rangle \coloneqq \langle\, F \mid \quad \rangle \ltimes \langle\, A, B, C\,\rangle$$

with F acting as an automorphism of $\langle\, A, B, C\,\rangle$ of the form $R^{a_1}L^{b_1}R^{a_2}L^{b_2}\cdots R^{a_p}L^{b_p}$, where $(a_1, b_1, a_2, b_2, \ldots, a_p, b_p)$ is a non-empty, even-length, positive-integer sequence, and R and L denote the automorphisms of $\langle\, A, B, C\,\rangle$ determined by ${}^R(A, B, C) = (A, BCB, B)$ and ${}^L(A, B, C) = (B, BAB, C)$. Any non-empty, even-length, positive-integer sequence can be realized in this way. The index-two subgroup $\pi_1(\mathbf{T}^*_\varphi)$ of $\pi_1(\mathbf{O}^*_\varphi)$ is mapped under the isomorphism to either $\langle\, F \mid \quad \rangle \ltimes \langle\, CB, AB \mid \quad \rangle$ or its sister, $\langle\, DF \mid \quad \rangle \ltimes \langle\, CB, AB \mid \quad \rangle$.

We then have homomorphisms $\rho_0\colon \langle\, A, B, C\,\rangle \to \mathrm{PSL}_2(\mathbb{R})$, $\rho_F\colon \langle\, A, B, C, F\,\rangle \to \mathrm{PSL}_2(\mathbb{C})$, and a continuous surjective $\langle\, A, B, C\,\rangle$-map $\mathrm{CT}\colon \hat{\mathbb{R}}_{\rho_0} \twoheadrightarrow \hat{\mathbb{C}}_{\rho_F}$, where $\hat{\mathbb{R}}_{\rho_0}$ denotes $\hat{\mathbb{R}}$ endowed with the conformal $\langle\, A, B, C\,\rangle$-action determined by ρ_0, and $\hat{\mathbb{C}}_{\rho_F}$ denotes $\hat{\mathbb{C}}$ endowed with the conformal $\langle\, A, B, C, F\,\rangle$-action determined by ρ_F. Without loss of generality for our purposes, we may normalize ρ_0 and ρ_F, and, in particular, we assume throughout that

$$\text{(1.1)} \qquad \rho_0(A) = \pm\left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right),\ \rho_0(B) = \pm\left(\begin{smallmatrix} 1 & -1 \\ 2 & -1 \end{smallmatrix}\right),\ \rho_0(C) = \pm\left(\begin{smallmatrix} 1 & -2 \\ 1 & -1 \end{smallmatrix}\right),\ \rho_0(D) = \pm\left(\begin{smallmatrix} 1 & 3 \\ 0 & 1 \end{smallmatrix}\right),$$

$\rho_F(D) = \pm\left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$, and $\rho_F(F) = \pm\left(\begin{smallmatrix} 1 & s \\ 0 & 1 \end{smallmatrix}\right)$ for some $s \in \mathbb{C}$ with $\mathrm{Im}(s) > 0$. Then the intervals $[A]_{\hat{\mathbb{R}}} \coloneqq [-\infty, 0]$, $[B]_{\hat{\mathbb{R}}} \coloneqq [0, 1]$, and $[C]_{\hat{\mathbb{R}}} \coloneqq [1, \infty]$ are easily seen to satisfy the ping-pong conditions

$$\text{(1.2)} \quad \rho_0(A)([A]_{\hat{\mathbb{R}}}) = [B]_{\hat{\mathbb{R}}} \cup [C]_{\hat{\mathbb{R}}}, \rho_0(B)([B]_{\hat{\mathbb{R}}}) = [C]_{\hat{\mathbb{R}}} \cup [A]_{\hat{\mathbb{R}}}, \rho_0(C)([C]_{\hat{\mathbb{R}}}) = [A]_{\hat{\mathbb{R}}} \cup [B]_{\hat{\mathbb{R}}}.$$

Hence the images $[A]_{\hat{\mathbb{C}}} \coloneqq \mathrm{CT}([A]_{\hat{\mathbb{R}}})$, $[B]_{\hat{\mathbb{C}}} \coloneqq \mathrm{CT}([B]_{\hat{\mathbb{R}}})$ and $[C]_{\hat{\mathbb{C}}} \coloneqq \mathrm{CT}([C]_{\hat{\mathbb{R}}})$ cover $\hat{\mathbb{C}}$ and satisfy the ping-pong conditions

$$\text{(1.3)} \quad \rho_F(A)([A]_{\hat{\mathbb{C}}}) = [B]_{\hat{\mathbb{C}}} \cup [C]_{\hat{\mathbb{C}}}, \rho_F(B)([B]_{\hat{\mathbb{C}}}) = [C]_{\hat{\mathbb{C}}} \cup [A]_{\hat{\mathbb{C}}}, \rho_F(C)([C]_{\hat{\mathbb{C}}}) = [A]_{\hat{\mathbb{C}}} \cup [B]_{\hat{\mathbb{C}}}.$$

The present article is the fourth in a numbered series on the map $\mathrm{CT}\colon \hat{\mathbb{R}}_{\rho_0} \twoheadrightarrow \hat{\mathbb{C}}_{\rho_F}$, and before stating our objectives it might be helpful to review the three numbered articles preceding this one.

In the first article, Cannon and Dicks [14] used a result of Bowditch [13] to show that the ping-pong subsets $[A]_{\hat{\mathbb{C}}}$, $[B]_{\hat{\mathbb{C}}}$ and $[C]_{\hat{\mathbb{C}}}$ are closed Jordan disks that meet each other only on the boundaries. For each $W \in \{A, B, C\}$, the image in $[W]_{\hat{\mathbb{C}}}$ of the initial, resp. terminal, point of the interval $[W]_{\hat{\mathbb{R}}}$ is called the *initial*, resp. *terminal*, point of the Jordan disk $[W]_{\hat{\mathbb{C}}}$. The boundary ∂W of $[W]_{\hat{\mathbb{C}}}$, oriented to keep the disk $[W]_{\hat{\mathbb{C}}}$ on the left, is partitioned into two arcs: $\partial^- W$ travels from the initial point of $[W]_{\hat{\mathbb{C}}}$ to the terminal point of $[W]_{\hat{\mathbb{C}}}$; and, $\partial^+ W$ travels from the terminal point of $[W]_{\hat{\mathbb{C}}}$ to the initial point of $[W]_{\hat{\mathbb{C}}}$. It was shown in [14] that $\partial^- A \cap \partial^+ B = \{\mathrm{CT}(0), \mathrm{CT}(1), \mathrm{CT}(\infty)\}$ and that $\partial^- A \cup \partial^+ B$ is a theta-shaped graph that marks out the break-up of $\hat{\mathbb{C}}$ into $[A]_{\hat{\mathbb{C}}}$, $[B]_{\hat{\mathbb{C}}}$ and $[C]_{\hat{\mathbb{C}}}$. See Figure 1.1(1), (2) for the case

$F = RL^3$. It was also shown in [14] that each of the arcs $\partial^- W$, $\partial^+ W$ is the limit set of a certain finitely generated subsemigroup of $\rho_F(\langle A, B, C, F \rangle)$, and, moreover, the arc endowed with the semigroup action was seen to be isomorphic to a real line segment endowed with the action of a semigroup of affine transformations. These results had been obtained by Alperin, Dicks and Porti [7] for $F = RL$ by elementary methods.

Independently, Mumford, Scorza, Series, Wright, and others discovered and studied more general lightning curves from the completely different viewpoint of limits of sequences of finite chains of round disks in $\hat{\mathbb{C}}$; see, for example, [31, Chapter 10] and [36].

In the second article in the sequence, Cannon and Dicks [15] defined the *Cannon-Thurston planar tessellation* $\mathrm{CW}(F)$ of $\mathbb{C}$ whose one-skeleton is formed by deleting ∞ from the union of all the $\rho_F(\langle D, F \rangle)$-translates of $\partial^- A \cup \partial^+ B$. They colored the tessellation with alternating gray L-columns and white R-columns; see Figure 1.1(3) for the case $F = RL^3$. The path $\mathrm{CT}\colon \hat{\mathbb{R}}_{\rho_0} \twoheadrightarrow \hat{\mathbb{C}}_{\rho_F}$ fills in the columns of $\mathrm{CW}(F)$ from left to right, proceeding down white R-columns and up gray L-columns. A mnemonic is that 'white', 'right' and 'down' are longer than 'gray', 'left' and 'up', respectively; also 'white' and 'right' rhyme and have five letters. We think of the pen tracing out the path CT as switching ink-color between gray and white each time the pen passes through ∞ in $\hat{\mathbb{C}}$. In [15], it was shown how to read from $\mathrm{CW}(F)$ the same bi-infinite word $\prod_{\mathbb{Z}}(R^{a_1}L^{b_1}R^{a_2}L^{b_2}\cdots R^{a_p}L^{b_p})$ that is classically read from an ending-lamination pair as described in [30]. Also in [15], the column-permuting-symmetry group of $\mathrm{CW}(F)$ was calculated, and then the referee of [18] showed that the column-permuting-symmetry group equals the whole symmetry group; see [18, Remark 8.16].

In the third article, Dicks and Sakuma [18] studied the connection between two planar $\rho_F(\langle D, F \rangle)$-tessellations of $\mathbb{C}$, namely the Cannon-Thurston tessellation $\mathrm{CW}(F)$ and the Jørgensen triangulation $\Delta(F)$, defined in Notation 2.3 below, that codifies the set of those ideal tetrahedra in the Jørgensen tessellation of $\mathbf{H}^3$ which have ∞ as an ideal vertex. Dicks and Sakuma showed that $\mathrm{CW}(F)$ and $\Delta(F)$ have the same vertex set and that the combinatorics of each tessellation can be reconstructed from the other; see Figure 1.1(3) for the case $F = RL^3$.

In this, the fourth article, we find that classic Adler-Weiss automata can be used to codify $\partial^- A$ and $\partial^+ B$, and we show that the Hausdorff dimensions of $\partial^- A$ and $\partial^+ B$ are equal. In detail, the article has the following structure.

In Section 2, we fix our main notational conventions.

In Section 3, we give detailed descriptions of automata for $\partial^- A$ and $\partial^+ B$. We define a finite subset S of $\langle A, B, C, F \rangle$ called the set of ∂^+*-syllables*, and we call S^{-1} the set of ∂^-*-syllables*. The $\partial^+ B$-automaton distinguishes a tree of words in the finite alphabet S, and codifies $\partial^+ B$ in terms of ends of the tree. The $\partial^- A$-automaton distinguishes a tree of words in the finite alphabet S^{-1}, and codifies $\partial^- A$ in terms of ends of the tree. We show how the automata allow depth-first searches which give drawings of $\partial^- A$ and $\partial^+ B$ that, while

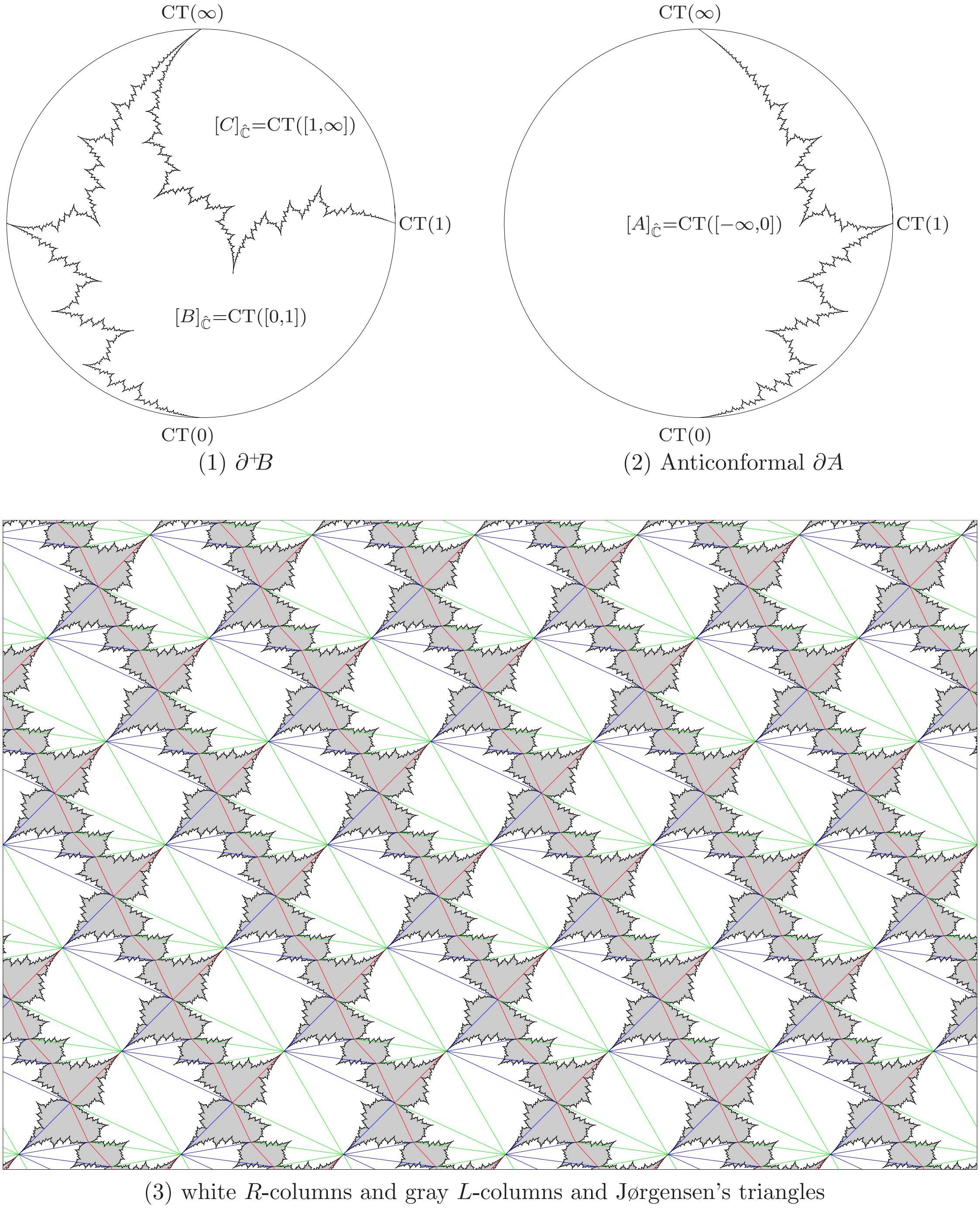


Figure 1.1: Theta-shaped graph $\partial^- A \cup \partial^+ B$ for $F = RL^3$ and $\mathrm{CW}(RL^3) \cup \Delta(RL^3)$

requiring less computer time and memory, are more detailed than those that have hitherto been obtained; compare Figure 1.1(1) with [14, Figure 5] and [31, Figure 10.14]. We can then use these to improve the drawings of $\mathrm{CW}(F)$.

In Section 4, we prepare for the proofs concerning the automata by reviewing the construction of the Cannon-Thurston model. We recall Nielsen's homomorphism ρ_0 from $\mathrm{Aut}\langle\, A, B, C\,\rangle$ to the group of all self-homeomorphisms of $\hat{\mathbb{C}}$ which carry $\hat{\mathbb{R}}$ to itself, and we then consider our previous usage of ρ_0 to be superseded. We let $\hat{\mathbb{C}}_{\rho_0}$ denote $\hat{\mathbb{C}}$ endowed with the Nielsen action of $\mathrm{Aut}\langle\, A, B, C\,\rangle$. Restricting, we let $\hat{\mathbb{R}}_{\rho_0}$ denote $\hat{\mathbb{R}}$ endowed with the Nielsen action of $\mathrm{Aut}\langle\, A, B, C\,\rangle$. Recall that $\hat{\mathbb{C}}_{\rho_F}$ denotes $\hat{\mathbb{C}}$ endowed with the Jørgensen action of $\langle\, A, B, C, F\,\rangle$. Minsky, McMullen and Bowditch proved that, as conjectured by Cannon-Thurston, there exists a continuous, surjective, foliation-collapsing, $\langle\, A, B, C, F\,\rangle$-map $\mathrm{CT}\colon \hat{\mathbb{C}}_{\rho_0} \to \hat{\mathbb{C}}_{\rho_F}$ such that the restriction $\mathrm{CT}\colon \hat{\mathbb{R}}_{\rho_0} \to \hat{\mathbb{C}}_{\rho_F}$ is the quotient $\langle\, A, B, C, F\,\rangle$-map which identifies all elements of $\hat{\mathbb{R}}$ fixed by $\rho_0(F)$; it is also the quotient $\langle\, A, B, C\,\rangle$-map which identifies all elements of $\hat{\mathbb{R}}$ fixed by $\rho_0(F)$. The map $\mathrm{CT}\colon \hat{\mathbb{C}}_{\rho_0} \to \hat{\mathbb{C}}_{\rho_F}$ provides a model of Jørgensen's geometry in which each point of $\hat{\mathbb{C}}_{\rho_F}$ is blown up to a point or a line or a 'spider' in $\hat{\mathbb{C}}_{\rho_0}$. We then obtain a manageable model of both $\partial^- A$ and $\partial^+ B$.

In Section 5, we verify all the details that were presented in Section 3 concerning the automata. The lower half-plane $\hat{\mathbb{C}}^-$ is the universal cover of a once-punctured torus $\rho_0(\langle\, BC, BA\,\rangle)\backslash\hat{\mathbb{C}}^-$ on which $\rho_0(F)$ acts as a pseudo-Anosov self-homeomorphism. The once-punctured torus has two Adler-Weiss CW-structures, called Markov partitions, and the closed two-cells are called rows in one and columns in the other; moreover $\rho_0(F)$ carries each column to a row. We lift the two Adler-Weiss CW-structures to a certain fundamental $\rho_0(\langle\, BC, BA\,\rangle)$-domain in $\hat{\mathbb{C}}^-$, and find that $\rho_0(F)$ carries each lifted column to a (unique) $\rho_0(\langle\, BC, BA\,\rangle)$-translate of a (unique) lifted row. Thus each lifted column is a (unique) $\rho_0(F^{-1}\langle\, BC, BA\,\rangle)$-translate of a (unique) lifted row; see Figure 1.2.

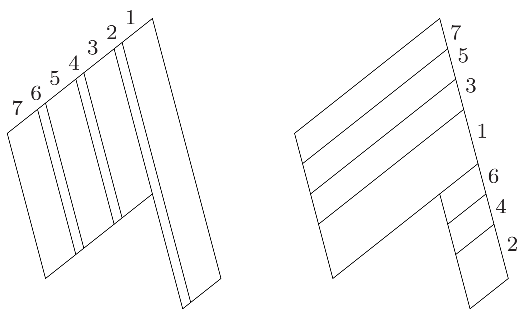


Figure 1.2: The numbering indicates how F transforms columns to rows for $F = RL^3$.

In this way, each column determines an element of $F^{-1}\langle\, BC, BA\,\rangle$; these elements, one for each column, are the ∂^--*syllables*. The classic Adler-Weiss automaton that is used in [1] to codify the dynamics of the action of F^{-1} on the unpunctured torus gives the $\partial^{-}A$-automaton. The ∂^{-}-syllables generate a subsemigroup of $\langle\, A, B, C, F\,\rangle$ whose limit set in $\hat{\mathbb{C}}_{\rho_F}$ is $\partial^{-}A$; this can also be deduced from results in [14]. The inverses of the ∂^{-}-syllables are the ∂^{+}-*syllables*, and they generate a subsemigroup of $\langle\, A, B, C, F\,\rangle$ whose limit set in $\hat{\mathbb{C}}_{\rho_F}$ is $\partial^{+}B$; this can also be deduced from results in [14]. The Adler-Weiss automaton that codifies the action of F on the unpunctured torus gives the $\partial^{+}B$-automaton.

Let us emphasize that classic automata that codify a pseudo-Anosov action on the torus now also codify lightning curves associated to a punctured sphere with three index-two cone points.

In Section 6, we use the fact that $\partial^{-}A$ and $\partial^{+}B$ are limit sets of mutually inverse semigroups to show that the Hausdorff dimensions of $\partial^{-}A$ and $\partial^{+}B$ are equal. We raise the problem of whether or not these Hausdorff dimensions can be calculated by applying a celebrated technique of McMullen to the $\partial^{-}A$-automaton. We present the results of some computer experiments and note that these experimental results agree nicely with results proved by Dicks-Porti [17] in the case $F = RL$.

At this stage we will have completed our main objectives in the article and we conclude with some divulgatory expositions.

In Section 7, we record a proof of the folklore implication, traditionally attributed to Riley, that if the hyperbolic once-punctured-torus bundle $\rho_F(\langle\, CB, AB, F\,\rangle)\backslash\mathbf{H}^3$ admits an orientation-reversing isometry then the actions of $\rho_F(D)$ and $\rho_F(F)$ on the plane $\mathbb{C}$ are mutually orthogonal. It is not known whether or not the converse holds. By work of McCullough [26], Riley's implication can be translated to the assertion that the columns of $\mathrm{CW}(F)$ are completely vertical if some odd-length-cyclic shift carries the sequence $(a_1, b_1, a_2, b_2, \ldots, a_p, b_p)$ to itself or some even-length-cyclic shift carries the sequence to its reverse.

In Section 8, we review, with some simplifications, the computation by Helling [22] of the quadruple $\rho_{RL^n}(A, B, C, RL^n)$ and the matrix-level description of the limit homomorphism ρ_{RL^∞} of Hodgson-Meyerhoff-Weeks [23]. We invent a definition for $\mathrm{CW}(RL^\infty)$ and we draw $\mathrm{CW}(RL^\infty)$ and $\mathrm{CW}(RL^{100})$ together.

2 Notation

In this section we introduce the main notation that will be used throughout.

2.1 Notation. For two subsets $\mathbb{A}$, $\mathbb{B}$ of a set $\mathbb{X}$, the complement of $\mathbb{A}\cap\mathbb{B}$ in $\mathbb{A}$ will be denoted by $\mathbb{A}-\mathbb{B}$ (and not by $\mathbb{A}\setminus\mathbb{B}$ since, when we have a multiplicative group π acting on a set $\mathbb{Y}$ on the left, we let $\pi\backslash\mathbb{Y}$ denote the set of π-orbits in $\mathbb{Y}$).

We will find it useful to have notation for intervals in $\mathbb{Z}$ that is different from the notation for intervals in $\mathbb{R}$. Let $i,\ j \in \mathbb{Z}$. We define the sequence

$$[\![i{\uparrow}j]\!] := \begin{cases} (i, i+1, \ldots, j-1, j) \in \mathbb{Z}^{j-i+1} & \text{if } i \leqslant j, \\ () \in \mathbb{Z}^0 & \text{if } i > j. \end{cases}$$

The subset of $\mathbb{Z}$ underlying $[\![i{\uparrow}j]\!]$ is denoted $[i{\uparrow}j] := \{i, i+1, \ldots, j-1, j\}$.

Also, $[\![i{\uparrow}\infty[\![\; := (i, i+1, i+2, \ldots)$ and $[i{\uparrow}\infty[\; := \{i, i+1, i+2, \ldots\}$.

We define $[\![j{\downarrow}i]\!]$ to be the reverse of the sequence $[\![i{\uparrow}j]\!]$, that is, $(j, j-1, \ldots, i+1, i)$.

Suppose we have a set $\mathbb{X}$ and a map $[i{\uparrow}j] \to \mathbb{X}$, $\ell \mapsto v_\ell$. We define the corresponding sequence in $\mathbb{X}$ as

$$v_{[\![i\uparrow j]\!]} := \begin{cases} (v_i, v_{i+1}, \cdots, v_{j-1}, v_j) \in \mathbb{X}^{i-j+1} & \text{if } i \leqslant j, \\ () & \text{if } i > j. \end{cases}$$

By abuse of notation, we shall also express this sequence as $(v_\ell \mid \ell \in [\![i{\uparrow}j]\!])$, although "$\ell \in [\![i{\uparrow}j]\!]$" on its own will not be assigned a meaning. The set of terms of $v_{[\![i\uparrow j]\!]}$ is denoted $v_{[i\uparrow j]}$. Also, $v_{[\![i\uparrow\infty[\![} := (v_i, v_{i+1}, v_{i+2}, \ldots)$ and $v_{[i\uparrow\infty[} := \{v_i, v_{i+1}, v_{i+2}, \ldots\}$. We define $v_{[\![j\downarrow i]\!]}$ to be the reverse of the sequence $v_{[\![i\uparrow j]\!]}$.

Suppose we have a multiplicative group π and a map $[i{\uparrow}j] \to \pi$, $\ell \mapsto v_\ell$. We write

$$\prod_{\ell \in [\![i\uparrow j]\!]} v_\ell := \Pi v_{[\![i\uparrow j]\!]} := \begin{cases} v_i v_{i+1} \cdots v_{j-1} v_j \in \pi & \text{if } i \leqslant j, \\ 1 \in \pi & \text{if } i > j. \end{cases}$$

$$\prod_{\ell \in [\![j\downarrow i]\!]} v_\ell := \Pi v_{[\![j\downarrow i]\!]} := \begin{cases} v_j v_{j-1} \cdots v_{i+1} v_i \in \pi & \text{if } j \geqslant i, \\ 1 \in \pi & \text{if } j < i. \end{cases}$$

By a π-*space* we shall mean a topological space specified with a π-action with the property that each element of π acts as a self-homeomorphism of the space. □

Throughout, we fix the following.

2.2 Notation. For any division ring $\mathbb{K}$, we let $\hat{\mathbb{K}} := \mathbb{K} \cup \{\infty\} := (\mathbb{K}^2 - \{(0,0)\})/(\mathbb{K} - \{0\})$, the projective line over $\mathbb{K}$. Each one-dimensional subspace $(x,y)\mathbb{K}$ of $\mathbb{K}^2$ corresponds to its *inverse-slope* $\frac{x}{y} \in \hat{\mathbb{K}}$. We let $\mathrm{PGL}_2(\mathbb{K}) := \mathrm{GL}_2(K)/(\mathrm{center}(\mathbb{K}) - \{0\})$, the projective linear group over $\mathbb{K}$. We let $\mathrm{PGL}_2(\mathbb{K})$ have the left Möbius action induced on $\hat{\mathbb{K}}$ by matrix multiplication, where the elements of $\mathbb{K}^2$ are thought of as 2×1 matrices that are expressed as row vectors to save space.

Let $\mathbb{H} := \{x + y\mathbf{i} + z\mathbf{j} + t\mathbf{k} \mid x, y, z, t \in \mathbb{R}\}$, the quaternion division ring. For each $w = x + y\mathbf{i} + z\mathbf{j} + t\mathbf{k} \in \mathbb{H}$, we let $\overline{w} := x - y\mathbf{i} - z\mathbf{j} - t\mathbf{k}$; then $w\overline{w} = x^2 + y^2 + z^2 + t^2 \in [0, \infty[$, and we define $|w| := \sqrt{w\overline{w}} \in [0, \infty[$.

We view $\hat{\mathbb{R}} \subseteq \hat{\mathbb{C}} \subseteq \hat{\mathbb{H}}$ and $\mathrm{PSL}_2(\mathbb{R}) \leqslant \mathrm{PSL}_2(\mathbb{C}) \leqslant \mathrm{PGL}_2(\mathbb{H})$. Here $\hat{\mathbb{R}}$ is a circle, $\hat{\mathbb{C}}$ is the Riemann two-sphere, and $\hat{\mathbb{H}}$ is a four-sphere.

An element of $\mathrm{PSL}_2(\mathbb{C})$ is said to be *parabolic* if it has a unique fixed point in $\hat{\mathbb{C}}$.

Let $\hat{\mathbb{C}}^+ := \{z \in \mathbb{C} \mid \mathrm{Im}(z) > 0\}$ and $\hat{\mathbb{C}}^- := \{z \in \mathbb{C} \mid \mathrm{Im}(z) < 0\}$.

Except in some figures, we understand that *hyperbolic three-space* is

$$\mathbf{H}^3 := \{x + y\mathbf{i} + z\mathbf{j} \in \mathbb{H} \mid z > 0\}$$

with the metric given by

$$\text{(2.1)} \qquad \mathrm{dist}(x_1 + y_1\mathbf{i} + z_1\mathbf{j},\ x_2 + y_2\mathbf{i} + z_2\mathbf{j}) = \operatorname{arccosh}\left(\tfrac{(x_1-x_2)^2+(y_1-y_2)^2+z_1^2+z_2^2}{2z_1z_2}\right);$$

see, for example, [34, Theorem 4.6.1]. The boundary of $\mathbf{H}^3$ in $\hat{\mathbb{H}}$ is $\hat{\mathbb{C}}$. For $w = x + y\mathbf{i} + z\mathbf{j}$ and any complex numbers a, b, c, d, we have

$$(aw + b)(\overline{w}\,\overline{c} + \overline{d}) = a\,|w|^2\,\overline{c} + aw\overline{d} + b\overline{w}\,\overline{c} + b\overline{d} \ \in\ \mathbb{C} + az\mathbf{j}\overline{d} - bz\mathbf{j}\overline{c} = \mathbb{C} + (ad - bc)z\mathbf{j};$$

this can be used to show that the action of $\mathrm{PGL}_2(\mathbb{H})$ on $\hat{\mathbb{H}}$ induces an action of $\mathrm{PSL}_2(\mathbb{C})$ on $\mathbf{H}^3$. Moreover, $\mathrm{PSL}_2(\mathbb{C})$ is then the orientation-preserving-isometry group of $\mathbf{H}^3$; see, for example, [34, Corollary 4.6.2 and Exercise 4.3.5].

Except in some figures, we understand that *the hyperbolic plane* is

$$\mathbf{H}^2 := \{x + z\mathbf{j} \in \mathbb{H} \mid z > 0\}.$$

The boundary of $\mathbf{H}^2$ in $\hat{\mathbb{H}}$ is $\hat{\mathbb{R}}$, and $\mathrm{PSL}_2(\mathbb{R})$ acts on $\mathbf{H}^2$ giving the orientation-preserving-isometry group. □

Throughout, we fix the following.

2.3 Notation. Let $\langle\, A, B, C\,\rangle := \langle\, A, B, C, D \mid A^2 = B^2 = C^2 = ABCD = 1\,\rangle$.

Let $\mathrm{Aut}\langle\, A, B, C\,\rangle$ denote the group of automorphisms of $\langle\, A, B, C\,\rangle$ acting on the left as exponents; thus, for $G \in \mathrm{Aut}\langle\, A, B, C\,\rangle$, we write $G\colon \langle\, A, B, C\,\rangle \to \langle\, A, B, C\,\rangle$, $W \mapsto {}^{G}W$. We shall use the triple $({}^{G}A, {}^{G}B, {}^{G}C)$ to denote G. We view $\langle\, A, B, C\,\rangle$ as a subgroup of $\mathrm{Aut}\langle\, A, B, C\,\rangle$ with ${}^{U}W := UWU^{-1}$ for all U, W in $\langle\, A, B, C\,\rangle$.

In $\mathrm{Aut}\langle\, A, B, C\,\rangle$, let $R := (A, BCB, B)$ and $L := (B, BAB, C)$; these are 'braid automorphisms' sometimes denoted σ_2^{-1} and σ_1, respectively.

Fix $p \in [1{\uparrow}\infty[$ and fix sequences $a_{[\![1\uparrow p]\!]}$ and $b_{[\![1\uparrow p]\!]}$ in $[1{\uparrow}\infty[$.

Let $F := \prod\limits_{i\in[\![1\uparrow p]\!]} (R^{a_i} L^{b_i}) \in \mathrm{Aut}\langle\, A, B, C\,\rangle$, and $\left(\begin{smallmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{smallmatrix}\right) := \prod\limits_{i\in[\![1\uparrow p]\!]} \left(\left(\begin{smallmatrix} 1 & a_i \\ 0 & 1 \end{smallmatrix}\right)\left(\begin{smallmatrix} 1 & 0 \\ b_i & 1 \end{smallmatrix}\right)\right) \in \mathrm{SL}_2(\mathbb{Z})$.

Here the subgroup $\langle\, A, B, C, F\,\rangle$ of $\mathrm{Aut}\langle\, A, B, C\,\rangle$ can be expressed as a semidirect product, $\langle F \mid \quad \rangle \ltimes \langle\, A, B, C\,\rangle$, and the subgroup $\langle\, D, F\,\rangle$ is free abelian of rank two.

Let $|F| := \sum\limits_{i\in[1\uparrow p]} (a_i + b_i)$, the length of F as a word in the alphabet $\{R, L\}$. Let $\mathfrak{f}_{\mathbb{Z}}$ denote the unique $\mathbb{Z}$-indexed sequence in the alphabet $\{R, L\}$ which has the properties that

$\Pi\mathfrak{f}_{[\![1\uparrow|F|]\!]} = \prod_{i\in[\![1\uparrow p]\!]}(R^{a_i}L^{b_i}) = F$ and, for each $n \in \mathbb{Z}$, $\mathfrak{f}_{n+|F|} = \mathfrak{f}_n$, that is, $|F|$ is a period of $\mathfrak{f}_{\mathbb{Z}}$. Let $\mathfrak{F}_{\mathbb{Z}}$ denote the unique $\mathbb{Z}$-indexed sequence in $\mathrm{Aut}\langle A, B, C\rangle$ which has the properties that $\mathfrak{F}_0 = 1$ and, for each $n \in \mathbb{Z}$, $\mathfrak{F}_n = \mathfrak{F}_{n-1}\mathfrak{f}_n$. In summary, we are interested in the unique $\mathbb{Z}$-indexed sequence $\mathfrak{F}_{\mathbb{Z}}$ in $\mathrm{Aut}\langle A, B, C\rangle$ which has the properties that, for each $n \in \mathbb{Z}$, $\mathfrak{F}_{n-1}^{-1}\mathfrak{F}_n \in \{R, L\}$ and $\mathfrak{F}_{n|F|} = F^n$.

We now recall Gueritaud's description of Jørgensen's representation as in [20] and [18, Section 8]. There exists a unique representation $\rho_F \colon \langle A, B, C, F\rangle \to \mathrm{PSL}_2(\mathbb{C})$ which has the properties that, firstly, there exists $(x, y, z, s) \in \mathbb{C}^4$ (unique up to changing the sign of two of the terms of (x, y, z)) such that $x^2 + y^2 + z^2 = xyz$ and $y \neq 0$ and $\mathrm{Im}(s) > 0$, and

$$\rho_F(A) = \pm\left(\begin{smallmatrix} -z/y & (x-yz)/y^2 \\ x & z/y \end{smallmatrix}\right),\ \rho_F(B) = \pm\left(\begin{smallmatrix} 0 & -1/y \\ y & 0 \end{smallmatrix}\right),\ \rho_F(C) = \pm\left(\begin{smallmatrix} x/y & (z-xy)/y^2 \\ z & -x/y \end{smallmatrix}\right),\ \rho_F(F) = \pm\left(\begin{smallmatrix} 1 & s \\ 0 & 1 \end{smallmatrix}\right),$$

and (hence) $\rho_F(D) = \pm\left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$, and, secondly, the function

$$\mathbb{Z}^2 \to \hat{\mathbb{C}}, \quad (m, n) \mapsto p_{m,n} \coloneqq \rho_F({}^{D^m\mathfrak{F}_n}B)(\infty),$$

is injective and has as its image a subset of $\mathbb{C}$ which forms the vertex set of a $\rho_F(\langle D, F\rangle)$-invariant triangulation $\Delta(F)$ of $\mathbb{C}$ made up of polygonal regions as indicated in Figure 2.1. Thus each polygon is a $(2a + 4)$-gon indexed by a quadruple $(m, a, b, c) \in \mathbb{Z}^4$ where a and b are positive and $\mathfrak{F}_{a+b+c+1} = \mathfrak{F}_{c-1}LR^aL^bR$, that is, (a, b, c) indexes a 'syllable' R^aL^b in $\Pi\mathfrak{f}_{\mathbb{Z}} = \prod_{\mathbb{Z}} F$. See Figure 1.1(3) for $\Delta(RL^3)$, and see Section 8 below for Helling's description of ρ_{RL^n}, $n \in [1\uparrow\infty[$.

We write $\hat{\mathbb{C}}_{\rho_F}$ to denote $\hat{\mathbb{C}}$ made into an $\langle A, B, C, F\rangle$-space via ρ_F. $\square$

We would like to draw attention to a rule of thumb that has been useful in our calculations.

2.4 An Open Problem. Let us use the lifts of ρ_F to $\mathrm{SL}_2(\mathbb{C})$. Here we write

$$\langle \hat{A}, \hat{B}, \hat{C}\rangle \coloneqq \langle \hat{A}, \hat{B}, \hat{C}, -1 \mid \hat{A}^2 = \hat{B}^2 = \hat{C}^2 = -1, (-1)^2 = 1\rangle,$$

${}^{\hat{R}}(\hat{A}, \hat{B}, \hat{C}) \coloneqq (\hat{A}, -\hat{B}\hat{C}\hat{B}, \hat{B})$, ${}^{\hat{L}}(\hat{A}, \hat{B}, \hat{C}) \coloneqq (\hat{B}, -\hat{B}\hat{A}\hat{B}, \hat{C})$, $\hat{F} \coloneqq \prod_{i\in[\![1\uparrow p]\!]}(\hat{R}^{a_i}\hat{L}^{b_i})$. Let $(\hat{A}', \hat{B}', \hat{C}')$ be the normal-form expression of ${}^{\hat{F}}(\hat{A}, \hat{B}, \hat{C})$.

Let $(x, y, z, s) \in \mathbb{C}^4$ with $y \neq 0$, and write

$$\hat{\rho}(\hat{A}) = \left(\begin{smallmatrix} -z/y & (x-yz)/y^2 \\ x & z/y \end{smallmatrix}\right),\ \hat{\rho}(\hat{B}) = \left(\begin{smallmatrix} 0 & -1/y \\ y & 0 \end{smallmatrix}\right),\ \hat{\rho}(\hat{C}) = \left(\begin{smallmatrix} x/y & (z-xy)/y^2 \\ z & -x/y \end{smallmatrix}\right),\ \hat{\rho}(\hat{F}) = \left(\begin{smallmatrix} 1 & s \\ 0 & 1 \end{smallmatrix}\right),$$

and interpret $\hat{\rho}(\hat{A}', \hat{B}', \hat{C}')$ in the natural way. It is an open problem to decide if, among the finite set of solutions of the algebraic system

$$x^2 + y^2 + z^2 = xyz,\ {}^{\hat{\rho}(\hat{F})}\hat{\rho}(A) = \hat{\rho}(\hat{A}'),\ {}^{\hat{\rho}(\hat{F})}\hat{\rho}(B) = \hat{\rho}(\hat{B}'),$$

the solutions (x, y, z, s) which *maximize* $\mathrm{Re}(y)$ necessarily give *discrete* representations of $\langle \hat{F} \mid \ \rangle \ltimes \langle \hat{A}, \hat{B}, \hat{C}\rangle$ in $\mathrm{SL}_2(\mathbb{C})$. Then taking $\mathrm{Im}(s) > 0$ would give (lifts of) ρ_F. $\square$

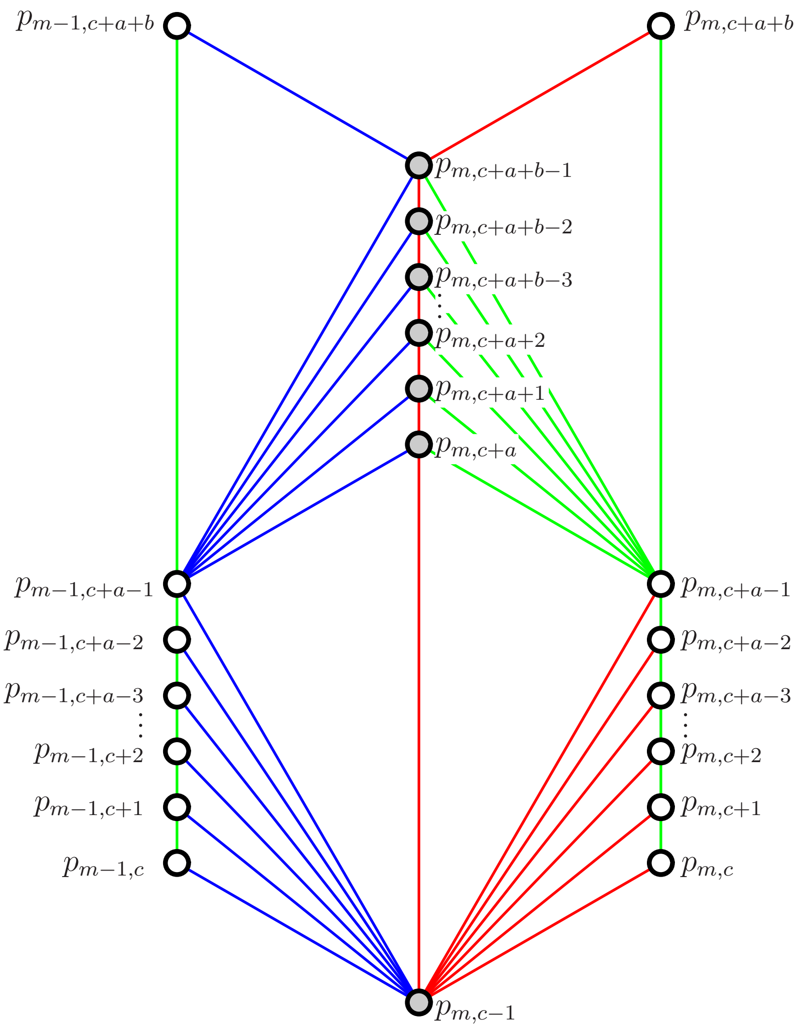


Figure 2.1: Schematic of Jørgensen's triangles for $\mathcal{F}_{c+a+b+1} = \mathcal{F}_{c-1}LR^aL^bR$

3 Summary of the automata

In this section, we present the main results; they will be proved in the two subsequent sections. We describe tree structures for $\partial^- A$, for $\partial^+ B$, and for the one-skeleton of $\mathrm{CW}(F)$. We show how the tree structures are suitable for programming. We discuss the example $F = RL^3$.

3.1 Definitions. We use some terminology that is not standard. A *path* is a continuous map whose domain is a connected subspace of $\hat{\mathbb{R}}$. A *curve* is a topological space which is the image of a path, sometimes understood to be provided with the information about the linear-or-cyclic order in which the points are visited by the path. By an *arc* we mean an oriented topological space that is homeomorphic to the interval $[0,1]$. We say that a subspace of $\hat{\mathbb{C}}$ is a *Jordan curve* if it is homeomorphic to $\hat{\mathbb{R}}$, not necessarily given an orientation.

For a given arc $\mathfrak{a}$, by a *fracturing* of $\mathfrak{a}$ we mean any finite sequence $\mathfrak{b}_{[\![1\uparrow n]\!]}$ of subarcs of $\mathfrak{a}$ such that $\mathfrak{a} = \bigcup\limits_{i\in[1\uparrow n]} \mathfrak{b}_i$ and, for each $i \in [2\uparrow n]$, the intersection of $\mathfrak{b}_{i-1}$ and $\mathfrak{b}_i$ consists of a single point which is the terminal point of $\mathfrak{b}_{i-1}$ and the initial point of $\mathfrak{b}_i$. By abuse of notation, we will then say that the expression $\mathfrak{a} = \bigcup\limits_{i\in[1\uparrow n]} \mathfrak{b}_i$ is a fracturing of $\mathfrak{a}$.

For $x \in \mathbb{R}$, $[x]$ denotes the greatest integer in the interval $]-\infty, x]$. □

3.2 Profile of $\partial^- A$. We use Notation 2.3.

Let $\mathfrak{a}_1 := \partial^- A$, $\mathfrak{a}_2 := \mathrm{reverse}(\partial^- C)$, $X_1 := BC$, $X_2 := BA$, $f_1 := f_{1\,1} + f_{2\,1}$, $f_2 := f_{1\,2} + f_{2\,2}$, and $\mu_+ := \frac{f_{1\,1} - f_{2\,2} + \sqrt{(f_{1\,1}+f_{2\,2})^2 - 4}}{2 f_{2\,1}}$. For each $\ell \in [1\uparrow f_1]$, let $\mathbf{s}_\ell := [\frac{\ell+1}{1+\mu_+}] - [\frac{\ell}{1+\mu_+}] + 1$. In Notation 5.1 and Lemma 5.3, we shall see that $\mathbf{s}_\ell \in \{1,2\}$. Let $U_\ell := F^{-1}\Pi X_{\mathbf{s}_{[\![1\uparrow(\ell-1)]\!]}}$.

We call $U_{[\![1\uparrow f_1]\!]}$ the ∂^--*syllable sequence*. The subsemigroup of $\langle\, A, B, C, F\,\rangle$ generated by $U_{[1\uparrow f_1]}$ is called the ∂^--*semigroup*. In Theorem 5.4, we shall see that the limit set in $\hat{\mathbb{C}}_{\rho_F}$ of the ∂^--semigroup is $\mathfrak{a}_1$, and that, for each $i \in \{1,2\}$, $(\rho_F(U_\ell)\mathfrak{a}_{\mathbf{s}_\ell} \mid \ell \in [\![1\uparrow f_i]\!])$, called the sequence of *columns* of $\mathfrak{a}_i$, is a fracturing of $\mathfrak{a}_i$. Informally, we have fracturings

$$\mathfrak{a}_1 = \bigcup_{\ell\in[1\uparrow f_1]} U_\ell \mathfrak{a}_{\mathbf{s}_\ell} \quad \text{and} \quad \mathfrak{a}_2 = \bigcup_{\ell\in[1\uparrow f_2]} U_\ell \mathfrak{a}_{\mathbf{s}_\ell}. \tag{3.1}$$

Starting from the symbol $\mathfrak{a}_1$, recursive substitution using (3.1) generates an ordered set of infinite words in $U_{[1\uparrow f_1]}$. Also (3.1) gives an automaton that accepts precisely this set of infinite words. We shall discuss this in Definitions 3.6.

In Notation 5.1 and Lemma 5.3, we shall see that $f_2 < f_1$ and ${}^F X_1 = \Pi X_{\mathbf{s}_{[\![1\uparrow f_1]\!]}}$ and ${}^F X_2 = \Pi X_{\mathbf{s}_{[\![1\uparrow f_2]\!]}}$. This gives an alternative method of calculating f_1 and $\mathbf{s}_{[\![1\uparrow f_1]\!]}$ and f_2. □

3.3 Profile of $\partial^+ B$. We use Notation 2.3.

Let $\mathfrak{a}_{-1} := \mathrm{reverse}(\partial^+ B)$, $\mathfrak{a}_{-2} := \mathrm{reverse}(\partial^+ BC)$, $G := AF^{-1}A$, $X_{-1} := CA$, $X_{-2} := BA$, $f_{-1} := f_{1\,1} + f_{2\,1}$, $f_{-2} := f_{1\,1}$, and $\mu_- := \frac{f_{1\,1} - f_{2\,2} - \sqrt{(f_{1\,1}+f_{2\,2})^2 - 4}}{2 f_{2\,1}}$. Let $\mathbf{t}_{f_{-1}} := -1$, and, for each

$\ell \in [1{\uparrow}(f_{-1}-1)]$, let $\mathbf{t}_\ell \coloneqq [(\ell)(-\mu_{-})] - [(\ell-1)(-\mu_{-})] - 2$. In Notation 5.1 and Lemma 5.5, we shall see that $\mathbf{t}_\ell \in \{-2,-1\}$. For each $\ell \in [1{\uparrow}f_{-1}]$, let $V_\ell \coloneqq G^{-1}\Pi X_{\mathbf{t}_{[\![1\uparrow(\ell-1)]\!]}}$.

We call $V_{[\![1\uparrow f_{-1}]\!]}$ the ∂^{+}-*syllable sequence*. The subsemigroup of $\langle\, A,B,C,F\,\rangle$ generated by $V_{[1\uparrow f_{-1}]}$ is called the ∂^{+}-*semigroup*. In Theorem 5.6, we shall see that the limit set in $\hat{\mathbb{C}}_{\rho_F}$ of the ∂^{+}-semigroup is $\mathfrak{a}_{-1}$, and that, for each $i \in \{-1,-2\}$, $(\rho_F(V_\ell)\mathfrak{a}_{\mathbf{s}_\ell} \mid \ell \in [\![1{\uparrow}f_i]\!])$, called the sequence of *rows* of $\mathfrak{a}_i$, is a fracturing of $\mathfrak{a}_i$. Informally, we have fracturings

$$\mathfrak{a}_{-1} = \bigcup_{\ell\in[1\uparrow f_{-1}]} V_\ell \mathfrak{a}_{\mathbf{s}_\ell} \quad\text{and}\quad \mathfrak{a}_{-2} = \bigcup_{\ell\in[1\uparrow f_{-2}]} V_\ell \mathfrak{a}_{\mathbf{s}_\ell}.$$

Again, these fracturings give an automaton and, starting from the symbol $\mathfrak{a}_{-1}$, recursive substitution generates an ordered set of infinite words in $V_{[1\uparrow f_{-1}]}$. We shall discuss this in Definitions 3.6.

In Notation 5.1 and Lemma 5.5, we shall see that $f_{-2} < f_{-1}$ and ${}^{G}X_{-1} = \Pi X_{\mathbf{t}_{[\![1\uparrow f_{-1}]\!]}}$ and ${}^{G}X_{-2} = \Pi X_{\mathbf{t}_{[\![1\uparrow f_{-2}]\!]}}$; this gives an alternative way of calculating f_{-1}, $\mathbf{t}_{[\![1\uparrow f_{-1}]\!]}$ and f_{-2}. □

3.4 Profile of the syllables. We use the notation of Profiles 3.2 and 3.3.

Clearly $f_{-1} = f_1$. In Notation 5.1, we shall see that $f_2 f_{-2} \equiv 1 (\mathrm{mod}\ f_1)$.

In Proposition 5.7, we shall see that, for each $\ell \in [1{\uparrow}f_1]$, $V_\ell = U^{-1}_{(1-\ell)f_2\,(\mathrm{mod}\ f_1)}$ and, hence, $U_\ell = V^{-1}_{1-(\ell f_{-2})\,(\mathrm{mod}\ f_1)}$.

In particular, $V_{[1\uparrow f_1]} = (U_{[1\uparrow f_1]})^{-1}$. In [14, Section 6.1], it was seen that the limit set in $\hat{\mathbb{C}}_{\rho_F}$ of the semigroup generated by $(U_{[1\uparrow f_1]})^{-1}$ is $\partial^{+}B$. In [14, Section 6.4], it was seen that the limit set in $\hat{\mathbb{C}}_{\rho_F}$ of the semigroup generated by $(V_{[1\uparrow f_1]})^{-1}$ is $\partial^{-}A$.

In Proposition 5.7, we shall see that $U_1 = V^{-1}_{f_2 1+1}$, $U_{f_2} = V^{-1}_{f_1}$ and $U_{f_1} = V^{-1}_1$ are the parabolic ∂^{-}-syllables, where we say an element of $\langle\, A,B,C,F\,\rangle$ is *parabolic* if it has a unique fixed point in $\hat{\mathbb{C}}_{\rho_F}$. In Corollary 6.8, we shall see that the parabolic elements of the ∂^{-}-semigroup are the positive powers of the three parabolic ∂^{-}-syllables. □

3.5 Example. Let $F = RL^3 = ((BCBA)^2BCB,\ (BCBA)^3BCB,\ B) \in \mathrm{Aut}\langle\, A,B,C\,\rangle$.

(i). We calculate

$${}^{F}(BC,BA) = ((BCBA)^3BC, BCBA).$$

Here, $f_1 = 7$, $f_2 = 2$, $\mathbf{s}_{[\![1\uparrow7]\!]} = (1,2,1,2,1,2,1)$. Now $\mathfrak{a}_1$ $(= \partial^{-}A)$ and $\mathfrak{a}_2$ $(= \mathrm{reverse}(\partial^{-}C))$ have fracturings

$$\begin{aligned}\mathfrak{a}_1 &= U_1\mathfrak{a}_1 \cup U_2\mathfrak{a}_2 \cup U_3\mathfrak{a}_1 \cup U_4\mathfrak{a}_2 \cup U_5\mathfrak{a}_1 \cup U_6\mathfrak{a}_2 \cup U_7\mathfrak{a}_1,\\ \mathfrak{a}_2 &= U_1\mathfrak{a}_1 \cup U_2\mathfrak{a}_2,\end{aligned}$$

where the action is via ρ_F. We give explicit expressions for the elements of $U_{[1\uparrow7]}$ in the following table which can be reconstructed from the sequence $(1,2,1,2,1,2,1)$.

$U_1\mathfrak{a}_1 = F^{-1}\mathfrak{a}_1$	$= F^{-1}\mathfrak{a}_1$	$\subseteq \mathfrak{a}_2$	$\subseteq \mathfrak{a}_1$
$U_2\mathfrak{a}_2 = F^{-1}X_1\mathfrak{a}_2$	$= F^{-1}BC\mathfrak{a}_2$	$\subseteq \mathfrak{a}_2$	$\subseteq \mathfrak{a}_1$
$U_3\mathfrak{a}_1 = F^{-1}X_1X_2\mathfrak{a}_1$	$= F^{-1}BCBA\mathfrak{a}_1$		$\subseteq \mathfrak{a}_1$
$U_4\mathfrak{a}_2 = F^{-1}X_1X_2X_1\mathfrak{a}_2$	$= F^{-1}BCBABC\mathfrak{a}_2$		$\subseteq \mathfrak{a}_1$
$U_5\mathfrak{a}_1 = F^{-1}X_1X_2X_1X_2\mathfrak{a}_1$	$= F^{-1}(BCBA)^2\mathfrak{a}_1$		$\subseteq \mathfrak{a}_1$
$U_6\mathfrak{a}_2 = F^{-1}X_1X_2X_1X_2X_1\mathfrak{a}_2$	$= F^{-1}(BCBA)^2BC\mathfrak{a}_2$		$\subseteq \mathfrak{a}_1$
$U_7\mathfrak{a}_1 = F^{-1}X_1X_2X_1X_2X_1X_2\mathfrak{a}_1$	$= F^{-1}(BCBA)^3\mathfrak{a}_1$		$\subseteq \mathfrak{a}_1$

(ii). We calculate $F^{-1} = ((AB)^3A, C, C(AB)^2AC) \in \mathrm{Aut}\langle\, A, B, C\,\rangle$ and, hence,

$$^{F^{-1}}(AC, AB) = ((AB)^3AC(AB)^2AC, (AB)^3AC),$$

and, hence, for $G = AF^{-1}A$,

$$^{G}(CA, BA) = ((BA)^3CA(BA)^2CA, (BA)^3CA).$$

Here, $f_{-1} = 7$, $f_{-2} = 4$, $\mathfrak{t}_{[\![1\uparrow 7]\!]} = (-2,-2,-2,-1,-2,-2,-1)$. Now $\mathfrak{a}_{-1}$ ($=$ reverse($\partial^{+}B$)) and $\mathfrak{a}_{-2}$ ($=$ reverse($\partial^{+}BC$)) have fracturings

$$\begin{aligned} \mathfrak{a}_{-1} &= V_1\mathfrak{a}_{-2} \cup V_2\mathfrak{a}_{-2} \cup V_3\mathfrak{a}_{-2} \cup V_4\mathfrak{a}_{-1} \cup V_5\mathfrak{a}_{-2} \cup V_6\mathfrak{a}_{-2} \cup V_7\mathfrak{a}_{-1}, \\ \mathfrak{a}_{-2} &= V_1\mathfrak{a}_{-2} \cup V_2\mathfrak{a}_{-2} \cup V_3\mathfrak{a}_{-2} \cup V_4\mathfrak{a}_{-1}, \end{aligned}$$

where the action is via ρ_F. We give explicit expressions for the elements of $V_{[1\uparrow 7]}$ in the following table which can be reconstructed from the sequence $(-2,-2,-2,-1,-2,-2,-1)$.

$V_1\mathfrak{a}_{-2} = G^{-1}\mathfrak{a}_{-2}$	$= AFA\mathfrak{a}_{-2}$	$\subseteq \mathfrak{a}_{-2}$	$\subseteq \mathfrak{a}_{-1}$
$V_2\mathfrak{a}_{-2} = G^{-1}X_{-2}\mathfrak{a}_{-2}$	$= AFABA\mathfrak{a}_{-2}$	$\subseteq \mathfrak{a}_{-2}$	$\subseteq \mathfrak{a}_{-1}$
$V_3\mathfrak{a}_{-2} = G^{-1}X_{-2}X_{-2}\mathfrak{a}_{-2}$	$= AFA(BA)^2\mathfrak{a}_{-1}$	$\subseteq \mathfrak{a}_{-2}$	$\subseteq \mathfrak{a}_{-1}$
$V_4\mathfrak{a}_{-1} = G^{-1}X_{-2}X_{-2}X_{-2}\mathfrak{a}_{-1}$	$= AFA(BA)^3\mathfrak{a}_{-2}$	$\subseteq \mathfrak{a}_{-2}$	$\subseteq \mathfrak{a}_{-1}$
$V_5\mathfrak{a}_{-2} = G^{-1}X_{-2}X_{-2}X_{-2}X_{-1}\mathfrak{a}_{-2}$	$= AFA(BA)^3BC\mathfrak{a}_{-2}$		$\subseteq \mathfrak{a}_{-1}$
$V_6\mathfrak{a}_{-2} = G^{-1}X_{-2}X_{-2}X_{-2}X_{-1}X_{-2}\mathfrak{a}_{-2}$	$= AFA(BA)^3BCBA\mathfrak{a}_{-2}$		$\subseteq \mathfrak{a}_{-1}$
$V_7\mathfrak{a}_{-1} = G^{-1}X_{-2}X_{-2}X_{-2}X_{-1}X_{-2}X_{-2}\mathfrak{a}_{-1}$	$= AFA(BA)^3BC(BA)^2\mathfrak{a}_{-1}$		$\subseteq \mathfrak{a}_{-1}$

(iii). We have

$$\begin{array}{lcr}
V_1 = AFA & = & (ABCB)^3F = U_7^{-1} \\
V_2 = AFABA & = & (ABCB)^2F = U_5^{-1} \\
V_3 = AFA(BA)^2 & = & ABCBF = U_3^{-1} \\
V_4 = AFA(BA)^3 & = & F = U_1^{-1} \\
V_5 = AFA(BA)^3CA & = & CB(ABCB)^2F = U_6^{-1} \\
V_6 = AFA(BA)^3CABA & = & CBABCBF = U_4^{-1} \\
V_7 = AFA(BA)^3CA(BA)^2 & = & CBF = U_2^{-1}.
\end{array}$$

If we interpret the index set as $\mathbb{Z}/7\mathbb{Z}$, then $f_2 f_{-2} = (2)(4) = 1$ and for each $\ell \in \mathbb{Z}/7\mathbb{Z}$, $V_\ell = U^{-1}_{(1-\ell)2}$, $U_\ell = V^{-1}_{1-(4\ell)}$. $\square$

3.6 Definitions. Let $\mathbf{z}_{[\![1\uparrow\infty[\![}$ be a sequence in $[1\uparrow f_1]$.

Then $(\rho_F(\Pi U_{\mathbf{z}_{[\![1\uparrow\ell]\!]}})\mathfrak{a}_1 \mid \ell \in [\![0\uparrow\infty[\![\ )$ is a strictly decreasing sequence of subarcs of $\mathfrak{a}_1$. It was seen in [14] that there exists a homeomorphism from $\mathfrak{a}_1$ to a closed interval in $\mathbb{R}$ such that on the closed interval each ∂^--syllable acts affinely with $\frac{1}{\lambda}$ as the scaling factor. Since $\frac{1}{\lambda} \in \,]0,1[$, it follows that $\bigcap\limits_{\ell\in[0\uparrow\infty[} (\rho_F(\Pi U_{\mathbf{z}_{[\![1\uparrow\ell]\!]}})\mathfrak{a}_1)$ consists of a single element of $\mathfrak{a}_1$. We then have a map to $\mathfrak{a}_1$ from the set of infinite sequences in $[1\uparrow f_1]$.

Let us say that the sequence $\mathbf{z}_{[\![1\uparrow\infty[\![}$ is $\mathfrak{a}_1$-*acceptable* if, for each $\ell \in [1\uparrow\infty[$ such that $\mathbf{s}_{\mathbf{z}_\ell} = 2$, we have $\mathbf{z}_{\ell+1} \in [1\uparrow f_2]$. In this event, by (3.1), $(\rho_F(\Pi U_{\mathbf{z}_{[\![1\uparrow\ell]\!]}})\mathfrak{a}_{\mathbf{s}_{\mathbf{z}_\ell}} \mid \ell \in [\![1\uparrow\infty[\![\ )$ is decreasing, and, hence, $\bigcap\limits_{\ell\in[1\uparrow\infty[} (\rho_F(\Pi U_{\mathbf{z}_{[\![1\uparrow\ell]\!]}})\mathfrak{a}_{\mathbf{s}_{\mathbf{z}_\ell}})$ is non-empty and, hence, it must equal the one-point superset $\bigcap\limits_{\ell\in[0\uparrow\infty[} (\rho_F(\Pi U_{\mathbf{z}_{[\![1\uparrow\ell]\!]}})\mathfrak{a}_1)$. Since (3.1) gives fracturings, the map to $\mathfrak{a}_1$ from the set of $\mathfrak{a}_1$-acceptable infinite sequences in $[1\uparrow f_1]$ is surjective, and one-to-one on the non-break-points, and two-to-one on the break-points. The lexicographic order on the set of $\mathfrak{a}_1$-acceptable infinite sequences in $[1\uparrow f_1]$ agrees with the arc order on $\mathfrak{a}_1$. This type of codification of points of an arc by sequences has as an ancestor the codification by decimal expansion of the points in the interval $[0,1]$.

In a natural way, the set of $\mathfrak{a}_1$-acceptable infinite sequences in $[1\uparrow f_1]$ is the set of ends of a tree generated by a finite-state automaton constructed from (3.1). From state 1 we can go to state 1 after reading any one of $f_{1\,1}$ different syllables, and we can go to state 2 after reading any one of $f_{2\,1}$ different syllables. From state 2 we can go to state 1 after reading any one of $f_{1\,2}$ different syllables, and we can go to state 2 after reading any one of $f_{2\,2}$ different syllables; see [1].

One advantage of the tree structure is that it permits depth-first searches that are useful for precise drawing. Let us review this well-known method, following [31, pp. 141–150]. We use a suitable Möbius transformation to move $\mathfrak{a}_1$ into $\hat{\mathbb{C}} - \{\infty\}$ and then use the $\mathbb{R}^2$-metric on $\mathbb{C}$. We choose an $\varepsilon \in \,]0,1[$. For $(M,i) \in \mathrm{PSL}_2(\mathbb{C}) \times \{1,2\}$, we define the *weight of* (M,i), denoted weight(M,i), to be the distance between the two end-points of $M\mathfrak{a}_i$, and we define the *sequence of descendants of* (M, i) to be $((M\rho(U_\ell), \mathbf{s}_\ell) \mid \ell \in [\![1{\uparrow}f_i]\!])$. We now construct certain finite sequences $((M_m, i_m) \mid m \in [\![1{\uparrow}n]\!])$ in $\mathrm{PSL}_2(\mathbb{C}) \times \{1, 2\}$, $n \in [1{\uparrow}\infty[$. The starting sequence is $((\pm\mathbf{I}_2, 1))$. Suppose that we have recursively constructed some sequence $((M_m, i_m) \mid m \in [\![1{\uparrow}n]\!])$. If $\mathrm{weight}(M_m, i_m) < \varepsilon$ for all $m \in [1{\uparrow}n]$, then the recursion terminates. Otherwise, we let m be the smallest element of $[1{\uparrow}n]$ such that $\mathrm{weight}(M_m, i_m) \geqslant \varepsilon$, and we redefine our sequence by replacing the pair (M_m, i_m) with the terms of its sequence of descendants. Now we have a new sequence of pairs and we repeat the process. This procedure terminates with a well-defined sequence $((M_m, i_m) \mid m \in [\![1{\uparrow}n]\!])$ and all of its terms then have weight less than ε. By (3.1), $(M_m\mathfrak{a}_{i_m} \mid m \in [\![1{\uparrow}n]\!])$ is a fracturing of $\mathfrak{a}_1$. Approximating each $M_m\mathfrak{a}_{i_m}$ with the straight line segment joining its end-points gives a piecewise linear approximation of $\mathfrak{a}_1$ which is quite reasonable if ε is small enough. This procedure is more efficient than the techniques previously available for drawing these arcs; compare Figure 1.1(1) with [14, Figure 5] and [31, Figure 10.14].

For the case $F = RL$, [7, Lemma 8.1] gave a more efficient, simpler automaton that involved orientation-reversing maps; see [7, Fig. 13]. □

3.7 Definitions. Let us now construct $\mathrm{CW}(F)$ using the proof of [15, Lemma 7.10(ii)]. Recall that the one-skeleton of $\mathrm{CW}(F)$ is constructed by deleting ∞ from the union of a certain $\mathbb{Z}$-indexed sequence $\partial_{\mathbb{Z}}$ of $\rho_F(F)$-invariant Jordan curves in $\hat{\mathbb{C}}$ which contain ∞.

Here $\bigcup\limits_{\ell \in [1{\uparrow}f_1]} \rho_F(U_\ell)\mathfrak{a}_{\mathbf{s}_\ell} = \mathfrak{a}_1 = \partial^- A$ is a subarc of the Jordan curve ∂_0 in $\hat{\mathbb{C}}$, and $\mathfrak{a}_1 = \partial^- A$ joins ∞ to $\rho_F(A)(\infty)$. Observe that $\rho_F(U_1)\mathfrak{a}_{\mathbf{s}_1} = \rho_F(F^{-1})\mathfrak{a}_1$ is the subarc of $\mathfrak{a}_1$ joining $\rho_F(F^{-1})(\infty) = \infty$ to $\rho_F(F^{-1}A)(\infty)$. Let $\mathfrak{a}'_1 := \bigcup\limits_{\ell \in [2{\uparrow}f_1]} \rho_F(U_\ell)\mathfrak{a}_{\mathbf{s}_\ell}$. Then $\mathfrak{a}'_1$ is the subarc of $\mathfrak{a}_1$ joining $\rho_F(F^{-1}A)(\infty)$ to $\rho_F(A)(\infty)$. Hence, $\mathfrak{a}'_1$ is a fundamental $\rho_F(\langle F \rangle)$-domain in $\partial_0 - \{\infty\}$, and, therefore, $\mathfrak{a}'_1$ is a fundamental $\rho_F(\langle D, F \rangle)$-domain in the even part of the one-skeleton of $\mathrm{CW}(F)$. Hence, $\rho_F(F)\mathfrak{a}'_1 = \bigcup\limits_{\ell \in [2{\uparrow}f_1]} \rho_F(FU_\ell)\mathfrak{a}_{\mathbf{s}_\ell}$ is a subarc of $\rho_F(F)\partial_0 = \partial_0$ joining $\rho_F(FF^{-1}A)(\infty) = \rho_F(A)(\infty)$ to $\rho_F(FA)(\infty)$, and it too is a fundamental $\rho_F(\langle D, F \rangle)$-domain in the even part of the one-skeleton of $\mathrm{CW}(F)$.

Recall that $\bigcup\limits_{\ell \in [1{\uparrow}f_1]} \rho_F(V_\ell)\mathfrak{a}_{\mathbf{t}_\ell} = \mathfrak{a}_{-1} = \mathrm{reverse}(\partial^+ B)$ is a subarc of the union of the Jordan curves ∂_{-1} and $\mathrm{reverse}(\partial_1)$ in $\hat{\mathbb{C}}$, and $\mathfrak{a}_{-1} = \mathrm{reverse}(\partial^+ B)$ joins $\rho_F(A)(\infty)$ to $\rho_F(C)(\infty)$. Now $\rho_F(V_{f_{2\,1}+1})\mathfrak{a}_{\mathbf{t}_{f_{2\,1}+1}} = \rho_F(F)\mathfrak{a}_{-1}$ is the subarc of $\mathfrak{a}_{-1}$ joining $\rho_F(FA)(\infty)$ to $\rho_F(FC)(\infty)$. Let $\mathfrak{a}'_{-1} = \bigcup\limits_{\ell \in [1{\uparrow}f_{2\,1}]} V_\ell\mathfrak{a}_{\mathbf{t}_\ell}$. Then $\mathfrak{a}'_{-1}$ is the subarc of $\mathfrak{a}_{-1}$ joining $\rho_F(A)(\infty)$ to $\rho_F(FA)(\infty)$. Hence $\mathfrak{a}'_{-1}$ is a fundamental $\rho_F(\langle F \rangle)$-domain in $\partial_{-1} - \{\infty\}$, and, therefore, $\mathfrak{a}'_{-1}$ is a fundamental $\rho_F(\langle D, F \rangle)$-domain in the odd part of the one-skeleton of $\mathrm{CW}(F)$.

Together, $\mathrm{reverse}(\mathfrak{a}'_{-1}) \cup \rho_F(F)\mathfrak{a}'_1$ is a pinched Jordan curve that is a fundamental $\rho_F(\langle D, F \rangle)$-domain in the one-skeleton of $\mathrm{CW}(F)$. The finite chain of Jordan disks which has $\mathrm{reverse}(\mathfrak{a}'_{-1}) \cup \rho_F(F)\mathfrak{a}'_1$ as oriented boundary is a fundamental $\rho_F(\langle D, F \rangle)$-domain in the gray L-region of $\mathrm{CW}(F)$.

By using the automata for $(\mathfrak{a}_{-1}, \mathfrak{a}_{-2})$ and $(\mathfrak{a}_1, \mathfrak{a}_2)$, it is straightforward to create an automaton to generate an ordered list of points on the pinched Jordan curve

$$\text{reverse}\Big(\bigcup_{\ell\in[1\uparrow f_{2\,1}]} \rho_F(V_\ell)\mathfrak{a}_{\mathbf{t}_\ell}\Big) \cup \bigcup_{\ell\in[2\uparrow f_1]} \rho_F(FU_\ell)\mathfrak{a}_{\mathbf{s}_\ell} = \text{reverse}(\mathfrak{a}'_{-1}) \cup \rho_F(F)\mathfrak{a}'_1.$$

Since this space already lies in $\hat{\mathbb{C}} - \{\infty\}$, we can use the $\mathbb{R}^2$-metric and depth-first searches to generate a piecewise-linear approximation. □

4 The Cannon-Thurston model

In the foregoing sections we have reviewed Jørgensen's geometry and stated our results in programmable detail. For the convenience of the reader, we devote this section to a review of information that we shall be using about the Cannon-Thurston model of Jørgensen's geometry; in the next section we shall prove our results within the context of the model. We now build the model in stages, starting with work of Nielsen. We define the $\mathrm{Aut}\langle\, A, B, C\,\rangle$-action on the set of ends of $\langle\, A, B, C\,\rangle$, and then choose an $\langle\, A, B, C\,\rangle$-action on $\hat{\mathbb{C}}$ which leaves invariant $\hat{\mathbb{R}}$ and successively gives $\mathrm{Aut}\langle\, A, B, C\,\rangle$-actions on $\hat{\mathbb{R}}$, $\hat{\mathbb{Q}}$, $\mathbb{Z}^2$, $\mathbb{R}^2$, and, finally, $\hat{\mathbb{C}}$. We let $\hat{\mathbb{C}}_{\rho_0}$ denote $\hat{\mathbb{C}}$ endowed with Nielsen's $\mathrm{Aut}\langle\, A, B, C\,\rangle$-action. Work of Cannon, Thurston, Minsky, McMullen, Bowditch and others gives an $\langle\, A, B, C, F\,\rangle$-map $\mathrm{CT}\colon \hat{\mathbb{C}}_{\rho_0} \to \hat{\mathbb{C}}_{\rho_F}$ which provides a model of Jørgensen's geometry in which each point of $\hat{\mathbb{C}}_{\rho_F}$ is blown up to a point or a line or a 'spider' in $\hat{\mathbb{C}}_{\rho_0}$. This gives a manageable model of both $\partial^- A$ and $\partial^+ B$.

4.1 Review. An *end of* $\langle\, A, B, C\,\rangle$ is a sequence $E_{[\![1\uparrow\infty[\![}$ in the alphabet $\{A, B, C\}$ which has the property that, for each $i \in [1\uparrow\infty[\,$, $E_{i+1} \neq E_i$. We then think of $E_{[\![1\uparrow\infty[\![}$ as a right-infinite reduced product $\Pi E_{[\![1\uparrow\infty[\![}$, and, for each $n \in [0\uparrow\infty[\,$, we say that $\Pi E_{[\![1\uparrow\infty[\![}$ *begins with* $\Pi E_{[\![1\uparrow n]\!]} \in \langle\, A, B, C\,\rangle$. The set of ends of $\langle\, A, B, C\,\rangle$ is denoted $\mathfrak{E}\langle\, A, B, C\,\rangle$. For each $W \in \langle\, A, B, C\,\rangle$, we let $(W\blacktriangleleft)$ denote the set of elements of $\mathfrak{E}\langle\, A, B, C\,\rangle$ that begin with W; for example, $(CBA)^\infty \in (CBAC\blacktriangleleft)$. Then $\mathfrak{E}\langle\, A, B, C\,\rangle$ is a topological space in which the $(W\blacktriangleleft)$ form a basis of the open sets, and $\langle\, A, B, C\,\rangle \cup \mathfrak{E}\langle\, A, B, C\,\rangle$ is a compactification of the discrete space $\langle\, A, B, C\,\rangle$.

In a natural way, $\langle\, A, B, C\,\rangle \cup \mathfrak{E}\langle\, A, B, C\,\rangle$ is an $\mathrm{Aut}\langle\, A, B, C\,\rangle$-space, with left-exponent action. For $G \in \mathrm{Aut}\langle\, A, B, C\,\rangle$ and $E \in \mathfrak{E}\langle\, A, B, C\,\rangle$, it will sometimes be useful to have the notation $GE \coloneqq {}^G E$. Notice that in $\mathfrak{E}\langle\, A, B, C\,\rangle$, $(A\blacktriangleleft)$ and $(B\blacktriangleleft)$ and $(C\blacktriangleleft)$ satisfy ping-pong conditions analogous to (1.2) and (1.3); we consider the $\mathfrak{E}\langle\, A, B, C\,\rangle$-ping-pong triple to be the primordial example from which all other examples arise.

We next recall the affine $\mathrm{Aut}\langle\, A, B, C\,\rangle$-action on $\mathbb{Z}^2$ obtained by distinguishing the pair (CB, AB).

It is a classic result of Nielsen that every automorphism of $\langle\, A, B, C\,\rangle$ maps D to a conjugate of D or D^{-1}. It then follows that $\langle\, A, B, C\,\rangle D^\infty \cup \langle\, A, B, C\,\rangle D^{-\infty}$ is an

$\mathrm{Aut}\langle A,B,C\rangle$-subset of $\mathfrak{E}\langle A,B,C\rangle$. We shall be using the quotient space obtained from $\mathfrak{E}\langle A,B,C\rangle$ by identifying $W(D^{-\infty})$ and $W(D^{\infty})$ to form a single point denoted $W(D^{\pm\infty})$, for each $W \in \langle A,B,C\rangle$. This quotient space is a quotient $\mathrm{Aut}\langle A,B,C\rangle$-space and $\langle A,B,C\rangle D^{\pm\infty}$ is an $\mathrm{Aut}\langle A,B,C\rangle$-subset.

Let $\langle\!\langle D^2\rangle\!\rangle$ denote the smallest normal subgroup of $\langle A,B,C\rangle$ containing D^2; notice that $D^2 = (CBA)^2 = (CB)(AB)(CB)^{-1}(AB)^{-1}$ and that $\langle\!\langle D^2\rangle\!\rangle\backslash\langle A,B,C\rangle D^{\pm\infty}$ is a quotient $\mathrm{Aut}\langle A,B,C\rangle$-set. We have presentations

$$\begin{aligned}\langle A,B,C\rangle &= \langle CB, AB, D \mid {}^{D}(CB) = {}^{(CB)(AB)}(CB)^{-1}, {}^{D}(AB) = {}^{CB}(AB)^{-1},\\ &\qquad (CB)(AB)(CB)^{-1}(AB)^{-1} = D^2\rangle,\\ \langle\!\langle D^2\rangle\!\rangle\backslash\langle A,B,C\rangle &= \langle CB, AB, D \mid {}^{D}(CB) = (CB)^{-1}, {}^{D}(AB) = (AB)^{-1},\\ &\qquad (CB)(AB)(CB)^{-1}(AB)^{-1} = D^2 = 1\rangle\\ &= \langle D \mid D^2\rangle \ltimes \langle CB, AB \mid (CB)(AB) = (AB)(CB)\rangle.\end{aligned}$$

We then have a bijective map $\mathbb{Z}^2 \to \langle\!\langle D^2\rangle\!\rangle\backslash\langle A,B,C\rangle/\langle D\rangle$, $(x,y) \mapsto \langle\!\langle D^2\rangle\!\rangle(CB)^x(AB)^y\langle D\rangle$. We have another bijection $\langle A,B,C\rangle/\langle D\rangle \simeq \langle A,B,C\rangle D^{\pm\infty}$. Hence we have a bijective map $\mathbb{Z}^2 \to \langle\!\langle D^2\rangle\!\rangle\backslash\langle A,B,C\rangle D^{\pm\infty}$, and the latter is an $\mathrm{Aut}\langle A,B,C\rangle$-set. In particular, we have endowed $\mathbb{Z}^2$ with the structure of an $\mathrm{Aut}\langle A,B,C\rangle$-set. Here, for each $G \in \mathrm{Aut}\langle A,B,C\rangle$, there exist a, b, c, d, e, $f \in \mathbb{Z}$ such that $|ad-bc| = 1$ and, for all $(x,y)\in\mathbb{Z}^2$,

$$G(\langle\!\langle D^2\rangle\!\rangle(CB)^x(AB)^yD^{\pm\infty}) = \langle\!\langle D^2\rangle\!\rangle(CB)^{ax+by+e}(AB)^{cx+dy+f}D^{\pm\infty}.$$

In summary, the $\mathrm{Aut}\langle A,B,C\rangle$-set $\langle\!\langle D^2\rangle\!\rangle\backslash\langle A,B,C\rangle D^{\pm\infty}$, when identified with $\mathbb{Z}^2$ by using the pair (CB, AB), gives an explicit representation of $\mathrm{Aut}\langle A,B,C\rangle$ in the group of affine automorphisms of $\mathbb{Z}^2$, with kernel $\langle\!\langle D^2\rangle\!\rangle$. □

4.2 Review. In $\mathrm{Aut}\langle A,B,C\rangle$, let $F_0 \coloneqq L^{-1}R = {}^{C}(C,A,B)$, and notice that

$$\langle A,B,C,F_0\rangle = \langle CF_0 \mid (CF_0)^3\rangle \ltimes \langle A,B,C\rangle.$$

We have a classic representation $\rho_0\colon \langle A,B,C,F_0\rangle \xrightarrow{\sim} \mathrm{PSL}_2(\mathbb{Z}) \leqslant \mathrm{PSL}_2(\mathbb{C})$ with

$$\rho_0(A) = \pm\left(\begin{smallmatrix}0 & -1\\ 1 & 0\end{smallmatrix}\right), \rho_0(B) = \pm\left(\begin{smallmatrix}1 & -1\\ 2 & -1\end{smallmatrix}\right), \rho_0(C) = \pm\left(\begin{smallmatrix}1 & -2\\ 1 & -1\end{smallmatrix}\right), \rho_0(D) = \pm\left(\begin{smallmatrix}1 & 3\\ 0 & 1\end{smallmatrix}\right), \rho_0(F_0) = \pm\left(\begin{smallmatrix}1 & 1\\ 0 & 1\end{smallmatrix}\right);$$

see (1.1). The representation ρ_0 is similar to the representations described in Notation 2.3 and corresponds to taking $(x,y,z,s) = (3,6,3,\frac{1}{3})$ and then conjugating by $\pm\left(\begin{smallmatrix}6 & 1\\ 0 & 2\end{smallmatrix}\right)$.

The Möbius action of $\mathrm{PSL}_2(\mathbb{Z})$ on $\hat{\mathbb{C}}$ leaves invariant $\hat{\mathbb{Q}}$, $\hat{\mathbb{R}}$, $\hat{\mathbb{C}}^+$ and $\hat{\mathbb{C}}^-$. The $\mathrm{PSL}_2(\mathbb{Z})$-orbit of $\mathbf{i}\hat{\mathbb{R}}$ divides $\hat{\mathbb{C}}^+ \cup \hat{\mathbb{C}}^- \cup \hat{\mathbb{Q}}$ into a collection of triangles called the *Farey tessellation of* $\hat{\mathbb{C}}$; here, $\hat{\mathbb{Q}}$ forms the set of vertices and $\hat{\mathbb{R}} - \hat{\mathbb{Q}}$ is omitted. Let $\mathbf{\Delta}^+$ denote the Farey triangle in $\hat{\mathbb{C}}^+ \cup \hat{\mathbb{Q}}$ with vertices 0, 1 and ∞. Then $\mathbf{\Delta}^+$ is a fundamental $\rho_0(\langle A,B,C\rangle)$-domain in $\hat{\mathbb{C}}^+ \cup \hat{\mathbb{Q}}$. Let $\mathbf{\Delta}^-$ denote the complex conjugate of $\mathbf{\Delta}^+$. Then $\mathbf{\Delta}^-$ is a fundamental $\rho_0(\langle A, B, C \rangle)$-domain in $\hat{\mathbb{C}}^- \cup \hat{\mathbb{Q}}$.

For $W \in \{A, B, C\}$, we label with the letter W each point of $\hat{\mathbb{C}}_{\rho_0}$ fixed by an $\langle A, B, C \rangle$-conjugate of W, and we think of the label as being associated with the edge of the Farey tessellation on which the point lies. Thus the edge of $\mathbf{\Delta}^+$ with vertices ∞ and 0 has label A, the edge of $\mathbf{\Delta}^+$ with vertices 0 and 1 has label B, and the edge of $\mathbf{\Delta}^+$ with vertices 1 and ∞ has label C, and the same holds for the $\langle A, B, C \rangle$-translates of these lines. See Figure 4.1(1).

For each $W \in \langle A, B, C \rangle$, starting at $\mathbf{\Delta}^+$ and crossing the edges indicated by reading the letters of W leads to $\rho_0(W)\mathbf{\Delta}^+$.

The quotient $\mathbf{O} \coloneqq \langle A, B, C \rangle \backslash (\hat{\mathbb{C}}^+ \cup \hat{\mathbb{Q}})$ is a $(2, 2, 2, \infty)$-orbifold, a sphere with three C_2-points (index-two cone points labelled with C_2, the cyclic group of order two) and one C_∞-point. The image in $\mathbf{O}$ of the Farey triangulation of $\hat{\mathbb{C}}^+ \cup \hat{\mathbb{Q}}$ endows $\mathbf{O}$ with three labelled tethers [40, Section 5] joining the C_∞-point to each of the three C_2-points. □

4.3 Review. We now restrict the foregoing $\langle A, B, C \rangle$-action to $\hat{\mathbb{R}}$. We want to extend this to an $\mathrm{Aut}\langle A, B, C \rangle$-action.

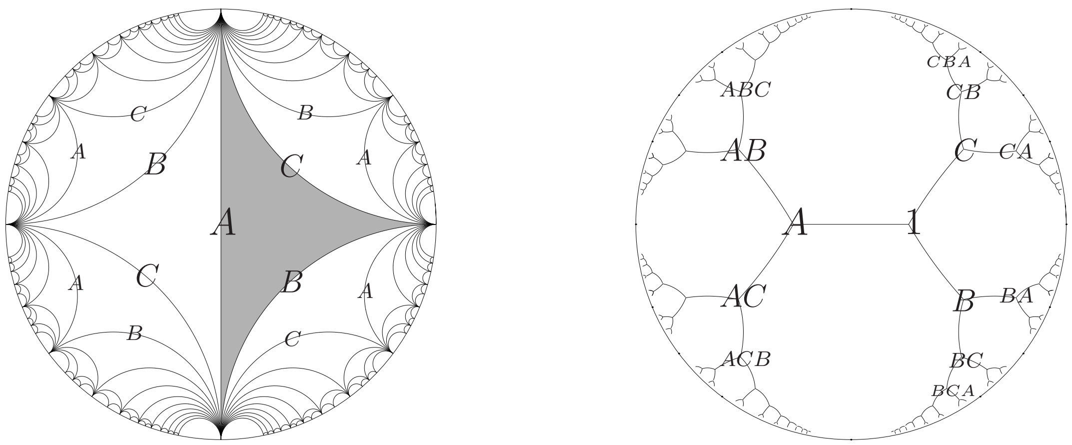
For each $W \in \mathfrak{E}\langle A, B, C \rangle$, starting at $\mathbf{\Delta}^+$ and crossing the edges indicated by reading the letters of W gives an infinite sequence of triangles in $\hat{\mathbb{C}}^+ \cup \hat{\mathbb{Q}}$. It is well known and not difficult to prove that each such sequence converges in the chord metric to a point of $\hat{\mathbb{R}}$, inducing a map $\widehat{\rho}_0 \colon \mathfrak{E}\langle A, B, C \rangle \to \hat{\mathbb{R}}$. Moreover, each point of $\hat{\mathbb{R}} - \hat{\mathbb{Q}}$ is reached by a unique such sequence, and each point of $\hat{\mathbb{Q}}$ is reached by exactly two such sequences. Further, $\hat{\mathbb{R}}$ can be viewed as the quotient space obtained from $\mathfrak{E}\langle A, B, C \rangle$ by identifying $W(D^{-\infty})$ and $W(D^{\infty})$ to form a single point denoted $W(D^{\pm\infty})$, for each $W \in \langle A, B, C \rangle$. It then follows that $\hat{\mathbb{R}}$ is an $\mathrm{Aut}\langle A, B, C \rangle$-quotient of $\mathfrak{E}\langle A, B, C \rangle$. See Figure 4.1(2).

We note that $\widehat{\rho}_0 \colon \mathfrak{E}\langle A, B, C \rangle \twoheadrightarrow \hat{\mathbb{R}}$ carries $(A\blacktriangleleft)$ onto $[A]_{\hat{\mathbb{R}}} \coloneqq [-\infty, 0]$, $(B\blacktriangleleft)$ onto $[B]_{\hat{\mathbb{R}}} \coloneqq [0, 1]$, and $(C\blacktriangleleft)$ onto $[C]_{\hat{\mathbb{R}}} \coloneqq [1, \infty]$, and, hence, the ping-pong conditions (1.2) are quotients of the primordial ping-pong conditions.

We shall be considering certain arcs in $\hat{\mathbb{C}}^+ \cup \hat{\mathbb{Q}}$ that start in $\mathbf{\Delta}^+$, travel in $\hat{\mathbb{C}}^+$, and end in $\hat{\mathbb{Q}}$ after crossing only *finitely* many edges, without backtracking, arriving in some $\rho_0(W)\mathbf{\Delta}^+$ and then heading towards $\widehat{\rho}_0(W(ABC)^{\pm\infty})$ through the channel between the A- and C-labelled edges of $\rho_0(W)\mathbf{\Delta}^+$, if W ends in B or $W = 1$, or heading towards $\widehat{\rho}_0(W(BCA)^{\pm\infty})$ through the channel between the B- and A-labelled edges of $\rho_0(W)\mathbf{\Delta}^+$, if W ends in C or $W = 1$, or heading towards $\widehat{\rho}_0(W(CAB)^{\pm\infty})$ through the channel between the C- and B-labelled edges of $\rho_0(W)\mathbf{\Delta}^+$, if W ends in A or $W = 1$. □

4.4 Review. Endow $\mathbb{R}^2$ with the orbifold structure in which each element of $\mathbb{Z}^2$ is given the structure of a C_∞-point. Give $\mathbb{R}^2$ the triangulated CW-structure that is induced from the set of all straight lines of inverse-slope 0, -1 or ∞ that pass through a point of $\mathbb{Z}^2$. Recall that Review 4.1 gives an affine action of $\mathrm{Aut}\langle A, B, C \rangle$ on $\mathbb{Z}^2$. As $\mathrm{Aut}\langle A, B, C \rangle$-sets,

$$\hat{\mathbb{Q}}_{\rho_0} = \rho_0(\langle A, B, C \rangle)\infty \simeq \langle A, B, C \rangle D^{\pm\infty},$$



(1) Disk model of triangulated $\hat{\mathbb{C}}^+ \cup \hat{\mathbb{R}}$. (2) $\widehat{\rho}_0 \colon \mathfrak{E}\langle\, A, B, C\,\rangle \twoheadrightarrow \hat{\mathbb{R}}$

Figure 4.1: Steps to an $\mathrm{Aut}\langle\, A, B, C\,\rangle$-action on $\hat{\mathbb{C}}$

and also

$$\langle\!\langle D^2 \rangle\!\rangle \backslash \hat{\mathbb{Q}}_{\rho_0} = \langle\!\langle D^2 \rangle\!\rangle \backslash \rho_0(\langle\, A, B, C\,\rangle)\infty \simeq \langle\!\langle D^2 \rangle\!\rangle \backslash \langle\, A, B, C\,\rangle D^{\pm\infty} \simeq \mathbb{Z}^2$$

where $\langle\!\langle D^2 \rangle\!\rangle (CB)^x (AB)^y D^{\pm\infty} \leftrightarrow (x, y)$. Since the group of affine automorphisms of $\mathbb{Z}^2$ acts on $\mathbb{R}^2$, we get an affine action of $\mathrm{Aut}\langle\, A, B, C\,\rangle$ on $\mathbb{R}^2$. We let $\bar{\rho}_0$ denote the homomorphism from $\mathrm{Aut}\langle\, A, B, C\,\rangle$ to the group of affine transformations of $\mathbb{R}^2$.

In summary, $\mathrm{Aut}\langle\, A, B, C\,\rangle$ acts on $\hat{\mathbb{R}}$ and then $\hat{\mathbb{Q}}$ and, after a choice of basis, then $\mathbb{Z}^2$ and then $\mathbb{R}^2$ and then on the set of inverse-slopes of the straight lines in $\mathbb{R}^2$, that is, $\hat{\mathbb{R}}$ again. We then understand that $\bar{\rho}_0(\mathrm{Aut}\langle\, A, B, C\,\rangle)$ acts on $\mathbb{Z}^2$, $\mathbb{R}^2$ and $\hat{\mathbb{R}}$. Explicitly, for all $(x, y) \in \mathbb{R}^2$, $\bar{\rho}_0(A)(x, y) = (-1 - x, -y)$, $\bar{\rho}_0(B)(x, y) = (-1 - x, -1 - y)$, $\bar{\rho}_0(C)(x, y) = (-x, -1 - y)$, $\bar{\rho}_0(D)(x, y) = (-x, -y)$, $\bar{\rho}_0(R)(x, y) = (x + y, y)$, and $\bar{\rho}_0(L)(x, y) = (x, x + y)$. Hence, for all $x \in \hat{\mathbb{R}}$, $\bar{\rho}_0(A)(x) = \bar{\rho}_0(B)(x) = \bar{\rho}_0(C)(x) = \bar{\rho}_0(D)(x) = x$, $\bar{\rho}_0(R)(x) = x + 1$, $\bar{\rho}_0(L)(x) = \frac{x}{x+1}$. In the background here is Nielsen's isomorphism $\mathrm{Aut}\langle\, A, B, C\,\rangle / \langle\, A, B, C\,\rangle \simeq \mathrm{PGL}_2(\mathbb{Z})$.

Let $\mathbf{\Delta} \coloneqq \mathrm{hull}\{(0, 0), (-1, 0), (0, -1)\}$, the triangle in $\mathbb{R}^2$ with C_∞-vertices $(0, 0)$, $(-1, 0)$ and $(0, -1)$. Then $\mathbf{\Delta}$ is a fundamental $\bar{\rho}_0(\langle\, A, B, C\,\rangle)$-domain in $\mathbb{R}^2$.

For $W \in \{A, B, C\}$, we label with the letter W each point of $\mathbb{R}^2_{\bar{\rho}_0}$ fixed by an $\langle\, A, B, C\,\rangle$-conjugate of W, and we think of the label as being associated with the line segment on which the point lies. Thus $\mathrm{hull}\{(0, 0), (-1, 0)\}$ has label A, $\mathrm{hull}\{(-1, 0), (0, -1)\}$ has label B, $\mathrm{hull}\{(0, -1), (0, 0)\}$ has label C, and the same holds for the $\langle\, A, B, C\,\rangle$-translates. See Figure 4.2. As in $\hat{\mathbb{C}}^+ \cup \hat{\mathbb{Q}}$, from reduced sequences of adjacent triangles in $\mathbb{R}^2_{\bar{\rho}_0}$ starting with $\mathbf{\Delta}$, we can read elements of $\langle\, A, B, C\,\rangle \cup \mathfrak{E}\langle\, A, B, C\,\rangle$.

Recall that $\mathbf{O}$ denotes $\rho_0(\langle\, A, B, C\,\rangle) \backslash (\hat{\mathbb{C}}^+ \cup \hat{\mathbb{Q}})$, which is a sphere with three C_2-points and a C_∞-point joined to each C_2-point by a labelled tether. The latter description also

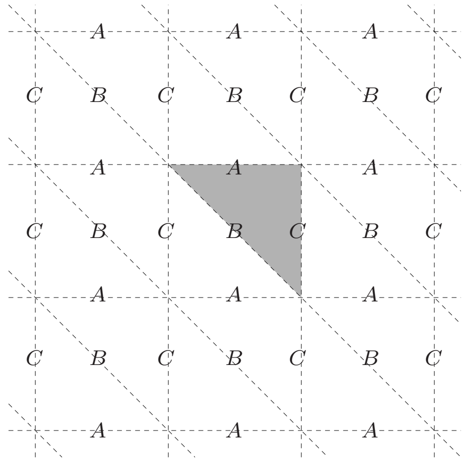


Figure 4.2: Labelled edges

applies to the quotient orbifold $\bar{\rho}_0(\langle A,B,C\rangle)\backslash\mathbb{R}^2$, and, hence, we may assume that we have a specified homeomorphism $\rho_0(\langle A,B,C\rangle)\backslash(\hat{\mathbb{C}}^+\cup\hat{\mathbb{Q}}) \xrightarrow{\sim} \bar{\rho}_0(\langle A,B,C\rangle)\backslash\mathbb{R}^2$ of orbifolds with labelled tethers. Now, $\mathbb{R}^2_{\bar{\rho}_0}$ is an orbifold cover of $\mathbf{O}$, and $(\hat{\mathbb{C}}^+\cup\hat{\mathbb{Q}})_{\rho_0}$ is the universal orbifold cover of $\mathbf{O}$. It follows that $(\hat{\mathbb{C}}^+\cup\hat{\mathbb{Q}})_{\rho_0}$ is the universal orbifold cover of $\mathbb{R}^2_{\bar{\rho}_0}$, and we have a specified $\langle A,B,C\rangle$-homeomorphism $\langle\!\langle D^2\rangle\!\rangle\backslash(\hat{\mathbb{C}}^+\cup\hat{\mathbb{Q}})_{\rho_0} \xrightarrow{\sim} \mathbb{R}^2_{\bar{\rho}_0}$ that carries the image of $\mathbf{\Delta}^+$ to $\mathbf{\Delta}$ and respects the labelled triangulations. The orbifold covering $(\hat{\mathbb{C}}^+\cup\hat{\mathbb{Q}})_{\rho_0}\to\mathbb{R}^2_{\bar{\rho}_0}$ will be used frequently. By precomposing with complex conjugation, we get an orbifold covering $(\hat{\mathbb{C}}^-\cup\hat{\mathbb{Q}})_{\rho_0}\to\mathbb{R}^2_{\bar{\rho}_0}$ which will be equally useful. □

We will now see that the $\mathrm{Aut}\langle A,B,C\rangle$-actions on $\mathbb{R}^2_{\bar{\rho}_0}$ and $\hat{\mathbb{R}}_{\rho_0}$ give an $\mathrm{Aut}\langle A,B,C\rangle$-action on $\hat{\mathbb{C}}$.

4.5 Review. Consider any $G\in\mathrm{Aut}\langle A,B,C\rangle$.

Since $\bar{\rho}_0(G)$ maps the unique fixed point of $\bar{\rho}_0(A)$ in $\mathbb{R}^2$ to the unique fixed point of $\bar{\rho}_0({}^GA)$ in $\mathbb{R}^2$, the action of $\bar{\rho}_0(G)$ on $\mathbb{R}^2$ has a unique continuous lifting to an action $\tilde{G}$ on $\hat{\mathbb{C}}^+\cup\hat{\mathbb{Q}}$ that maps the unique fixed point of $\rho_0(A)$ in $\hat{\mathbb{C}}^+\cup\hat{\mathbb{Q}}$ to the unique fixed point of $\rho_0({}^GA)$ in $\hat{\mathbb{C}}^+\cup\hat{\mathbb{Q}}$.

Let us say that G is a *well behaved* element of $\mathrm{Aut}\langle A,B,C\rangle$ if $\tilde{G}$ maps the unique fixed point of $\rho_0(B)$ in $\hat{\mathbb{C}}^+\cup\hat{\mathbb{Q}}$ to the unique fixed point of $\rho_0({}^GB)$ in $\hat{\mathbb{C}}^+\cup\hat{\mathbb{Q}}$, and maps the unique fixed point of $\rho_0(C)$ in $\hat{\mathbb{C}}^+\cup\hat{\mathbb{Q}}$ to the unique fixed point of $\rho_0({}^GC)$ in $\hat{\mathbb{C}}^+\cup\hat{\mathbb{Q}}$.

If G is well behaved then, for each $W\in\langle A,B,C\rangle$, $\tilde{G}(\rho_0(W)\mathbf{\Delta})=\rho_0({}^GW)(\tilde{G}\mathbf{\Delta})$, and, in particular, $\tilde{G}(\rho_0(W)\infty)=\rho_0({}^GW)(\tilde{G}\infty)$. It follows that $\tilde{G}$ and $\rho_0(G)$ have the same action on $\hat{\mathbb{Q}}$.

It is not difficult to show that, in $\mathrm{Aut}\langle A,B,C\rangle$, all of the elements of $\langle A,B,C\rangle$, as well as $R=(A,BCB,B)$, $L=(B,BAB,C)$ and $M=(C,B,A)$ are well behaved. Also, the composition of two well-behaved elements is well behaved. It is a classic result of Nielsen that $\{A, R, M\}$ is a generating set of $\mathrm{Aut}\langle A, B, C\rangle$. Hence, every element of $\mathrm{Aut}\langle A, B, C\rangle$ is well behaved.

We now have an action of $\mathrm{Aut}\langle A, B, C\rangle$ on $\hat{\mathbb{C}}^+ \cup \hat{\mathbb{R}}$ which extends all of the following: the $\langle A, B, C\rangle$-action on $\hat{\mathbb{C}}_{\rho_0}$; the $\mathrm{Aut}\langle A, B, C\rangle$-action on $\hat{\mathbb{R}}_{\rho_0}$; and, the $\mathrm{Aut}\langle A, B, C\rangle$-action on $\langle\!| D^2 |\!\rangle \backslash (\hat{\mathbb{C}}^+ \cup \hat{\mathbb{Q}})_{\rho_0} = \mathbb{R}^2_{\bar{\rho}_0}$.

Using complex conjugation, we get an action of $\mathrm{Aut}\langle A, B, C\rangle$ on all of $\hat{\mathbb{C}}$. We let ρ_0 denote the associated homomorphism from $\mathrm{Aut}\langle A, B, C\rangle$ to the group of all self-homeomorphisms of $\hat{\mathbb{C}}$, and this notation henceforth supersedes the previous usage of ρ_0. We denote by $\hat{\mathbb{C}}_{\rho_0}$ the Riemann sphere endowed with the foregoing $\mathrm{Aut}\langle A, B, C\rangle$-action, and similarly for the $\mathrm{Aut}\langle A, B, C\rangle$-invariant subsets. $\square$

By restriction, $\hat{\mathbb{C}}_{\rho_0}$ is an $\langle A, B, C, F\rangle$-space. To construct the Cannon-Thurston model we first partition $\hat{\mathbb{C}}^+$ and $\hat{\mathbb{C}}^-$ using $\rho_0(F)$-invariant foliations and here we will begin with the quotient $\mathbb{R}^2_{\bar{\rho}_0}$.

4.6 Review. Recall that $p \in [1{\uparrow}\infty[$ and that $a_{[\![1\uparrow p]\!]}$ and $b_{[\![1\uparrow p]\!]}$ are sequences in $[1{\uparrow}\infty[$, and $F := \prod_{i\in[\![1\uparrow p]\!]} (R^{a_i} L^{b_i}) \in \mathrm{Aut}\langle A, B, C\rangle$, and $\left(\begin{smallmatrix} f_{11} & f_{12} \\ f_{21} & f_{22}\end{smallmatrix}\right) := \prod_{i\in[\![1\uparrow p]\!]} (\left(\begin{smallmatrix} 1 & a_i \\ 0 & 1\end{smallmatrix}\right)\left(\begin{smallmatrix} 1 & 0 \\ b_i & 1\end{smallmatrix}\right)) \in \mathrm{SL}_2(\mathbb{Z})$. Then $\bar{\rho}_0(F)(x, y) = (f_{11}x + f_{12}y, f_{21}x + f_{22}y)$ for all $(x, y) \in \mathbb{R}^2$. In $\mathbb{R}$, let

$$\lambda := \tfrac{f_{11}+f_{22}+\sqrt{(f_{11}+f_{22})^2-4}}{2}, \quad \mu_+ := \tfrac{f_{11}-f_{22}+\sqrt{(f_{11}+f_{22})^2-4}}{2f_{21}}, \quad \mu_- := \tfrac{f_{11}-f_{22}-\sqrt{(f_{11}+f_{22})^2-4}}{2f_{21}}.$$

Then $\lambda \in]1, \infty[$, and λ, $\frac{1}{\lambda}$ are the eigenvalues for the action of $\bar{\rho}_0(F)$ on $\mathbb{R}^2$; also, the eigenlines have inverse-slopes μ_+ and μ_-.

Notice that $\bar{\rho}_0(L)([0, \infty]) = [0, 1] \subseteq [0, \infty]$ and $\bar{\rho}_0(R)([0, \infty]) = [1, \infty] \subseteq [0, \infty]$. (In $[0, \infty]$, it is usual to think of $[0, 1]$ as being on the left and $[1, \infty]$ as being on the right. On the exterior of $\hat{\mathbb{C}}$, after entering $\boldsymbol{\Delta}^+$ through the $(\infty, 0)$-edge, it is usual to think of the $(0, 1)$-edge as being on the right and the $(1, \infty)$-edge as being on the left. The relation between L and R and left and right depends on the context.) Now μ_+, resp. μ_-, is the attractive fixed point of the action of $\bar{\rho}_0(F)$, resp. $\bar{\rho}_0(F^{-1})$, on $\hat{\mathbb{R}}$. We see that $\mu_+ = \bar{\rho}_0(F)(\mu_+) \in \bar{\rho}_0(F)([0, \infty]) \subseteq \bar{\rho}_0(R)([0, \infty]) = [1, \infty]$ and, similarly, $\mu_- = \bar{\rho}_0(F^{-1})(\mu_-) \in \bar{\rho}_0(F^{-1})([-\infty, 0]) \subseteq \bar{\rho}_0(L^{-1})([-\infty, 0]) = [-1, 0]$. Hence, $\mu_+ \in]1, \infty[$ and $\mu_- \in]-1, 0[$.

The straight lines in $\mathbb{R}^2$ of inverse-slope μ_+ will be called *plus-lines*, and the straight lines of inverse-slope μ_- will be called *minus-lines*. These lines give two foliations of $\mathbb{R}^2$. We will use terms such as *width* and *row* and *from right to left* for the plus-line directions, and terms such as *height* and *column* and *from top to bottom* for the minus-line directions. Deleting $\mathbb{Z}^2$ from $\mathbb{R}^2$ converts some of the plus-lines into two half-lines called *half-plus-lines*. We define *half-minus-lines* similarly. A *half-plus-eigenline* is a half-plus-line incident to the origin, and similarly for the other concepts.

The foliation of $\mathbb{R}^2 - \mathbb{Z}^2$ by minus-lines and half-minus-lines can be lifted, via the chosen covering $\hat{\mathbb{C}}^-_{\rho_0} \to (\mathbb{R}^2 - \mathbb{Z}^2)_{\bar{\rho}_0}$, to a foliation of $\hat{\mathbb{C}}^-$ called the *minus foliation*. In detail, each point of $\hat{\mathbb{C}}^-$ has passing through it a unique minus-foliation curve which is mapped homeomorphically to a minus-line or a half-minus-line by the covering $\hat{\mathbb{C}}^-_{\rho_0} \to (\mathbb{R}^2 - \mathbb{Z}^2)_{\bar{\rho}_0}$; the minus-foliation curve has two limit points in $\hat{\mathbb{R}}$.

Similarly, the foliation of $\mathbb{R}^2 - \mathbb{Z}^2$ by plus-lines and half-plus-lines lifts to a foliation of the other hemisphere $\hat{\mathbb{C}}^+$, and we call this the *plus foliation*.

Now all of $\hat{\mathbb{C}} - \hat{\mathbb{R}}$ has been partitioned into foliation curves, and the closure of each foliation curve joins together two elements of $\hat{\mathbb{R}}$; thus the foliation curves on $\hat{\mathbb{C}} - \hat{\mathbb{R}}$ embody a relation on $\hat{\mathbb{R}}$, and we shall be interested in the resulting equivalence relation on $\hat{\mathbb{R}}$. The closures of two different foliation curves on $\hat{\mathbb{C}} - \hat{\mathbb{R}}$ are either disjoint or meet in a point of $\hat{\mathbb{Q}}$, and each point of $\hat{\mathbb{Q}}$ is the head of a *spider* with one C_∞-orbit of legs in each hemisphere. The closures of the foliation curves on $\hat{\mathbb{C}} - \hat{\mathbb{R}}$ result in a partition of $\hat{\mathbb{C}}$ and each element of the partition of $\hat{\mathbb{C}}$ is one of the following: a spider joining together countably many elements of $\hat{\mathbb{R}}$; the closure of a foliation curve joining two points of $\hat{\mathbb{R}}$; a point of $\hat{\mathbb{R}}$.

Collapsing the closure of every foliation curve gives a quotient $\langle\, A, B, C, F\,\rangle$-space of $\hat{\mathbb{C}}_{\rho_0}$ and of $\hat{\mathbb{R}}_{\rho_0}$; we let $\mathbb{S}^2_F$ denote this quotient space. Cannon-Thurston [16], [14, Appendix], showed that $\mathbb{S}^2_F$ is a topological sphere. McMullen [29], using a model developed by Minsky [30], constructed a continuous, surjective $\langle\, A, B, C, F\,\rangle$-map $\mathbb{S}^2_F \to \hat{\mathbb{C}}_{\rho_F}$, and Bowditch [13] showed that the latter map is a homeomorphism, thus proving that the Cannon-Thurston model represented the Jørgensen geometry. The case where $F = RL$ had been obtained by Alperin-Dicks-Porti [7].

The consequence of these results that interests us is that there exists a (unique) continuous (surjective) $\langle\, A, B, C, F\,\rangle$-map $\mathrm{CT}\colon \hat{\mathbb{C}}_{\rho_0} \to \hat{\mathbb{C}}_{\rho_F}$ which is the quotient map that collapses the closures in $\hat{\mathbb{C}}$ of the foliation lines in $\hat{\mathbb{C}} - \hat{\mathbb{R}}$. In particular, the map $\mathrm{CT}\colon \hat{\mathbb{C}}_{\rho_0} \to \hat{\mathbb{C}}_{\rho_F}$ is constant along each foliation line in $\hat{\mathbb{C}} - \hat{\mathbb{R}}$.

The following results, proved by Cannon-Dicks [14], provide all the information we shall require concerning $\partial^+ B$ and $\partial^- A$:

• the A-labelled edge of $\boldsymbol{\Delta}^-$ joining ∞ to 0 is mapped homeomorphically to $\partial^- A$ under $\mathrm{CT}\colon \hat{\mathbb{C}}_{\rho_0} \to \hat{\mathbb{C}}_{\rho_F}$;

• the C-labelled edge of $\boldsymbol{\Delta}^-$ joining 1 to ∞ is mapped homeomorphically to the subarc $\partial^- C$ of reverse($\partial^- A$) under $\mathrm{CT}\colon \hat{\mathbb{C}}_{\rho_0} \to \hat{\mathbb{C}}_{\rho_F}$;

• the B-labelled edge of $\boldsymbol{\Delta}^+$ joining 0 to 1 is mapped homeomorphically to reverse($\partial^+ B$) under $\mathrm{CT}\colon \hat{\mathbb{C}}_{\rho_0} \to \hat{\mathbb{C}}_{\rho_F}$;

• the C-labelled edge of $\boldsymbol{\Delta}^+$ joining 1 to ∞ is mapped homeomorphically to the subarc reverse($\partial^+ C$) of $\partial^+ B$ under $\mathrm{CT}\colon \hat{\mathbb{C}}_{\rho_0} \to \hat{\mathbb{C}}_{\rho_F}$.

• Since the map $\mathrm{CT}\colon \hat{\mathbb{C}}_{\rho_0} \to \hat{\mathbb{C}}_{\rho_F}$ is constant along foliation lines, it follows that $\mathrm{CT}(\boldsymbol{\Delta}^+) = \partial^+ B$ and $\mathrm{CT}(\boldsymbol{\Delta}^-) = \partial^- A$. □

5 Markov partitions

In this section, we give proofs of the results described in Section 3, using slightly different notation that will be seen by the end of this section to agree with all the notation used in Section 3. We study the continuous action of $\rho_0(F)$ on $\hat{\mathbb{C}}$ by using lifts of famous Markov partitions described in 1967 by Adler-Weiss.

5.1 Notation. Recall that we have $p \in [1{\uparrow}\infty[$ and sequences $a_{[\![1\uparrow p]\!]}, b_{[[\![1\uparrow p]\!]]}$ in $[1{\uparrow}\infty[$, and that, in $\mathrm{Aut}\langle\, A, B, C\,\rangle$, $R \coloneqq (A, BCB, B)$, $L \coloneqq (B, BAB, C)$, $F \coloneqq \prod\limits_{i\in[\![1\uparrow p]\!]} (R^{a_i} L^{b_i})$. Let $G \coloneqq AF^{-1}A$, $X_1 \coloneqq BC$, $X_2 \coloneqq BA$, $X_{-1} \coloneqq CA$, $X_{-2} \coloneqq BA$. It is straightforward to calculate that, for any a, b in $[1{\uparrow}\infty[$,

$$^{R^aL^b}(X_1, X_2) = ((X_1^a X_2)^b X_1, X_1^a X_2), \tag{5.1}$$

$$^{AL^{-b}R^{-a}A}(X_{-1}, X_{-2}) = (X_{-2}(X_{-2}^{b-1} X_{-1})^{a+1}, X_{-2}(X_{-2}^{b-1} X_{-1})^a). \tag{5.2}$$

By (5.1), F acts on the *semigroup* freely generated by X_1 and X_2, and ${}^F X_2$ is an initial segment of ${}^F X_1$, and ${}^F X_1$ begins and ends in X_1, and ${}^F X_2$ begins in X_1 and ends in X_2. By abelianizing, we see that the matrix of F with respect to (X_1, X_2) is

$$\left(\begin{smallmatrix} f_{1\,1} & f_{1\,2} \\ f_{2\,1} & f_{2\,2} \end{smallmatrix}\right) \coloneqq \prod_{i\in[\![1\uparrow p]\!]} ((\begin{smallmatrix} 1 & a_i \\ 0 & 1 \end{smallmatrix})(\begin{smallmatrix} 1 & 0 \\ b_i & 1 \end{smallmatrix})).$$

Let $f_1 \coloneqq f_{1\,1} + f_{2\,1}$, $f_2 \coloneqq f_{1\,2} + f_{2\,2}$, and let $\mathbf{s}_{[\![1\uparrow f_1]\!]}$ denote the sequence in $\{1, 2\}$ such that ${}^F X_1 = \Pi X_{\mathbf{s}_{[\![1\uparrow f_1]\!]}}$. Then ${}^F X_2 = \Pi X_{\mathbf{s}_{[\![1\uparrow f_2]\!]}}$, $\mathbf{s}_1 = 1$, $\mathbf{s}_{f_2} = 2$, and $\mathbf{s}_{f_1} = 1$. For each $\ell \in [1{\uparrow}f_1]$, we define the ℓth ∂^-*-syllable* to be

$$U_\ell \coloneqq F^{-1}\Pi X_{\mathbf{s}_{[\![1\uparrow(\ell-1)]\!]}} \in \langle\, A, B, C, F\,\rangle.$$

We call $U_{[\![1\uparrow f_1]\!]}$ the ∂^-*-syllable sequence*, and we call the subsemigroup of $\langle\, A, B, C, F\,\rangle$ generated by $U_{[1\uparrow f_1]}$ the ∂^-*-semigroup*.

By (5.2), G acts on the semigroup freely generated by X_{-1} and X_{-2}, and ${}^G X_{-2}$ is an initial segment of ${}^G X_{-1}$, and ${}^G X_{-1}$ and ${}^G X_{-2}$ both begin in X_{-2} and end in X_{-1}. By abelianizing, we see that the matrix of F^{-1} and G with respect to (X_{-1}, X_{-2}) is

$$\begin{pmatrix} -1 & 0 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} f_{2,2} & -f_{1\,2} \\ -f_{2\,1} & f_{1\,1} \end{pmatrix}\begin{pmatrix} -1 & 0 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} f_{1\,2}+f_{2\,2} & f_{1\,2} \\ f_{1\,1}-f_{1\,2}+f_{2\,1}-f_{2\,2} & f_{1\,1}-f_{1\,2} \end{pmatrix}.$$

Let $f_{-1} \coloneqq f_{1\,1} + f_{2\,1} = f_1$, $f_{-2} \coloneqq f_{1\,1}$, and let $\mathbf{t}_{[\![1\uparrow f_{-1}]\!]}$ denote the sequence in $\{-1, -2\}$ such that ${}^G X_{-1} = \Pi X_{\mathbf{t}_{[\![1\uparrow f_{-1}]\!]}}$. Then ${}^G X_{-2} = \Pi X_{\mathbf{t}_{[\![1\uparrow f_{-2}]\!]}}$, $\mathbf{t}_1 = -2$, $\mathbf{t}_{f_{-2}} = -1$, and $\mathbf{t}_{f_{-1}} = -1$. For each $\ell \in [1{\uparrow}f_{-1}]$, we define the ℓth ∂^+*-syllable* to be

$$V_\ell \coloneqq G^{-1}\Pi X_{\mathbf{t}_{[\![1\uparrow(\ell-1)]\!]}} \in \langle\, A, B, C, F\,\rangle.$$

We call $V_{[\![1\uparrow f_{-1}]\!]}$ the ∂^{+}-*syllable sequence*, and we call the subsemigroup of $\langle\, A, B, C, F\,\rangle$ generated by $V_{[1\uparrow f_{-1}]}$ the ∂^{+}-*semigroup*.

Notice that

$$f_{-2}f_2 - f_{1\,2}f_1 = f_{1\,1}(f_{1\,2} + f_{2\,2}) - f_{1\,2}(f_{1\,1} + f_{2\,1}) = f_{1\,1}f_{2\,2} - f_{1\,2}f_{2\,1} = 1.$$

For each $\ell \in [1\uparrow f_1]$, let $F(\ell)$ denote the unique element of $[1\uparrow f_1]$ with the property that $1 - \ell f_{-2} \equiv F(\ell)(\text{mod } f_1)$. For example, $F(1) = f_{2\,1} + 1$, $F(f_2) = f_1$, $F(f_1) = 1$. Let $F^{-1}(\ell)$ denote the unique element of $[1\uparrow f_1]$ such that $(1-\ell)f_2 \equiv F^{-1}(\ell)(\text{mod } f_1)$. Then F and F^{-1} have mutually inverse actions on $[1\uparrow f_1]$.

We now describe how to tessellate $\mathbb{R}^2$ with an L-shaped region as illustrated in Figure 5.1(4). Deleting from $\mathbb{R}^2$ the (three) minus-lines which pass through $(-1,0)$, $(0,0)$ and $(0,-1)$, and the (three) plus-lines which pass through $(-1,0)$, $(-1,-1)$, and $(0,-1)$, leaves a region of $\mathbb{R}^2$ with sixteen components, four of which are interiors of closed parallelograms. Let $\mathbf{P}_{[\![1\uparrow 4]\!]}$ denote the sequence of these four closed parallelograms in clockwise order with $\mathbf{P}_1$ being the parallelogram containing $(-1,0)$. We let $\mathbf{L} \coloneqq \bigcup \mathbf{P}_{[1\uparrow 3]}$; see Figure 5.1(1) for the case $F = RL^3$. It is not difficult to see that $\mathbf{L}$ is a fundamental $\mathbb{Z}^2$-domain in $\mathbb{R}^2$; see Figure 5.1(4) for the case $F = RL^3$. We let $\tilde{\mathbf{L}} \coloneqq \bigcup \mathbf{P}_{[1\uparrow 4]}$, the smallest parallelogram containing $\mathbf{L}$.

We let $\mathbf{L}_1 \coloneqq \bigcup \mathbf{P}_{[1\uparrow 2]}$, colored gray, and $\mathbf{L}_2 \coloneqq \mathbf{P}_3$, colored white; see Figure 5.1(3). Hence, we get an $\mathbf{L}_{[1\uparrow 2]}$-tessellation of $\mathbb{R}^2$; see Figure 5.1(4).

We now want to choose a copy of most of $\mathbb{R}^2$ in $\hat{\mathbb{C}}^{+} \cup \hat{\mathbb{Q}}$.

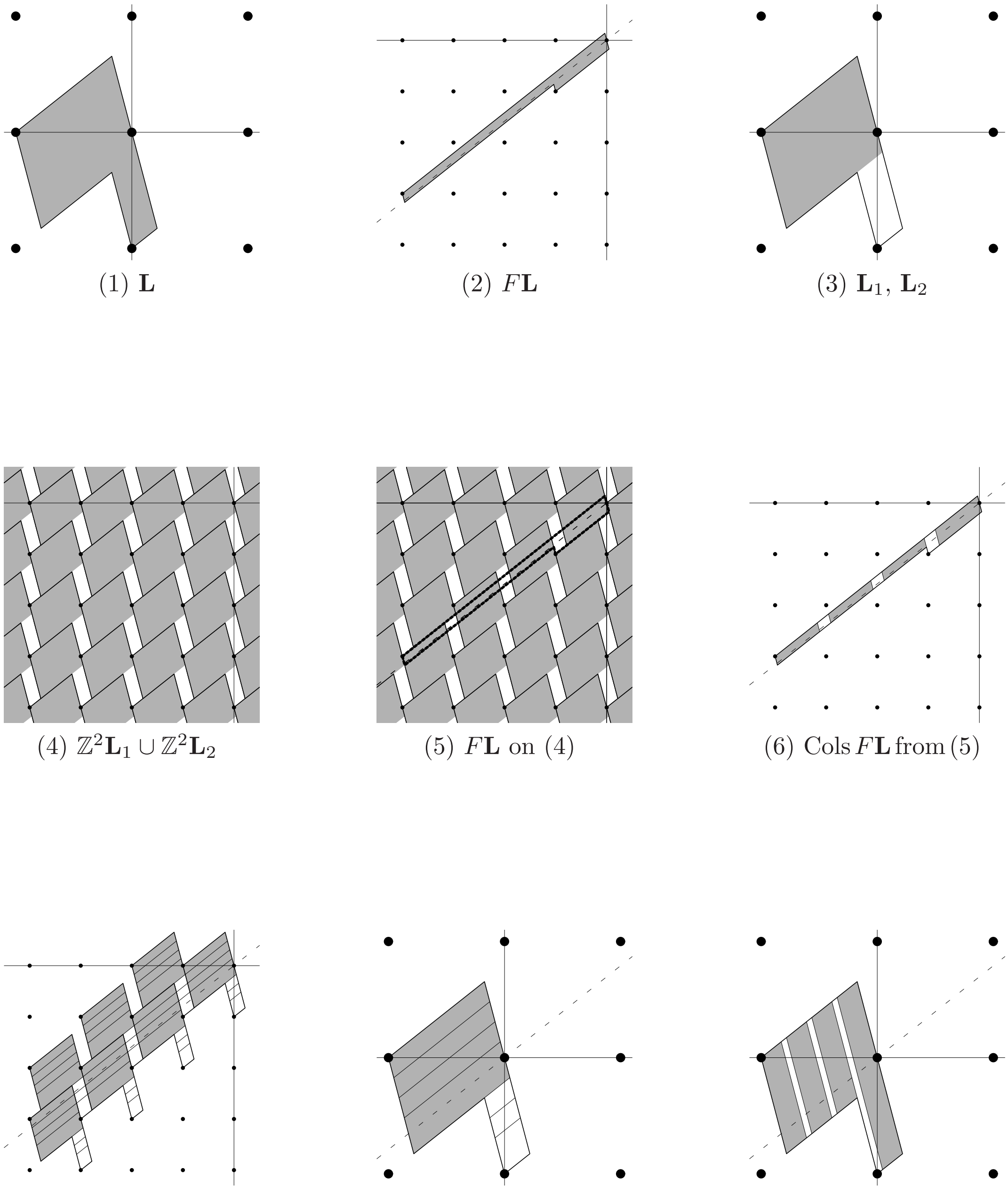
Let $\mathfrak{l}_-$ denote the half-minus-line incident to $(-1,0)$ in the lower-half-plane, oriented from top to bottom. We define a region $\mathbf{M}_-$ in $\mathbb{R}^2$ as follows. We start by removing from $\mathbb{R}^2$ the second-quadrant portion of the minus-line incident to $(-1,0)$. We also remove all the half-plus-lines which do not meet the minus-line incident to $(-1,0)$. We do not remove any element of $\mathbb{Z}^2$. This completes the description of $\mathbf{M}_-$. For each element of $\mathbb{Z}^2$, we have removed exactly one of the four incident half-lines, no two of which meet. Then $\mathbf{M}_-$ and $\mathbf{M}_- - \mathbb{Z}^2$ are contractible, and, hence, we have a well-defined lift $\mathbf{M}_- \to \hat{\mathbb{C}}^{+} \cup \hat{\mathbb{Q}}$, $() \mapsto ()^{+}$, such that $(0,0)^{+} = \infty$; notice that $(0,0)$ is a fixed point of $\bar{\rho}_0(\langle\, D, F\,\rangle)$ and ∞ is a fixed point of $\rho_0(\langle\, D, F\,\rangle)$. Notice that $\tilde{\mathbf{L}}^{+}$, $\mathbf{L}^{+}$, $\mathbf{L}_1^{+}$ and $\mathbf{L}_2^{+}$ are all defined. Here, $\mathbf{L}^{+}$ is a fundamental $\rho_0(\langle\, CA, BA\,\rangle)$-domain in $\hat{\mathbb{C}}^{+} \cup \hat{\mathbb{Q}}$, and we get an $\mathbf{L}^{+}_{[1\uparrow 2]}$-tessellation of $\hat{\mathbb{C}}^{+} \cup \hat{\mathbb{Q}}$. Notice that $\mathbf{M}_-$ is $\bar{\rho}_0(AFA)$-invariant, since $\bar{\rho}_0(AFA)$ leaves invariant both of the half-minus-lines incident to $(-1,0)$ and leaves invariant the set of half-plus-lines which do not meet the minus-line incident to $(-1,0)$. The action of $\bar{\rho}_0(AFA)$ on $\mathbf{M}_-$ exactly models the action of $\rho_0(AFA)$ on $\mathbf{M}_-^{+}$.

We let $\mathbf{L}_{-2} \coloneqq \mathbf{P}_1$, colored white, and $\mathbf{L}_{-1} \coloneqq \bigcup \mathbf{P}_{[2\uparrow 3]}$, colored gray; see Figure 5.2(2). Hence, we get an $\mathbf{L}_{[(-2)\uparrow(-1)]}$-tessellation of $\mathbb{R}^2$; see Figure 5.2(4).

We now want to choose a copy of most of $\mathbb{R}^2$ in $\hat{\mathbb{C}}^{-} \cup \hat{\mathbb{Q}}$.

Let $\mathfrak{l}_+$ denote the half-plus-line incident to $(0,0)$ in the lower-half-plane, oriented from right to left. We define a region $\mathbf{M}_+$ in $\mathbb{R}^2$ as follows. We start by removing from

$\mathbb{R}^2$ the first-quadrant portion of the plus-line incident to $(0,0)$. We also remove all the half-minus-lines which do not meet the plus-line incident to $(0,0)$. We do not remove any

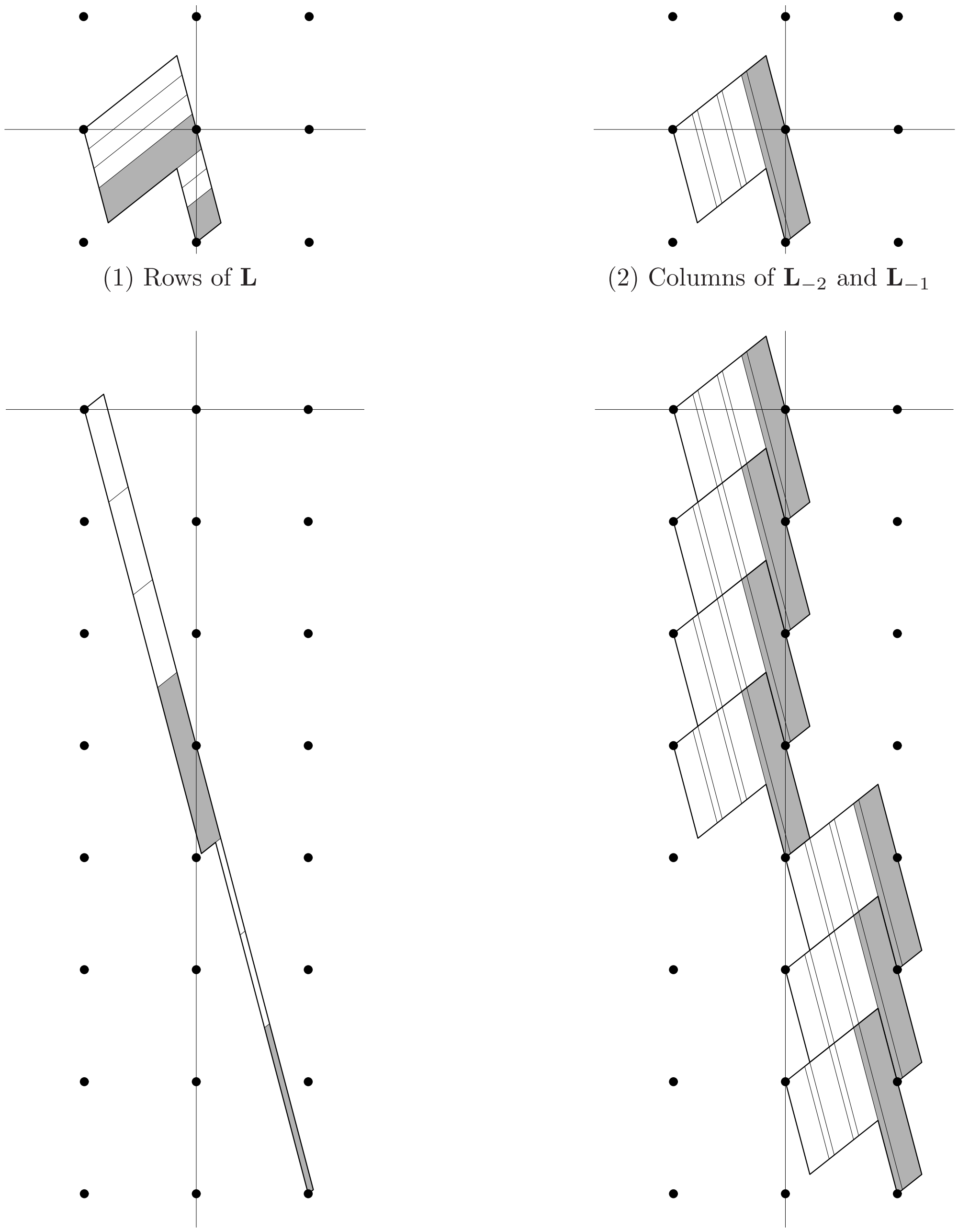


(1) $\mathbf{L}$ (2) $F\mathbf{L}$ (3) $\mathbf{L}_1$, $\mathbf{L}_2$

(4) $\mathbb{Z}^2\mathbf{L}_1 \cup \mathbb{Z}^2\mathbf{L}_2$ (5) $F\mathbf{L}$ on (4) (6) Cols $F\mathbf{L}$ from (5)

(7) $\mathbb{Z}^2$Cols $F\mathbf{L}$ from (6) (8) Rows $\mathbf{L}$ from (7) (9) Cols $\mathbf{L}$ from F^{-1}(6)

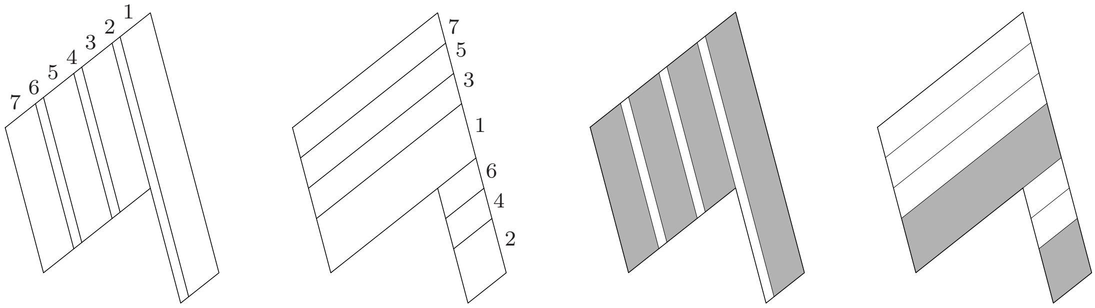
Figure 5.1: F, columns to rows: broad = gray = $BC = X_1$, thin = white = $BA = X_2$



(1) Rows of $\mathbf{L}$ (2) Columns of $\mathbf{L}_{-2}$ and $\mathbf{L}_{-1}$

(3) $G = AF^{-1}A$ applied to (1) (4) $\mathbb{Z}^2$-translates of (2) or (3)

Figure 5.2: F^{-1}, rows to columns: squat = white = $BA = X_{-2}$, tall = gray = $CA = X_{-1}$



(1) The numbering indicates how F transforms columns to rows.

(2) Reading columns from right to left gives ${}^{F}(BC)=(BCBA)^3BC$, ${}^{F}(BA)=BCBA$.
Reading rows from top to bottom gives ${}^{F^{-1}}(AC)=(AB)^3AC(AB)^2AC$, ${}^{F^{-1}}(AB)=(AB)^3AC$.

Figure 5.3: The Markov partitions for F and F^{-1} in the case $F = RL^3$

element of $\mathbb{Z}^2$. This completes the description of $\mathbf{M}_+$. For each element of $\mathbb{Z}^2$, we have removed exactly one of the four incident half-lines, no two of which meet. Then $\mathbf{M}_+$ and $\mathbf{M}_+ - \mathbb{Z}^2$ are contractible, and, hence, we have a well-defined lift $\mathbf{M}_+ \to \hat{\mathbb{C}}^- \cup \hat{\mathbb{Q}}$, $() \mapsto ()^-$, such that $(0,0)^- = \infty$; again, $(0,0)$ is a fixed point of $\bar{\rho}_0(\langle D, F \rangle)$ and ∞ is a fixed point of $\rho_0(\langle D, F \rangle)$. Notice that $\tilde{\mathbf{L}}^-$, $\mathbf{L}^-$, $\mathbf{L}^-_{-1}$ and $\mathbf{L}^-_{-2}$ are all defined. Notice that $\mathbf{L}^-$ is a fundamental $\rho_0(\langle BC, BA \rangle)$-domain in $\hat{\mathbb{C}}^- \cup \hat{\mathbb{Q}}$, and we get an $\mathbf{L}^-_{[(-2)\uparrow(-1)]}$-tessellation of $\hat{\mathbb{C}}^- \cup \hat{\mathbb{Q}}$. The action of $\bar{\rho}_0(F)$ on $\mathbf{M}_+$ exactly models the action of $\rho_0(F)$ on $\mathbf{M}^-_+$.

For each $\ell \in [1\uparrow f_1]$, the *ℓth column* of $\mathbf{L}$ is defined as $\mathrm{col}_\ell(\mathbf{L}) := \mathbf{L} \cap \bar{\rho}_0(U_\ell)\mathbf{L}$ and the *ℓth row* of $\mathbf{L}$ is defined as $\mathrm{row}_\ell(\mathbf{L}) := \mathbf{L} \cap \bar{\rho}_0(V_\ell)\mathbf{L}$; see Figures 5.1, 5.3. For $W \in \langle A, B, C, F \rangle$, we define $\mathrm{col}_\ell(\rho_0(W)\mathbf{L}^+) := \rho_0(W)((\mathrm{col}_\ell(\mathbf{L}))^+)$ and $\mathrm{row}_\ell(\rho_0(W)\mathbf{L}^+) := \rho_0(W)((\mathrm{row}_\ell(\mathbf{L}))^+)$ and similarly for $\rho_0(W)\mathbf{L}^-$.

We read $\mathbf{L}^-$ from right to left following its plus-foliation, and, collapsing along the minus-foliation, we define

$$\mathfrak{a}_1 := \mathrm{CT}(\mathbf{L}^-_1) = \mathrm{CT}(\mathbf{L}^-) = CT(\mathbf{\Delta}^-) = \partial^- A,$$

$$\mathfrak{a}_2 := \mathrm{CT}(\mathbf{L}^-_2) = \mathrm{reverse}(\partial^- C);$$

see Review 4.6.

We read $\mathbf{L}^+$ from top to bottom following its minus-foliation, and, collapsing along the plus-foliation, we define

$$\mathfrak{a}_{-1} := \mathrm{CT}(\mathbf{L}^+_{-1}) = \mathrm{CT}(\mathbf{L}^+) = \mathrm{CT}(\mathbf{\Delta}^+) = \mathrm{reverse}(\partial^+ B),$$

$$\mathfrak{a}_{-2} := \mathrm{CT}(\mathbf{L}^+_{-2}) = \rho_F(B)\,\mathrm{reverse}(\partial^+ C) = \mathrm{reverse}(\partial^+ BC);$$

see Review 4.6. □

5.2 Remarks. The sequence $\mathrm{col}_{[\![1\uparrow f_1]\!]}(\mathbf{L})$ is called *the Markov partition* for F; see Figure 5.3. It was described explicitly by Adler-Weiss [2, Section 5], [3, Figure 14], [1, Figure 6], in 1967, in connection with their analysis of the entropy of a hyperbolic automorphism of $\mathbb{R}^2/\mathbb{Z}^2$; see also [9], [10] for earlier, related studies. The sequence $\mathrm{row}_{[\![1\uparrow f_1]\!]}(\mathbf{L})$ is called *the Markov partition* for F^{-1}. The Markov partition was subsequently lifted to the punctured case in the manner we shall recall in this section; see, for example, [11, Section 3.4].

The novelty of this section lies in the made-to-measure application to lightning curves at the covering level.

We shall see the following well-known facts. The union of the columns of $\mathbf{L}^-$ is all of $\mathbf{L}^-$, the intersections of consecutive columns lie in minus-foliation curves, and distinct non-consecutive columns are disjoint. Each column of $\mathbf{L}^-$ is mapped by $\rho_0(F)$ into a translate $\rho_0(W)\mathbf{L}^-$ for a unique $W \in \langle\, BC, BA\,\rangle$ and the various resulting $\rho_0(F^{-1}W)$ together give a bijective map from the disjoint union of the rows of $\mathbf{L}^-$ to the disjoint union of the columns of $\mathbf{L}^-$. This bijective map then induces the action of $\rho_0(F^{-1})$ on the once-punctured torus $\rho_0(\langle\, BC, BA\,\rangle)\backslash\hat{\mathbb{C}}^-$ which carries rows to columns contracting in the plus-direction and expanding in the minus-direction. $\square$

We now describe the tessellation of $\rho_0(F)\mathbf{L}^-$ that is induced by the $\mathbf{L}^-_{[1\uparrow 2]}$-tessellation of $\hat{\mathbb{C}}^- \cup \hat{\mathbb{Q}}$; see Figure 5.1(5) and (6).

5.3 Lemma. *Let* Notation 5.1 *hold.*

(i) *Let* $\mathrm{CW}(\rho_0(F)\mathbf{L}^-)$ *denote* $\rho_0(F)\mathbf{L}^-$ *endowed with the CW-structure that is induced by the* $\mathbf{L}^-_{[1\uparrow 2]}$*-tessellation of* $\hat{\mathbb{C}}^- \cup \hat{\mathbb{Q}}$*. Here, the two-cells of* $\mathrm{CW}(\rho_0(F)\mathbf{L}^-)$ *are the terms of the sequence* $(\rho_0(F)\mathbf{L}^- \cap \rho_0(FU_\ell)\mathbf{L}^- \mid \ell \in [\![1\uparrow f_1]\!])$*, consecutive terms overlap on a minus-foliation one-cell of* $\mathrm{CW}(\rho_0(F)\mathbf{L}^-)$*, and non-consecutive distinct terms are disjoint. Each minus-foliation one-cell of* $\mathrm{CW}(\rho_0(F)\mathbf{L}^-)$ *lies in the boundary of an* $\mathbf{L}^-$*-tile. Each plus-foliation one-cell of* $\mathrm{CW}(\rho_0(F)\mathbf{L}^-)$ *lies in the boundary of* $\rho_0(F)\mathbf{L}^-$*.*

(ii) *Let* $\ell \in [1\uparrow f_1]$ *and define* $i_\ell := -2$ *for* $\ell \in [1\uparrow f_2]$*, and* $i_\ell := -1$ *for* $\ell \in [(f_2+1)\uparrow f_1]$*. Then* $\rho_0(F)\mathbf{L}^- \cap \rho_0(FU_\ell)\mathbf{L}^- = \rho_0(F)\mathbf{L}^-_{i_\ell} \cap \rho_0(FU_\ell)\mathbf{L}^-_{\mathbf{s}_\ell} = (\bar{\rho}_0(F)\mathbf{L}_{i_\ell} \cap \bar{\rho}_0(FU_\ell)\mathbf{L}_{\mathbf{s}_\ell})^-$ *and* $\bar{\rho}_0(F)\mathbf{L}_{i_\ell} \cap \bar{\rho}_0(FU_\ell)\mathbf{L}_{\mathbf{s}_\ell}$ *is a parallelogram. The left and right borders of* $\rho_0(F)\mathbf{L}^-_{i_\ell} \cap \rho_0(FU_\ell)\mathbf{L}^-_{\mathbf{s}_\ell}$ *lie in the left and right borders of* $\rho_0(FU_\ell)\mathbf{L}^-_{\mathbf{s}_\ell}$*, respectively. The top and bottom borders of* $\rho_0(F)\mathbf{L}^-_{i_\ell} \cap \rho_0(FU_\ell)\mathbf{L}^-_{\mathbf{s}_\ell}$ *lie in the top and bottom borders of* $\rho_0(F)\mathbf{L}^-_{i_\ell}$*, respectively. The bottom border of* $\rho_0(F)\mathbf{L}^-_{i_\ell} \cap \rho_0(FU_\ell)\mathbf{L}^-_{\mathbf{s}_\ell}$ *lies in the bottom border of* $\rho_0(FU_\ell)\mathbf{L}^-_{\mathbf{s}_\ell}$ *if and only if* $\ell \in [(f_2-1)\uparrow f_2]$*. The top border of* $\rho_0(F)\mathbf{L}^-_{i} \cap \rho_0(FU_\ell)\mathbf{L}^-_{\mathbf{s}_\ell}$ *lies in the top border of* $\rho_0(FU_\ell)\mathbf{L}^-_{\mathbf{s}_\ell}$ *if and only if* $\ell \in [(f_1-1)\uparrow f_1]$*. Also,* $\mathbf{s}_\ell = [\frac{\ell+1}{1+\mu_+}] - [\frac{\ell}{1+\mu_+}] + 1$*.*

Proof. For the current purpose of studying the $\mathbf{L}^-_{[1\uparrow 2]}$-tessellation of $\rho_0(F)\mathbf{L}^-$, we shall treat $\mathbf{M}_+$ and $\mathbf{M}_+^-$ as essentially interchangeable. We understand that the following discussion takes place in $\hat{\mathbb{C}}^- \cup \hat{\mathbb{Q}}$ but with the notation of $\mathbb{R}^2$.

The half-plus-line $\mathfrak{l}_+$ starts from within $\mathbf{\Delta}$, and the first labelled line segment it crosses has the label B. Let $\mathbf{s}'_{[\![1\uparrow\infty[\![}$ denote the sequence in $\{1,2\}$ with the property that the element of $\mathfrak{E}\langle\, A, B, C\,\rangle$ read by $\mathfrak{l}_+$ is $\Pi X_{\mathbf{s}'_{[\![1\uparrow\infty[\![}}$.

Let $\ell \in [1\uparrow\infty[$. Define $W_\ell \coloneqq \Pi X_{\mathbf{s}'_{[\![1\uparrow(\ell-1)]\!]}}$.

We will now show that $\mathbf{s}'_\ell = [\frac{\ell+1}{1+\mu_+}] - [\frac{\ell}{1+\mu_+}] + 1$. Consider the line in $\mathbb{R}^2$ with equation $x + y = -\ell$, which is a concatenation of B-labelled line segments. Consider also the line with equation $x = \mu_+ y$, which is the plus-eigenline. Let $(x_\ell, y_\ell\,)$ denote the point where these two lines intersect, that is, the point where $\mathfrak{l}_+$ reads B for the ℓth time. Then $(x_\ell, y_\ell\,) = (-\ell - \frac{-\ell}{1+\mu_+}, \frac{-\ell}{1+\mu_+})$. As $\mathfrak{l}_+$ passes through (x_ℓ, y_ℓ) it reads B, and then either $\mathfrak{l}_+$ reads C and reaches $(x_{\ell+1}, y_{\ell+1})$ and we have $[y_{\ell+1}] = [y_\ell]$, or $\mathfrak{l}_+$ reads A and reaches $(x_{\ell+1}, y_{\ell+1})$ and we have $[y_{\ell+1}] = [y_\ell] - 1$; see Figures 4.2 and 5.1(3). Thus, either $[y_\ell] - [y_{\ell+1}] = 0$ and $\mathbf{s}'_\ell = 1$, or $[y_\ell] - [y_{\ell+1}] = 1$ and $\mathbf{s}'_\ell = 2$. It follows that $\mathbf{s}'_\ell = [y_\ell] - [y_{\ell+1}] + 1 = -[\frac{\ell}{1+\mu_+}] + [\frac{\ell+1}{1+\mu_+}] + 1$, as desired.

We shall now prove that the sequence of $\mathbf{L}^-_{[1\uparrow 2]}$-tiles cut through by $\mathfrak{l}^-_+$ from right to left is $(\rho_0(W_\ell)\mathbf{L}^-_{\mathbf{s}'_\ell} \mid \ell \in [\![1\uparrow\infty[\![\ )$.

Now, either $\ell = 1$ and $\mathfrak{l}_+$ originates in $\bar\rho_0(W_\ell)\mathbf{\Delta}$, or $\ell \geqslant 2$ and $\mathfrak{l}_+$ enters $\bar\rho_0(W_\ell)\mathbf{\Delta}$. Then, in either event, $\mathfrak{l}_+$ exits $\bar\rho_0(W_\ell)\mathbf{\Delta}$ reading a B-labelled line segment. And then, either $\mathbf{s}'_\ell = 1$ and $\mathfrak{l}_+$ reads a C-labelled line segment and enters $\bar\rho_0(W_\ell BC)\mathbf{\Delta} = \bar\rho_0(W_\ell X_1)\mathbf{\Delta} = \bar\rho_0(W_{\ell+1})\mathbf{\Delta}$, or $\mathbf{s}'_\ell = 2$ and $\mathfrak{l}_+$ reads an A-labelled line segment and enters $\bar\rho_0(WBA)\mathbf{\Delta} = \bar\rho_0(W_\ell X_2)\mathbf{\Delta} = \bar\rho_0(W_{\ell+1})\mathbf{\Delta}$. From Figures 4.2 and 5.1(3), we see that, in both cases, the initial point of $\mathfrak{l}_+ \cap \bar\rho_0(W_\ell)\mathbf{\Delta}$ lies in $\bar\rho_0(W_\ell)\mathbf{L}_{\mathbf{s}'_\ell}$, and $\mathfrak{l}_+$ cuts through $\bar\rho_0(W_\ell)\mathbf{L}_{\mathbf{s}'_\ell}$ from right to left, even for $\ell = 1$. It follows that $\bar\rho_0(W_\ell)\mathbf{L}_{\mathbf{s}'_\ell}$ is the ℓth $\mathbf{L}_{[1\uparrow 2]}$-tile cut by $\mathfrak{l}_+$, as desired. Since $X_1(x,y) = (x-1, y)$ and $X_2(x,y) = (x, y-1)$, we see that if $\bar\rho_0(W_\ell X_{\mathbf{s}'_\ell})(0,0) = (-p,-q)$ then $p + q = \ell$. Notice that the left border of $\bar\rho_0(W_\ell)\mathbf{L}_{\mathbf{s}'_\ell}$ contains $\bar\rho_0(W_\ell X_{\mathbf{s}'_\ell})(0,0)$ as one of its end-points.

We shall now prove that $\bar\rho_0(F)\tilde{\mathbf{L}}$ cuts through $(\bar\rho_0(W_\ell)\mathbf{L}_{\mathbf{s}'_\ell} \mid \ell \in [\![1\uparrow f_1]\!])$ in the same way that $\mathfrak{l}_+$ does.

Notice that $\tilde{\mathbf{L}} \cap \mathbb{Z}^2$, which is $\{(0,0), (-1,0), (0,-1)\}$, lies in the boundary of $\tilde{\mathbf{L}}$. Hence the boundary of $\bar\rho_0(F)\tilde{\mathbf{L}}$ contains $\bar\rho_0(F)(\tilde{\mathbf{L}} \cap \mathbb{Z}^2) = \bar\rho_0(F)(\tilde{\mathbf{L}}) \cap \bar\rho_0(F)(\mathbb{Z}^2) = \bar\rho_0(F)(\tilde{\mathbf{L}}) \cap \mathbb{Z}^2$. In particular, the interior of $\bar\rho_0(F)\tilde{\mathbf{L}}$ does not meet $\mathbb{Z}^2$.

Since $\mathfrak{l}_+$ cuts through $\tilde{\mathbf{L}}$ from right to left, we see that $\bar\rho_0(F)\mathfrak{l}_+ = \mathfrak{l}_+$ cuts through $\bar\rho_0(F)\tilde{\mathbf{L}}$ from right to left. Since the interior of $\bar\rho_0(F)\tilde{\mathbf{L}}$ does not meet $\mathbb{Z}^2$, we see that any labelled line segment that cuts through $\mathfrak{l}_+ \cap \bar\rho_0(F)\tilde{\mathbf{L}}$ from top to bottom must cut through $\bar\rho_0(F)\tilde{\mathbf{L}}$ from top to bottom.

Notice that the top left corner of $\tilde{\mathbf{L}}$ is $(-1,0)$; see Figure 5.1(1). Hence the top left corner of $\bar\rho_0(F)\tilde{\mathbf{L}}$ is $\bar\rho_0(F)(-1,0) = (-f_{11}, -f_{21})$. The A-labelled line segment joining $\bar\rho_0(F)(-1,0) = (-f_{11}, -f_{21})$ to $(-f_{11}+1, -f_{21})$ cuts through an $\mathbf{L}_1$-tile incident to $\bar\rho_0(F)(-1,0)$ from left to right, and cuts through $\bar\rho_0(F)\tilde{\mathbf{L}}$ from top to bottom, and cuts through $\mathfrak{l}_+$. Hence $\mathfrak{l}_+$ cuts through this $\mathbf{L}_1$-tile from right to left, and this tile must be the $(f_{11}+f_{21})(=f_1)$th tile cut by $\mathfrak{l}_+$, that is, the tile $\bar\rho_0(W_{f_1})\mathbf{L}_{\mathbf{s}'_{f_1}}$. In particular, $\mathbf{s}'_{f_1}=1$ and the top left corner of $\bar\rho_0(F)\mathbf{L}_1$ is $\bar\rho_0(W_{f_1})(-1,0)$. It follows that the top boundary of $\bar\rho_0(F)\tilde{\mathbf{L}}\cap\bar\rho_0(W_{f_1})\mathbf{L}_{\mathbf{s}_{f_1}}$ lies in both the top boundary of $\bar\rho_0(F)\tilde{\mathbf{L}}$ and the top boundary of $\bar\rho_0(W_{f_1})\mathbf{L}_{\mathbf{s}_{f_1}}$. Also the left boundary of $\bar\rho_0(F)\tilde{\mathbf{L}}\cap\bar\rho_0(W_{f_1})\mathbf{L}_{\mathbf{s}_{f_1}}$ lies in both the left boundary of $\bar\rho_0(F)\tilde{\mathbf{L}}$ and the left boundary of $\bar\rho_0(W_{f_1})\mathbf{L}_{\mathbf{s}_{f_1}}$. It can be seen that the bottom boundary of $\bar\rho_0(F)\tilde{\mathbf{L}}\cap\bar\rho_0(W_{f_1})\mathbf{L}_{\mathbf{s}_{f_1}}$ lies in the bottom boundary of $\bar\rho_0(F)\tilde{\mathbf{L}}$ and does not touch the bottom boundary of $\bar\rho_0(W_{f_1})\mathbf{L}_{\mathbf{s}_{f_1}}$. It can also be seen that the right boundary of $\bar\rho_0(F)\tilde{\mathbf{L}}\cap\bar\rho_0(W_{f_1})\mathbf{L}_{\mathbf{s}_{f_1}}$ lies in the right boundary of $\bar\rho_0(W_{f_1})\mathbf{L}_{\mathbf{s}_{f_1}}$. See Figure 5.1(5).

Now consider $\ell\in[1{\uparrow}(f_1-1)]$. We shall show that the left and right borders of $\bar\rho_0(F)\tilde{\mathbf{L}}\cap\bar\rho_0(W_\ell)\mathbf{L}_{\mathbf{s}_\ell}$ lie in the left and right borders of $\bar\rho_0(W_\ell)\mathbf{L}_{\mathbf{s}_\ell}$, respectively, and that the top and bottom borders of $\bar\rho_0(F)\tilde{\mathbf{L}}\cap\bar\rho_0(W_\ell)\mathbf{L}_{\mathbf{s}_\ell}$ lie in the top and bottom borders of $\bar\rho_0(F)\tilde{\mathbf{L}}$, respectively. We shall also show that if the top border of $\bar\rho_0(F)\tilde{\mathbf{L}}\cap\bar\rho_0(W_\ell)\mathbf{L}_{\mathbf{s}_\ell}$ lies in the top border of $\bar\rho_0(F)\tilde{\mathbf{L}}$ then $\ell=f_1-1$ and $\mathbf{s}_{f_1-1}=2$. We consider two cases.

Case 1. $\mathbf{s}_\ell=1$.

Here, $\mathfrak{l}_+$ cuts through $\bar\rho_0(W_\ell)\mathbf{L}_{\mathbf{s}'_\ell}=\bar\rho_0(W_\ell)\mathbf{L}_1$ from right to left and subsequently crosses the incident C-labelled line segment joining $\bar\rho_0(W_\ell)(-1,0)$ to $\bar\rho_0(W_\ell)(-1,-1)$; see Figure 5.1(3). Hence this C-labelled line segment cuts through $\bar\rho_0(F)\tilde{\mathbf{L}}$ from top to bottom. It follows that the left and right borders of $\bar\rho_0(F)\tilde{\mathbf{L}}\cap\bar\rho_0(W_\ell)\mathbf{L}_{\mathbf{s}_\ell}$ lie in the left and right borders of $\bar\rho_0(W_\ell)\mathbf{L}_{\mathbf{s}_\ell}$, respectively, and that the top and bottom borders of $\bar\rho_0(F)\tilde{\mathbf{L}}\cap\bar\rho_0(W_\ell)\mathbf{L}_{\mathbf{s}_\ell}$ lie in the top and bottom borders of $\bar\rho_0(F)\tilde{\mathbf{L}}$, respectively; see Figure 5.1(3).

If the top border of $\bar\rho_0(W_\ell)\mathbf{L}_1$ meets (the top border of) $\bar\rho_0(F)\tilde{\mathbf{L}}$, then $\bar\rho_0(W_\ell)(-1,0)$ lies in $\bar\rho_0(F)\tilde{\mathbf{L}}$ and this implies that $\ell=f_1$, a contradiction; hence the top border of $\bar\rho_0(F)\tilde{\mathbf{L}}$ does not meet $\bar\rho_0(W_\ell)\mathbf{L}_1$.

Case 2. $\mathbf{s}_\ell=2$.

Here, $\mathfrak{l}_+$ cuts through $\bar\rho_0(W_\ell)\mathbf{L}_{\mathbf{s}_\ell}=\bar\rho_0(W_\ell)\mathbf{L}_2$ from right to left and subsequently crosses the incident A-labelled line segment joining $\bar\rho_0(W_\ell)(-1,-1)$ to $\bar\rho_0(W_\ell)(0,-1)$; see Figure 5.1(3). Hence this A-labelled line segment cuts through $\bar\rho_0(F)\tilde{\mathbf{L}}$ from top to bottom. It follows that the left and right borders of $\bar\rho_0(F)\tilde{\mathbf{L}}\cap\bar\rho_0(W_\ell)\mathbf{L}_{\mathbf{s}_\ell}$ lie in the left and right borders of $\bar\rho_0(W_\ell)\mathbf{L}_{\mathbf{s}_\ell}$, respectively, and that the top and bottom borders of $\bar\rho_0(F)\tilde{\mathbf{L}}\cap\bar\rho_0(W_\ell)\mathbf{L}_{\mathbf{s}_\ell}$ lie in the top and bottom borders of $\bar\rho_0(F)\tilde{\mathbf{L}}$, respectively; see Figure 5.1(3).

If the top border of $\bar\rho_0(W_\ell)\mathbf{L}_2$ meets (the top border of) $\bar\rho_0(F)\tilde{\mathbf{L}}$, then $\bar\rho_0(W_\ell)(-1,-1)$ lies in $\bar\rho_0(F)\tilde{\mathbf{L}}$, and this implies that $\ell=f_1-1$.

Thus the desired result holds in both cases.

Notice that $(0,-1)$ is the bottom left corner of $\mathbf{L}_2$; see Figure 5.1(3). It follows that the bottom left corner of $\bar\rho_0(F)\mathbf{L}_2$ coincides with the bottom left corner of an $\mathbf{L}_2$-tile incident to $\bar\rho_0(F)(0,-1)=(-f_{12},-f_{22})$, and it must be the $(f_{12}+f_{22})(=f_2)$th tile cut by $\bar\rho_0(F)\tilde{\mathbf{L}}$

and $\mathfrak{l}_+$, that is, the tile $\bar{\rho}_0(W_{f_2})\mathbf{L}_{\mathbf{s}'_{f_2}}$. In particular, $\mathbf{s}'_{f_2} = 2$ and the bottom left corner of $\bar{\rho}_0(F)\mathbf{L}_2$ is $\bar{\rho}_0(W_{f_2})(0,-1)$, that is, $\bar{\rho}_0(F)(0,-1) = (-f_{12}, -f_{22}) = \bar{\rho}_0(W_{f_2})(0,-1)$.

It is now clear that, for all $\ell \in [1\!\uparrow\! f_1]$, $\bar{\rho}_0(F)\mathbf{L} \cap \bar{\rho}_0(W_\ell)\mathbf{L} = \bar{\rho}_0(F)\mathbf{L}_{i_\ell} \cap \bar{\rho}_0(W_\ell)\mathbf{L}_{\mathbf{s}_\ell}$, and the borders are as claimed.

It remains to show that $\mathbf{s}_{[\![1\uparrow f_1]\!]} = \mathbf{s}'_{[\![1\uparrow f_1]\!]}$, that is, ${}^F X_1 = W_{f_1}$. Recall that $X_1 = BC$ and $X_2 = BA$. The A-labelled line segment in $\hat{\mathbb{C}}^- \cup \hat{\mathbb{Q}}$ joining $(0,0)^- = \infty = \widehat{\rho}_0((CBA)^{\pm\infty})$ to $(-1,0)^- = 0 = \widehat{\rho}_0((ACB)^{\pm\infty}) = \widehat{\rho}_0((BC)(CBA)^{\pm\infty})$ lies in $\mathbf{L}_1^-$ and is carried by $\rho_0(F)$ to an arc from $(0,0)^- = \widehat{\rho}_0((CBA)^{\pm\infty})$ to $(-f_{11}, -f_{21})^- = \widehat{\rho}_0({}^F(BC)(CBA)^{\pm\infty})$. This arc reads the word $W_{f_1-2}B$ and then, inside $\rho_0(W_{f_1-2}B)\mathbf{\Delta}^-$, the arc reaches the vertex between the C-labelled edge and the A-labelled edge, that is, it then reads $(CBA)^{\pm\infty}$; see Figure 5.1(6). Hence, ${}^F(BC)(CBA)^{\pm\infty} = W_{f_1-2}B(CBA)^{\pm\infty}$. Also, $W_{f_1} = W_{f_1-2}BABC$; see Figure 5.1(6), again. Hence,

$${}^F(BC)(CBA)^{\pm\infty} = W_{f_1-2}B(CBA)^{\pm\infty} = W_{f_1-2}BABC(CBA)^{\pm\infty} = W_{f_1}(CBA)^{\pm\infty}.$$

Thus, $\Pi X_{\mathbf{s}'_{[\![1\uparrow f_1]\!]}}((CBA)^{\pm\infty}) = \Pi X_{\mathbf{s}_{[\![1\uparrow f_1]\!]}}((CBA)^{\pm\infty})$. Since $\mathbf{s}'_{f_1} = 1 = \mathbf{s}_{f_1}$, it follows that $\Pi X_{\mathbf{s}'_{[\![1\uparrow f_1]\!]}} = \Pi X_{\mathbf{s}_{[\![1\uparrow f_1]\!]}}$, and, hence, $\mathbf{s}'_{[\![1\uparrow f_1]\!]} = \mathbf{s}_{[\![1\uparrow f_1]\!]}$, as desired. □

The following gives all the claims in Profile 3.2.

5.4 Theorem. *Let* Notation 5.1 *hold. For each $i \in \{1,2\}$, $(\rho_F(U_\ell)\mathfrak{a}_{\mathbf{s}_\ell} \mid \ell \in [\![1\!\uparrow\! f_i]\!])$ is a fracturing of $\mathfrak{a}_i$, the elements of $\mathfrak{a}_1$ are codified as the ends of the $\mathfrak{a}_1$-accepting tree of ∂^--syllables, and the limit set of the ∂^--semigroup acting on $\hat{\mathbb{C}}_{\rho_F}$ is $\mathfrak{a}_1$.*

Proof. Notice that $\mathrm{CT}(\rho_0(F)\mathbf{L}^-) = \rho_F(F)\,\mathrm{CT}(\mathbf{L}^-) = \rho_F(F)\mathfrak{a}_1$ is an arc.

Lemma 5.3(i) implies that $(\mathrm{CT}(\rho_0(F)\mathbf{L}^- \cap \rho_0(FU_\ell)\mathbf{L}^-_{\mathbf{s}_\ell}) \mid \ell \in [\![1\!\uparrow\! f_1]\!])$ is a fracturing of $\mathrm{CT}(\rho_0(F)\mathbf{L}^-) = \rho_F(F)\mathfrak{a}_1$. Since CT is constant on the closures of minus-foliation curves in $\hat{\mathbb{C}}^-$, Lemma 5.3(ii) implies that, for each $\ell \in [1\!\uparrow\! f_1]$,

$$\mathrm{CT}(\rho_0(F)\mathbf{L}^- \cap \rho_0(FU_\ell)\mathbf{L}^-_{\mathbf{s}_\ell}) = \mathrm{CT}(\rho_0(FU_\ell)\mathbf{L}^-_{\mathbf{s}_\ell}) = \rho_F(FU_\ell)\,\mathrm{CT}(\mathbf{L}^-_{\mathbf{s}_\ell}) = \rho_F(FU_\ell)\mathfrak{a}_{\mathbf{s}_\ell}.$$

Hence $(\rho_F(FU_\ell)\mathfrak{a}_{\mathbf{s}_\ell} \mid \ell \in [\![1\!\uparrow\! f_1]\!])$ is a fracturing of $\rho_F(F)\mathfrak{a}_1$. On applying $\rho_F(F^{-1})$, we see that $(\rho_F(U_\ell)\mathfrak{a}_{\mathbf{s}_\ell} \mid \ell \in [\![1\!\uparrow\! f_1]\!])$ is a fracturing of $\mathfrak{a}_1$.

Hence, $(\rho_F(FU_\ell)\mathfrak{a}_{\mathbf{s}_\ell} \mid \ell \in [\![1\!\uparrow\! f_2]\!])$ is a fracturing of $\bigcup\limits_{\ell\in[1\uparrow f_2]} \mathrm{CT}(\rho_0(F)\mathbf{L}^- \cap \rho_0(FU_\ell)\mathbf{L}^-_{\mathbf{s}_\ell})$.

Lemma 5.3(ii) implies that

$$\bigcup_{\ell\in[1\uparrow f_2]} \mathrm{CT}(\rho_0(F)\mathbf{L}^- \cap \rho_0(FU_\ell)\mathbf{L}^-_{\mathbf{s}_\ell}) = \bigcup_{\ell\in[1\uparrow f_2]} \mathrm{CT}(\rho_0(F)\mathbf{L}^-_{-1} \cap \rho_0(FU_\ell)\mathbf{L}^-_{\mathbf{s}_\ell}) \subseteq \mathrm{CT}(\rho_0(F)\mathbf{L}^-_{-1}),$$

$$\bigcup_{\ell\in[(f_2+1)\uparrow f_1]} \mathrm{CT}(\rho_0(F)\mathbf{L}^- \cap \rho_0(FU_\ell)\mathbf{L}^-_{\mathbf{s}_\ell}) = \bigcup_{\ell\in[1\uparrow f_2]} \mathrm{CT}(\rho_0(F)\mathbf{L}^-_{-2} \cap \rho_0(FU_\ell)\mathbf{L}^-_{\mathbf{s}_\ell}) \subseteq \mathrm{CT}(\rho_0(F)\mathbf{L}^-_{-2}).$$

Since the subarcs $\mathrm{CT}(\rho_0(F)\mathbf{L}^-_{-1})$ and $\mathrm{CT}(\rho_0(F)\mathbf{L}^-_{-2})$ overlap in a single point, it follows that $(\rho_F(FU_\ell)\mathfrak{a}_{\mathbf{s}_\ell} \mid \ell \in [\![1{\uparrow}f_2]\!])$ is a fracturing of $\mathrm{CT}(\rho_0(F)\mathbf{L}^-_{-1}) = \mathrm{CT}(\rho_0(F)\mathbf{L}^-_2) = \rho_F(F)\mathfrak{a}_2$. Thus $(\rho_F(U_\ell)\mathfrak{a}_{\mathbf{s}_\ell} \mid \ell \in [\![1{\uparrow}f_2]\!])$ is a fracturing of $\mathfrak{a}_2$.

As in Definitions 3.6, we can use a tree to codify the elements of $\mathfrak{a}_1$ as infinite words in the ∂^--syllables.

To show that the limit set of the ∂^--semigroup is $\mathfrak{a}_1$, it remains to show that, for each $\ell \in [1{\uparrow}f_1]$, $\rho_F(U_\ell)\mathfrak{a}_1 \subseteq \mathfrak{a}_1$, or, equivalently, $\rho_F(FU_\ell)\mathfrak{a}_1 \subseteq \rho_F(F)\mathfrak{a}_1$. If $s_\ell = 1$, then

$$\rho_F(FU_\ell)\mathfrak{a}_1 = \rho_F(FU_\ell)\mathfrak{a}_{\mathbf{s}_\ell} \subseteq \rho_F(F)\mathfrak{a}_1.$$

Notice that $\mathfrak{a}_2 \cup \rho_F(X_2)\mathfrak{a}_1 \supseteq \mathfrak{a}_1$ since $\mathrm{CT}(\mathbf{L}^-_2 \cup \rho_F(X_2)\mathbf{L}^-) \supseteq \mathrm{CT}(\mathbf{L}^-)$; see Figure 5.1(5). If $s_\ell = 2$, then $\ell \in [1{\uparrow}(f_1 - 1)]$, and

$$\rho_F(FU_\ell)\mathfrak{a}_1 \subseteq \rho_F(FU_\ell)\mathfrak{a}_2 \cup \rho_F(FU_\ell X_2)\mathfrak{a}_1 = \rho_F(FU_\ell)\mathfrak{a}_{\mathbf{s}_\ell} \cup \rho_F(FU_{\ell+1})\mathfrak{a}_1,$$

and, by a reverse induction hypothesis, this lies in $\rho_F(F)\mathfrak{a}_1$. □

We now turn to $\mathfrak{a}_{-1}$ and apply a very similar argument.

5.5 Lemma. *Let* Notation 5.1 *hold.*

(i) *Let* $\mathrm{CW}(\rho_0(G)\mathbf{L}^+)$ *denote* $\rho_0(G)\mathbf{L}^+$ *endowed with the CW-structure that is induced by the* $\mathbf{L}^+_{[(-2){\uparrow}(-1)]}$*-tessellation of* $\hat{\mathbb{C}}^+ \cup \hat{\mathbb{Q}}$. *Here, the two-cells of* $\mathrm{CW}(\rho_0(G)\mathbf{L}^+)$ *are the terms of the sequence* $(\rho_0(G)\mathbf{L}^+ \cap \rho_0(GV_\ell)\mathbf{L}^+ | \ell \in [\![1{\uparrow}f_{-1}]\!])$, *consecutive terms overlap on a plus-foliation one-cell, and non-consecutive distinct terms are disjoint. Each plus-foliation one-cell of* $\mathrm{CW}(\rho_0(G)\mathbf{L}^+)$ *lies in the boundary of an* $\mathbf{L}^+$*-tile. Each minus-foliation one-cell of* $\mathrm{CW}(\rho_0(G)\mathbf{L}^+)$ *lies in the boundary of* $\rho_0(G)\mathbf{L}^+$*.*

(ii) *Let* $\ell \in [1{\uparrow}f_{-1}]$ *and define* $j_\ell \coloneqq 1$ *for* $\ell \in [1{\uparrow}f_{-2}]$, *and* $j_\ell \coloneqq 2$ *for* $\ell \in [(f_{-2}+1){\uparrow}f_{-1}]$. *Then* $\rho_0(G)\mathbf{L}^+ \cap \rho_0(GV_\ell)\mathbf{L}^+ = \rho_0(G)\mathbf{L}^+_{j_\ell} \cap \rho_0(GV_\ell)\mathbf{L}^+_{\mathbf{t}_\ell} = (\bar\rho_0(G)\mathbf{L}_{j_\ell} \cap \bar\rho_0(GV_\ell)\mathbf{L}_{\mathbf{t}_\ell})^+$ *and* $\bar\rho_0(G)\mathbf{L}_{j_\ell} \cap \bar\rho_0(GV_\ell)\mathbf{L}_{\mathbf{t}_\ell}$ *is a parallelogram. The top and bottom borders of* $\rho_0(G)\mathbf{L}^+_{j_\ell} \cap \rho_0(GV_\ell)\mathbf{L}^+_{\mathbf{t}_\ell}$ *lie in the top and bottom borders of* $\rho_0(GV_\ell)\mathbf{L}^+_{\mathbf{t}_\ell}$*, respectively. The left and right borders of* $\rho_0(G)\mathbf{L}^+_{j_\ell} \cap \rho_0(GV_\ell)\mathbf{L}^+_{\mathbf{t}_\ell}$ *lie in the left and right borders of* $\rho_0(G)\mathbf{L}^+_{j_\ell}$*, respectively. If* $\ell \in [1{\uparrow}(f_{-1}-1)]$*, then* $\mathbf{t}_\ell = [\ell(-\mu_-)] - [(\ell-1)(-\mu_-)] - 2$*; also,* $\mathbf{t}_{f_{-1}} = -1$*.*

Proof. For the current purpose of studying the $\mathbf{L}^+_{[(-2){\uparrow}(-1)]}$-tessellation of $\rho_0(G)\mathbf{L}^+$, we shall treat $\mathbf{M}_-$ and $\mathbf{M}^+_-$ as essentially interchangeable. We understand that the following discussion takes place in $\hat{\mathbb{C}}^+ \cup \hat{\mathbb{Q}}$ but with the notation of $\mathbb{R}^2$.

The half-minus-line $\mathfrak{l}_-$ starts from within $\mathbf{\Delta}$ and exits $\mathbf{\Delta}$ through the infinitesimal beginning of a B-labelled line segment. Let $\mathbf{t}''_{[\![1{\uparrow}\infty]\!]}$ denote the sequence in $\{-2,-1\}$ with the property that the element of $\mathfrak{C}\langle\, A, B, C\,\rangle$ read by $\mathfrak{l}_-$ is $\Pi X_{\mathbf{t}''_{[\![1{\uparrow}\infty[\![}}$. Thus $\mathbf{t}''_1 = -2$.

We will now show that,

(5.3) for each $\ell \in [1{\uparrow}\infty[$, $\mathbf{t}''_\ell = [\ell(-\mu_-)] - [(\ell-1)(-\mu_-)] - 2.$

Let $\ell \in [0{\uparrow}\infty[$. Consider the line in $\mathbb{R}^2$ with equation $y = -\ell$, which is a concatenation of A-labelled line segments. Consider also the line with equation $x = \mu_- y - 1$, which is the minus-line through $(-1, 0)$. Let (x_ℓ, y_ℓ) denote the point where these two lines intersect. Then $(x_\ell, y_\ell) = (-\ell\mu_- - 1, -\ell)$. Thus $\mathfrak{l}_-$ begins at (x_0, y_0), and, if $\ell \geqslant 1$, then (x_ℓ, y_ℓ) is the point where $\mathfrak{l}_-$ reads A for the ℓth time. After leaving (x_ℓ, y_ℓ), either $\mathfrak{l}_-$ reads B and $\mathfrak{l}_-$ reaches $(x_{\ell+1}, y_{\ell+1})$ and reads A and we have $[x_{\ell+1}] = [x_\ell]$, or $\mathfrak{l}_-$ reads C and $\mathfrak{l}_-$ reaches $(x_{\ell+1}, y_{\ell+1})$ and reads A and we have $[x_{\ell+1}] = [x_\ell] + 1$; see Figures 4.2 and 5.2(2). Thus, either $[x_{\ell+1}] - [x_\ell] = 0$ and $\mathbf{t}''_{\ell+1} = -2$, or $[x_{\ell+1}] - [x_\ell] = 1$ and $\mathbf{t}''_{\ell+1} = -1$; hence, $\mathbf{t}''_{\ell+1} = [x_{\ell+1}] - [x_\ell] - 2$. Now (5.3) follows.

Let $\ell \in [1{\uparrow}\infty[$ and redefine $W_\ell \coloneqq \Pi X_{\mathbf{t}''_{[\![1\uparrow(\ell-1)]\!]}}$.

We shall now prove that the sequence of $\mathbf{L}_{[(-2)\uparrow(-1)]}$-tiles cut through by $\mathfrak{l}_-$ from top to bottom is $(\bar\rho_0(W_\ell)\mathbf{L}_{\mathbf{t}''_\ell} | \ell \in [\![1{\uparrow}\infty[\![\,)$.

Now, either $\ell = 1$ and $\mathfrak{l}_-$ originates in $\bar\rho_0(W_\ell)\mathbf{\Delta}$, or $\ell \geqslant 2$ and $\mathfrak{l}_-$ enters $\bar\rho_0(W_\ell)\mathbf{\Delta}$ reading an A-labelled line segment. In either event, if $\mathbf{t}''_\ell = -2$, then $\mathfrak{l}_-$ exits $\bar\rho_0(W_\ell)\mathbf{\Delta}$ reading a B-labelled line segment, and if $\mathbf{t}''_\ell = -1$, then $\mathfrak{l}_-$ exits $\bar\rho_0(W_\ell)\mathbf{\Delta}$ reading a C-labelled line segment. From Figures 4.2 and 5.2(2), we see that, in all cases, the initial point of $\mathfrak{l}_- \cap \bar\rho_0(W_\ell)\mathbf{\Delta}$ lies in $\bar\rho_0(W_\ell)\mathbf{L}_{\mathbf{t}''_\ell}$, and $\mathfrak{l}_-$ cuts through $\bar\rho_0(W_\ell)\mathbf{L}_{\mathbf{t}''_\ell}$ from top to bottom. It follows that $\bar\rho_0(W_\ell)\mathbf{L}_{\mathbf{t}''_\ell}$ is the ℓth $\mathbf{L}_{[(-2)\uparrow(-1)]}$-tile cut through by $\mathfrak{l}_-$, as desired. Since $\bar\rho_0(X_{-1})(x, y) = (x+1, y-1)$ and $\bar\rho_0(X_{-2})(x, y) = (x, y-1)$, we see that both $\bar\rho_0(W_\ell)(-1, 0)$ and $\bar\rho_0(W_\ell)(0, 0)$ have $1 - \ell$ as the second coordinate.

Define $\mathbf{t}'_{[\![1\uparrow f_1]\!]}$ to be the concatenation of $\mathbf{t}''_{[\![1\uparrow(f_1-1)]\!]}$ and (-1).

We shall now prove that $\bar\rho_0(G)\tilde{\mathbf{L}}$ cuts through $(\bar\rho_0(W_\ell)\mathbf{L}_{\mathbf{t}'_\ell} \mid \ell \in [\![1{\uparrow}f_1]\!])$.

Notice that $\tilde{\mathbf{L}} \cap \mathbb{Z}^2$, which is $\{(0,0), (-1,0), (0,-1)\}$, lies in the boundary of $\tilde{\mathbf{L}}$. Hence the boundary of $\bar\rho_0(G)\tilde{\mathbf{L}}$ contains $\bar\rho_0(G)(\tilde{\mathbf{L}} \cap \mathbb{Z}^2) = \bar\rho_0(G)(\tilde{\mathbf{L}}) \cap \bar\rho_0(G)(\mathbb{Z}^2) = \bar\rho_0(G)(\tilde{\mathbf{L}}) \cap \mathbb{Z}^2$. In particular, the interior of $\bar\rho_0(G)\tilde{\mathbf{L}}$ does not meet $\mathbb{Z}^2$.

Since $\mathfrak{l}_-$ contains the left border of $\tilde{\mathbf{L}}$, we see that $\bar\rho_0(G)\mathfrak{l}_- = \mathfrak{l}_-$ contains the left border of $\bar\rho_0(G)\tilde{\mathbf{L}}$. Since the interior of $\bar\rho_0(G)\tilde{\mathbf{L}}$ does not meet $\mathbb{Z}^2$, we see that any labelled line segment whose interior cuts through $\mathfrak{l}_- \cap \bar\rho_0(G)\tilde{\mathbf{L}}$ from right to left must cut through $\bar\rho_0(G)\tilde{\mathbf{L}}$ from right to left.

Notice that the bottom border of $\tilde{\mathbf{L}}$ contains $(0, -1)$; see Figure 5.2(2). Hence, the bottom border of $\bar\rho_0(G)\tilde{\mathbf{L}}$ contains $\bar\rho_0(G)(0, -1) = (f_2 - 1, -f_1)$ which is an end-point of the B-labelled line segment joining $(f_2, -f_1 + 1)$ to $(f_2 - 1, -f_1)$. The other end-point of the B-labelled line segment does not lie in $\bar\rho_0(G)\tilde{\mathbf{L}}$, and, hence, the B-labelled line segment must exit the left border of $\bar\rho_0(G)\tilde{\mathbf{L}}$. Hence, the left border of $\bar\rho_0(G)\tilde{\mathbf{L}}$ cuts through an $\mathbf{L}_{-2}$-tile incident to $(f_2, -f_1 + 1)$, and this must then be the f_1st $\tilde{\mathbf{L}}_{[(-2)\uparrow(-1)]}$-tile cut by $\mathfrak{l}_-$, that is, $\bar\rho_0(W_{f_1})\mathbf{L}_{\mathbf{t}''_{f_1}}$, In particular, $\mathbf{t}''_{f_1} = -2$. Now $\bar\rho_0(G)(0, -1)$ is the lower left corner of $\bar{\rho}_0(W_{f_1})\mathbf{L}_{-1} = \bar{\rho}_0(W_{f_1})\mathbf{L}_{\mathbf{t}'_{f_1}}$, and is also the lower left corner of $\bar{\rho}_0(G)\tilde{\mathbf{L}}$. Hence $\bar{\rho}_0(G)\tilde{\mathbf{L}} \cap \bar{\rho}_0(W_{f_1})\mathbf{L} = \bar{\rho}_0(G)\mathbf{L}_{-1} \cap \bar{\rho}_0(W_{f_1})\mathbf{L}_{\mathbf{t}'_{f_1}}$, and the bottom border of $\bar{\rho}_0(W_{f_1})\mathbf{L}_{\mathbf{t}'_{f_1}}$ contains the bottom border of $\bar{\rho}_0(G)\mathbf{L}_{-1}$. Moreover, the top and bottom borders of $\bar{\rho}_0(G)\mathbf{L}_{-1} \cap \bar{\rho}_0(W_{f_1})\mathbf{L}_{\mathbf{t}'_{f_1}}$ lie in the top and bottom borders of $\bar{\rho}_0(W_{f_1})\mathbf{L}_{\mathbf{t}'_{f_1}}$, respectively, and the right and left borders of $\bar{\rho}_0(G)\mathbf{L}_{-1} \cap \bar{\rho}_0(W_{f_1})\mathbf{L}_{\mathbf{t}'_{f_1}}$ lie in the right and left borders of $\bar{\rho}_0(G)\mathbf{L}_{-1}$, respectively; see Figure 5.2(3).

Now consider $\ell \in [1{\uparrow}(f_1 - 1)]$. We shall show that the left and right borders of $\bar{\rho}_0(G)\tilde{\mathbf{L}} \cap \bar{\rho}_0(W_\ell)\mathbf{L}_{\mathbf{t}''_\ell}$ lie in the left and right borders of $\bar{\rho}_0(G)\tilde{\mathbf{L}}$, respectively, and that the top and bottom borders of $\bar{\rho}_0(G)\tilde{\mathbf{L}} \cap \bar{\rho}_0(W_\ell)\mathbf{L}_{\mathbf{t}''_\ell}$ lie in the top and bottom borders of $\bar{\rho}_0(W_\ell)\mathbf{L}_{\mathbf{t}''_\ell}$, respectively. We consider two cases.

Case 1. $\mathbf{t}''_\ell = -1$.

Here, $\mathfrak{l}_-$ cuts through $\bar{\rho}_0(W_\ell)\mathbf{L}_{\mathbf{t}''_\ell} = \bar{\rho}_0(W_\ell)\mathbf{L}_{-1}$ from top to bottom and crosses the embedded C-labelled line segment joining $\bar{\rho}_0(W_\ell)(0,0)$ to $\bar{\rho}_0(W_\ell)(0,-1)$; see Figure 5.2(2). Hence this C-labelled line segment cuts through $\bar{\rho}_0(G)\tilde{\mathbf{L}}$ from right to left. It follows that the right and left borders of $\bar{\rho}_0(G)\tilde{\mathbf{L}} \cap \bar{\rho}_0(W_\ell)\mathbf{L}_{\mathbf{t}''_\ell}$ lie in the right and left borders of $\bar{\rho}_0(G)\tilde{\mathbf{L}}$, respectively, and that the top and bottom borders of $\bar{\rho}_0(G)\tilde{\mathbf{L}} \cap \bar{\rho}_0(W_\ell)\mathbf{L}_{\mathbf{t}''_\ell}$ lie in the top and bottom borders of $\bar{\rho}_0(W_\ell)\mathbf{L}_{\mathbf{t}''_\ell}$, respectively; see Figure 5.2(2).

Case 2. $\mathbf{t}''_\ell = -2$.

Here, $\mathfrak{l}_-$ cuts through $\bar{\rho}_0(W_\ell)\mathbf{L}_{\mathbf{t}''_\ell} = \bar{\rho}_0(W_\ell)\mathbf{L}_{-2}$ from top to bottom and crosses the incident B-labelled line segment joining $\bar{\rho}_0(W_\ell)(-1,-1)$ to $\bar{\rho}_0(W_\ell)(0,-1)$; see Figure 5.2(2). Hence this B-labelled line segment cuts through $\bar{\rho}_0(G)\tilde{\mathbf{L}}$ from right to left. It follows that the top and bottom borders of $\bar{\rho}_0(G)\tilde{\mathbf{L}} \cap \bar{\rho}_0(W_\ell)\mathbf{L}_{\mathbf{t}''_\ell}$ lie in the top and bottom borders of $\bar{\rho}_0(W_\ell)\mathbf{L}_{\mathbf{t}''_\ell}$, respectively, and that the right and left borders of $\bar{\rho}_0(G)\tilde{\mathbf{L}} \cap \bar{\rho}_0(W_\ell)\mathbf{L}_{\mathbf{t}''_\ell}$ lie in the right and left borders of $\bar{\rho}_0(G)\tilde{\mathbf{L}}$, respectively; see Figure 5.2(2).

Thus the desired result holds in both cases.

Notice that the lower left corner of $\bar{\rho}_0(BC)\mathbf{L}_{-1}$ is $(-1,-1)$ and the right border of $\bar{\rho}_0(BC)\mathbf{L}_{-1}$ is contained in $\mathfrak{l}_-$. Hence, the lower left corner of $\bar{\rho}_0(GBC)\mathbf{L}_{-1}$ is $\bar{\rho}_0(G)(-1,-1) = (f_{1\,2} - 1, -f_{1\,1})$ and the right border of $\bar{\rho}_0(GBC)\mathbf{L}_{-1}$ is contained in $\bar{\rho}_0(G)\mathfrak{l}_- = \mathfrak{l}_-$. Hence, the C-labelled line segment joining $(f_{1\,2}-1, -f_{1\,1}+1)$ to $(f_{1\,2}-1, -f_{1\,1})$ crosses $\bar{\rho}_0(GBC)\mathbf{L}_{-1}$. It follows that the $\mathbf{L}_{-1}$-tile containing the C-labelled line segment is the $f_{1\,1} = f_{-2}$th $\mathbf{L}_{[(-2)\uparrow(-1)]}$-tile cut by $\mathfrak{l}_-$, that is, $\bar{\rho}_0(W_{f_{-2}})\mathbf{L}_{\mathbf{t}''_{f_{-2}}}$. The minus-line through $(-1,-1)$ contains the bottom borders of $\bar{\rho}_0(BC)\mathbf{L}_{-1}$ and $\mathbf{L}_1$. Hence, the minus-line through $\bar{\rho}_0(G)(-1,-1)$ contains the bottom borders of $\bar{\rho}_0(GBC)\mathbf{L}_{-1}$ and $\bar{\rho}_0(G)\mathbf{L}_1$. Hence $\bar{\rho}_0(G)\mathbf{L}_1$ lies in $\bigcup\limits_{[1\uparrow f_{-2}]} \bar{\rho}_0(W_\ell)\mathbf{L}_{\mathbf{t}'_\ell}$ and $\bar{\rho}_0(G)\mathbf{L}_2$ lies in $\bigcup\limits_{[(f_{-2}+1)\uparrow f_{-1}]} \bar{\rho}_0(W_\ell)\mathbf{L}_{\mathbf{t}'_\ell}$.

It is now clear that, for all $\ell \in [1{\uparrow}f_{-1}]$, $\bar{\rho}_0(G)\mathbf{L} \cap \bar{\rho}_0(W_\ell)\mathbf{L} = \bar{\rho}_0(G)\mathbf{L}_{j_\ell} \cap \bar{\rho}_0(W_\ell)\mathbf{L}_{\mathbf{t}'_\ell}$, and the borders are as claimed.

It remains to show that $\mathbf{t}_{[\![1\uparrow f_1]\!]} = \mathbf{t}'_{[\![1\uparrow f_1]\!]}$, that is, ${}^{G}X_{-1} = W_{f_1-1}X_{-1}$. Recall that $X_{-1} = CA$ and $X_{-2} = BA$. The B-labelled line segment in $\hat{\mathbb{C}}^{+} \cup \hat{\mathbb{Q}}$ that joins

$$(-1,0)^{+} = 0 = \widehat{\rho}_0((ACB)^{\pm\infty}) \quad \text{to} \quad (0,-1)^{+} = 1 = \widehat{\rho}_0((BAC)^{\pm\infty}) = \widehat{\rho}_0(CA(ACB)^{\pm\infty})$$

lies in $\tilde{\mathbf{L}}^{+}$ and is carried by $\rho_0(G)$ to an arc from

$$(-1,0)^{+} = \widehat{\rho}_0((ACB)^{\pm\infty}) \quad \text{to} \quad (f_2-1,-f_1)^{+} = \widehat{\rho}_0({}^{G}(CA)(ACB)^{\pm\infty}).$$

The arc reads W_{f_1-1} and then in $\rho_0(W_{f_1-1})\mathbf{\Delta}^{+}$ it enters the vertex between the edges labelled B and C, and, hence, then reads $(BAC)^{\pm\infty}$; see Figure 5.2(3). Thus

$${}^{G}(CA)((ACB)^{\pm\infty}) = W_{f_1-1}((BAC)^{\pm\infty}) = W_{f_1-1}CA((ACB)^{\pm\infty}).$$

Since ${}^{G}X_{-1}$ has length f_1 and ends in X_{-1}, we see that ${}^{G}X_{-1} = W_{f_1-1}X_{-1}$, as desired. □

The following gives all the claims in Profile 3.2.

5.6 Theorem. *Let* Notation 5.1 *hold. For each* $j \in \{-2,-1\}$, $(\rho_F(V_\ell)\mathfrak{a}_{\mathbf{t}_\ell} \mid \ell \in [\![1\uparrow f_j]\!])$ *is a fracturing of* $\mathfrak{a}_j$*, the elements of* $\mathfrak{a}_{-1}$ *are codified as the ends of the* $\mathfrak{a}_{-1}$*-accepting tree of* ∂^{+}*-syllables, and the limit set of the* ∂^{+}*-semigroup acting on* $\hat{\mathbb{C}}_{\rho_F}$ *is* $\mathfrak{a}_{-1}$.

Proof. Notice that $\mathrm{CT}(\rho_0(G)\mathbf{L}^{+}) = \rho_F(G)\,\mathrm{CT}(\mathbf{L}^{+}) = \rho_F(G)\mathfrak{a}_{-1}$ is an arc.

Lemma 5.5(i) implies that $(\mathrm{CT}(\rho_0(G)\mathbf{L}^{+} \cap \rho_0(GV_\ell)\mathbf{L}^{+}_{\mathbf{t}_\ell}) \mid \ell \in [\![1\uparrow f_{-1}]\!])$ is a fracturing of $\mathrm{CT}(\rho_0(G)\mathbf{L}^{+}) = \rho_F(G)\mathfrak{a}_{-1}$. Since CT is constant on the closures of plus-foliation curves in $\hat{\mathbb{C}}^{+} \cup \hat{\mathbb{R}}$, Lemma 5.5(ii) implies that, for each $\ell \in [1\uparrow f_{-1}]$,

$$\mathrm{CT}(\rho_0(G)\mathbf{L}^{+} \cap \rho_0(GV_\ell)\mathbf{L}^{+}_{\mathbf{t}_\ell}) = \mathrm{CT}(\rho_0(GV_\ell)\mathbf{L}^{+}_{\mathbf{t}_\ell}) = \rho_F(GV_\ell)\,\mathrm{CT}(\mathbf{L}^{+}_{\mathbf{t}_\ell}) = \rho_F(GV_\ell)\mathfrak{a}_{\mathbf{t}_\ell}.$$

Hence, $(\rho_F(GV_\ell)\mathfrak{a}_{\mathbf{t}_\ell} \mid \ell \in [\![1\uparrow f_{-1}]\!])$ is a fracturing of $\rho_F(G)\mathfrak{a}_{-1}$. On applying $\rho_F(G^{-1})$, we see that $(\rho_F(V_\ell)\mathfrak{a}_{\mathbf{t}_\ell} \mid \ell \in [\![1\uparrow f_{-1}]\!])$ is a fracturing of $\mathfrak{a}_{-1}$.

Hence, $(\rho_F(GV_\ell)\mathfrak{a}_{\mathbf{t}_\ell} \mid \ell \in [\![1\uparrow f_{-2}]\!])$ is a fracturing of $\bigcup\limits_{\ell \in [1\uparrow f_{-2}]} \mathrm{CT}(\rho_0(G)\mathbf{L}^{+} \cap \rho_0(GV_\ell)\mathbf{L}^{+}_{\mathbf{t}_\ell})$.
Lemma 5.5(ii) implies that

$$\begin{aligned}
\bigcup_{\ell \in [1\uparrow f_{-2}]} \mathrm{CT}(\rho_0(G)\mathbf{L}^{+} \cap \rho_0(GV_\ell)\mathbf{L}^{+}_{\mathbf{t}_\ell}) &= \bigcup_{\ell \in [1\uparrow f_{-2}]} \mathrm{CT}(\rho_0(G)\mathbf{L}^{+}_1 \cap \rho_0(GV_\ell)\mathbf{L}^{+}_{\mathbf{t}_\ell}) \\
&\subseteq \mathrm{CT}(\rho_0(G)\mathbf{L}^{+}_1), \\
\bigcup_{\ell \in [(f_{-2}+1)\uparrow f_{-1}]} \mathrm{CT}(\rho_0(G)\mathbf{L}^{+} \cap \rho_0(GV_\ell)\mathbf{L}^{+}_{\mathbf{t}_\ell}) &= \bigcup_{\ell \in [1\uparrow f_{-2}]} \mathrm{CT}(\rho_0(G)\mathbf{L}^{+}_2 \cap \rho_0(GV_\ell)\mathbf{L}^{+}_{\mathbf{t}_\ell}) \\
&\subseteq \mathrm{CT}(\rho_0(G)\mathbf{L}^{+}_2).
\end{aligned}$$

Since the subarcs $\mathrm{CT}(\rho_0(G)\mathbf{L}_1^+)$ and $\mathrm{CT}(\rho_0(G)\mathbf{L}_2^+)$ overlap in a single point, it follows that $(\rho_F(GV_\ell)\mathfrak{a}_{\mathbf{t}_\ell} \mid \ell \in [\![1{\uparrow}f_{-2}]\!])$ is a fracturing of $\mathrm{CT}(\rho_0(G)\mathbf{L}_1^+)$ which equals $\mathrm{CT}(\rho_0(G)\mathbf{L}_{-2}^+)$ which equals $\rho_F(G)\mathfrak{a}_{-2}$. Thus $(\rho_F(V_\ell)\mathfrak{a}_{\mathbf{t}_\ell} \mid \ell \in [\![1{\uparrow}f_{-2}]\!])$ is a fracturing of $\mathfrak{a}_{-2}$.

As in Definitions 3.6, we can use a tree to codify the elements of $\mathfrak{a}_{-1}$ as infinite words in the ∂^+-syllables. To show that the limit set of the ∂^+-semigroup in $\hat{\mathbb{C}}_{\rho_F}$ is $\mathfrak{a}_{-1}$, it remains to show that, for each $\ell \in [1{\uparrow}f_{-1}]$, $\rho_F(V_\ell)\mathfrak{a}_{-1} \subseteq \mathfrak{a}_{-1}$, or, equivalently, $\rho_F(GV_\ell)\mathfrak{a}_{-1} \subseteq \rho_F(G)\mathfrak{a}_{-1}$. If $\mathbf{t}_\ell = -1$, then

$$\rho_F(GV_\ell)\mathfrak{a}_{-1} = \rho_F(GV_\ell)\mathfrak{a}_{\mathbf{t}_\ell} \subseteq \rho_F(G)\mathfrak{a}_{-1}.$$

Notice that $\mathfrak{a}_{-2}\cup\rho_F(X_{-2})\mathfrak{a}_{-1} \supseteq \mathfrak{a}_{-1}$ since $\mathrm{CT}(\mathbf{L}_{-2}^+\cup\rho_0(X_{-2})\mathbf{L}^+) \supseteq \mathrm{CT}(\mathbf{L}^+)$; see Figure 5.2(4). If $\mathbf{t}_\ell = -2$, then $\ell \in [(f_{-1}-1){\downarrow}1]$, and

$$\rho_F(GV_\ell)\mathfrak{a}_{-1} \subseteq \rho_F(GV_\ell)\mathfrak{a}_{-2} \cup \rho_F(GV_\ell X_{-2})\mathfrak{a}_{-1} = \rho_F(GV_\ell)\mathfrak{a}_{\mathbf{t}_\ell} \cup \rho_F(GV_{\ell+1})\mathfrak{a}_{-1},$$

and, by a reverse induction hypothesis, this lies in $\rho_F(G)\mathfrak{a}_{-1}$. □

We now verify most of Profile 3.4.

5.7 Proposition. *Let* Notation 5.1 *hold. The following hold.*

(i). $U_{[1\uparrow f_1]} = \{W \in \langle A,B,C\rangle F^{-1} \mid$ *the interior of* $\rho_0(W)\mathbf{L}^-$ *meets the interior of* $\mathbf{L}^-\}$.

(ii). $V_{[1\uparrow f_1]} = \{W \in \langle A,B,C\rangle F \mid$ *the interior of* $\rho_0(W)\mathbf{L}^+$ *meets the interior of* $\mathbf{L}^+\}$.

(iii). $(U_{[1\uparrow f_1]})^{-1} = V_{[1\uparrow f_1]}$, *and the inversion involution on* $\langle A,B,C,F\rangle$ *carries the* ∂^-*-semigroup bijectively to the* ∂^+*-semigroup.*

(iv). *For each* $\ell \in [1{\uparrow}f_1]$, $U_\ell^{-1} = V_{F(\ell)}$ *and* $\rho_0(U_\ell^{-1})(\mathrm{col}_\ell(\mathbf{L}^-)) = \mathrm{row}_{F(\ell)}(\mathbf{L}^-)$.

Proof. (i)–(iii) are clear from the preceding results of this section and the fact that complex conjugation carries $\mathbf{L}^+$ to $\mathbf{L}^-$.

(iv). It can be seen that $\mathbf{L}_{-1}^- = \bigcup\limits_{\ell\in[1\uparrow f_2]} \mathrm{col}_\ell(\mathbf{L}^-)$ and $\mathbf{L}_{-2}^- = \bigcup\limits_{\ell\in[(f_2+1)\uparrow f_1]} \mathrm{col}_\ell(\mathbf{L}^-)$; see Figure 5.2(2). The top border of $\mathbf{L}^- = \mathbf{L}_{-1}^- \cup \mathbf{L}_{-2}^-$ is equal to the bottom border of $\rho_0(X_{-2}^{-1})\mathbf{L}_{-2}^- \cup \rho_0(X_{-1}^{-1})\mathbf{L}_{-1}^-$. Hence, the top border of $\bigcup\limits_{\ell\in[1\uparrow f_1]} \mathrm{col}_\ell(\mathbf{L}^-)$ is equal to the bottom border of $\bigcup\limits_{\ell\in[(f_2+1)\uparrow f_1]} \mathrm{col}_\ell(\rho_0(X_{-2}^{-1})\mathbf{L}^-) \cup \bigcup\limits_{\ell\in[1\uparrow f_2]} \mathrm{col}_\ell(\rho_0(X_{-1}^{-1})\mathbf{L}^-)$, and we can see that there is a cyclic shift by f_2. It follows from Lemma 5.3(ii) that the top borders of the first f_1-2 columns of $\mathbf{L}^-$ equal the bottom borders of the columns of $\rho_0(X_{-2}^{-1})\mathbf{L}_{-2}^-$ and $\rho_0(X_{-1}^{-1})\mathbf{L}_{-1}^-$. For each $\ell \in [1{\uparrow}(f_1-f_2-1)]$, the top border of $\mathrm{col}_\ell(\mathbf{L}^-)$ is equal to the bottom border of $\mathrm{col}_{\ell+f_2}(X_{-2}^{-1}\mathbf{L}^-)$, and, for each $\ell \in [(f_1-f_2){\uparrow}(f_1-2)]$, the top border of $\mathrm{col}_\ell(\mathbf{L}^-)$ is equal to the bottom border of $\mathrm{col}_{\ell+f_2-f_1}(\rho_0(X_{-1}^{-1})\mathbf{L}^-)$, and the top border of $\mathrm{col}_{f_1-1}(\mathbf{L}^-)$ is equal to a right segment of the bottom border of $\mathrm{col}_{f_2-1}(\rho_0(X_2^{-1}X_1)\mathbf{L}^-)$.

Let $\ell \in [1{\uparrow}f_1]$. Then $\rho_0(U_\ell^{-1})$ carries $\mathrm{col}_\ell(\mathbf{L}^-) = \mathbf{L}^- \cap \rho_0(U_\ell)\mathbf{L}^-$ bijectively to $\rho_0(U_\ell^{-1})\mathbf{L}^- \cap \mathbf{L}^- \subseteq \mathbf{L}^-$, and this must be one of the rows of $\mathbf{L}^-$. In particular, $\rho_0(U_{f_1}^{-1})$, which is $\rho_0(AFA)$, carries the f_1st column to a row containing 0, and this is the first row. The bottom border of the f_1st column of $\mathbf{L}$ is the top border of the $(f_1 - f_2)$th column of $\rho_0(X_{-2})\mathbf{L}^-$. Hence, $\rho_0(U_{f_2}^{-1})$ carries the $(f_1 - f_2)$th column of $\mathbf{L}^-$ to the second row. At each step, we increase the number of the row by one and decrease the number of the column by f_2 mod f_1. Hence, if $\ell' = F^{-1}(\ell) \equiv -f_2(\ell-1)(\text{mod } f_1)$, then $\rho_0(U_{\ell'}^{-1})$ carries the ℓ'th column of $\mathbf{L}^-$ to the ℓth row of $\mathbf{L}^-$; see Figure 5.1(9) and (8). This proves (iv). $\square$

6 Hausdorff dimension

In this section, we consider techniques that were developed by Dicks-Porti [17] for the case $F = RL$ and adapt them to the case $F = \prod\limits_{i \in [\![1\uparrow p]\!]} (R^{a_i} L^{b_i})$. We use results of Bishop-Jones and Beardon-Maskit to prove that the critical exponent of the ∂^--semigroup equals the Hausdorff dimension of $\partial^- A$. Similarly, the critical exponent of the ∂^+-semigroup equals the Hausdorff dimension of $\partial^+ B$. These four numbers are then equal.

6.1 Definitions. (i). The Möbius action of $\left(\begin{smallmatrix} -1 & \mathbf{j} \\ \mathbf{j} & -1 \end{smallmatrix}\right) \in \mathrm{GL}_2(\mathbb{H})$ on $\hat{\mathbb{H}}$ induces a map

$$\mathbf{H}^3 \cup \hat{\mathbb{C}} \to \mathbb{R}^3, \quad x + y\mathbf{i} + z\mathbf{j} \mapsto \tfrac{(2x,\, 2y,\, x^2+y^2+z^2-1)}{x^2+y^2+z^2+2z+1}, \quad \infty \mapsto (0,0,1),$$

which identifies $\mathbf{H}^3 \cup \hat{\mathbb{C}}$ with the closed ball in $\mathbb{R}^3$ of radius one centered at the origin. The Euclidean metric on $\mathbb{R}^3$ pulls back to the chord metric on $\hat{\mathbb{C}}$, where the distance between elements w and v of $\hat{\mathbb{C}}$ is

$$\operatorname{chord}(w,v) := \tfrac{2|w-v|}{\sqrt{(1+|w|^2)(1+|v|^2)}}\,,$$

with the natural interpretations if one of the points is ∞; see, for example, [34, Theorem 4.2.1].

The stabilizer of $\mathbf{j}$ for the action of $\mathrm{PSL}_2(\mathbb{C})$ on $\hat{\mathbb{H}}$ is $\mathrm{PSU}_2(\mathbb{C}) \simeq \mathrm{SO}_3(\mathbb{R})$, and the action of $\mathrm{PSU}_2(\mathbb{C})$ on $\hat{\mathbb{C}}$ gives the orientation-preserving isometry group; see, for example, [34, Exercises 5.2.11–12].

By (2.1), for any $x \in \,]0,1]$, $\operatorname{dist}(\mathbf{j}, x\mathbf{j}) = \operatorname{arccosh}(\tfrac{1+x^2}{2x}) = \operatorname{arccosh}(\tfrac{2}{\operatorname{chord}(-x,x)})$, and, moreover, among all the points on the geodesic curve in $\mathbf{H}^3$ which has limiting end-points x and $-x$, the point closest to $\mathbf{j}$ is $x\mathbf{j}$.

It then follows that, for any distinct w, v in $\hat{\mathbb{C}}$, if σ denotes the geodesic curve in $\mathbf{H}^3$ which has limiting end-points w and v, then $\operatorname{dist}(\mathbf{j}, \sigma) = \operatorname{arccosh}(\tfrac{2}{\operatorname{chord}(w,v)})$.

(ii). Let $\mathbf{A}$ be a subset of $\mathbb{R}^3$ endowed with the induced metric, for example, $\mathbf{A}$ could be a subset of the sphere with center the origin and radius one, viewed as a subset of $\hat{\mathbb{C}}$ with the chord metric.

Let $\mathcal{R}\mathbf{A}$ denote the set consisting of those sequences $\mathbf{r} = \mathbf{r}_{[\![1\uparrow\infty[\![}$ in $[0,\infty[$ for which there exists some sequence $z_{[\![1\uparrow\infty[\![}$ in $\mathbb{R}^3$ such that, for each $a \in \mathbf{A}$, there exists some $n \in [1\uparrow\infty[$

such that $\operatorname{dist}_{\mathbb{R}^3}(z_n, a) \leqslant \mathbf{r}_n$. The *Hausdorff dimension of* $\mathbf{A}$ is defined as

$$\operatorname{Hdim}(\mathbf{A}) \coloneqq \inf\{\ t \in [0,\infty[\ : (\inf \textstyle\sum \mathcal{R}\mathbf{A}^t) = 0\} \ \in\ [0,\infty],$$

where $\inf \sum \mathcal{R}\mathbf{A}^t \coloneqq \inf\limits_{\mathbf{r}\in\mathcal{R}\mathbf{A}} (\sum \mathbf{r}^t_{[\![1\uparrow\infty[\![}) \ \in\ [0,\infty]$.

(iii). Let $\mathbf{X}$ be any infinite discrete subset of $\mathbf{H}^3$. By enumerating the elements of $\mathbf{X}$, we can think of $\mathbf{X}$ as an infinite sequence.

Let $j \in \mathbf{H}^3$.

The *critical exponent of* $\mathbf{X}$ is defined as

$$\operatorname{critical}(\mathbf{X}) \coloneqq \operatorname{critical}(\mathbf{X}, j) \coloneqq \inf\{t \in [0,\infty[\ : (\sum_{x\in\mathbf{X}} \mathbf{e}^{-\operatorname{dist}(x,j)\cdot t}) < \infty\} \in [0,\infty];$$

this definition is independent of the choice of j.

Let $z \in \hat{\mathbb{C}}$ and let $\sigma(j,z)$ denote the geodesic curve in $\mathbf{H}^3$ which starts at j and has limiting end-point z. We say that z is a *conical limit point* for $\mathbf{X}$ if the sequence $(\operatorname{dist}(x, \sigma(j,z)) \mid x \in \mathbf{X})$ in $[0,\infty[$ does not diverge to ∞; this definition is independent of the choice of j. The set of conical limit points for $\mathbf{X}$ is denoted $\operatorname{conical}(\mathbf{X})$. Every conical limit point for $\mathbf{X}$ is a limit point for $\mathbf{X}$, that is, an accumulation point of $\mathbf{X}$. The limit points for $\mathbf{X}$ that are not conical limit points for $\mathbf{X}$ are called *non-conical limit points* for $\mathbf{X}$.

(iv). Let S be an infinite discrete subset of $\mathrm{PSL}_2(\mathbb{C})$.

Let $k \in \mathbf{H}^3$. Let $S(k)$ denote the orbit $\{s(k) \mid s \in S\}$, an infinite discrete subset of $\mathbf{H}^3$.

We define the *critical exponent* of S to be the critical exponent of the orbit $S(k)$ and write $\operatorname{critical}(S) \coloneqq \operatorname{critical}(S(k))$; this definition is independent of the choice of k. If $j \in \mathbf{H}^3$, then $\operatorname{critical}(S(k), j) = \operatorname{critical}(S^{-1}(j), k)$, and, hence, $\operatorname{critical}(S) = \operatorname{critical}(S^{-1})$.

We define the set of *conical limit points* of S to be the set of conical limit points of the orbit $S(k)$, and write $\operatorname{conical}(S) \coloneqq \operatorname{conical}(S(k))$; this definition is independent of the choice of k.

(v). A discrete subgroup π of $\mathrm{PSL}_2(\mathbb{C})$ is said to be *geometrically finite* if there exists a fundamental π-domain in $\mathbf{H}^3$ whose closure in $\mathbf{H}^3 \cup \hat{\mathbb{C}}$ is a polyhedron with finitely many faces. □

Recall the following result from [8].

6.2 The Beardon-Maskit Theorem. *If π is a geometrically finite, discrete subgroup of* $\mathrm{PSL}_2(\mathbb{C})$*, then the set of non-conical limit points of π equals the set of points of $\hat{\mathbb{C}}$ fixed by parabolic elements of π.* □

We cannot apply the Beardon-Maskit Theorem directly to semigroups. The following remarks and two lemmas will be useful for our semigroups.

6.3 Remarks. Let Notation 5.1 hold. Then the following hold:

$U_1 = V_{f_{21}+1}^{-1} = F^{-1}$ and $\rho_F(U_1)$ is parabolic with fixed point $\mathrm{CT}(\infty) = \infty = \mathrm{CT}(\widehat{\rho}_0((CBA)^{\pm\infty}))$;

$U_{f_2} = V_{f_1}^{-1} = F^{-1}({}^F X_2)X_2^{-1} = BAF^{-1}AB = CF^{-1}C$ and $\rho_F(U_{f_2})$ is parabolic with fixed point $\mathrm{CT}(1) = \rho_F(C)(\infty) = \mathrm{CT}(\widehat{\rho}_0((BAC)^{\pm\infty}))$; and,

$U_{f_1} = V_1^{-1} = F^{-1}({}^F X_1)X_1^{-1} = BCF^{-1}CB = AF^{-1}A$ and $\rho_F(U_{f_1})$ is parabolic with fixed point $\mathrm{CT}(0) = \rho_F(A)(\infty) = \mathrm{CT}(\widehat{\rho}_0((ACB)^{\pm\infty}))$. □

6.4 Lemma. *Let* Notation 5.1 *hold. Suppose that $\ell \in [1{\uparrow}f_1]$ and that x and y are elements of $\{\infty, \rho_F(A)(\infty), \rho_F(C)(\infty)\}$.*

If $\rho_F(V_\ell)(x) = y$, then either $(x, y, \ell) = (C(\infty), \infty, f_{21})$, or $y = x$ and $\rho_F(V_\ell)$ is parabolic.

If $\rho_F(U_\ell)(x) = y$, then either $(x, y, \ell) = (\infty, C(\infty), f_2 + 1)$, or $y = x$ and $\rho_F(U_\ell)$ is parabolic.

Proof. By Proposition 5.7 and Theorem 5.6, the following hold:

- $\rho_F(A)(\infty)$ is the initial point of the first row $\rho_F(V_1)\mathfrak{a}_{-2}$ and the terminal point of the last column $\rho_F(U_{f_1})\mathfrak{a}_1$;
- ∞ is an interior point of the $(f_{21} + 1)$th row $\rho_F(V_{f_{21}+1})\mathfrak{a}_{-1}$ and the initial point of the first column $\rho_F(U_1)\mathfrak{a}_1$; and,
- $\rho_F(C)(\infty)$ is the terminal point of the last row $\rho_F(V_{f_1})\mathfrak{a}_{-1}$ and the terminal point of the f_2th column $\rho_F(U_{f_2})\mathfrak{a}_2$.

If $x = \rho_F(A)(\infty)$ and $y = \rho_F(V_\ell)(x)$, then y is the initial point of the ℓth row $\rho_F(V_\ell)\mathfrak{a}_{\mathbf{t}_\ell}$, and we see that $\ell = 1$, and $y = x$.

If $x = \infty$ and $y = \rho_F(V_\ell)(x)$, then y is an interior point of $\mathfrak{a}_{-1}$, and we see that $y = x$. Then, $\rho_F(U_{F^{-1}(\ell)})(y) = \rho_F(V_\ell^{-1})(y) = x$ is the initial point of the $F^{-1}(\ell)$th column $\rho_F(U_{F^{-1}(\ell)})\mathfrak{a}_{\mathbf{s}_{F^{-1}(\ell)}}$ and $F^{-1}(\ell) = 1$. Hence $\ell = F(1) = f_{21} + 1$.

If $x = \rho_F(C)(\infty)$ and $y = \rho_F(V_\ell)(x)$, then y is the terminal point of $\rho_F(V_\ell)\mathfrak{a}_{-1}$. If $\ell \neq f_1$, then we must have $y = \infty$. Here, $\rho_F(U_{F^{-1}(\ell)})(y) = \rho_F(V_\ell^{-1})(y) = x$ is the initial point of the $F^{-1}(\ell)$th column $\rho_F(U_{F^{-1}(\ell)})\mathfrak{a}_{\mathbf{s}_{F^{-1}(\ell)}}$. Hence $F^{-1}(\ell) = f_2 + 1$ and $\ell = F(f_2 + 1) = f_{21}$.

The result is now clear. □

6.5 Lemma. *Let* Notation 5.1 *hold. Let $\mathbf{z}_{[\![1\uparrow\infty[\![}$ be a sequence in $[1{\uparrow}f_1]$ that does not converge to an element of $\{1, f_2, f_1\}$ in the discrete topology of $[1{\uparrow}f_1]$. Let z_1 denote the element of $\mathfrak{a}_1$ such that $\bigcap\limits_{\ell\in[0\uparrow\infty[} (\rho_F(\Pi U_{\mathbf{z}_{[\![1\uparrow\ell]\!]}})\mathfrak{a}_1) = \{z_1\}$. Then z_1 is a conical limit point of the infinite, discrete subset $\{\rho_F(\Pi U_{\mathbf{z}_{[\![1\uparrow\ell]\!]}}) \mid \ell \in [0{\uparrow}\infty[\ \}$ of* $\mathrm{PSL}_2(\mathbb{C})$.

Proof. In [14, Theorem 5.6(ii)], it was shown that

$$\partial^- A \cap \partial^+ B = \{\mathrm{CT}(0), \mathrm{CT}(1), \mathrm{CT}(\infty)\} = \{\infty, \rho_F(A)(\infty), \rho_F(C)(\infty)\}. \tag{6.1}$$

Let us choose some element z_{-1} of $\mathfrak{a}_{-1} - \{\infty, \rho_F(A)(\infty), \rho_F(C)(\infty)\}$. By (6.1), $z_{-1} \notin \mathfrak{a}_1$. By Theorem 5.4, z_{-1} is not a limit point of the ∂^--semigroup.

Let S denote the set of accumulation points of the sequence

$$(\rho_F((\Pi U_{\mathbf{z}_{[\![1\uparrow\ell]\!]}})^{-1})(z_{-1}) \mid \ell \in [\![1\uparrow\infty]\!]\ ).$$

Notice that S is nonempty.

We show first that there exists some $z_{-2} \in S - \mathfrak{a}_1$.

By Proposition 5.7, $\{(\Pi U_{\mathbf{z}_{[\![1\uparrow\ell]\!]}})^{-1} \mid \ell \in [1\uparrow\infty[\ \}$ lies in the ∂^+-semigroup, and, by Theorem 5.6, the ∂^+-semigroup acts on $\mathfrak{a}_{-1}$ in $\hat{\mathbb{C}}_{\rho_F}$. Hence, $\{\rho_F((\Pi U_{\mathbf{z}_{[\![1\uparrow\ell]\!]}})^{-1})(z_{-1}) \mid \ell \in [1\uparrow\infty[\ \}$ lies in $\mathfrak{a}_{-1}$. Since $\mathfrak{a}_{-1}$ is closed, $S \subseteq \mathfrak{a}_{-1}$.

By (6.1), $S - \mathfrak{a}_1 = S - \{\infty, \rho_F(A)(\infty), \rho_F(C)(\infty)\}$.

Fact 1. If $\infty \in S$, then there exists some $z_{-2} \in S - \{\infty, \rho_F(A)(\infty), \rho_F(C)(\infty)\}$.

Proof of Fact 1. Here, there exists an infinite subsequence N_1 of $[\![1\uparrow\infty[\![$ with the property that the sequence $(\rho_F((\Pi U_{\mathbf{z}_{[\![1\uparrow\ell]\!]}})^{-1})(z_{-1}) \mid \ell \in N_1)$ converges to ∞ in $\mathfrak{a}_{-1}$.

Now $\rho_F(U_1^{-1})$ acts on $\mathfrak{a}_{-1}$, and $\rho_F(U_1^{-1})$ is parabolic with fixed point ∞. It follows that, for each $n \in [0\uparrow\infty[$, $(\rho_F(((\Pi U_{\mathbf{z}_{[\![1\uparrow\ell]\!]}})U_1^n)^{-1})(z_{-1}) \mid \ell \in N_1)$ converges to $\rho_F(U_1^{-n})(\infty) = \infty$, and does so more and more rapidly, in the $\mathfrak{a}_{-1}$ arc order, as n increases.

For each $\ell \in [1\uparrow\infty[$, let ℓ' denote the least element of $[(\ell+1)\uparrow\infty[$ such that $\mathbf{z}_{\ell'} \neq 1$; this is well-defined since $\mathbf{z}_{[1\uparrow\infty[}$ does not converge to 1.

It then follows that $(\rho_F((\Pi U_{\mathbf{z}_{[\![1\uparrow(\ell'-1)]\!]}})^{-1})(z_{-1}) \mid \ell \in N_1)$ also converges to ∞.

There exists some $m \in [2\uparrow f_1]$ such that $N_2 := \{\ell \in N_1 \mid \ell' = m\}$ is infinite.

Then $(\rho_F((\Pi U_{\mathbf{z}_{[\![1\uparrow\ell']\!]}})^{-1})(z_{-1}) \mid \ell \in N_2)$ converges to $z_{-2} := \rho_F(U_m^{-1})(\infty) \in S$.

By Lemma 6.4, $z_{-2} \in S - \{\infty, \rho_F(A)(\infty), \rho_F(C)(\infty)\}$, and Fact 1 is now proved. □

By the same type of argument, we prove the following.

Fact 2. If $\rho_F(A)(\infty) \in S$, then there exists some $z_{-2} \in S - \{\infty, \rho_F(A)(\infty), \rho_F(C)(\infty)\}$. □

Fact 3. If $\rho_F(C)(\infty) \in S$, then there exists some $z_{-2} \in S - \{\infty, \rho_F(A)(\infty), \rho_F(C)(\infty)\}$.

Proof of Fact 3. If we apply the same type of argument again, we find there exists some $z_{-3} \in S - \{\rho_F(A)(\infty), \rho_F(C)(\infty)\}$. Together with Fact 1, this implies Fact 3. □

In summary, we have proved that there exists some $z_{-2} \in S - \mathfrak{a}_1$.

Let σ denote the geodesic curve in $\mathbf{H}^3$ with limiting end-points z_{-1} and z_1.

Let $\varepsilon = \text{chord}(z_{-2}, \mathfrak{a}_1)$; see Definitions 6.1(i). Since $z_{-2} \notin \mathfrak{a}_1$, $\varepsilon > 0$.

Let $N_3 := \{\ell \in [0\uparrow\infty[\ : \text{chord}(z_{-2}, \rho_F((\Pi U_{\mathbf{z}_{[\![1\uparrow\ell]\!]}})^{-1})(z_{-1})) < \frac{\varepsilon}{2}\}$. Since $z_{-2} \in S$, N_3 is infinite.

Let $\ell \in N_3$. By the choice of ε, $\text{chord}(\rho_F((\Pi U_{\mathbf{z}_{[\![1\uparrow\ell]\!]}})^{-1})(z_{-1}), \mathfrak{a}_1) > \frac{\varepsilon}{2}$. We no longer need z_{-2}, and we argue as in [8]. We know that $z_1 \in \rho_F(\Pi U_{\mathbf{z}_{[\![1\uparrow\ell]\!]}})\mathfrak{a}_1$. Hence, $\rho_F((\Pi U_{\mathbf{z}_{[\![1\uparrow\ell]\!]}})^{-1})(z_1) \in \mathfrak{a}_1$. In particular,

$$\text{chord}(\ \rho_F((\Pi U_{\mathbf{z}_{[\![1\uparrow\ell]\!]}})^{-1})(z_{-1}),\ \rho_F((\Pi U_{\mathbf{z}_{[\![1\uparrow\ell]\!]}})^{-1})(z_1)\ ) > \tfrac{\varepsilon}{2}.$$

The geodesic curve in $\mathbf{H}^3$ which has as its limiting end-points $\rho_F((\Pi U_{\mathbf{z}_{[\![1\uparrow\ell]\!]}})^{-1})(z_{-1})$ and $\rho_F((\Pi U_{\mathbf{z}_{[\![1\uparrow\ell]\!]}})^{-1})(z_1)$ is $\rho_F((\Pi U_{\mathbf{z}_{[\![1\uparrow\ell]\!]}})^{-1})(\sigma)$. By Definitions 6.1(i),

$$\operatorname{arccosh}(\tfrac{4}{\varepsilon}) \quad > \quad \operatorname{dist}(\mathbf{j}, \rho_F((\Pi U_{\mathbf{z}_{[\![1\uparrow\ell]\!]}})^{-1})(\sigma)) \quad = \quad \operatorname{dist}(\rho_F(\Pi U_{\mathbf{z}_{[\![1\uparrow\ell]\!]}})(\mathbf{j}), \sigma).$$

Since N_3 is infinite, the foregoing implies that z_{-1} or z_1 is a conical limit point of $(\rho_F(\Pi U_{\mathbf{z}_{[\![1\uparrow\ell]\!]}})(\mathbf{j}) \mid \ell \in [\![0\uparrow\infty[\![\ )$. Since z_{-1} is not a limit point of the ∂^--semigroup, the result follows. □

6.6 Theorem. *Let* Notation 5.1 *hold. The set of conical limit points of the ∂^--semigroup equals $\mathfrak{a}_1 - \rho_F(\langle\, A, B, C\,\rangle)(\infty)$.*

Proof. Jørgensen showed that $\rho_F(\langle\, A, B, C, F\,\rangle)$ is geometrically finite. By the Beardon-Maskit Theorem 6.2, or simply [8, Proposition 3], every fixed point of a parabolic element of $\rho_F(\langle\, A, B, C, F\,\rangle)$ is a non-conical limit point of $\rho_F(\langle\, A, B, C, F\,\rangle)$, and, in particular, is not a conical limit point of the ∂^--semigroup. Thus, every conical limit point of the ∂^--semigroup lies in $\mathfrak{a}_1 - \rho_F(\langle\, A, B, C\,\rangle)(\infty)$.

It remains to show that if $z \in \mathfrak{a}_1 - \rho_F(\langle\, A, B, C\,\rangle)(\infty)$, then z is a conical limit point of the ∂^--semigroup. Let $\mathbf{z}_{[\![1\uparrow\infty[\![}$ be a sequence in $[1\uparrow f_1]$ such that $\bigcap\limits_{\ell\in[1\uparrow\infty[} (\rho_F(\Pi U_{\mathbf{z}_{[\![1\uparrow(\ell-1)]\!]}})\mathfrak{a}_1) = \{z\}$. Since $z \notin \rho_F(\langle\, A, B, C\,\rangle)(\infty)$, $\mathbf{z}_{[\![1\uparrow\infty[\![}$ does not converge to 1 or f_2 or f_1. By Lemma 6.5, z is a conical limit point of the ∂^--semigroup. □

A similar argument shows the following.

6.7 Theorem. *Let* Notation 5.1 *hold. The set of conical limit points of the ∂^+-semigroup equals $\mathfrak{a}_{-1} - \rho_F(\langle\, A, B, C\,\rangle)(\infty)$.* □

Using Lemma 6.5 again, we can deduce the following.

6.8 Corollary. *Let* Notation 5.1 *hold, let $N \in [1\uparrow\infty[$ and let $\mathbf{z}_{[\![1\uparrow N]\!]}$ be a sequence in $[1\uparrow f_1]$.*
If $\rho_F(\Pi U_{\mathbf{z}_{[\![1\uparrow N]\!]}})$ is parabolic, then $\mathbf{z}_{[\![1\uparrow N]\!]}$ is a constant sequence in $\{1, f_2, f_1\}$.
If $\rho_F(\Pi V_{\mathbf{z}_{[\![1\uparrow N]\!]}})$ is parabolic, then $\mathbf{z}_{[\![1\uparrow N]\!]}$ is a constant sequence in $\{1, f_{2\,1}+1, f_1\}$. □

In [12, Section 2], a succinct, self-contained proof of the following result is given, although the result itself is not stated.

6.9 The Bishop-Jones Theorem. *If S is a subsemigroup of a discrete subgroup of* $\mathrm{PSL}_2(\mathbb{C})$ *such that S does not have a global fixed point in $\hat{\mathbb{C}}$, then* $\mathrm{critical}(S) = \mathrm{Hdim}(\mathrm{conical}(S))$. □

6.10 Theorem. *Let* Notation 5.1 *hold. The following four numbers are equal:*
the Hausdorff dimension of $\partial^- A$;
the critical exponent of the ∂^--semigroup;
the critical exponent of the ∂^+-semigroup;

the Hausdorff dimension of $\partial^+ B$.

Proof. By Theorem 6.6, $\mathrm{Hdim}(\partial^- A)$ equals the Hausdorff dimension of the set of conical limit points of the ∂^--semigroup which, by the Bishop-Jones Theorem 6.9, equals the critical exponent of the ∂^--semigroup. By Theorem 6.7, $\mathrm{Hdim}(\partial^+ B)$ equals the Hausdorff dimension of the set of conical limit points of the ∂^+-semigroup which, by Theorem 6.9, equals the critical exponent of the ∂^+-semigroup. By Proposition 5.7(iii) and Definitions 6.1(iv), the critical exponent of the ∂^--semigroup equals the critical exponent of the ∂^+-semigroup. □

6.11 Remark. For $F = RL$, $\mathrm{Hdim}(\partial^- A)$ seems to be near 1.27910. Recall that for the sequence $b_{[\![0\uparrow\infty[\![}$ of [17, Calculations 5.2], $\mathrm{Hdim}(\partial^- A)$ lies between the liminf and the limsup of the sequence $(\log \frac{b_{n+1}-b_n}{b_n - b_{n-1}} \mid n \in [\![1\uparrow\infty[\![)$. Jaume Amoros and Javier Vindel very kindly computed

$$b_{[\![21\uparrow 25]\!]} = (547206912858, 2002063012565, 7324919842341, 26799555593731, 98051175064382),$$

and, hence, $(\log \frac{b_{n+1}-b_n}{b_n - b_{n-1}} \mid n \in [\![22\uparrow 24]\!]) = (1.2971031\ldots,\ 1.2971027\ldots,\ 1.2971046\ldots)$. □

In [28], McMullen described an eigenvalue algorithm and applied a theorem of Sullivan [38] to prove that this eigenvalue algorithm gives good estimates for the Hausdorff dimension of the limit set of a Kleinian group generated by reflections in pairwise disjoint circles; see also [31, p.156]. It would be very interesting to know if some variant of the algorithm gives estimates for $\mathrm{Hdim}(\partial^- A)$. Let us propose the following.

6.12 Some Open Problems. We have sequences of positive integers $a_{[\![1\uparrow p]\!]}, b_{[\![1\uparrow p]\!]}$, and, on $\langle\, X_1, X_2 \mid \quad \rangle$, ${}^R(X_1, X_2) = (X_1, X_1X_2)$, ${}^L(X_1, X_2) = (X_2X_1, X_2)$, and $F = \prod\limits_{i\in[\![1\uparrow p]\!]} (R^{a_i} L^{b_i})$, and $\langle\, X_1, X_2, F \,\rangle$ acts on $\hat{\mathbb{C}}$ via a normalized discrete faithful representation ρ_F in $\mathrm{PSL}_2(\mathbb{C})$, and, in $\mathrm{SL}_2(\mathbb{Z})$, $\left(\begin{smallmatrix} f_{1\,1} & f_{1\,2} \\ f_{2\,1} & f_{2\,2}\end{smallmatrix}\right) := \prod\limits_{i\in[\![1\uparrow p]\!]} (\left(\begin{smallmatrix} 1 & a_i \\ 0 & 1\end{smallmatrix}\right)\left(\begin{smallmatrix} 1 & 0 \\ b_i & 1\end{smallmatrix}\right))$.

Also, $f_1 := f_{1\,1} + f_{2\,1}$, $f_2 := f_{1\,2} + f_{2\,2}$, $\mu_+ := \frac{f_{1\,1} - f_{2\,2} + \sqrt{(f_{1\,1}+f_{2\,2})^2 - 4}}{2 f_{2\,1}}$, and, for each $\ell \in [1\uparrow f_1]$, $\mathbf{s}_\ell := [\frac{\ell+1}{1+\mu_+}] - [\frac{\ell}{1+\mu_+}] + 1 \in [1\uparrow 2]$, and $U_\ell := F^{-1} \Pi X_{\mathbf{s}_{[\![1\uparrow(\ell-1)]\!]}}$.

In Notation 5.1 and Lemma 5.3, we saw that $f_2 < f_1$, and $\mathbf{s}_{[\![1\uparrow f_1]\!]}$ is a sequence in $[1\uparrow 2]$, and ${}^F X_1 = \Pi X_{\mathbf{s}_{[\![1\uparrow f_1]\!]}}$, and ${}^F X_2 = \Pi X_{\mathbf{s}_{[\![1\uparrow f_2]\!]}}$. In Theorem 5.4, we saw that we have arcs $\mathfrak{a}_1$ and $\mathfrak{a}_2$ in $\hat{\mathbb{C}}$ with fracturings

$$\mathfrak{a}_1 = \bigcup_{\ell\in[1\uparrow f_1]} U_\ell \mathfrak{a}_{\mathbf{s}_\ell} \quad \text{and} \quad \mathfrak{a}_2 = \bigcup_{\ell\in[1\uparrow f_2]} U_\ell \mathfrak{a}_{\mathbf{s}_\ell}; \tag{6.2}$$

throughout this subsection, we understand that $\langle\, X_1, X_2, F \,\rangle$ acts on $\hat{\mathbb{C}}$ via ρ_F.

For each subarc $\mathfrak{b}$ of $\mathfrak{a}_1$, let weight($\mathfrak{b}$) denote the chord-metric distance between the two end-points of $\mathfrak{b}$.

Fix $\varepsilon \in\ ]0,\ \min\{\text{weight}(U_\ell \mathfrak{a}_1) \mid \ell \in [1\!\uparrow\! f_1]\ \}\,]$.

Let $\mathcal{Z}_\varepsilon$ denote the set of all $\mathfrak{a}_1$-acceptable sequences in $[1\!\uparrow\! f_1]$ of the form $\mathbf{z}_{[\![1\uparrow N]\!]}$ with $N \in [2\!\uparrow\!\infty[$ satisfying $\text{weight}(\mathfrak{a}_{\mathbf{z}_{[\![1\uparrow(N-1)]\!]}}) \geqslant \varepsilon$ and $\text{weight}(\mathfrak{a}_{\mathbf{z}_{[\![1\uparrow N]\!]}}) < \varepsilon$ where $\mathfrak{a}_{\mathbf{z}_{[\![1\uparrow N]\!]}}$ denotes $(\Pi U_{\mathbf{z}_{[\![1\uparrow N]\!]}})\mathfrak{a}_{\mathbf{s}_{\mathbf{z}_N}}$, a subarc of $\mathfrak{a}_1$. Recall that $\mathbf{z}_{[\![1\uparrow N]\!]}$ is $\mathfrak{a}_1$-acceptable if, for each $\ell \in [1\!\uparrow\!(N{-}1)]$ with $\mathbf{s}_{\mathbf{z}_\ell} = 2$, $\mathbf{z}_{\ell+1} \in [1\!\uparrow\! f_2]$. Repeated substitution in (6.2) gives a fracturing $\mathfrak{a}_1 = \bigcup\limits_{\mathbf{z}_{[\![1\uparrow N]\!]} \in \mathcal{Z}_\varepsilon} \mathfrak{a}_{\mathbf{z}_{[\![1\uparrow N]\!]}}$ where $\mathcal{Z}_\varepsilon$ has been given the lexicographic order.

Let $\mathbf{M}_\varepsilon$ denote the real square matrix with rows and columns indexed by $\mathcal{Z}_\varepsilon$ such that

$$\begin{aligned}&\text{the coordinate corresponding to } (\mathbf{y}_{[\![1\uparrow M]\!]}, \mathbf{z}_{[\![1\uparrow N]\!]}) \in \mathcal{Z}_\varepsilon \times \mathcal{Z}_\varepsilon\\ &\qquad\qquad\text{equals } \frac{\text{weight}(U_{\mathbf{y}_1}\mathfrak{a}_{\mathbf{z}_{[\![1\uparrow N]\!]}})}{\text{weight}(\mathfrak{a}_{\mathbf{z}_{[\![1\uparrow N]\!]}})} \quad \text{if } M{-}1 \leqslant N \text{ and } \mathbf{y}_{[\![2\uparrow M]\!]} = \mathbf{z}_{[\![1\uparrow(M-1)]\!]},\\ &\qquad\text{and equals zero otherwise.}\end{aligned}$$

For each $x \in [0,\infty[$, let $\mathbf{M}_\varepsilon^{\wedge x}$ denote the matrix obtained from $\mathbf{M}_\varepsilon$ by raising each entry to the power x.

It is an open problem to decide whether there exists a unique $x_\varepsilon \in [0,\infty[$ such that the spectral radius of $\mathbf{M}_\varepsilon^{\wedge x_\varepsilon}$ equals 1. If the x_ε exist, then the next open problem is to decide whether $\lim\limits_{\varepsilon\to 0} x_\varepsilon$ exists and equals the Hausdorff dimension of $\mathfrak{a}_1$. If so, the open problem after that is to describe the error term.

For $F = RL$ and $\varepsilon = 0.00018$, we have found that $|\mathcal{Z}_\varepsilon| = 937517$ and that x_ε exists and equals $1.29710693\ldots$; the latter value is quite close to the terms of the sequence mentioned in Remark 6.11. We have computed the following cases.

F	ε	$\lvert\mathcal{Z}_\varepsilon\rvert$	x_ε
RL	0.00018	937517	$1.29710693\ldots$
RL^2	0.00025	871775	$1.29536\ldots$
RL^3	0.0003	870188	$1.29159\ldots$
R^2L^2	0.0003	1012285	$1.30790\ldots$
$RL^2R^3L^4$	0.0012	884553	$1.32636\ldots$

Recall that McMullen [28] computed that the Hausdorff dimension of the Apollonian gasket is about 1.305688 with an analogue of $\varepsilon = 0.0005$ and $|\mathcal{Z}_\varepsilon| = 1397616$. The Apollonian gasket and $CW(RL^{100})$ appear together in Figure 8.2 □

6.13 Background. To motivate the above open problems, let us sketch some of the theory behind McMullen's algorithm and its relation with our situation.

Let Γ be a subsemigroup of a discrete subgroup of $\mathrm{PSL}_2(\mathbb{C})$ and let $\Lambda(\Gamma)$ be the limit set of Γ. Let $\hbar \in [0,\infty[$. Let μ be a measure on $\hat{\mathbb{C}}$. We say that μ is a Γ-*invariant probability* *measure of dimension* $\hbar$ if $\mu(\Lambda(\Gamma)) = \mu(\hat{\mathbb{C}}) = 1$, and, for each $\gamma \in \Gamma$ and each Borel subset E of $\hat{\mathbb{C}}$, $\mu(\gamma(E)) = \int_{z\in E}(\gamma'(z))^{\hbar}\,\mathrm{d}\mu$, where, for $\gamma(z) = \frac{az+b}{cz+d}$, $\gamma'(z) = \frac{|z|^2+1}{|az+b|^2+|cz+d|^2}$, the local dilation with respect to the chord metric. In cases where Γ is a group, the existence and uniqueness of such a pair $(\mu, \hbar)$ have been studied by Bowen, Patterson, Sullivan, and others. In the case where Γ is a geometrically finite group, Sullivan [37], [38] showed that there exists a unique pair $(\mu, \hbar)$ such that μ is a Γ-invariant probability measure of dimension $\hbar$, and, moreover, $\hbar$ then equals the Hausdorff dimension of $\Lambda(\Gamma)$.

Consider now the case where Γ is the subsemigroup of $\langle X_1, X_2, F\rangle$ generated by $U_{[1\uparrow f_1]}$, acting on $\hat{\mathbb{C}}_{\rho_F}$. By Theorem 5.4, $\Lambda(\Gamma) = \mathfrak{a}_1$. Let us suppose that there exists a unique pair $(\mu, \hbar)$ such that μ is a Γ-invariant probability measure of dimension $\hbar$, and let us also suppose that $\hbar$ equals the Hausdorff dimension of $\mathfrak{a}_1$, and let us also suppose that, for each $\mathbf{y}_{[\![1\uparrow M]\!]} \in \mathcal{Z}_\varepsilon$, $\mathrm{weight}(\mathfrak{a}_{\mathbf{y}_{[\![2\uparrow(M-1)]\!]}}) \geqslant \varepsilon$, and, hence,

$$\begin{aligned}(6.3)\qquad \mu(\mathfrak{a}_{\mathbf{y}_{[\![1\uparrow M]\!]}}) &= \mu(U_{\mathbf{s}_{\mathbf{y}_1}}\mathfrak{a}_{\mathbf{y}_{[\![2\uparrow M]\!]}}) \\ &= \sum_{\{\mathbf{z}_{[\![1\uparrow N]\!]}\in\mathcal{Z}_\varepsilon \mid \mathbf{z}_{[\![1\uparrow(M-1)]\!]}=\mathbf{y}_{[\![2\uparrow M]\!]}\}} \mu(U_{\mathbf{s}_{\mathbf{y}_1}}\mathfrak{a}_{\mathbf{z}_{[\![1\uparrow N]\!]}}) \\ &= \sum_{\{\mathbf{z}_{[\![1\uparrow N]\!]}\in\mathcal{Z}_\varepsilon \mid \mathbf{z}_{[\![1\uparrow(M-1)]\!]}=\mathbf{y}_{[\![2\uparrow M]\!]}\}} \frac{\mu(U_{\mathbf{y}_1}\mathfrak{a}_{\mathbf{z}_{[\![1\uparrow N]\!]}})}{\mu(\mathfrak{a}_{\mathbf{z}_{[\![1\uparrow N]\!]}})}\mu(\mathfrak{a}_{\mathbf{z}_{[\![1\uparrow N]\!]}}).\end{aligned}$$

Let $\mathbf{A}_\varepsilon$ denote the real square matrix with rows and columns indexed by $\mathcal{Z}_\varepsilon$ such that

$$\begin{aligned}&\text{the coordinate corresponding to } (\mathbf{y}_{[\![1\uparrow M]\!]}, \mathbf{z}_{[\![1\uparrow N]\!]}) \in \mathcal{Z}_\varepsilon \times \mathcal{Z}_\varepsilon \\ &\qquad \text{equals } \frac{\mu(U_{\mathbf{y}_1}\mathfrak{a}_{\mathbf{z}_{[\![1\uparrow N]\!]}})}{\mu(\mathfrak{a}_{\mathbf{z}_{[\![1\uparrow N]\!]}})} \quad \text{if } M-1 \leqslant N \text{ and } \mathbf{y}_{[\![2\uparrow M]\!]} = \mathbf{z}_{[\![1\uparrow(M-1)]\!]}, \\ &\quad \text{and equals zero otherwise.}\end{aligned}$$

Let $\mathbf{v}_\varepsilon$ denote the real column vector with coordinates indexed by $\mathcal{Z}_\varepsilon$ where the coordinate corresponding to $\mathbf{z}_{[\![1\uparrow N]\!]} \in \mathcal{Z}_\varepsilon$ equals $\mu(\mathfrak{a}_{\mathbf{z}_{[\![1\uparrow N]\!]}})$. Then $\mathbf{v}_\varepsilon$ is a positive vector with coordinate sum equal to 1 and (6.3) shows that $\mathbf{A}_\varepsilon\mathbf{v}_\varepsilon = \mathbf{v}_\varepsilon$. By Perron-Frobenius Theory, the spectral radius of $\mathbf{A}_\varepsilon$ equals 1 (since $\mathbf{A}_\varepsilon$ is non-zero).

If $\ell \in [1\uparrow f_1]$, and U'_ℓ denotes the local dilation for the action of U_ℓ on $\hat{\mathbb{C}}_{\rho_F}$, and $\mathfrak{b}$ is a sufficiently small subarc of $\mathfrak{a}_1$, then the subinterval $U'_\ell(\mathfrak{b})$ of $[0,\infty[$ can be approximated by a single value and we choose the 'mean weight-value' $\frac{\mathrm{weight}(U_\ell\mathfrak{b})}{\mathrm{weight}(\mathfrak{b})}$ as an approximation. (McMullen uses an evaluation of U'_ℓ at some chosen point, but in the presence of parabolics some choices may not work well for what we have in mind.) The key fact now about the measure μ is that $(U'_\ell(\mathfrak{b}))^{\hbar}$ can be approximated by the 'mean μ-value' $\frac{\mu(U_\ell\mathfrak{b})}{\mu(\mathfrak{b})}$. Hence, $\frac{\mu(U_\ell\mathfrak{b})}{\mu(\mathfrak{b})} \simeq \left(\frac{\mathrm{weight}(U_\ell\mathfrak{b})}{\mathrm{weight}(\mathfrak{b})}\right)^{\hbar}$ since both sides are approximations for the interval $(U'_\ell(\mathfrak{b}))^{\hbar}$. Thus, for sufficiently small ε, $\mathbf{A}_\varepsilon \simeq \mathbf{M}_\varepsilon^{\wedge\hbar}$ and, hence, the spectral radius of $\mathbf{M}_\varepsilon^{\wedge\hbar}$ is near 1. In the different circumstances of a Kleinian group generated by reflections in pairwise disjoint circles, McMullen shows that there exists a unique $x_\varepsilon \in [0, \infty[$ for which the spectral radius of $\mathbf{M}_\varepsilon^{\wedge x_\varepsilon}$ is equal to 1, and that x_ε is a good approximation of $\hbar$. $\square$

7 When are the columns of $\mathrm{CW}(F)$ vertical?

This section is an addendum to [15] that records the fact that the columns of $\mathrm{CW}(F)$ are vertical whenever some odd-length-cyclic shift carries our sequence $(a_1, b_1, a_2, b_2, \ldots, a_p, b_p)$ to itself, or some even-length-cyclic shift carries our sequence to its reverse, or both. It is an open problem to decide whether the converse is true. In the cases where our sequence is $(1,1)$, $(2,2)$, $(1,1,2,2)$, or $(1,2,3,1,2,3)$, verticality is illustrated in [15, Section 14].

7.1 Discussion. Recall that $(a_1, b_1, a_2, b_2, \ldots, a_p, b_p)$ is an even-length, nonempty, positive-integer sequence, and, in $\mathrm{Aut}\langle A, B, C\rangle$, $R := (A, BCB, B)$ and $L := (B, BAB, C)$ and $F := R^{a_1}L^{b_1}R^{a_2}L^{b_2}\cdots R^{a_p}L^{b_p}$, and for the action of $\langle A, B, C, F\rangle$ on $\hat{\mathbb{C}}_{\rho_F}$, D acts as $z \mapsto z+1$ and F acts as $z \mapsto z+s$, with $\mathrm{Im}(s) > 0$.

For the following four conditions, it is known that (a) $\Leftrightarrow$ (b) $\Rightarrow$ (c) $\Leftrightarrow$ (d):

(a) At least one of the following holds.
  (i) Some odd-length-cyclic shift carries $(a_1, b_1, a_2, b_2, \ldots, a_p, b_p)$ to itself.
  (ii) Some even-length-cyclic shift carries $(a_1, b_1, a_2, b_2, \ldots, a_p, b_p)$ to its reverse, $(b_p, a_p, \ldots, b_2, a_2, b_1, a_1)$.

(b) The hyperbolic once-punctured-torus bundle $\rho_F(\langle CB, AB, F\rangle)\backslash\mathbf{H}^3$ has an orientation-reversing isometry.

(c) $\mathrm{Re}(s) = 0$.

(d) The (colored) columns of the Cannon-Thurston tessellation $\mathrm{CW}(F)$ are (completely) vertical.

For example, (a)(i) holds for the sequences $(1,1)$ and $(1,2,3,1,2,3)$.

For example, (a)(ii) holds for the sequences $(1,1)$ and $(1,1,2,2)$.

For example, (c) fails for the sequence $(1,2,3,2)$, where one can show that $\mathrm{Re}(s) < -0.03$; here s satisfies $s^6 + s^5 - s^4 - s^3 - 12s^2 - 12s + 8 = 0$.

(a) $\Leftrightarrow$ (b) was proved by McCullough [26]; we shall use the main ideas of his argument in a proof of (a) $\Rightarrow$ (c) below.

(b) $\Rightarrow$ (c) is a folklore result, traditionally attributed to Robert F. Riley. We have not found a statement or proof of Riley's implication in the literature; below, we record a proof that (a) $\Rightarrow$ (c).

(c) $\overset{?}{\Rightarrow}$ (b) is an open question.

(c) $\Leftrightarrow$ (d) holds since each column of $\mathrm{CW}(F)$ is invariant under $\rho_F(F)$, which is translation by s.

Proof that (a)(i) $\Rightarrow$ (c). Suppose that some odd-length-cyclic shift carries the sequence $(a_1, b_1, a_2, b_2, \ldots, a_p, b_p)$ to itself. Thus there exists some odd divisor i of p such that

$$F = ((R^{a_1} L^{b_1} R^{a_2} L^{b_2} \cdots R^{a_{\frac{i-1}{2}}} L^{b_{\frac{i-1}{2}}} R^{a_{\frac{i+1}{2}}})(L^{a_1} R^{b_1} L^{a_2} R^{b_2} \cdots L^{a_{\frac{i-1}{2}}} R^{b_{\frac{i-1}{2}}} L^{a_{\frac{i+1}{2}}}))^{\frac{p}{i}}.$$

It is not difficult to show that for $M := (C, B, A) \in \mathrm{Aut}\langle A, B, C \rangle$, we have ${}^{M}R = L$ and ${}^{M}L = R$ and $M^2 = 1$ and ${}^{M}D = D^{-1}$. Let $G := R^{a_1} L^{b_1} R^{a_2} L^{b_2} \cdots R^{a_{\frac{i-1}{2}}} L^{b_{\frac{i-1}{2}}} R^{a_{\frac{i+1}{2}}} M$. Then $G^{\frac{2p}{i}} = F$ and ${}^{G}D = D^{-1}$. Since G normalizes $\langle A, B, C, F \rangle$, conjugation by G determines an automorphism of $\langle A, B, C, F \rangle$; by Mostow's Rigidity Theorem, there exists a unique conformal-or-anti-conformal automorphism η of $\hat{\mathbb{C}}$ such that ρ_F pre-composed with the conjugation-by-G automorphism of $\langle A, B, C, F \rangle$ equals ρ_F post-composed with the conjugation-by-η automorphism of $\mathrm{PSL}_2(\mathbb{C})$. We can extend ρ_F to $\langle A, B, C, G \rangle$ by $\rho_G(G) = \eta$. If $\rho_G(G)$ is conformal, resp. anti-conformal, we can express $\rho_G(G)$ as $z \mapsto \frac{az+b}{cz+b}$, resp. $z \mapsto \frac{a\overline{z}+b}{c\overline{z}+d}$, where $ad - bc = 1$. Since ${}^{G}D = D^{-1}$ and $G^2 \neq 1$, we see that $\rho_G(G)(\infty) = \infty$ and $c = 0$ and $a = -d$ and $\rho_G(G)$ is $z \mapsto \overline{z} + b$. Since $G^{\frac{2p}{i}} = F$, we see that $s = \rho_F(F)(0) = \frac{p}{i}(b + \overline{b})$ and $\mathrm{Re}(s) = 0$, that is, (c) holds.

Proof that (a)(ii) $\Rightarrow$ (c). Suppose that some even-length-cyclic shift carries the sequence $(a_1, b_1, a_2, b_2, \ldots, a_p, b_p)$ to its reverse. Thus there exists some $i \in [0{\uparrow}(p-1)]$ such that both

$$(a_1, b_1, a_2, b_2, \ldots, a_{i-1}, b_{i-1}, a_i, b_i) \text{ and } (a_{i+1}, b_{i+1}, a_{i+2}, b_{i+2}, \ldots, a_{p-1}, b_{p-1}, a_p, b_p)$$

are 'palindromes', equal to their own reverses. It is not difficult to show that for $P := (C, CBC, CBABC) \in \mathrm{Aut}\langle A, B, C \rangle$ we have ${}^{P}R = L^{-1}$ and ${}^{P}L = R^{-1}$ and $P^2 = D$. Let $G := R^{a_1} L^{b_1} R^{a_2} L^{b_2} \cdots R^{a_{i-1}} L^{b_{i-1}} R^{a_i} L^{b_i} P$. Then $G^2 = D$ and ${}^{G}F = F^{-1}$. Since G normalizes $\langle A, B, C, F \rangle$, conjugation by G determines an automorphism of $\langle A, B, C, F \rangle$; by Mostow's Rigidity Theorem, there exists a unique conformal-or-anti-conformal automorphism η of $\hat{\mathbb{C}}$ such that ρ_F pre-composed with the conjugation-by-G automorphism of $\langle A, B, C, F \rangle$ equals ρ_F post-composed with the conjugation-by-η automorphism of $\mathrm{PSL}_2(\mathbb{C})$. We can extend ρ_F to $\langle A, B, C, G, F \rangle$ by $\rho_G(G) = \eta$. If $\rho_G(G)$ is conformal, resp. anti-conformal, we can express $\rho_G(G)$ as $z \mapsto \frac{az+b}{cz+b}$, resp. $z \mapsto \frac{a\overline{z}+b}{c\overline{z}+d}$, where $ad - bc = 1$. Since ${}^{G}F = F^{-1}$ and $G^2 \neq 1$, we see that $\rho_G(G)(\infty) = \infty$ and $c = 0$ and $a\overline{s} = -sd$ and $\rho_G(G)$ is $z \mapsto -\frac{s}{\overline{s}}\overline{z} + t$. Since $G^2 = D$, we see that $1 = \rho_F(D)(0) = -\frac{s}{\overline{s}}\overline{t} + t$ and $s = -\overline{s}t + \overline{t}s$ and $\mathrm{Re}(s) = 0$, that is, (c) holds. $\square$

7.2 Remarks. In [15], the column-permuting-symmetry group of $\mathrm{CW}(F)$ was calculated; the referee of [18] showed that the column-permuting-symmetry group equals the whole symmetry group; see [18, Remark 8.16]. It then follows that the type of the planar tessellation $\mathrm{CW}(F)$ is **p 1** (the orbifold quotient is a torus) or **p 2** (the orbifold quotient is a sphere with four C_2-points) or **p g** (the orbifold quotient is a Klein bottle) or **p g g** (the orbifold quotient is a projective plane with two C_2-points). Moreover, $\mathrm{CW}(F)$ is of type **p g g** if and only if both (a)(i) and (a)(ii) hold. Also, CW(F) is of type **p g** if and only if (a)(i) holds or (a)(ii) holds, but not both.

It is well known that a planar group of type **p g** or **p g g** contains two mutually orthogonal planar translations. The traditional proof of this fact includes an argument that shows that if $\mathbf{G}$ and $\mathbf{T}$ are symmetries of the Euclidean plane such that $\mathbf{G}$ is a glide reflection, and $\mathbf{T}$ is a translation, and $\mathbf{G}^{-1}\mathbf{T}\mathbf{G} = \mathbf{T}^{-1}$, then the translations $\mathbf{G}^2$ and $\mathbf{T}$ are mutually orthogonal. Above, we have used this argument twice to verify that if CW(F) is of type **p g** or **p g g** then the planar translations D and F are mutually orthogonal. □

8 RL^n and RL^∞

In this section, we simplify slightly some arguments of Helling [22] that, firstly, give an explicit description of Jørgensen's representation ρ_{RL^n} specified in Notation 2.3, and secondly, on letting n tend to ∞, realize the homomorphism ρ_{RL^∞} described by Hodgson-Meyerhoff-Weeks [23]. We invent a definition of CW(RL^∞) and compare the drawings of CW(RL^∞) and CW(RL^{100}).

Recall that the meromorphic map $\hat{\mathbb{C}} \to \hat{\mathbb{C}}$, $w \mapsto w + w^{-1}$, induces a homeomorphism of open annuli $\{w \in \mathbb{C} : |w| > 1\} \xrightarrow{\sim} \mathbb{C} - [-2, 2]$. Most of Helling's calculations concern $\mathbb{C} - [-2, 2]$ while we feel that the calculations become a little simpler when lifted to $\{w \in \mathbb{C} : |w| > 1\}$. We begin by translating an interesting result and proof of Helling's that showed that, for a certain w, $w^2 + w^{-2}$ is restricted to a small region, while the following shows that w is restricted to a small *easily described* region.

8.1 Lemma (Helling [22])**.** *For each $n \in [1{\uparrow}\infty[$, there exists a unique $w \in \mathbb{C}$ with the properties that $|w| \in \;]1, \sqrt[n]{3}[$ and $\arg(w) \in \;]0, \frac{\pi}{2n+4}[$ and $w^{n+2} + w^{-n-2} = w^2 - w^{-2}$.*

Proof. We begin with a variation of an argument often used to prove Rouché's Theorem; see [4, p.153]. Let $\mathfrak{H} := \{z \in \mathbb{C} : \; |z| \in \;]0, \sqrt[n]{3}[\;, \; \arg(z) \in \;]0, \frac{\pi}{n+2}[\;\}$ and consider the boundary, $\partial\mathfrak{H} = [0, \sqrt[n]{3}] \cup [0, \sqrt[n]{3}]\mathbf{e}^{\frac{\pi \mathbf{i}}{n+2}} \cup \sqrt[n]{3}\mathbf{e}^{\frac{\pi \mathbf{i}}{n+2}[0,1]}$. Let $z \in \partial\mathfrak{H}$. We claim that either $\left|\frac{z^n - z^{n+4}}{z^{2n+4}+1}\right| < 1$ or $\frac{z^n - z^{n+4}}{z^{2n+4}+1} = \mathbf{i}$.

Consider first the case that $z \in \sqrt[n]{3}\mathbf{e}^{\frac{\pi \mathbf{i}}{n+2}[0,1]}$. Then $|z|^{-4} + 1 + |z|^{-n-4} < 1 + 1 + 1 = |z|^n$ and $|z^{-4} - 1| \leqslant |z|^{-4} + 1 < |z|^n - |z|^{-n-4} \leqslant |z^n + z^{-n-4}|$. Thus, $1 > \left|\frac{z^{-4}-1}{z^n + z^{-n-4}}\right| = \left|\frac{z^n - z^{n+4}}{z^{2n+4}+1}\right|$.

Consider now the case that $z \in [0, \sqrt[n]{3}] \cup [0, \sqrt[n]{3}]\mathbf{e}^{\frac{\pi \mathbf{i}}{n+2}}$. In $\mathbb{R}$, $(|z|^n - 1)(|z|^{n+4} - 1)$ achieves its minimum value, 0, where $|z| = 1$, that is, where $z = 1$ or $z = \mathbf{e}^{\frac{\pi \mathbf{i}}{n+2}}$. Thus, $|z|^n + |z|^{n+4} \leqslant |z|^{2n+4} + 1$ and equality holds only if $z = 1$ or $z = \mathbf{e}^{\frac{\pi \mathbf{i}}{n+2}}$. Now,

$$|z^n - z^{n+4}| \leqslant |z^n| + |-z^{n+4}| = |z|^n + |z|^{n+4} \leqslant |z|^{2n+4} + 1 = |z^{2n+4} + 1|,$$

and equality holds throughout only if $z = \mathbf{e}^{\frac{\boldsymbol{\pi}\mathbf{i}}{n+2}}$ and $n = 2$, where we then have $z^2 = \mathbf{i}$. Thus, $1 \geqslant \left|\frac{z^n - z^{n+4}}{z^{2n+4}+1}\right|$, and equality holds only if $n = 2$ and $z = \mathbf{e}^{\frac{\boldsymbol{\pi}\mathbf{i}}{4}}$, where we then have $\frac{z^n - z^{n+4}}{z^{2n+4}+1} = \mathbf{i}$.

We have now proved the claim.

Consider the meromorphic map $\mathfrak{f}\colon \hat{\mathbb{C}} \to \hat{\mathbb{C}}, z \mapsto 1 + \frac{z^n - z^{n+4}}{z^{2n+4}+1}$. If $z \in \partial\mathfrak{H}$, then we have $\mathfrak{f}(z) \in \,]0, 2] + \mathbf{i}[-1, 1]$, since the preceding argument shows that either $|\mathfrak{f}(z) - 1| < 1$ or $\mathfrak{f}(z) - 1 = \mathbf{i}$; hence, $\arg(\mathfrak{f}(z)) \in [-\frac{\boldsymbol{\pi}}{2}, \frac{\boldsymbol{\pi}}{2}]$. It is straightforward to show that when z travels a complete counter-clockwise circuit of $\partial\mathfrak{H}$, the total change in $\arg(z^{2n+4} + 1)$ is $2\boldsymbol{\pi}$, and, hence, $2\boldsymbol{\pi}$ is also the total change in $\arg(z^{2n+4} + 1) + \arg(\mathfrak{f}(z))$, that is, $\arg((z^{2n+4} + 1)\mathfrak{f}(z))$, that is, $\arg(z^{2n+4} + 1 + z^n - z^{n+4})$. Now, by Cauchy's argument principle, there exists a unique $w \in \mathfrak{H}$ such that $w^{2n+4} + 1 + w^n - w^{n+4} = 0$; see [4, p.131 or p.152]. In summary, there exists a unique $w \in \mathbb{C}$ with the properties that $|w| \in \left]0, \sqrt[n]{3}\right[$ and $\arg(w) \in \left]0, \frac{\boldsymbol{\pi}}{n+2}\right[$ and $w^{n+2} + w^{-n-2} = w^2 - w^{-2}$. It remains to show that w has all the desired properties.

Here, $\mathrm{Im}(w^2) > 0$ and $\mathrm{Im}(-w^{-2}) > 0$ and $\mathrm{Im}(w^{n+2} + w^{-n-2}) = \mathrm{Im}(w^2 - w^{-2}) > 0$ and $\mathrm{Im}(w^{n+2}) > 0$. It follows that $|w| > 1$.

Let $\theta \coloneqq \arg(w)$. We have $(n + 2)\theta \in \,]0, \boldsymbol{\pi}[$ and we wish to show that $(n + 2)\theta \in \left]0, \frac{\boldsymbol{\pi}}{2}\right[$, or, equivalently, $\cos((n + 2)\theta) > 0$.

If $n = 1$, $((w - w^{-1})^2 + 1) - (w - w^{-1}) = \frac{w^3 + w^{-3}}{w + w^{-1}} - \frac{w^2 - w^{-2}}{w + w^{-1}} = 0$, and $\mathrm{Im}(w - w^{-1}) > 0$, and we see that $w - w^{-1} = \frac{1}{2} + \frac{\sqrt{3}}{2}\mathbf{i}$, and, hence, $w^3 - w^{-3} = \frac{1}{2} + \frac{3\sqrt{3}}{2}\mathbf{i}$. On equating real parts, we find that $(|w|^3 - |w|^{-3})\cos(3\theta) = \frac{1}{2} > 0$, and, hence, $\cos(3\theta) > 0$.

If $n \geqslant 2$, we have $\cos(2\theta) > 0$, and, on equating the real parts of $w^{n+2} + w^{-n-2} = w^2 - w^{-2}$, we find that $(|w|^{n+2} + |w|^{-n-2})\cos((n + 2)\theta) = (|w|^2 - |w|^{-2})\cos(2\theta) > 0$.

For all $n \in [1, \infty[$, we then have $\cos((n + 2)\theta) > 0$, and w has all the desired properties. $\square$

8.2 Notation. Let $n \in [1{\uparrow}\infty[$. We consider Notation 2.3 and take $F = RL^n$. Recall that

$$^{R}(A, B, C) = (A, BCB, B) \quad \text{and} \quad ^{L}(A, B, C) = (B, BAB, C).$$

Notice that $^{L}D = D$ and $^{L}C = C$ and $^{L}A = B = (CD)A$. Hence, $^{L^n}D = D$ and $^{L^n}C = C$ and $^{L^n}A = (CD)^nA$ and $^{L^n}B = {}^{L^n}(CDA) = (CD)^{n+1}A$. Now, $^{RL^n}D = D$ and $^{RL^n}C = B$ and $^{RL^n}A = (BD)^nA$ and $^{RL^n}B = (BD)^{n+1}A$. Thus,

$$\begin{aligned}\langle\, A, B, C, RL^n \,\rangle = \langle A, B, C, D, F \mid A^2 = B^2 = C^2 = ABCD = 1,& \\ {}^{F}\!A = (BD)^nA, \quad {}^{F}\!B = (BD)^{n+1}A, \quad {}^{F}C = B\rangle.&\end{aligned}$$

Let w denote the unique element of $\mathbb{C}$ with the properties that $|w| \in \left]1, \sqrt[n]{3}\right[$ and $\arg(w) \in \left]0, \frac{\pi}{2n+4}\right[$ and $w^{n+2} + w^{-n-2} = w^2 - w^{-2}$; see Lemma 8.1.

Since $\arg(w) \in \left]0, \frac{\pi}{2n+4}\right[$, we have $\mathrm{Im}(w^n) > 0$ and, hence, $\mathrm{Im}(-w^{-n}) > 0$ and, hence, $\mathrm{Im}(-w^n + 2 + w^{-n}) < 0$. Since $|w| > 1$, we have, for each $i \in [1{\uparrow}\infty[$, $w^i \neq \pm w^{-i}$.

Let $x \coloneqq -w^n + 2 + w^{-n}$ (and, thus, $\mathrm{Im}(x) < 0$), $y \coloneqq z \coloneqq w^2 + w^{-2} \neq 0$, and $s \coloneqq \frac{w^n - w^{-n}}{-w^n+2+w^{-n}} = \frac{2}{x} - 1$ (and, thus, $\mathrm{Im}(s) > 0$).

For the purposes of the following discussion, it will be convenient to rewrite expressions in terms of $u \coloneqq -w^{n+2} + w^2 = w^{-2} + w^{-n-2}$. Notice that

$$u^2 = (-w^{n+2} + w^2)(w^{-2} + w^{-n-2}) = -w^n + w^{-n} = x - 2.$$

In particular, $u^2 + 2 = x$ and $u(u^2+2) \neq 0$. Notice also that

$$uw^{-2} - 1 = -w^n \qquad \text{and} \qquad uw^2 - 1 = w^{-n}. \tag{8.1}$$

On multiplying these two equations, we see that $u^2 - uw^2 - uw^{-2} + 1 = -1$, and then $u + 2u^{-1} = w^2 + w^{-2} = y = z$. Also, $s = \frac{2}{x} - 1 = -\frac{u^2}{u^2+2}$. Now

$$x^2 + y^2 + z^2 = (u^2+2)^2 + (\tfrac{u^2+2}{u})^2 + (\tfrac{u^2+2}{u})^2 = (u^2+1+1)(\tfrac{u^2+2}{u})^2 = xyz \neq 0.$$

We then have elements of $\mathrm{SL}_2(\mathbb{C})$,

$$\tilde{A} \coloneqq \begin{pmatrix} -z/y & (x-yz)/y^2 \\ x & z/y \end{pmatrix} = \begin{pmatrix} -1 & -2/(u^2+2) \\ u^2+2 & 1 \end{pmatrix}, \quad \tilde{B} \coloneqq \begin{pmatrix} 0 & -1/y \\ y & 0 \end{pmatrix} = \begin{pmatrix} 0 & -u/(u^2+2) \\ (u^2+2)/u & 0 \end{pmatrix},$$
$$\tilde{C} \coloneqq \begin{pmatrix} x/y & (z-xy)/y^2 \\ z & -x/y \end{pmatrix} = \begin{pmatrix} u & -(u^3+u)/(u^2+2) \\ (u^2+2)/u & -u \end{pmatrix}, \quad \tilde{D} \coloneqq \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad \tilde{F} \coloneqq \begin{pmatrix} 1 & s \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & -u^2/(u^2+2) \\ 0 & 1 \end{pmatrix},$$

satisfying $\tilde{A}^2 = \tilde{B}^2 = \tilde{C}^2 = \tilde{A}\tilde{B}\tilde{C}\tilde{D} = -\mathbf{I}_2$ and ${}^{\tilde{F}}\tilde{D} = \tilde{D}$. Straightforward calculations show that $\tilde{F}\tilde{C} = \begin{pmatrix} 0 & -u/(u^2+2) \\ (u^2+2)/u & -u \end{pmatrix} = \tilde{B}\tilde{F}$, and that $\tilde{B}\tilde{D} = \begin{pmatrix} 0 & -u/(u^2+2) \\ (u^2+2)/u & (u^2+2)/u \end{pmatrix}$, and that

$$(u\tilde{B}\tilde{D} - \mathbf{I}_2)^2 = \begin{pmatrix} -u^2+1 & -u^4/(u^2+2) \\ u^4+2u^2 & u^4+u^2+1 \end{pmatrix} = \tilde{A}\tilde{F}\tilde{A}^{-1}\tilde{F}^{-1}. \tag{8.2}$$

Now $\tilde{B}\tilde{D} = \begin{pmatrix} 0 & -u/(u^2+2) \\ (u^2+2)/u & (u^2+2)/u \end{pmatrix} = \begin{pmatrix} 0 & -1/(w^2+w^{-2}) \\ w^2+w^{-2} & w^2+w^{-2} \end{pmatrix}$, which is then in a form that is easily diagonalized, and, for $P \coloneqq \begin{pmatrix} -w^{-2} & -w^2 \\ w^2+w^{-2} & w^2+w^{-2} \end{pmatrix} \in \mathrm{GL}_2(\mathbb{C})$, we have

$$\tilde{B}\tilde{D}P = \begin{pmatrix} -1 & -1 \\ w^4+1 & 1+w^{-4} \end{pmatrix} = P\begin{pmatrix} w^2 & 0 \\ 0 & w^{-2} \end{pmatrix} \quad \text{and, hence,} \quad \tilde{B}\tilde{D} = {}^{P}\!\begin{pmatrix} w^2 & 0 \\ 0 & w^{-2} \end{pmatrix}. \tag{8.3}$$

Altogether,

$$\tilde{F}\tilde{A}\tilde{F}^{-1}\tilde{A}^{-1} \overset{(8.2)}{=} (u\tilde{B}\tilde{D} - \mathbf{I}_2)^{-2} \overset{(8.3)}{=} {}^{P}\!\begin{pmatrix} (uw^2-1)^{-2} & 0 \\ 0 & (uw^{-2}-1)^{-2} \end{pmatrix} \overset{(8.1)}{=} {}^{P}\!\begin{pmatrix} w^{2n} & 0 \\ 0 & w^{-2n} \end{pmatrix} \overset{(8.3)}{=} (\tilde{B}\tilde{D})^n.$$

We then have ${}^{\tilde{F}}\tilde{D} = \tilde{D}$ and ${}^{\tilde{F}}\tilde{C} = \tilde{B}$ and ${}^{\tilde{F}}\tilde{A} = (\tilde{B}\tilde{D})^n\tilde{A}$.

We have a representation $\rho'_{RL^n} \colon \langle\, A, B, C, F \,\rangle \to \mathrm{PSL}_2(\mathbb{C})$ determined by $\rho'_{RL^n}(W) \coloneqq \pm\, \tilde{W}$ for all $W \in \{A, B, C, F\}$. We next prove that ρ'_{RL^n} is equal to the ρ_{RL^n} of Notation 2.3. $\square$

8.3 Theorem (Helling [22]). *With* Notation 8.2, $\rho'_{RL^n} = \rho_{RL^n}$.

Proof. To simplify notation, we let $\langle A, B, C, F \rangle$ act on $\hat{\mathbb{C}}$ via ρ'_{RL^n}.

We shall use the following.

$$(8.4) \quad x = (-w^n + w^{-n}) + 2 = (-w^n + w^{-n})(\tfrac{w^2-w^{-2}}{w^{n+2}+w^{-n-2}}) + 2 = \tfrac{w^{n+2}+w^{-n+2}+w^{n-2}+w^{-n-2}}{w^{n+2}+w^{-n-2}}.$$

By (8.3), the columns of $P = \begin{pmatrix} -w^{-2} & -w^{2} \\ w^2+w^{-2} & w^2+w^{-2} \end{pmatrix}$ are eigenvectors for $\tilde{B}\tilde{D}$. For each $j \in [(-1)\uparrow(n+1)]$, we shall also be using

$$(8.5) \qquad q_j \coloneqq (BD)^j A(\infty) = (BD)^j(\tfrac{-1}{x}) \overset{(8.4)}{=} (BD)^j(\tfrac{w^n(-w^2)+w^{-n}(-w^{-2})}{w^n(w^2+w^{-2})+w^{-n}(w^2+w^{-2})})$$

$$(8.6) \qquad \overset{(8.3)}{=} \tfrac{w^{-2j}w^n(-w^2)+w^{2j}w^{-n}(-w^{-2})}{w^{-2j}w^n(w^2+w^{-2})+w^{2j}w^{-n}(w^2+w^{-2})} = -\tfrac{w^{n-2j+2}+w^{-n+2j-2}}{(w^2+w^{-2})(w^{n-2j}+w^{-n+2j})}.$$

For $i \in \mathbb{Z}$, $j \in [1\uparrow(n+1)]$, we calculate

$$\begin{aligned}
p_{i,j} &= ({}^{D^iRL^{j-1}}B)(\infty) = ({}^{D^i}((BD)^jA))(\infty) = D^i(BD)^jA(\infty) = D^i(q_j) = i + q_j, \\
p_{i,0} &= ({}^{D^i}B)(\infty) = D^iB(\infty) = D^i(0) = i, \\
p_{i,-1} &= ({}^{D^iL^{-1}}B)(\infty) = ({}^{D^i}A)(\infty) = D^iA(\infty) = D^i(q_0) = i + q_0, \\
p_{i,-n-1} &= ({}^{D^iL^{-n}R^{-1}}B)(\infty) = ({}^{D^iL^{-n}}C)(\infty) = ({}^{D^i}C)(\infty) = D^iCD(\infty) = D^iBA(\infty) \\
&= D^{i+1}(BD)^{-1}A(\infty) = i + 1 + q_{-1}.
\end{aligned}$$

See Figure 8.1. Also, see Figure 1.1(3) for the case $F = RL^3$.

Let $\theta \coloneqq \arg(w)$. Then $\theta \in \left]0, \frac{\pi}{2n+4}\right[$, and, for $t \in [0, n+2]$, $t\theta \in [0, \frac{\pi}{2}[$. Since $|w| > 1$, it follows that, for all $t \in [0, n{+}2]$, $\arg(w^t + w^{-t}) \in [0, \frac{\pi}{2}[$, and the same holds for all $t \in [-n{-}2, n{+}2]$. Further,

$$[0, n{+}2] \to \mathbb{R}, \quad t \mapsto \tan(\arg(w^t + w^{-t})) = \tfrac{\sin(t\theta)(|w|^t - |w|^{-t})}{\cos(t\theta)(|w|^t + |w|^{-t})} = \tan(t\theta)(1 - \tfrac{2}{|w|^{2t}+1})$$

is an increasing function, since it is the product of two positive, increasing functions. For all $t \in [0, n{+}2[$, $0 \leqslant \arg(w^t + w^{-t}) < \arg(w^{n+2} + w^{-n-2}) = \arg(w^2 - w^{-2}) < \frac{\pi}{2}$, and, hence, $\arg(\frac{w^2-w^{-2}}{w^t+w^{-t}}) \in \left]0, \frac{\pi}{2}\right[$, and, hence, $0 < \operatorname{Im}(\big(\frac{w^2-w^{-2}}{w^t+w^{-t}}\big)^2) = \operatorname{Im}(1 + \big(\frac{w^2-w^{-2}}{w^t+w^{-t}}\big)^2)$. The same holds for all $t \in \left]-n{-}2, n{+}2\right[$.

For $j \in \mathbb{Z}$,

$$\tfrac{q_j}{q_{j+1}} \overset{(8.6)}{=} \tfrac{(w^{n-2j+2}+w^{-n+2j-2})(w^{n-2j-2}+w^{-n+2j+2})}{(w^{n-2j}+w^{-n+2j})^2} = \tfrac{w^{2n-4j}+w^{-4}+w^4+w^{-2n+4j}}{w^{2n-4j}+2+w^{-2n+4j}} = 1 + \big(\tfrac{w^2-w^{-2}}{w^{n-2j}+w^{-n+2j}}\big)^2.$$

If $j \in [0\uparrow n]$, then $n - 2j \in [-n, n]$, and we see that $\operatorname{Im}(\frac{q_j}{q_{j+1}}) > 0$. Taking $j = -1$, we see that $q_{-1} = 2q_0$. Notice $q_0 = -\frac{1}{x} = -\frac{s+1}{2}$, $q_{-1} = 2q_0 = -s - 1$. Thus, 0, q_0 and q_{-1} are collinear.

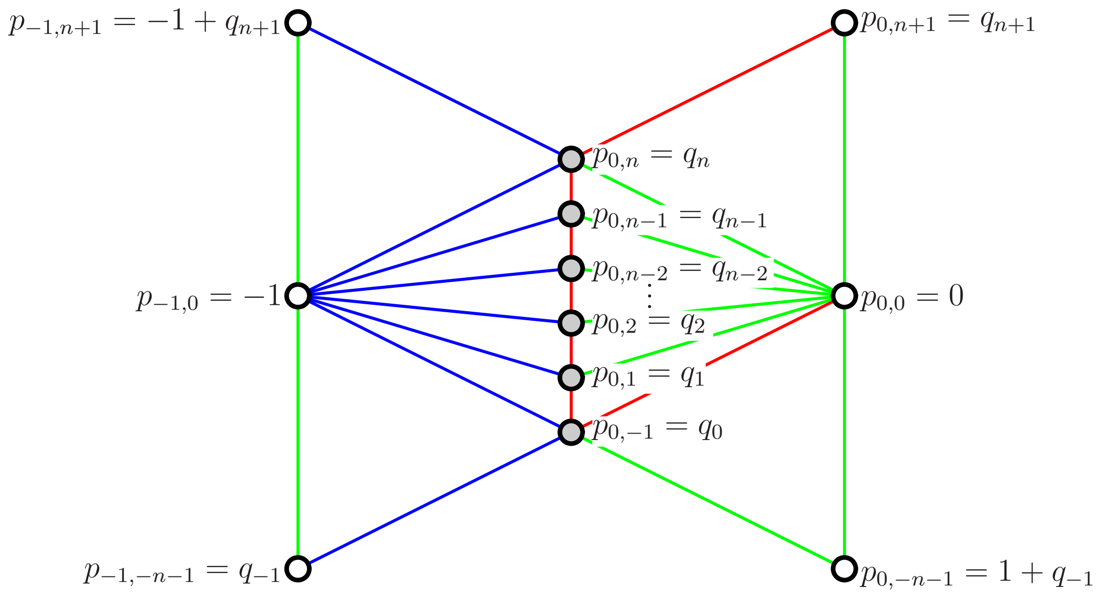


Figure 8.1: Schematic depiction of part of the RL^n-triangulation

For each $j \in [0{\uparrow}(n{+}1)]$, $-(w^2{+}w^{-2})q_j \overset{(8.6)}{=} \frac{w^{n-2j+2}+w^{-n+2j-2}}{w^{n-2j}+w^{-n+2j}}$; in this expression the numerator is $w^t + w^{-t}$ for $t = n{-}2j{+}2 \in [-n, n{+}2]$, and the denominator is $w^t + w^{-t}$ for $t = n{-}2j \in [-n{-}2, n]$. Hence the arguments of the numerator and denominator of $-(w^2{+}w^{-2})q_j$ lie in $[0, \frac{\pi}{2}[$, and, hence, $\mathrm{Re}(-(w^2{+}w^{-2})q_j) > 0$. Thus, $-(w^2{+}w^{-2})q_{[0\uparrow(n+1)]}$ lies in one of the half-planes marked out by $\mathbf{i}\mathbb{R}$, and, hence, $q_{[0\uparrow(n+1)]}$ lies in one of the half-planes marked out by $\frac{\mathbf{i}}{w^2+w^{-2}}\mathbb{R}$. Together with the fact that, for each $j \in [0{\uparrow}n]$, $\mathrm{Im}(\frac{q_j}{q_{j+1}}) > 0$, this shows that $q_{\llbracket 0\uparrow(n+1)\rrbracket}$ forms a clockwise arrangement in one of the quadrants marked out by $q_0\mathbb{R} \cup q_{n+1}\mathbb{R}$ $(= q_{-1}\mathbb{R} \cup q_{n+1}\mathbb{R})$. We then have a strip of $n{+}1$ triangles $(\mathrm{hull}\{0, q_j, q_{j+1}\} \mid j \in \llbracket 0{\uparrow}n\rrbracket)$ in the distinguished quadrant.

$$\text{For all } j \in \mathbb{Z},\ -q_j - q_{n-j} \overset{(8.6)}{=} \frac{w^{n-2j+2}+w^{-n+2j-2}}{(w^2+w^{-2})(w^{n-2j}+w^{-n+2j})} + \frac{w^{n-2(n-j)+2}+w^{-n+2(n-j)-2}}{(w^2+w^{-2})(w^{n-2(n-j)}+w^{-n+2(n-j)})}$$
$$= \frac{w^{n-2j+2}+w^{-n+2j-2}}{(w^2+w^{-2})(w^{n-2j}+w^{-n+2j})} + \frac{w^{-n+2j+2}+w^{n-2j-2}}{(w^2+w^{-2})(w^{-n+2j}+w^{n-2j})} = 1,$$

and, hence, $-q_j - 1 = q_{n-j}$. Thus $q_{n+1} = -q_{-1} - 1 = s$, $q_n = -q_0 - 1 = \frac{s-1}{2}$, and -1, q_n, q_{n+1} are collinear. We have seen that $q_{[(-1)\uparrow(n+1)]}$ lies in one of the quadrants marked out by $q_{-1}\mathbb{R} \cup q_{n+1}\mathbb{R}$; we now see that $q_{[(-1)\uparrow(n+1)]}$ also lies in the image of this quadrant under the $\boldsymbol{\pi}$-rotation about $-\frac{1}{2}$. Hence, $q_{[(-1)\uparrow(n+1)]}$ lies in the intersection of the quadrant and its image, and it is easy to see that this intersection is the parallelogram $\mathrm{hull}\{0, q_{-1} = -s-1, q_{n+1} = s, q_{-1} + q_{n+1} = -1\}$. Also, the $\boldsymbol{\pi}$-rotation about $-\frac{1}{2}$ carries the parallelogram to itself and carries $(\mathrm{hull}\{0, q_j, q_{j+1}\} \mid j \in \llbracket 0{\uparrow}n\rrbracket)$ to

$(\mathrm{hull}\{-1, q_j, q_{j-1}\} \mid j \in [\![n{\downarrow}0]\!])$. These two sequences of triangles fit together to form the triangulation of the parallelogram, and we have all the conditions of Notation 2.3 for ρ_{RL^n}. $\square$

Gueritaud [20, Section 10] gives much useful geometric information in a similar spirit about the more general case $\rho_{R^aL^b}$.

We now recall the limit representation obtained by Hodgson-Meyerhoff-Weeks [23].

8.4 Definitions. For each $n \in [1{\uparrow}\infty[$, Theorem 8.3 gives an explicit description of Jørgensen's representation ρ_{RL^n} of

$$\langle\, A,B,C,RL^n \,\rangle = \langle A,B,C,D,F_n \mid A^2 = B^2 = C^2 = ABCD = 1, \\ {}^{F_n}D = D, \quad {}^{F_n}C = B, \quad {}^{F_n}A = (BD)^n A\rangle$$

in $\mathrm{PSL}_2(\mathbb{C})$. The tessellation $\mathrm{CW}(RL^3)$ is displayed in Figure 1.1(3). The tessellation $\mathrm{CW}(RL^{100})$ is displayed in Figure 8.2.

We now want to replace n with ∞. We formally define

$$\langle\, A,B,C,RL^\infty \,\rangle := \langle A,B,C,D,F_\infty \mid A^2 = B^2 = C^2 = ABCD = 1, {}^{F_\infty}D = D, {}^{F_\infty}C = B\rangle.$$

For each $n \in [1{\uparrow}\infty[$, $\langle\, A,B,C,RL^n \,\rangle$ is a quotient group of $\langle\, A,B,C,RL^\infty \,\rangle$, and we then have an induced homomorphism $\rho_{RL^n} \colon \langle\, A,B,C,RL^\infty \,\rangle \to \mathrm{PSL}_2(\mathbb{C})$. It follows from Notation 8.2 that, if n is large, then w is near 1, and, hence, $w^{n+2} + w^{-n-2}$, which equals $w^2 - w^{-2}$, is near 0, and, hence, w^{n+2} is near $\pm\mathbf{i}$, but, $\arg(w) \in \,]0, \frac{\pi}{2n+4}[$ and, hence, $\mathrm{Im}(w^{n+2}) > 0$, and, hence, w^{n+2} is near $\mathbf{i}$. Now, x equals $-w^n + 2 + w^{-n}$ which is near $2 - 2\mathbf{i}$, and y and z equal $w^2 + w^{-2}$ which is near 2. Then $\rho_{RL^n}(A) = \pm\left(\begin{smallmatrix} -1 & -2/x \\ x & 1 \end{smallmatrix}\right)$ is near $\pm\left(\begin{smallmatrix} -1 & -\frac12-\frac{\mathbf{i}}{2} \\ 2-2\mathbf{i} & 1 \end{smallmatrix}\right)$, $\rho_{RL^n}(B) = \pm\left(\begin{smallmatrix} 0 & -1/y \\ y & 0 \end{smallmatrix}\right)$ is near $\pm\left(\begin{smallmatrix} 0 & -\frac12 \\ 2 & 0 \end{smallmatrix}\right)$, $\rho_{RL^n}(C) = \pm\left(\begin{smallmatrix} x/y & (-x+1)/y \\ y & -x/y \end{smallmatrix}\right)$ is near $\pm\left(\begin{smallmatrix} 1-\mathbf{i} & -\frac12+\mathbf{i} \\ 2 & -1+\mathbf{i} \end{smallmatrix}\right)$, and $\rho_{RL^n}(F_n) = \pm\left(\begin{smallmatrix} 1 & -1+\frac{2}{x} \\ 0 & 1 \end{smallmatrix}\right)$ is near $\pm\left(\begin{smallmatrix} 1 & -\frac12+\frac{\mathbf{i}}{2} \\ 0 & 1 \end{smallmatrix}\right)$. Letting n tend to ∞, we recover Hodgson-Meyerhoff-Weeks' representation $\rho_{RL^\infty} \colon \langle\, A,B,C,RL^\infty \,\rangle \to \mathrm{PSL}_2(\mathbb{C})$ with $\rho_{RL^\infty}(A) = \pm\left(\begin{smallmatrix} -1 & -\frac12-\frac{\mathbf{i}}{2} \\ 2-2\mathbf{i} & 1 \end{smallmatrix}\right)$, $\rho_{RL^\infty}(B) = \pm\left(\begin{smallmatrix} 0 & -\frac12 \\ 2 & 0 \end{smallmatrix}\right)$, $\rho_{RL^\infty}(C) = \pm\left(\begin{smallmatrix} 1-\mathbf{i} & -\frac12+\mathbf{i} \\ 2 & -1+\mathbf{i} \end{smallmatrix}\right)$, $\rho_{RL^\infty}(D) = \pm\left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$, $\rho_{RL^\infty}(F_\infty) = \pm\left(\begin{smallmatrix} 1 & -\frac12+\frac{\mathbf{i}}{2} \\ 0 & 1 \end{smallmatrix}\right)$.

The Möbius action of $\mathrm{PSL}_2(\mathbb{Z}[\mathbf{i}])$ on $\mathbf{H}^3$ can be used to derive presentations of low-index subgroups of $\mathrm{PSL}_2(\mathbb{Z}[\mathbf{i}])$; see [42, Example 1]. Conjugation by $\left(\begin{smallmatrix} 2 & 0 \\ 0 & 1 \end{smallmatrix}\right) \in \mathrm{GL}_2(\mathbb{Q})$ carries the image of ρ_{RL^∞} to a subgroup of $\mathrm{PSL}_2(\mathbb{Z}[\mathbf{i}])$ which is known to have index 6 and to have exactly the presentation that we chose for $\langle\, A,B,C,RL^\infty \,\rangle$. Hence, ρ_{RL^∞} is discrete and faithful. (We remark that it is well known that conjugation by $\left(\begin{smallmatrix} \frac{3+\mathbf{i}\sqrt3}{2} & 0 \\ 0 & 1 \end{smallmatrix}\right) \in \mathrm{GL}_2(\mathbb{C})$ carries the image of ρ_{RL} to an index-six subgroup of $\mathrm{PSL}_2(\mathbb{Z}[\frac{-1+\mathbf{i}\sqrt3}{2}])$.)

The foregoing is an index-two extension of part of the exposition by Helling [22] of the limit that was discovered by Hodgson-Meyerhoff-Weeks [23, Proposition 3] through an analysis of surgeries on the complement of the Whitehead link that give once-punctured-torus bundles.

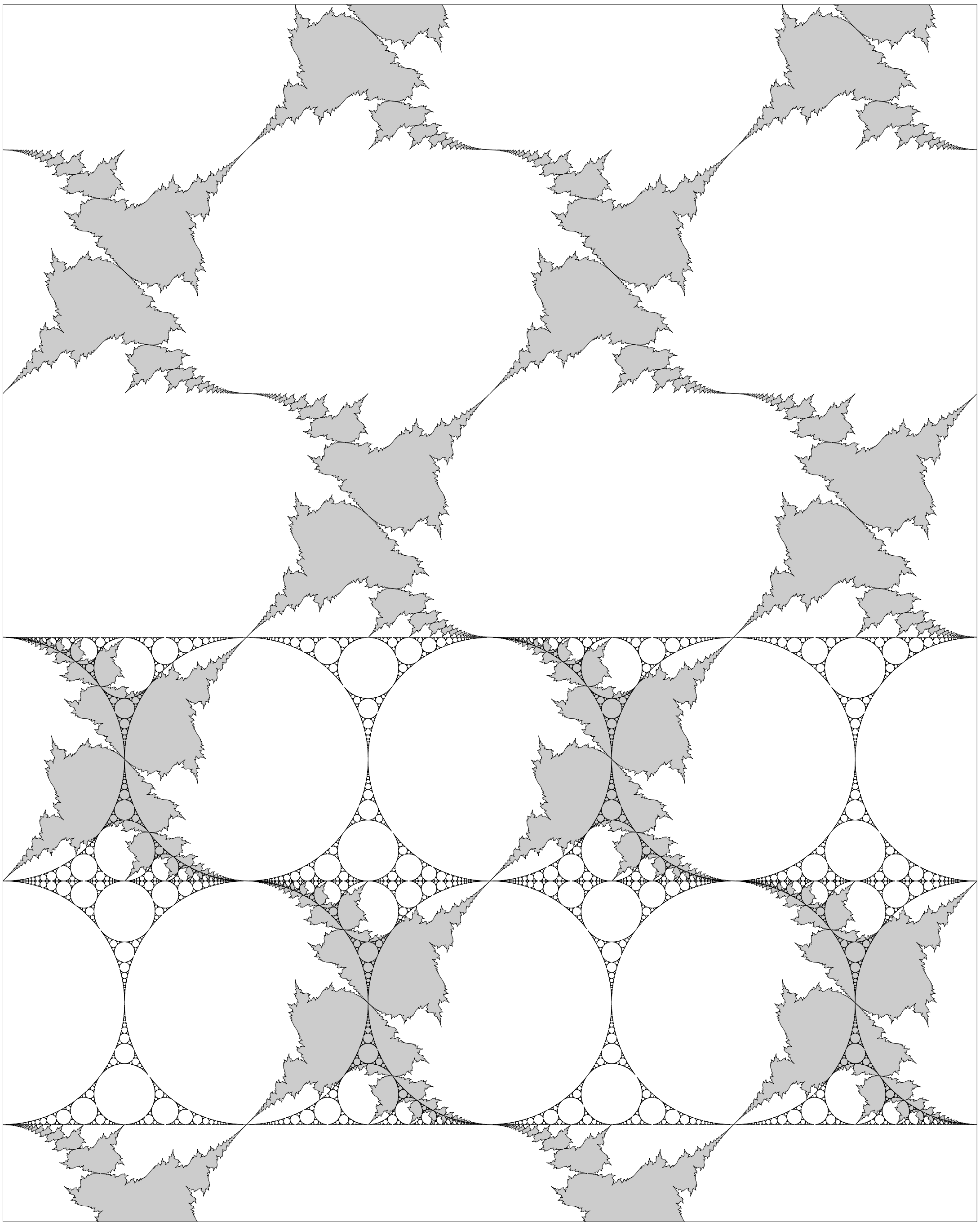


Figure 8.2: CW(RL^{100}) and two Apollonian gaskets from CW(RL^{∞})

The limit set of $\rho_{RL^\infty}(\langle A, B, C, RL^\infty \rangle)$ is $\hat{\mathbb{C}}$ while the limit set of $\rho_{RL^\infty}(\langle A, B, C \rangle)$ is a horizontal Apollonian gasket. We want to define a $\rho_{RL^\infty}(\langle D, F_\infty \rangle)$-invariant subset of $\mathbb{C}$ which suggests "$\lim\limits_{n\to\infty} \mathrm{CW}(RL^n)$". Let $\mathrm{CW}(RL^\infty)$ denote the stack of horizontal Apollonian gaskets obtained by applying $\rho_{RL^\infty}(\langle F_\infty \rangle)$ to the limit set of $\rho_{RL^\infty}(\langle A, B, C \rangle)$ and then deleting the point ∞. See Figure 8.2. The Cannon-Thurston path $\mathrm{CT}_{RL^{100}} : \hat{\mathbb{R}} \to \hat{\mathbb{C}}$ travels upward in gray columns and downward in white columns from left to right making an effort to respect Sakuma's alternating-gray-and-white-column structure on each horizontal gasket [35]; the white Apollonian columns seem to be decorations of the white Cannon-Thurston tiles. □

8.5 Frivolous remark. The gray regions of $\mathrm{CW}(RL^{100})$ in Figure 8.2 suggest to us chains of fleas preying on successively smaller fleas, therein reversing the well-known arrangement imagined by Jonathan Swift in *On Poetry: A Rhapsody* (1733):

So, naturalists observe, a flea
Has smaller fleas that on him prey;
And these have smaller still to bite 'em;
And so proceed *ad infinitum*.

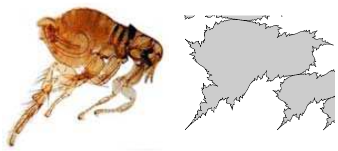


Figure 8.3: Flea and fractal

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