On the intersection of free subgroups in free products of groups with no 2-torsion

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To our friend and colleague Paul Schupp on the occasion of his retirement

Abstract

Let $(G_\ell \mid \ell \in L)$ be a family of groups and let F be a free group. Let G denote $F * \mathop{\ast}\limits_{\ell \in L} G_\ell$, the free product of F and all the G_ℓ. Let $\mathcal{F}$ denote the set of all finitely generated (free) subgroups H of G which have the property that, for each $g \in G$ and each $\ell \in L$, $H \cap G_\ell^g = \{1\}$. For each free group H, the *reduced rank* of H is defined as $\bar{\mathrm{r}}(H) := \max\{\mathrm{rank}(H) - 1, 0\} \in [0, \infty]$. Set

$$\theta \;:=\; \max\{\, \left|\tfrac{|D|}{|D|-2}\right| : D \text{ is a finite subgroup of } G \text{ with } |D| \neq 2\} \;\in\; [1,3],$$

$$\sigma := \inf\{\, s \in [0,\infty] \;:\; \text{for all } H, K \in \mathcal{F}, \sum_{HgK \in H\backslash G/K} \bar{\mathrm{r}}(H^g \cap K) \;\leqslant\; s\,\theta\, \bar{\mathrm{r}}(H)\,\bar{\mathrm{r}}(K)\,\} \;\in\; [0,\infty].$$

We are interested in precise bounds for σ. If every element of $\mathcal{F}$ is cyclic then $\sigma = 0$. We henceforth assume that some element of $\mathcal{F}$ has rank two.

In the case where $G = F$ and, hence, $\theta = 1$, Hanna Neumann and Walter Neumann proved that $\sigma \in [1,2]$ and it is a famous conjecture that $\sigma = 1$, called the Strengthened Hanna Neumann Conjecture.

For the general case, we proved that $\sigma \in [1,2]$ and if G has 2-torsion then $\sigma = 2$. We conjectured that if G is 2-torsion-free then $\sigma = 1$. In this article, we prove the following implications which show that under certain circumstances $\sigma < 2$.

If G is 2-torsion-free and has 3-torsion, then $\sigma \leqslant \frac{8}{7}$.

If G is 2-torsion-free and 3-torsion-free and has 5-torsion, then $\sigma \leqslant \frac{9}{5}$.

If p is an odd prime number and $G = C_p * C_p$, then $\sigma \leqslant 2 - \frac{(4+2\sqrt{3})p}{(2p-3+\sqrt{3})^2}$. In particular, if $G = C_3 * C_3$ then $\sigma = 1$, and if $G = C_5 * C_5$ then $\sigma \leqslant 1.52$.

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1 Conventions and notation

We now summarize our main general conventions and notation.

1.1 Conventions. Throughout this article, let G be a group.

All our G-sets will be left G-sets.

For two subsets A, B of a set, the complement of $A \cap B$ in A will be denoted by $A - B$ (and not by $A \setminus B$ since we let $G\backslash Y$ denote the set of G-orbits of a G-set Y).

To indicate disjoint unions, we shall use the symbols $\vee$, $\bigvee$ in place of $\cup$, $\bigcup$.

We shall use the totally ordered set $\{-\infty\} \vee \mathbb{R} \vee \{\infty\}$.

We will find it useful to have notation for intervals in $\mathbb{Z} \vee \{\infty\}$ analogous to the notation for intervals in $\mathbb{R} \vee \{\infty\}$. Let i, $j \in \mathbb{Z}$. We write

$$[i{\uparrow}j] \coloneqq \begin{cases} \{i, i+1, \ldots, j-1, j\} \subseteq \mathbb{Z} & \text{if } i \leqslant j, \\ \emptyset & \text{if } i > j. \end{cases}$$

Also, $[i{\uparrow}\infty[\ \coloneqq \{i, i+1, i+2, \ldots\}$ and $[i{\uparrow}\infty] \coloneqq [i{\uparrow}\infty[\ \vee\ \{\infty\}$.

For each set S, let $|S|$ denote the element of $[0{\uparrow}\infty]$ which is the cardinal of S if S is a finite set, and is ∞ if S is an infinite set.

For each $n \in [1{\uparrow}\infty]$, let C_n denote a multiplicative cyclic group of order n.

If a, b are elements of G, and S is a subset of G, we shall denote the inverse of a by $\overline{a}$, and we shall write $b^a \coloneqq \overline{a}ba$ and $S^a \coloneqq \{c^a \mid c \in S\}$.

We define the *rank* of G as $\operatorname{rank}(G) \coloneqq \min\{\,|S| : S$ is a generating set of $G\,\} \in [0{\uparrow}\infty]$. If G is a free group, we define the *reduced rank* of G as $\overline{\mathrm{r}}(G) \coloneqq \max\{\operatorname{rank}(G) - 1,\, 0\} \in [0{\uparrow}\infty]$; thus, $\overline{\mathrm{r}}(G) = b_1^{(2)}(G)$, the first L^2-Betti number of G; see Example 7.19 of **[6]**.

We define $\alpha_3(G) \coloneqq \inf\{\,|D| : D$ is a finite subgroup of G with $|D| \geqslant 3\,\} \in [3{\uparrow}\infty]$; it is understood that the infimum of the empty set is ∞.

We define $\theta(G) \coloneqq \frac{\alpha_3(G)}{\alpha_3(G)-2} \in [1, 3]$; it is understood that $\frac{\infty}{\infty-2} \coloneqq 1$. □

Let us fix the hypotheses and notation that will be used throughout the article.

1.2 Notation. Let L be a set, let $(G_\ell \mid \ell \in L)$ be a family of groups, let F be a free group, and suppose that $G = F * \mathop{\ast}\limits_{\ell \in L} G_\ell$, the free product of F and the members of $(G_\ell \mid \ell \in L)$.

Let $(t_e \mid e \in E_0)$ be a free-generating family of F, where E_0 is an index set of the correct size.

We now fix a graph of groups and a Bass-Serre tree, using Chapter I of **[1]** as our reference.

Let $Z \coloneqq VZ \vee EZ$ be the graph defined as follows.

The vertex set has the form $VZ \coloneqq \{z_\ell \mid \ell \in L\} \vee \{z_0\} = \{z_\ell \mid \ell \in L \vee \{0\}\}$, and the edge set has the form $EZ \coloneqq \{e_\ell \mid \ell \in L\} \vee E_0$, and the initial-vertex function $\iota_Z \colon EZ \to VZ$ maps each element of EZ to z_0, and the terminal-vertex function $\tau_Z \colon EZ \to VZ$ maps each element of E_0 to z_0 and, for each $\ell \in L$, maps e_ℓ to z_ℓ.

The graph Z has a unique maximal subtree, obtained from Z by deleting E_0.

Let $(\mathcal{G}, Z)$ be the graph of groups determined by the map of classes $\mathcal{G}\colon Z \to \mathit{Groups}$, $z \mapsto \mathcal{G}(z)$, defined as follows: $\mathcal{G}(z_0) = \{1\}$, and, for each $\ell \in L$, $\mathcal{G}(z_\ell) = G_\ell$, and, for each $e \in EZ$, $\mathcal{G}(e) = \{1\}$. Then G is the fundamental group of the graph of groups $(\mathcal{G}, Z)$ with respect to the unique maximal subtree of Z. For each $e \in EZ - E_0$, we define $t_e := 1$.

Let $T := T(\mathcal{G}, Z) := \bigvee_{z\in Z} Gz$, the Bass-Serre tree of $(\mathcal{G}, Z)$. Thus, for each $z \in Z$, the G-stabilizer of z is $\mathcal{G}(z)$,

$$VT := \bigvee_{z\in VZ} Gz, \qquad ET := \bigvee_{e\in EZ} Ge,$$

and, for each $e \in EZ$ and $g \in G$, the edge $ge \in ET$ joins $\iota(ge) := g\iota_Z e$ to $\tau(ge) := gt_e\tau_Z e$.

Notice that G acts freely on ET.

We identify $G\backslash T = Z$.

Let $\mathcal{F}$ denote the set of all finitely generated subgroups of G which act freely on T and are then free by Reidemeister's Theorem, Theorem I.8.3 of [**1**]. Alternatively, a finitely generated subgroup H of G belongs to $\mathcal{F}$ if and only if, for each $g \in G$ and each $\ell \in L$, $H \cap G_\ell^g = \{1\}$, and then one can see that H is free by the Kurosh Subgroup Theorem, Theorem I.7.8 of [**1**], see Theorem I.7.7 of [**1**].

We define

$$\sigma(\mathcal{F}) := \inf\{s \in [0,\infty] \ : \text{ for all } H, K \in \mathcal{F}, \sum_{HgK\in H\backslash G/K} \bar{\mathrm{r}}(H^g \cap K) \leqslant s\cdot \theta(G)\cdot \bar{\mathrm{r}}(H)\cdot \bar{\mathrm{r}}(K)\,\} \in [0,\infty];$$

notice that this differs from the notation in [**2**] by a factor of $\theta(G)$. We are interested in precise bounds for $\sigma(\mathcal{F})$. If every element of $\mathcal{F}$ is cyclic, then $\sigma(\mathcal{F}) = 0$. We henceforth assume that some element of $\mathcal{F}$ has rank two.

Let H and K be arbitrary elements of $\mathcal{F}$ and let $S \subseteq G$ be a set of representatives of $H\backslash G/K$. Notice that the value of $\sum_{g\in S} \bar{\mathrm{r}}(H^g \cap K)$ does not depend on the choice of S and is denoted $\sum_{HgK\in H\backslash G/K} \bar{\mathrm{r}}(H^g \cap K)$. We emphasize that $\sum_{HgK\in H\backslash G/K} \bar{\mathrm{r}}(H^g \cap K) \leqslant \sigma(\mathcal{F})\cdot \theta(G)\cdot \bar{\mathrm{r}}(H)\cdot \bar{\mathrm{r}}(K)$.

The *core* of $H\backslash T$, denoted $\mathrm{core}(H\backslash T)$, is the (finite) possibly empty subgraph of the quotient graph $H\backslash T$ consisting of all those vertices and edges which lie in cyclically reduced closed paths in $H\backslash T$.

Let $X := \mathrm{core}(H\backslash T)$, $Y := \mathrm{core}(K\backslash T)$, $W := \bigvee_{g\in S} \mathrm{core}((H^g \cap K)\backslash T)$.

For each $x \in VX$, $\deg_X(x)$ denotes the valence of x in X, the number of points in the link of x.

There are natural graph maps $X \to Z$, $Y \to Z$, $W \to Z$.

For each $\ell \in L \vee \{0\}$, we let $V_{/\ell}X$ denote the set of those vertices of X which are *over* z_ℓ, that is, map to z_ℓ under the map $VX \to VZ$. For each $i \in [0{\uparrow}\infty[$, we define

$$V_iX := \{x \in VX \mid \deg_X(x) = i\} \text{ and } V_{>i}X := \{x \in VX \mid \deg_X(x) > i\}.$$

Similarly, $V_{/\ell,i}X := V_iX \cap V_{/\ell}X$, $V_{/\ell,>i}X := V_{>i}X \cap V_{/\ell}X$.

We define $\bar{\mathrm{r}}X \coloneqq \sum\limits_{i\in[3\uparrow\infty[} (\frac{1}{2}(i-2)\,|V_iX|)$. Notice that

$$2\,|EX| = \sum_{i\in[2\uparrow\infty[} (i\,|V_iX|) = \sum_{i\in[2\uparrow\infty[} ((i-2)\,|V_iX|) + \sum_{i\in[2\uparrow\infty[} (2\,|V_iX|) = 2\,\bar{\mathrm{r}}X + 2\,|VX|\,,$$

and, hence, $\bar{\mathrm{r}}X = |EX| - |VX| = \bar{\mathrm{r}}(\pi_1(X)) = \bar{\mathrm{r}}(H)$.

Similar notation applies for Y and W, and we have $\bar{\mathrm{r}}Y = \bar{\mathrm{r}}(K)$, and $\bar{\mathrm{r}}W = \sum\limits_{HgK\in H\backslash G/K} \bar{\mathrm{r}}(H^g \cap K)$.

Since H and K are arbitrary elements of $\mathcal{F}$, if we show that $\bar{\mathrm{r}}W \leqslant s\,\bar{\mathrm{r}}X\,\bar{\mathrm{r}}Y$ for some $s \in [0,\infty]$, then we have $\sigma(\mathcal{F}) \leqslant \frac{s}{\theta(G)}$.

The following was seen in Section 6 of **[2]** which expanded an argument introduced by Sykiotis in the proof of Theorem 2.13(1) in **[11]**, which in turn expanded an argument introduced by Stallings.

We denote the pullback of the graph maps $X \to Z$ and $Y \to Z$ by $X \times_Z Y \subseteq X \times Y$. The graph $X \times_Z Y$ may have more than one component and may have vertices of valence less than 2. There exists a natural graph map $\psi\colon W \to X \times_Z Y$. The map ψ is injective on edges and also on vertices over the vertex z_0. Let $\ell \in L$. To each $x \in V_{/\ell}X$ there is associated a certain subset A_x of G_ℓ such that $\deg_X(x) = |A_x|$. To each $y \in V_{/\ell}Y$ there is associated a certain subset B_y of G_ℓ such that $\deg_Y(y) = |B_y|$; the inversion map gives a bijection between B_y and the set denoted B in Section 6 of **[2]**. To each $w \in V_\ell Y$, there is associated a pair $(x,y) = \psi(w)$, a certain element $c_w \in G_\ell$, and a certain subset

$$C_w \subseteq \mathrm{rep}(c_w, A_x \times B_y) \coloneqq \{(a,b) \in A_x \times B_y \mid ab = c_w\}$$

such that $\deg_W(w) = |C_w|$. Moreover, for each $(x,y) \in V_{/\ell}X \times_{\{z_\ell\}} V_{/\ell}Y$, the elements of $(C_w \mid w \in \psi^{-1}(\{(x,y)\})\ )$ are pairwise disjoint in $A_x \times B_y \subseteq G_\ell \times G_\ell$.

In particular, W is finite. □

2 Summary

Let Notation 1.2 hold.

In the case where $G = F$ and, hence, $\theta(G) = 1$, Walter Neumann **[9]**, generalizing results of Hanna Neumann **[7]**, **[8]**, proved that $\sigma(\mathcal{F}) \in [1,2]$; it is conjectured that $\sigma(\mathcal{F}) = 1$ and this is called the Strengthened Hanna Neumann Conjecture.

In **[2]**, which evolved from **[4]**, **[5]**, we proved that, for the general case, $\sigma(\mathcal{F}) \in [1,2]$ and if G has 2-torsion then $\sigma(\mathcal{F}) = 2$. We conjectured that if G is 2-torsion-free then $\sigma(\mathcal{F}) = 1$ (thus generalizing the Strengthened Hanna Neumann Conjecture); in this article, we obtain some partial results on this conjecture which show that it is possible to have $\sigma(\mathcal{F}) < 2$.

In outline, the article has the following structure.

In Section 3, we give a useful inequality which arises from the study of paths in the Bass-Serre tree and was suggested by an argument of Tardos.

In Section 4, we give some inequalities that arise from the study of representable products.

In Section 5, we show, in Theorem 5.3, that if G is 2-torsion-free, then

$$\sum_{HgK\in H\backslash G/K} \bar{\mathrm{r}}(H^g\cap K) \;\leqslant\; \tfrac{24}{7}\,\bar{\mathrm{r}}(H)\,\bar{\mathrm{r}}(K);$$

hence, if, moreover, G has 3-torsion then $\sigma(\mathcal{F}) \leqslant \frac{8}{7}$. We show also that if G is 2- and 3-torsion-free, then

$$\sum_{HgK\in H\backslash G/K} \bar{\mathrm{r}}(H^g\cap K) \;\leqslant\; 3\,\bar{\mathrm{r}}(H)\,\bar{\mathrm{r}}(K);$$

hence, if, moreover, G has 5-torsion then $\sigma(\mathcal{F}) \leqslant \frac{9}{5}$.

In Section 6, which does not use Section 5, we show, in Theorem 6.3, that if p is an odd prime number and $G = C_p{*}C_p$, then $\sigma(\mathcal{F}) \leqslant 2 - \frac{(4+2\sqrt{3})p}{(2p-3+\sqrt{3})^2}$. Here, $\theta(G) = \frac{p}{p-2}$ and $\mathcal{F}$ is the set of finitely generated free subgroups of G.

If $G = C_3{*}C_3$, then $\sigma(\mathcal{F}) = 1$, $\sigma(\mathcal{F})\,\theta(G) = 3$, and $\bar{\mathrm{r}}(H\cap K) \leqslant 3\,\bar{\mathrm{r}}(H)\,\bar{\mathrm{r}}(K)$; recall Example 2.8 of **[2]** in which $\bar{\mathrm{r}}(H\cap K) = 3\,\bar{\mathrm{r}}(H)\,\bar{\mathrm{r}}(K) = 3$.

If $G = C_5{*}C_5$, then $\sigma(\mathcal{F}) < 1.52$, $\sigma(\mathcal{F})\,\theta(G) < 2.52$, and $\bar{\mathrm{r}}(H\cap K) \leqslant 2.52\,\bar{\mathrm{r}}(H)\,\bar{\mathrm{r}}(K)$; recall Example 2.8 of **[2]** in which $\bar{\mathrm{r}}(H\cap K) = \frac{5}{3}\,\bar{\mathrm{r}}(H)\,\bar{\mathrm{r}}(K) = 15$.

In the Appendix, we give a proof of an inequality used in Section 6.

3 Paths in trees

In this section we examine paths in the Bass-Serre tree and deduce a useful inequality, by expanding an argument introduced by Tardos in the proof of Lemma 3 of **[12]**.

3.1 Notation. Let Notation 1.2 hold.

Let $\overline{E}X$ denote a copy of EX given with a bijection $EX \to \overline{E}X$, $e \mapsto \overline{e}$, and let the inverse $\overline{E}X \to EX$ also be denoted $e \mapsto \overline{e}$. For each $e \in EX$, define $\iota(\overline{e}) := \tau(e)$ and $\tau(\overline{e}) := \iota(e)$. Define $E^{\pm1}X := EX \vee \overline{E}X$. Let $E_+X := \{e \in E^{\pm1}X \mid \deg_X(\iota e) \geqslant 3\}$. Notice that $|E_+X| = \sum\limits_{i\in[3\uparrow\infty[} i\,|V_iX|$.

A *non-trivial reduced path in* X is a finite sequence $\mathbf{p} = (e_1, e_2, \ldots, e_{N-1}, e_N)$ in $E^{\pm1}X$ such that $N \in [1\uparrow\infty[$ and, for each $i \in [2\uparrow N]$, $\tau e_{i-1} = \iota e_i$ and $\overline{e_{i-1}} \neq e_i$. In this event, we call e_1 the *initial edge* of $\mathbf{p}$, ιe_1 the *initial vertex* of $\mathbf{p}$, e_N the *terminal edge* of $\mathbf{p}$ and τe_N the *terminal vertex* of $\mathbf{p}$. We define the *inverse* of $\mathbf{p}$ as $\overline{\mathbf{p}} := (\overline{e_N}, \overline{e_{N-1}}, \ldots, \overline{e_2}\,, \overline{e_1}\,)$. We call N the *length* of $\mathbf{p}$.

Let $\mathbf{P}(X)$ denote the set of non-trivial reduced paths in X.

The foregoing notation applies to any graph.

Since G acts on T, there is a natural G-action on $\mathbf{P}(T)$.

Let $\mathbf{p} \in \mathbf{P}(T)$.

We shall study $G\mathbf{p} \subseteq \mathbf{P}(T)$.

Recall that X is the core of $H\backslash T$. We define $\mathbf{P}(H\backslash T, \mathbf{p}) := H\backslash G\mathbf{p} \subseteq \mathbf{P}(H\backslash T)$, and $\mathbf{P}(X, \mathbf{p}) := (H\backslash G\mathbf{p}) \cap \mathbf{P}(X) \subseteq \mathbf{P}(X)$. Thus $\mathbf{P}(X, \mathbf{p})$ denotes the set of paths in X whose lifts to T lie in $G\mathbf{p}$. Since H acts freely on T, every path in X has exactly one lift to T for each lift of the initial vertex. Since G need not act freely on T, it is possible for two different elements of $G\mathbf{p}$ to have the same initial vertex. Since G acts freely on ET, two different elements of $G\mathbf{p}$ cannot have the same initial edge, and the same holds for $H\backslash G\mathbf{p}$ in $H\backslash T$. Let $E_+\mathbf{P}(X, \mathbf{p})$ denote the set of those elements of E_+X which occur as initial edges of elements of $\mathbf{P}(X, \mathbf{p})$.

Let e and e' be elements of E_+X. We say that e is *linked to* e' in X if there exists a non-trivial reduced path in X through vertices of valence 2 which has initial edge e and has terminal edge $\overline{e'}$; some authors call such a path a *super-edge*. This path is clearly unique and we say that the path *links* e to e'. The inverse of this path links e' to e. Since X is a core graph, each element of E_+X is linked to a unique element of E_+X.

The foregoing notation also applies with Y and W in place of X. □

3.2 Proposition. *Let* Notation 3.1 *hold and suppose that* G *is* 2*-torsion-free. Let* $\mathbf{p} \in \mathbf{P}(T)$. *Then the following hold.*

(i) *In* $\mathbf{P}(H\backslash T)$*, any two distinct elements of* $H\backslash G\mathbf{p}$ *have distinct initial edges.*

(ii) $|E_+\mathbf{P}(X, \mathbf{p})| + |E_+\mathbf{P}(X, \overline{\mathbf{p}})| \leqslant |E_+X|$.

(iii) $2\,|E_+W| \leqslant |E_+X|\,|E_+Y|$.

Proof. Let $e \in E^{\pm 1}T$ denote the initial edge of $\mathbf{p}$.

(i). Consider any $g_1, g_2 \in G$ such that $Hg_1e = Hg_2e$ in $H\backslash T$. Since G acts freely on $E^{\pm 1}T$, we see that $Hg_1 = Hg_2$ and, hence, $Hg_1\mathbf{p} = Hg_2\mathbf{p}$.

(ii). By induction on the length of $\mathbf{p}$, we may assume that the inequality holds for all paths that are strictly shorter than $\mathbf{p}$.

If $E_+\mathbf{P}(X, \mathbf{p}) \cap E_+\mathbf{P}(X, \overline{\mathbf{p}}) = \emptyset$, then,

$$|E_+\mathbf{P}(X, \mathbf{p})| + |E_+\mathbf{P}(X, \overline{\mathbf{p}})| = |E_+\mathbf{P}(X, \mathbf{p}) \cup E_+\mathbf{P}(X, \overline{\mathbf{p}})| \leqslant |E_+X|\,.$$

Thus, we may also assume that $E_+\mathbf{P}(X, \mathbf{p}) \cap E_+\mathbf{P}(X, \overline{\mathbf{p}}) \neq \emptyset$.

In particular, there exists a unique $g \in G$ such that $\mathbf{p}$ and $g\overline{\mathbf{p}}$ have the same initial edge, e. Let $\mathbf{q} \in \mathbf{P}(T)$ denote the longest common initial segment of $\mathbf{p}$ and $g\overline{\mathbf{p}}$.

It is clear that $E_+\mathbf{P}(X, \mathbf{p}) \subseteq E_+\mathbf{P}(X, \mathbf{q})$ and $E_+\mathbf{P}(X, \overline{\mathbf{p}}) \subseteq E_+\mathbf{P}(X, \mathbf{q})$. Hence, $|E_+\mathbf{P}(X, \mathbf{p}) \cup E_+\mathbf{P}(X, \overline{\mathbf{p}})| \leqslant |E_+\mathbf{P}(X, \mathbf{q})|$.

Also, we have factorizations $\mathbf{p} = \mathbf{q}{\cdot}\mathbf{x}$ and $g\overline{\mathbf{p}} = \mathbf{q}{\cdot}\mathbf{y}$, where the dot indicates concatenation of possibly empty sequences. Thus $\mathbf{q} \cdot \mathbf{x} = \mathbf{p} = \overline{g}\,\overline{\mathbf{y}} \cdot \overline{g}\,\overline{\mathbf{q}}$.

Consider first the case where $\mathbf{x}$ and $\mathbf{y}$ are strictly shorter than $\mathbf{q}$. Then there exists a factorization $\mathbf{p} = \overline{g}\,\overline{\mathbf{y}} \cdot \mathbf{z} \cdot \mathbf{x}$ with $\mathbf{z} \in \mathbf{P}(T)$, and, here, $\overline{g}\,\overline{\mathbf{y}} \cdot \mathbf{z} = \mathbf{q}$ and $\mathbf{z} \cdot \mathbf{x} = \overline{g}\,\overline{\mathbf{q}} = \overline{g}\,\overline{\mathbf{z}} \cdot \overline{g}^2\,\mathbf{y}$. Hence, $\mathbf{z} = \overline{g}\,\overline{\mathbf{z}}$ and, taking inverses, we see that $\overline{\mathbf{z}} = \overline{g}\,\mathbf{z}$, and hence $\mathbf{z} = \overline{g}\,\overline{\mathbf{z}} = \overline{g}^2\,\mathbf{z}$. Since G acts freely on ET, $g^2 = 1$. Since G is 2-torsion-free, $g = 1$. Thus $\overline{\mathbf{z}} = \mathbf{z}$. It follows that $\mathbf{z}$

has even length, and that the middle pair of terms are mutually inverse, which contradicts the fact that $\mathbf{z}$ is reduced. Hence, $\mathbf{x}$ and $\mathbf{y}$ must be at least as long as $\mathbf{q}$.

Thus, $\mathbf{q}$ is strictly shorter than $\mathbf{p}$ and there exists a factorization of the form $\mathbf{p} = \mathbf{q}\cdot\mathbf{r}\cdot\overline{g}\,\overline{\mathbf{q}}$, which gives $g\overline{\mathbf{p}} = \mathbf{q}\cdot g\overline{\mathbf{r}}\cdot g\overline{\mathbf{q}}$. It is possible that $\mathbf{r}$ is an empty sequence. Since $\mathbf{q}$ is the longest common initial segment of $\mathbf{p}$ and $g\overline{\mathbf{p}}$, we see that $\mathbf{r}\cdot\overline{g}\,\overline{\mathbf{q}}$ and $g\overline{\mathbf{r}}\cdot g\overline{\mathbf{q}}$ have the same initial vertex and different initial edges. For all $g_1 \in G$, there are factorizations $Hg_1\mathbf{p} = Hg_1\mathbf{q}\cdot Hg_1\mathbf{r}\cdot Hg_1\overline{g}\,\overline{\mathbf{q}}$ and $Hg_1g\,\overline{\mathbf{p}} = Hg_1\mathbf{q}\cdot Hg_1g\,\overline{\mathbf{r}}\cdot Hg_1g\,\overline{\mathbf{q}}$. Hence, in $H\backslash T$, $Hg_1\mathbf{p}$ and $Hg_1g\,\overline{\mathbf{p}}$ have the initial segment $Hg_1\mathbf{q}$ in common while $Hg_1\mathbf{r}\cdot Hg_1\overline{g}\,\overline{\mathbf{q}}$ and $Hg_1g\,\overline{\mathbf{r}}\cdot Hg_1g\,\overline{\mathbf{q}}$ have the same initial vertex, but, since H acts freely on T, they have different initial edges. Thus the two paths separate immediately after $Hg_1\mathbf{q}$ forming a vertex of $H\backslash T$ of valence at least 3.

The path $\mathbf{q}$ induces a natural embedding of $E_+\mathbf{P}(X,\mathbf{p})\cap E_+\mathbf{P}(X,\overline{\mathbf{p}})$ in $E_+\mathbf{P}(X,\overline{\mathbf{q}})$ as follows. Suppose that we have $g_1 \in G$ such that $Hg_1e \in E_+\mathbf{P}(X,\mathbf{p})\cap E_+\mathbf{P}(X,\overline{\mathbf{p}})$. We define the image of Hg_1e in $E_+\mathbf{P}(X,\overline{\mathbf{q}})$ to be the initial edge of $Hg_1\overline{\mathbf{q}}$, or, equivalently, the inverse of the terminal edge of $Hg_1\mathbf{q}$; this is an element of $E_+\mathbf{P}(X,\overline{\mathbf{q}})$, since, in X, the two paths $Hg_1\mathbf{p}$ and $Hg_1g\,\overline{\mathbf{p}}$ have the initial segment $Hg_1\mathbf{q}$ in common and then separate forming a vertex of X of valence at least 3 from which $Hg_1\overline{\mathbf{q}}$ returns. This gives the desired embedding. It follows that $|E_+\mathbf{P}(X,\mathbf{p})\cap E_+\mathbf{P}(X,\overline{\mathbf{p}})| \leqslant |E_+\mathbf{P}(X,\overline{\mathbf{q}})|$. Now,

$$\begin{aligned}|E_+\mathbf{P}(X,\mathbf{p})| + |E_+\mathbf{P}(X,\overline{\mathbf{p}})| &= |E_+\mathbf{P}(X,\mathbf{p})\cup E_+\mathbf{P}(X,\overline{\mathbf{p}})| + |E_+\mathbf{P}(X,\mathbf{p})\cap E_+\mathbf{P}(X,\overline{\mathbf{p}})|\\ &\leqslant |E_+\mathbf{P}(X,\mathbf{q})| + |E_+\mathbf{P}(X,\overline{\mathbf{q}})| \leqslant |E_+X|\,,\end{aligned}$$

by the induction hypothesis. This proves (ii).

(iii). We extend ψ to include an injective map $\psi\colon \overline{E}W \to \overline{E}X\times_{\overline{E}Z}\overline{E}Y \subseteq \overline{E}X\times\overline{E}Y$.

Let us suppose that $H\mathbf{p}$ links e to e' in X. Then $H\overline{\mathbf{p}}$ links e' to e.

Consider any $f \in E_+Y$. Suppose that $(e,f)\in\psi(E_+W)$. Let $\psi^{-1}(e,f)$ denote the unique element of E_+W mapped to (e,f) by ψ. Then $\psi^{-1}(e,f)$ is linked to an element of E_+W by some path which necessarily lies in $\mathbf{P}(W,\mathbf{p})$. It follows that f is the initial edge of some element of $\mathbf{P}(Y,\mathbf{p})$, that is, $f\in E_+\mathbf{P}(Y,\mathbf{p})$.

Thus, $\{f\in E_+Y : (e,f)\in\psi(E_+W)\}\subseteq E_+\mathbf{P}(Y,\mathbf{p})$.

Similarly, $\{f\in E_+Y : (e',f)\in\psi(E_+W)\}\subseteq E_+\mathbf{P}(Y,\overline{\mathbf{p}})$.

Using the analogue of (ii) for Y, we deduce that

$$|\{f\in E_+Y : (e,f)\in\psi(E_+W)\}| + |\{f\in E_+Y : (e',f)\in\psi(E_+W)\}| \leqslant |E_+Y|\,.$$

Since $e\in E_+X$ is arbitrary, we may sum over all $e\in E_+X$ and find that

$$2\,|\{(e,f)\in E_+X\times E_+Y : (e,f)\in\psi(E_+W)\}| \leqslant |E_+X|\,|E_+Y|\,,$$

and (iii) follows. □

We record the following restatement of (iii) in the vocabulary of Notation 1.2.

3.3 Corollary. *Let* Notation 1.2 *hold.*

If G is 2-torsion-free then $\sum\limits_{k\in[3\uparrow\infty[}(2k\,|V_kW|) \;\leqslant\; \sum\limits_{i,j\in[3\uparrow\infty[}(ij\,|V_iX|\,|V_jY|)$. □

4 Representable products

In this section we record some inequalities which arise from the study of representable products and which will be applied in the subsequent sections.

We shall frequently use the following observation.

4.1 Lemma. *If A and B are finite subsets of G, then*

$$\sum_{c\in G} \min(|A|, |\mathrm{rep}(c, A\times B)|) = \sum_{c\in G} |\mathrm{rep}(c, A\times B)| = |A\times B| = |A|\,|B|. \qquad \square$$

We shall frequently use the following, also.

4.2 Lemma. *Let* Notation 1.2 *hold, and let $r \in [0{\uparrow}\infty[$ and $\lambda \in [1,\infty[$ and $\mu \in [0,\infty]$. Suppose that, for all $\ell \in L$, and all $x \in V_{/\ell,>r}X$, and all $y \in V_{\ell,>r}Y$,*

$$\sum_{c\in G_\ell} \min(r, |\mathrm{rep}(c, A_x\times B_y)|) \geqslant \min(\mu,\ |A_x|\,|B_y| - \lambda(|A_x| - r)(|B_y| - r)).$$

Then $\displaystyle\sum_{k\in[(r+1)\uparrow\infty[} ((k-r)\,|V_kW|) \leqslant \sum_{i,j\in[(r+1)\uparrow\infty[} (\max(ij-\mu,\ \lambda(i-r)(j-r))\,|V_iX|\,|V_jY|).$

Proof. We first decompose the left-hand sum into an L part and a $\{0\}$ part.

$$\begin{aligned}
&\sum_{k\in[(r+1)\uparrow\infty[} ((k-r)\,|V_kW|)\\
&\quad= \sum_{k\in[0\uparrow\infty[} (\max(k-r,0)\,|V_kW|)\\
&\quad= \sum_{w\in VW} \max(\deg_W(w) - r, 0)\\
&\quad= \sum_{\ell\in L\vee\{0\}} \sum_{x\in V_{/\ell}X} \sum_{y\in V_{/\ell}Y} \sum_{w\in\psi^{-1}(\{(x,y)\})} \max(\deg_W(w) - r, 0)\\
&\quad= \sum_{\ell\in L\vee\{0\}} \sum_{x\in V_{/\ell,>r}X} \sum_{y\in V_{/\ell,>r}Y} \sum_{w\in\psi^{-1}(\{(x,y)\})} \max(\deg_W(w) - r, 0).
\end{aligned}$$

For the L part, we have

$$\begin{aligned}
&\sum_{\ell\in L} \sum_{x\in V_{/\ell,>r}X} \sum_{y\in V_{/\ell,>r}Y} \sum_{w\in\psi^{-1}(\{(x,y)\})} \max(\deg_W(w) - r, 0)\\
&= \sum_{\ell\in L} \sum_{x\in V_{/\ell,>r}X} \sum_{y\in V_{/\ell,>r}Y} \sum_{w\in\psi^{-1}(\{(x,y)\})} \max(|C_w| - r, 0)\\
&= \sum_{\ell\in L} \sum_{x\in V_{/\ell,>r}X} \sum_{y\in V_{/\ell,>r}Y} \sum_{c\in G_\ell} \sum_{\{w\in\psi^{-1}(\{(x,y)\})|c_w=c\}} \max(|C_w| - r, 0)\\
&\leqslant \sum_{\ell\in L} \sum_{x\in V_{/\ell,>r}X} \sum_{y\in V_{/\ell,>r}Y} \sum_{c\in G_\ell} \max(|\mathrm{rep}(c, A_x\times B_y)| - r, 0) \text{ as the } C_w \text{ are pairwise disjoint}\\
&= \sum_{\ell\in L} \sum_{x\in V_{/\ell,>r}X} \sum_{y\in V_{/\ell,>r}Y} \sum_{c\in G_\ell} (|\mathrm{rep}(c, A_x\times B_y)| - \min(r, |\mathrm{rep}(c, A_x\times B_y)|))
\end{aligned}$$

$$
\begin{aligned}
&= \sum_{\ell\in L}\ \sum_{x\in V_{/\ell,>r}X}\ \sum_{y\in V_{/\ell,>r}Y} (\,|A_x|\,|B_y| - \sum_{c\in G_\ell}\min(r, |\mathrm{rep}(c, A_x\times B_y)|\,))\ \text{by Lemma 4.1}\\
&\leqslant \sum_{\ell\in L}\ \sum_{x\in V_{/\ell,>r}X}\ \sum_{y\in V_{/\ell,>r}Y} \max(\,|A_x|\,|B_y| - \mu,\ \lambda(\,|A_x| - r)(\,|B_y| - r))\ \text{by hypothesis}\\
&= \sum_{\ell\in L}\ \sum_{x\in V_{/\ell,>r}X}\ \sum_{y\in V_{/\ell,>r}Y} \max(\deg_X(x)\deg_Y(y) - \mu,\ \lambda(\deg_X(x) - r)(\deg_Y(y) - r)).
\end{aligned}
$$

For the $\{0\}$ part, we have

$$
\sum_{x\in V_{/0,>r}X}\ \sum_{y\in V_{/0,>r}Y}\ \sum_{w\in\psi^{-1}(\{(x,y)\})} \max(\deg_W(w) - r, 0) \leqslant \sum_{x\in V_{/0,>r}X}\ \sum_{y\in V_{/0,>r}Y} (\deg_X(x) - r),
$$

since, over z_0, $\psi^{-1}(\{(x,y)\})$ has at most one element, and if there is an element, it is of degree at most $\deg_X(x)$. Since $\lambda \geqslant 1$, it follows that we have

$$
\begin{aligned}
&\sum_{k\in[(r+1)\uparrow\infty[} ((k-r)\,|V_kW|)\\
&\quad\leqslant \sum_{\ell\in L\vee\{0\}}\ \sum_{x\in V_{/\ell,>r}X}\ \sum_{y\in V_{/\ell,>r}Y} \max(\deg_X(x)\deg_Y(y) - \mu,\ \lambda(\deg_X(x) - r)(\deg_Y(y) - r))\\
&\quad\leqslant \sum_{x\in V_{>r}X}\ \sum_{y\in V_{>r}Y} \max(\deg_X(x)\deg_Y(y) - \mu,\ \lambda(\deg_X(x) - r)(\deg_Y(y) - r))\\
&\quad= \sum_{i,j\in[(r+1)\uparrow\infty[} (\max(ij - \mu,\ \lambda(i-r)(j-r))\,|V_iX|\,|V_jY|).
\end{aligned}
$$

□

We recall the following.

4.3 Theorem ([2])**.** *Let* Notation 1.2 *hold. Then the following hold.*

(i) *If A and B are finite subsets of G such that $|A| \geqslant 2$ and $|B| \geqslant 2$, then*

$$
\sum_{c\in G}\min(2, |\mathrm{rep}(c, A\times B)|\,)\ \geqslant\ \min(2\,\alpha_3(G),\ |A|\,|B| - (\,|A| - 2)(\,|B| - 2)).
$$

(ii) $\displaystyle\sum_{k\in[3\uparrow\infty[} ((k-2)\,|V_kW|) \leqslant \sum_{i,j\in[3\uparrow\infty[} (\max(ij - 2\,\alpha_3(G),\ (i-2)(j-2))\,|V_iX|\,|V_jY|).$

(iii) $\displaystyle\sum_{k\in[3\uparrow\infty[} ((k-2)\,|V_kW|) \leqslant \sum_{i,j\in[3\uparrow\infty[} ((ij-6)\,|V_iX|\,|V_jY|).$

(iv) *If $\alpha_3(G) \geqslant 4$, that is, G is 3-torsion-free, then*

$$
\sum_{k\in[3\uparrow\infty[} ((k-2)\,|V_kW|) \leqslant \sum_{i,j\in[3\uparrow\infty[} ((ij-8)\,|V_iX|\,|V_jY|).
$$

Proof. (i) is Theorem 5.10 of [**2**].

(ii), which is implicit in Section 6 of [**2**], follows by applying Lemma 4.2 with $(r, \lambda, \mu) = (2, 1, 2q)$ together with (i).

(iii) and (iv) follow from (ii) since we are considering expressions where $i + j \geqslant 6$. □

In the remaining sections, we shall be looking at various analogous assertions for $\sum\limits_{k\in[3\uparrow\infty[}((k-3)\,|V_kW|)$ obtained by studying $\sum\limits_{c\in G}\min(3,|\mathrm{rep}(c,A\times B)|\,)$, although work of Grynkiewicz [**3**] on abelian groups has shown that there can be no simple sharp bound.

5 3-torsion or 5-torsion in G

In this section, we show that if G is 2-torsion-free and $\alpha_3(G)=3$ then $\sigma(\mathcal{F})\leqslant\frac{8}{7}$, and if G is 2-torsion-free and $\alpha_3(G)=5$ then $\sigma(\mathcal{F})\leqslant\frac{9}{5}$.

5.1 Lemma. *Let A and B be finite subsets of G such that $|A|\geqslant 3$ and $|B|\geqslant 3$. Then exactly one of the following holds.*

(5.1.a) $\sum\limits_{c\in G}\min(3,|\mathrm{rep}(c,A\times B)|)\ \geqslant\ |A|\,|B|-\frac{5}{2}(|A|-3)(|B|-3)$.

(5.1.b) *There exists some subgroup H of G of order four such that A is a left coset of H and B is a right coset of H.*

Proof. If $|B|=3$ then, by Lemma 4.1, equality holds in (5.1.a).

Consider the case where $|B|\geqslant 5$. Let A' be a three-element subset of A. Then

$$\sum_{c\in G}(2\min(3,|\mathrm{rep}(c,A\times B)|\,))\geqslant\sum_{c\in G}(2\min(3,|\mathrm{rep}(c,A'\times B)|))=2\,|A'|\,|B|\ \text{ by Lemma 4.1}$$
$$=6\,|B|\ \geqslant\ 6\,|B|-3(\,|A|-3)(\,|B|-5)\ =\ -3\,|A|\,|B|+15\,|A|+15\,|B|-45,$$

and (5.1.a) holds.

Thus we may assume that $|B|=4$, and, by symmetry, we may also assume that $|A|=4$. We may further assume that (5.1.a) fails, that is,

$$\sum_{c\in G}\min(3,|\mathrm{rep}(c,A\times B)|\,)<|A|\,|B|-\tfrac{5}{2}(|A|-3)(|B|-3)=\tfrac{27}{2}.$$

By Lemma 4.1, $\sum\limits_{c\in G}(\min(4,|\mathrm{rep}(c,A\times B)|))=16$. Hence, there are at least three elements of G which appear four times in the $A\times B$ multiplication table, once in every row, and once in every column. Choose $a_0\in A$ and $b_0\in B$ such that a_0b_0 appears in every row and every column. By replacing A with $\overline{a}_0A$ and B with $B\overline{b}_0$, we may assume that 1 occurs in every row and every column, and that $1\in A\cap B$. A familiar argument then shows that $A=B$ and that A is a subgroup of G of order four, as follows. Let 1, a, b denote three distinct elements of G which appear in every row and every column. Let $c\in A-\{1,a,b\}$ and let $d\in B-\{1,a,b\}$. If $b=\overline{a}$, it follows that $cd=1$, and that $\overline{a}d=c\overline{a}=a$, and that $ca=ad=\overline{a}$, and that $a^4=1$. If $ab\neq 1$, then $ab\notin\{1,a,b\}$, and then either $(a^2,ad,ca)=(1,b,b)$ or $(a^2,ad,ca)=(b,1,1)$. If $(a^2,ad,ca)=(1,b,b)$, it follows that $b^2=1$, and that $bab=a$. If $(a^2,ad,ca)=(b,1,1)$, it follows that $a^4=1$. In all events, $A=B$ and A is a subgroup of order four, and (5.1.b) holds. Here, $\sum\limits_{c\in G}\min(3,|\mathrm{rep}(c,A\times B)|)=12<\frac{27}{2}$ and (5.1.a) fails. □

5.2 Corollary. *Let* Notation 1.2 *hold. Suppose that G has no subgroup of order* 4.

Then $\sum\limits_{k\in[4\uparrow\infty[}((k-3)\,|V_kW|) \leqslant \sum\limits_{i,j\in[4\uparrow\infty[}(\frac{5}{2}(i-3)(j-3)\,|V_iX|\,|V_jY|)$.

Proof. This follows by applying Lemma 4.2 with $(r,\lambda,\mu) = (3,\frac{5}{2},\infty)$ together with Lemma 5.1. □

5.3 Theorem. *Let* Notation 1.2 *hold. Suppose that G is* 2*-torsion-free. Then the following hold.*

(i) $\sum\limits_{HgK\in H\backslash G/K} \bar{\mathrm{r}}(H^g\cap K) \;\leqslant\; \frac{24}{7}\,\bar{\mathrm{r}}(H)\,\bar{\mathrm{r}}(K)$.

(ii) *If $\alpha_3(G)=3$, that is, G has* 3*-torsion, then $\sigma(\mathcal{F}) \leqslant \frac{8}{7}$.*

(iii) *If $\alpha_3(G)\geqslant 5$, that is, G is* 3*-torsion-free, then* $\sum\limits_{HgK\in H\backslash G/K} \bar{\mathrm{r}}(H^g\cap K) \;\leqslant\; 3\,\bar{\mathrm{r}}(H)\,\bar{\mathrm{r}}(K)$.

(iv) *If $\alpha_3(G)=5$, that is, G is* 3*-torsion-free and has* 5*-torsion, then $\sigma(\mathcal{F}) \leqslant \frac{9}{5}$.*

Proof. (i). We have

$$\begin{aligned}
14\,\bar{\mathrm{r}}W &= \sum_{k\in[3\uparrow\infty[}(7(k-2)\,|V_kW|)\\
&= \sum_{k\in[3\uparrow\infty[}(2k\,|V_kW|) + \sum_{k\in[3\uparrow\infty[}((k-2)\,|V_kW|) + \sum_{k\in[4\uparrow\infty[}(4(k-3)\,|V_kW|)\\
&\leqslant \sum_{i,j\in[3\uparrow\infty[}(ij\,|V_iX|\,|V_jY|) + \sum_{i,j\in[3\uparrow\infty[}((ij-6)\,|V_iX|\,|V_jY|)\\
&\qquad\qquad + \sum_{i,j\in[4\uparrow\infty[}(10(i-3)(j-3)\,|V_iX|\,|V_jY|)
\end{aligned}$$

by Corollary 3.3, Theorem 4.3(iii), and Corollary 5.2

$$\begin{aligned}
&= \sum_{i,j\in[3\uparrow\infty[}(6(2ij-5i-5j+14)\,|V_iX|\,|V_jY|)\\
&= \sum_{i,j\in[3\uparrow\infty[}(6(2(i-2)(j-2)-(i+j-6))\,|V_iX|\,|V_jY|)\\
&\leqslant \sum_{i,j\in[3\uparrow\infty[}(12(i-2)(j-2)\,|V_iX|\,|V_jY|) = 12\cdot 2\,\bar{\mathrm{r}}X\cdot 2\,\bar{\mathrm{r}}Y = 48\,\bar{\mathrm{r}}X\,\bar{\mathrm{r}}Y = 14\cdot\tfrac{24}{7}\cdot\bar{\mathrm{r}}X\,\bar{\mathrm{r}}Y.
\end{aligned}$$

(ii). If $\alpha_3(G)=3$, then $\theta(G)=3$, and, by (i), $\sigma(\mathcal{F}) \leqslant \frac{24}{7\,\theta(G)} = \frac{8}{7}$.

(iii). As before, we have

$$\begin{aligned}
20\,\bar{\mathrm{r}}W &= \sum_{k\in[3\uparrow\infty[}(10(k-2)\,|V_kW|)\\
&= \sum_{k\in[3\uparrow\infty[}(2k\,|V_kW|) + \sum_{k\in[3\uparrow\infty[}(4(k-2)\,|V_kW|) + \sum_{k\in[4\uparrow\infty[}(4(k-3)\,|V_kW|)
\end{aligned}$$

$$\leqslant \sum_{i,j\in[3\uparrow\infty[} (ij\,|V_iX|\,|V_jY|) + \sum_{i,j\in[3\uparrow\infty[} (4(ij-8)\,|V_iX|\,|V_jY|)$$
$$+ \sum_{i,j\in[4\uparrow\infty[} (10(i-3)(j-3)\,|V_iX|\,|V_jY|)$$

by Corollary 3.3, Theorem 4.3(iv), and Corollary 5.2.

$$= \sum_{i,j\in[3\uparrow\infty[} ((15ij-30i-30j+58)\,|V_iX|\,|V_jY|)$$
$$= \sum_{i,j\in[3\uparrow\infty[} ((15(i-2)(j-2)-2)\,|V_iX|\,|V_jY|)$$
$$\leqslant \sum_{i,j\in[3\uparrow\infty[} (15(i-2)(j-2)\,|V_iX|\,|V_jY|) = 15\cdot 2\,\bar{\mathrm{r}}X\cdot 2\,\bar{\mathrm{r}}Y = 60\,\bar{\mathrm{r}}X\,\bar{\mathrm{r}}Y = 20\cdot 3\cdot \bar{\mathrm{r}}X\,\bar{\mathrm{r}}Y.$$

(iv). If $\alpha_3(G)=5$, then $\theta(G)=\frac{5}{3}$, and, by (iii), $\sigma(\mathcal{F}) \leqslant \frac{3}{\theta(G)} = \frac{9}{5}$. □

5.4 Remarks. Let Notation 1.2 hold. Combining Theorem 6.5 of [**2**] with the above, we see the following.

If $\alpha_3(G)=3$ then $\bar{\mathrm{r}}W \leqslant 6.00\,\bar{\mathrm{r}}X\,\bar{\mathrm{r}}Y$, and if G is 2-torsion-free then $\bar{\mathrm{r}}W \leqslant 3.43\,\bar{\mathrm{r}}X\,\bar{\mathrm{r}}Y$.
If $\alpha_3(G)=4$ then $\bar{\mathrm{r}}W \leqslant 4.00\,\bar{\mathrm{r}}X\,\bar{\mathrm{r}}Y$, and G has 2-torsion.
If $\alpha_3(G)=5$ then $\bar{\mathrm{r}}W \leqslant 3.34\,\bar{\mathrm{r}}X\,\bar{\mathrm{r}}Y$, and if G is 2-torsion-free then $\bar{\mathrm{r}}W \leqslant 3.00\,\bar{\mathrm{r}}X\,\bar{\mathrm{r}}Y$.
If $\alpha_3(G)>5$ then $\bar{\mathrm{r}}W \leqslant 2.80\,\bar{\mathrm{r}}X\,\bar{\mathrm{r}}Y$. □

6 $C_p{*}C_p$

In this section, we show that if p is an odd prime number and $G = C_p{*}C_p$, then $\sigma(\mathcal{F}) \leqslant 2 - \frac{(4+2\sqrt{3})p}{(2p-3+\sqrt{3})^2}$.

For convenience, we recall a classic result.

6.1 Pollard's Theorem [10]. *Let p be a prime number, suppose that G has order p, let A and B be subsets of G, and let $r \in [0\uparrow \min(|A|,|B|)]$. Then*

$$\sum_{c\in G} \min(r, |\mathrm{rep}(c, A\times B)|) \;\geqslant\; r\cdot \min(p, |A|+|B|-r). \tag{6.1.1}$$

Proof (Pollard [**10**]). By induction on $|A|$, we may assume that the analogue of (6.1.1) holds for smaller A.

If $|A| \leqslant 1$, then (6.1.1) is easily verified. Thus we may assume that $|A| \geqslant 2$.

By replacing A with $\overline{a_0}A$ for any $a_0 \in A$, we may assume that $1 \in A$. If $AB = B$, then $\langle A\rangle B \subseteq B$, but $\langle A\rangle = G$ since $|A| \geqslant 2$, and, hence, either $B=\emptyset$ or $B=G$, and again (6.1.1) is easily verified. Thus we may assume that $AB \neq B$. Here, there exists some $a_1 \in A$ such that $a_1B \neq B$. There then exists some $b_1 \in B$ such that $a_1b_1 \notin B$. By replacing B with $B\overline{b_1}$

we may assume that $a_1 \in A - B$ and $1 \in A \cap B$. Thus the induction hypothesis applies to $(A \cap B, A \cup B)$ and to $(A - B, B - A)$.

Let $\mathbb{T}_i$, $i \in [1{\uparrow}5]$, denote the multiplication tables that arise as follows.

	$A \cap B$	$B - A$
$A \cap B$	$\mathbb{T}_1$	$\mathbb{T}_2$
$A - B$	$\mathbb{T}_3$	$\mathbb{T}_4$

	$A \cap B$	$B - A$	$A - B$
$A \cap B$	$\mathbb{T}_1$	$\mathbb{T}_2$	$\mathbb{T}_5$
$A - B$		$\mathbb{T}_4$	

We see that $\mathbb{T}_5$ is the transpose of $\mathbb{T}_3$ and, hence, for each $c \in G$,

$$\text{(6.1.2)} \qquad |\text{rep}(c, A \times B)| = |\text{rep}(c, (A \cap B){\times}(A \cup B))| + |\text{rep}(c, (A - B){\times}(B - A))|\,.$$

If $r < |A \cap B|$, then, by (6.1.2), replacing (A, B) with $(A \cap B, A \cup B)$ does not increase the left-hand side of the inequality in (6.1.1) and does not change the right-hand side of the inequality in (6.1.1), and the desired conclusion follows by the induction hypothesis applied to $(A \cap B, A \cup B)$. Thus we may assume that $r \geqslant |A \cap B|$.

By Lemma 4.1,

$$\text{(6.1.3)} \qquad \sum_{c \in G} \min(\,|A \cap B|\,, |\text{rep}(c, (A \cap B) \times (A \cup B)|\,) = |A \cap B| \cdot |A \cup B|\,.$$

Since $|A - B| + |B - A| - (r - |A \cap B|) = |A \cup B| - r \leqslant p$, the induction hypothesis applied to $(A - B, B - A)$ gives

$$\text{(6.1.4)} \quad \sum_{c \in G} \min(r - |A \cap B|\,,\ |\text{rep}(c, (A - B) \times (B - A)|\,) \ \geqslant\ (r - |A \cap B|)(\,|A \cup B| - r).$$

It follows from (6.1.2) that, for each $c \in G$, both $r = (\,|A \cap B|) + (r - |A \cap B|)$ and $|\text{rep}(c, A \times B)|$ are at least as big as

$$\min(\,|A \cap B|\,, |\text{rep}(c, (A \cap B) \times (A \cup B)|\,) + \min(r - |A \cap B|\,, |\text{rep}(c, (A - B) \times (B - A)|\,).$$

Summing (6.1.3) and (6.1.4), we find that

$$\begin{aligned} \sum_{c \in G} \min(r, |\text{rep}(c, A \times B)|\,) &\geqslant |A \cap B| \cdot |A \cup B| + (r - |A \cap B|){\cdot}(\,|A \cup B| - r) \\ &= r(\,|A \cap B| + |A \cup B| - r) = r(\,|A| + |B| - r). \end{aligned}$$

This completes the proof. □

6.2 Corollary. *Let* Notation 1.2 *hold. Let p be a prime number and suppose that every element of $(G_\ell \mid \ell \in L)$ has order p. Let $r \in [0{\uparrow}\infty[$. Then*

$$\sum_{k \in [(r+1)\uparrow\infty[} ((k - r)\,|V_k W|) \leqslant \sum_{i,j \in [(r+1)\uparrow\infty[} (\max(ij - rp,\ (i - r)(j - r))\,|V_i X|\,|V_j Y|).$$

Proof. This follows by applying Lemma 4.2 with the same r and $(\lambda,\mu) = (1, rp)$, together with Pollard's Theorem 6.1. $\square$

Let p be an odd prime number and suppose that $G = C_p{*}C_p$; here, $\theta(G) = \frac{p}{p-2}$ and $\mathcal{F}$ is the set of finitely generated free subgroups of G. The following result restricts $\sigma(\mathcal{F})$. For example:

$$\begin{array}{lllll} \text{for} & C_3{*}C_3, & \sigma(\mathcal{F}) = 1, & \sigma(\mathcal{F})\,\theta(G) = 3, & 2\,\theta(G) = 6; \\ \text{for} & C_5{*}C_5, & \sigma(\mathcal{F}) \leqslant 1.52, & \sigma(\mathcal{F})\,\theta(G) \leqslant 2.52, & 2\,\theta(G) = 3.\overline{3}; \\ \text{for} & C_7{*}C_7, & \sigma(\mathcal{F}) \leqslant 1.68, & \sigma(\mathcal{F})\,\theta(G) \leqslant 2.35, & 2\,\theta(G) = 2.8; \\ \text{for} & C_{11}{*}C_{11}, & \sigma(\mathcal{F}) \leqslant 1.81, & \sigma(\mathcal{F})\,\theta(G) \leqslant 2.22, & 2\,\theta(G) = 2.\overline{4}\,. \end{array}$$

If $G = C_3{*}C_3$, then $\bar{\mathrm{r}}(H \cap K) \leqslant 3\,\bar{\mathrm{r}}(H)\,\bar{\mathrm{r}}(K)$; Example 2.8 of **[2]** has $\bar{\mathrm{r}}(H \cap K) = 3\,\bar{\mathrm{r}}(H)\,\bar{\mathrm{r}}(K) = 3$.

If $G = C_5{*}C_5$, then $\bar{\mathrm{r}}(H \cap K) \leqslant 2.52\,\bar{\mathrm{r}}(H)\,\bar{\mathrm{r}}(K)$; Example 2.8 of **[2]** has $\bar{\mathrm{r}}(H \cap K) = 1.\overline{6}\,\bar{\mathrm{r}}(H)\,\bar{\mathrm{r}}(K) = 15$.

6.3 Theorem. *Let* Notation 1.2 *hold, let p be an odd prime number and suppose that L is nonempty and that each element of $(G_\ell \mid \ell \in L)$ has order p. Then the following hold.*

(i) $\sigma(\mathcal{F}) \leqslant 2 - \frac{(4+2\sqrt{3})p}{(2p-3+\sqrt{3})^2} < 2.$

(ii) *If $p = 3$ then $\sigma(\mathcal{F}) = 1$.*

Proof. (i). It suffices to prove the result in the case where L is finite and F is finitely generated. Then G embeds in $C_p{*}C_p$, and we may assume that $C_p{*}C_p$, that $|L| = 2$ and that $F = \{1\}$. Thus, Z has three vertices, and the vertex z_0 has valence two. Hence, in W, X, and Y, each vertex has valence at most p.

Now

$$2\,\bar{\mathrm{r}}W = \sum_{k\in[3\uparrow p]} ((k-2)\,|V_kW|) \leqslant \sum_{i,j\in[3\uparrow p]} (\max(ij-2p,\ (i-2)(j-2))\,|V_iX|\,|V_jY|),$$

by Corollary 6.2 with $r = 2$ or Theorem 4.3(ii), where $\alpha_3(G) = p$.

Putting $\mathrm{bound}_2 \coloneqq \sum\limits_{i,j\in[3\uparrow p]} (\max(6ij-12p,\ 6ij-12i-12j+24)\,|V_iX|\,|V_jY|)$, we see that $12\,\bar{\mathrm{r}}W \leqslant \mathrm{bound}_2$.

Also,

$$\begin{aligned} 12\,\bar{\mathrm{r}}W &= \sum_{k\in[3\uparrow p]} (6(k-2)\,|V_kW|) = \sum_{k\in[3\uparrow p]} (2k\,|V_kW|) + \sum_{k\in[4\uparrow p]} (4(k-3)\,|V_kW|) \\ &\leqslant \sum_{i,j\in[3\uparrow p]} (ij\,|V_iX|\,|V_jY|) + \sum_{i,j\in[3\uparrow p]} (4\max(ij-3p,\ (i-3)(j-3))\,|V_iX|\,|V_jY|), \end{aligned}$$

by Corollaries 3.3 and 6.2 with $r = 3$.

Putting $\mathrm{bound}_3 \coloneqq \sum\limits_{i,j\in[3\uparrow p]} (\max(5ij-12p,\ 5ij-12i-12j+36)\,|V_iX|\,|V_jY|)$, we see that

$12\,\bar{\mathrm{r}}W \leqslant \mathrm{bound}_3$.

Hence, $12\,\bar{\mathrm{r}}W \leqslant \min(\mathrm{bound}_2, \mathrm{bound}_3)$. Let $\kappa := \frac{2p(4p-6+3\sqrt{3})}{(2p-3+\sqrt{3})^2}$. By Theorem III.1 of the Appendix, $\min(\mathrm{bound}_2, \mathrm{bound}_3) \leqslant 3\kappa(2\,\bar{\mathrm{r}}X)(2\,\bar{\mathrm{r}}Y)$. It then follows that $12\,\bar{\mathrm{r}}W \leqslant 12\kappa\,\bar{\mathrm{r}}X\,\bar{\mathrm{r}}Y$, and, hence, $\sigma(\mathcal{F}) \leqslant \frac{\kappa}{\theta(G)}$. Since $\theta(G) = \frac{p}{p-2}$, it can be shown that

$$\sigma(\mathcal{F}) \leqslant \frac{\kappa}{\theta(G)} = \frac{2(p-2)(4p-6+3\sqrt{3})}{(2p-3+\sqrt{3})^2} = 2 - \frac{(4+2\sqrt{3})p}{(2p-3+\sqrt{3})^2} < 2.$$

This proves (i).

(ii). If $p = 3$, then (i) shows that $\sigma(\mathcal{F}) \leqslant 1$, and Proposition 2.10 of **[2]** shows that $\sigma(\mathcal{F}) \geqslant 1$. □

Appendix. A technical inequality

In this Appendix we prove an inequality that is used in the proof of Theorem 6.3.

I Statement of the inequality

I.1 Notation. Let p be an odd prime.

Let $\kappa := \frac{2p(4p-6+3\sqrt{3})}{(2p-3+\sqrt{3})^2}$.

Let $[0,\infty[^{\,[3\uparrow p]}$ denote the set of all functions of the form $\mathbf{x}\colon [3\uparrow p] \to [0,\infty[,\ \ i \mapsto \mathbf{x}(i)$.

For any $\mathbf{x}, \mathbf{y} \in [0,\infty[^{\,[3\uparrow p]}$, we define

$$\begin{aligned} \mathrm{s}(\mathbf{x}) &:= \sum_{i\in[3\uparrow p]} (i-2)\mathbf{x}(i), \\ \mathrm{bound}_2(\mathbf{x},\mathbf{y}) &:= \sum_{i,j\in[3\uparrow p]} (\max(6ij-12p,\ \ 6ij-12i-12j+24)\mathbf{x}(i)\mathbf{y}(j)), \\ \mathrm{bound}_3(\mathbf{x},\mathbf{y}) &:= \sum_{i,j\in[3\uparrow p]} (\max(5ij-12p,\ \ 5ij-12i-12j+36)\mathbf{x}(i)\mathbf{y}(j)). \end{aligned}$$ □

I.2 Lemma. *With* Notation I.1, $\frac{2p}{p-1} \leqslant \kappa$.

Proof. It is straightforward to check that $\frac{2p}{p-1} \leqslant \frac{2p}{p-1} + \frac{2(2-\sqrt{3})p(p-3)}{(p-1)(2p-3+\sqrt{3})^2} = \frac{2p(4p-6+3\sqrt{3})}{(2p-3+\sqrt{3})^2} = \kappa$. □

The purpose of this Appendix is to show that, for all $\mathbf{x}, \mathbf{y} \in [0,\infty[^{\,[3\uparrow p]}$,

$$\min(\mathrm{bound}_2(\mathbf{x},\mathbf{y}),\ \mathrm{bound}_3(\mathbf{x},\mathbf{y})) \leqslant 3\kappa\,\mathrm{s}(\mathbf{x})\,\mathrm{s}(\mathbf{y}).$$

If $\mathrm{s}(\mathbf{x}) = 0$ or $\mathrm{s}(\mathbf{y}) = 0$, then the inequality is easily seen to be true. For most of the argument, we shall think of $\mathbf{x}$ and $\mathrm{s}(\mathbf{y})$ as fixed.

II Keeping $\mathbf{x}$ and $\mathrm{s}(\mathbf{y})$ fixed

II.1 Notation. Let Notation I.1 hold.

We fix $s \in \,]0,\infty[$ and $\mathbf{x} \in [0,\infty[^{[3\uparrow p]}$.

Let $\Delta \coloneqq \{\mathbf{y} \in [0,\infty[^{[3\uparrow p]} : \mathrm{s}(\mathbf{y}) = s\}$.

Let $\mathrm{bound}_2(-)\colon \Delta \to [0,\infty[, \quad \mathbf{y} \mapsto \mathrm{bound}_2(\mathbf{x},\mathbf{y})$, and similarly for $\mathrm{bound}_3(-)$.

For each $j \in [3\uparrow p]$, we define

$$\mathbf{y}_j\colon [3\uparrow p] \to [0,\infty[, \quad m \mapsto \delta_{j,m}\tfrac{s}{j-2}.$$

Then $\{\mathbf{y}_j \mid j \in [3\uparrow p]\}$ is the set of vertices of the simplex Δ. □

II.2 Remarks. Let Notation II.1 hold.

Let $j \in [3\uparrow p]$.

We find that

$$\text{(II.2.1)} \qquad \mathrm{bound}_2(\mathbf{y}_j) = \tfrac{s}{j-2} \sum_{i\in[3\uparrow p]} \max(6ij-12p,\ 6ij-12i-12j+24)\mathbf{x}(i),$$

$$\text{(II.2.2)} \qquad \mathrm{bound}_3(\mathbf{y}_j) = \tfrac{s}{j-2} \sum_{i\in[3\uparrow p]} \max(5ij-12p,\ 5ij-12i-12j+36)\mathbf{x}(i).$$

We then have two expressions for $\frac{j-2}{s}\mathrm{bound}_2(\mathbf{y}_j)$:

$$\text{(II.2.3)} \quad \tfrac{j-2}{s}\mathrm{bound}_2(\mathbf{y}_j) = \sum_{i\in[3\uparrow(p-(j-1))]} (6ij-12i-12j+24)\mathbf{x}(i) + \sum_{i\in[(p-(j-2))\uparrow p]} (6ij-12p)\mathbf{x}(i)$$

and

$$\text{(II.2.4)} \quad \tfrac{j-2}{s}\mathrm{bound}_2(\mathbf{y}_j) = \sum_{i\in[3\uparrow(p-(j-2))]} (6ij-12i-12j+24)\mathbf{x}(i) + \sum_{i\in[(p-(j-3))\uparrow p]} (6ij-12p)\mathbf{x}(i).$$

Putting $j = p-1$ in (II.2.3), we see that

$$\text{(II.2.5)} \qquad \tfrac{p-3}{s}\mathrm{bound}_2(\mathbf{y}_{p-1}) = \sum_{i\in[3\uparrow p]} (6ip-6i-12p)\mathbf{x}(i).$$

Putting $j = 3$ in (II.2.4), we see that

$$\text{(II.2.6)} \qquad \tfrac{1}{s}\mathrm{bound}_2(\mathbf{y}_3) = \sum_{i\in[3\uparrow(p-1)]} (6i-12)\mathbf{x}(i) + 6p\mathbf{x}(p) = \sum_{i\in[3\uparrow p]} (6i-12+\delta_{i,p}12)\mathbf{x}(i).$$

Similarly we have two expressions for $\frac{j-2}{s}\mathrm{bound}_3(\mathbf{y}_j)$:

$$\text{(II.2.7)} \quad \tfrac{j-2}{s}\mathrm{bound}_3(\mathbf{y}_j) = \sum_{i\in[3\uparrow(p-(j-2))]} (5ij-12i-12j+36)\mathbf{x}(i) + \sum_{i\in[(p-(j-3))\uparrow p]} (5ij-12p)\mathbf{x}(i)$$

and

$$\text{(II.2.8)}\quad \tfrac{j-2}{s}\operatorname{bound}_3(\mathbf{y}_j) = \sum_{i\in[3\uparrow(p-(j-3))]} (5ij-12i-12j+36)\mathbf{x}(i) + \sum_{i\in[(p-(j-4))\uparrow p]} (5ij-12p)\mathbf{x}(i).$$

Putting $j=3$ in (II.2.8), we see that

$$\text{(II.2.9)}\qquad \tfrac{1}{s}\operatorname{bound}_3(\mathbf{y}_3) = \sum_{i\in[3\uparrow p]} 3i\mathbf{x}(i). \qquad \square$$

II.3 Lemma. *Let* Notation I.1 *hold. Then at least one of the following holds.*
(II.3.a) $\operatorname{bound}_3(\mathbf{x},\mathbf{y}) \leqslant \frac{6p}{p-1}\operatorname{s}(\mathbf{x})\operatorname{s}(\mathbf{y}) \leqslant 3\kappa\operatorname{s}(\mathbf{x})\operatorname{s}(\mathbf{y})$.
(II.3.b) $(p-3)\mathbf{x}(3) \quad > \sum_{i\in[4\uparrow p]} (pi-4p+i)\mathbf{x}(i)$, *and, hence,* $p \geqslant 5$.

Proof. Suppose that (II.3.b) fails. We shall verify that (II.3.a) holds.

The second inequality in (II.3.a) holds by Lemma I.2.

Let Notation II.1 hold.

By linearity, it suffices to show that, for each $j \in [3\uparrow p]$, $\frac{6p}{p-1}\operatorname{s}(\mathbf{x})s - \operatorname{bound}_3(\mathbf{y}_j)$ is non-negative.

Using the definition of $\operatorname{s}(\mathbf{x})$ together with (II.2.9), we find that

$$\begin{aligned}
&\tfrac{p-1}{s}\big(\tfrac{6p}{p-1}\operatorname{s}(\mathbf{x})s - \operatorname{bound}_3(\mathbf{y}_3)\big)\\
&= \sum_{i\in[3\uparrow p]} 6p(i-2)\mathbf{x}(i) - \sum_{i\in[3\uparrow p]} (p-1)3i\mathbf{x}(i) = \sum_{i\in[3\uparrow p]} 3(pi-4p+i)\mathbf{x}(i),
\end{aligned}$$

and this is non-negative since (II.3.b) fails.

Suppose that $j \in [4\uparrow p]$, and, hence, that $p \geqslant 5$. Using the definition of $\operatorname{s}(\mathbf{x})$ together with (II.2.8), we find that

$$\begin{aligned}
&\tfrac{(j-2)(p-1)}{s}\big(\tfrac{6p}{p-1}\operatorname{s}(\mathbf{x})s - \operatorname{bound}_3(\mathbf{y}_j)\big)\\
&= \sum_{i\in[3\uparrow p]} 6(j-2)p(i-2)\mathbf{x}(i)\\
&\qquad - \sum_{i\in[3\uparrow(p-(j-3))]} (p-1)(5ij-12i-12j+36)\mathbf{x}(i)\\
&\qquad - \sum_{i\in[(p-(j-4))\uparrow p]} (p-1)(5ij-12p)\mathbf{x}(i)\\
&= \sum_{i\in[3\uparrow(p-(j-3))]} (pij-12p+5ij-12i-12j+36)\mathbf{x}(i)\\
&\quad + \sum_{i\in[(p-(j-4))\uparrow p]} (pij+5ij-12pi-12pj+12p^2+12p)\mathbf{x}(i)\\
&= \sum_{i\in[3\uparrow(p-(j-3))]} (p(ij-12)+ij+4(i-3)(j-3))\mathbf{x}(i)
\end{aligned}$$

$$+ \sum_{i\in[(p-(j-4))\uparrow p]} ((p-5)(i-2)(j-2) + 10(p+1-i)(p+1-j) \\ + 2(p-1)(p-5))\mathbf{x}(i).$$

This is non-negative and (II.3.a) holds. □

II.4 Lemma. *Let* Notation II.1 *hold.*

Let $j, j' \in [3\uparrow p]$. *If* $j \leqslant j'$, *then* $\mathrm{bound}_2(\mathbf{y}_j) \leqslant \mathrm{bound}_2(\mathbf{y}_{j'})$.

Proof. It follows from (II.2.1) that $\frac{1}{s}\,\mathrm{bound}_2(\mathbf{y}_j) = 6\sum_{i\in[3\uparrow p]} \max(i - \frac{2p-2i}{j-2}, i-2)\mathbf{x}(i)$. Since $j \leqslant j'$, we see that $-2\frac{p-i}{j-2} \leqslant -2\frac{p-i}{j'-2}$, and, hence, $\max(i - \frac{2p-2i}{j-2}, i-2) \leqslant \max(i - 2\frac{2p-2i}{j'-2}, i-2)$. The result follows. □

II.5 Lemma. *Let* Notation II.1 *hold.*

If (II.3.b) *holds, then, for all* $j \in [4\uparrow p]$, $\mathrm{bound}_3(\mathbf{y}_j) < \mathrm{bound}_3(\mathbf{y}_3)$.

Proof. Using (II.2.9) and (II.2.8), we find that

$$\begin{aligned}
&\tfrac{j-2}{s}(\mathrm{bound}_3(\mathbf{y}_3) - \mathrm{bound}_3(\mathbf{y}_j))\\
&= \sum_{i\in[3\uparrow p]} 3i(j-2)\mathbf{x}(i) - \sum_{i\in[3\uparrow(p-(j-3))]} (5ij - 12i - 12j + 36)\mathbf{x}(i)\\
&\qquad - \sum_{i\in[(p-(j-4))\uparrow p]} (5ij - 12p)\mathbf{x}(i)\\
&= \sum_{i\in[3\uparrow(p-(j-3))]} (-2ij + 6i + 12j - 36)\mathbf{x}(i) + \sum_{i\in[(p-(j-4))\uparrow p]} (-2ij - 6i + 12p)\mathbf{x}(i)\\
&= (6j-18)\mathbf{x}(3) + \sum_{i\in[4\uparrow(p-(j-3))]} (-2ij + 6i + 12j - 36)\mathbf{x}(i)\\
&\qquad + \sum_{i\in[(p-(j-4))\uparrow p]} (-2ij - 6i + 12p)\mathbf{x}(i)\\
&> \tfrac{6j-18}{p-3} \sum_{i\in[4\uparrow p]} (pi - 4p + i)\mathbf{x}(i) + \sum_{i\in[4\uparrow(p-(j-3))]} (-2ij + 6i + 12j - 36)\mathbf{x}(i)\\
&\qquad + \sum_{i\in[(p-(j-4))\uparrow p]} (-2ij - 6i + 12p)\mathbf{x}(i) \text{ since (II.3.b) holds}\\
&= \tfrac{1}{p-3}\Big(\sum_{i\in[4\uparrow(p-(j-3))]} ((6j-18)(pi - 4p + i) + (p-3)(-2ij + 6i + 12j - 36))\mathbf{x}(i)\\
&\qquad + \sum_{i\in[(p-(j-4))\uparrow p]} ((6j-18)(pi - 4p + i) + (p-3)(-2ij - 6i + 12p))\mathbf{x}(i)\Big)\\
&= \tfrac{1}{p-3}\Big(\sum_{i\in[4\uparrow(p-(j-3))]} (12ij + 4pij - 12pi - 36i - 36j - 12pj + 36p + 108)\mathbf{x}(i)\\
&\qquad + \sum_{i\in[(p-(j-4))\uparrow p]} (12p^2 - 24pi + 4pij - 24pj + 36p + 12ij)\mathbf{x}(i)\Big)
\end{aligned}$$

$$= \tfrac{1}{p-3}\Big(\sum_{i\in[4\uparrow(p-(j-3))]} 4(p+3)(i-3)(j-3)\mathbf{x}(i)$$
$$\qquad + \sum_{i\in[(p-(j-4))\uparrow p]} (4p(i-3)(j-3)+12(p-i)(p-j))\mathbf{x}(i)\Big)$$
$$\geqslant 0. \qquad \square$$

II.6 Lemma. *Let* Notation II.1 *hold. If* (II.3.b) *holds, then* $\mathrm{bound}_2(\mathbf{y}_3) < \mathrm{bound}_3(\mathbf{y}_3)$.

Proof. We have

$$\tfrac{1}{s}(\mathrm{bound}_3(\mathbf{y}_3) - \mathrm{bound}_2(\mathbf{y}_3))$$
$$= \sum_{i\in[3\uparrow p]} 3i\mathbf{x}(i) - \sum_{i\in[3\uparrow p]} (6i-12+\delta_{i,p}12)\mathbf{x}(i) \quad \text{by (II.2.6) and (II.2.9)}$$
$$= 3\mathbf{x}(3) + \sum_{i\in[4\uparrow p]} 3(-i+4-\delta_{i,p}4)\mathbf{x}(i)$$
$$> \sum_{i\in[4\uparrow p]} \tfrac{3}{p-3}(pi-4p+i)\mathbf{x}(i) + \sum_{i\in[4\uparrow p]} 3(-i+4-\delta_{i,p}4)\mathbf{x}(i) \quad \text{since (II.3.b) holds}$$
$$= \tfrac{3}{p-3}\sum_{i\in[4\uparrow p]} ((pi-4p+i)+(p-3)(-i+4-\delta_{i,p}4))\mathbf{x}(i)$$
$$= \tfrac{3}{p-3}\sum_{i\in[4\uparrow p]} (pi-4p+i-pi+4p-4p\delta_{i,p}+3i-12+12\delta_{i,p})\mathbf{x}(i)$$
$$= \tfrac{12}{p-3}\sum_{i\in[4\uparrow(p-1)]} (i-3)\mathbf{x}(i) \geqslant 0. \qquad \square$$

II.7 Lemma. *Let* Notation II.1 *hold. For all* $j \in [4\uparrow p]$, $\mathrm{bound}_3(\mathbf{y}_j) \leqslant \mathrm{bound}_2(\mathbf{y}_j)$.

Proof. By (II.2.4) and (II.2.7), we see that $\frac{j-2}{s}(\mathrm{bound}_2(\mathbf{y}_j) - \mathrm{bound}_3(\mathbf{y}_j))$ is equal to $\sum\limits_{i\in[3\uparrow(p-(j-2))]} (ij-12)\mathbf{x}(i) + \sum\limits_{i\in[(p-(j-3))\uparrow p]} ij\mathbf{x}(i)$ which is non-negative. $\square$

II.8 Lemma. *Let* Notation II.1 *hold.*
If (II.3.b) *holds, then* $\mathrm{bound}_2(\mathbf{y}_{p-1}) < \frac{6p}{p-1}\,\mathrm{s}(\mathbf{x})s \leqslant 3\kappa\,\mathrm{s}(\mathbf{x})s$.

Proof. The second inequality follows from Lemma I.2.
Using (II.2.5) together with the definition of $\mathrm{s}(\mathbf{x})$, we find that

$$\tfrac{(p-1)(p-3)}{s}\big(\tfrac{6p}{p-1}\,\mathrm{s}(\mathbf{x})s - \mathrm{bound}_2(\mathbf{y}_{p-1})\big)$$
$$= \sum_{i\in[3\uparrow p]} (6p(p-3)(i-2)-(p-1)(6ip-6i-12p))\mathbf{x}(i)$$
$$= \sum_{i\in[3\uparrow p]} (6p(pi-3i-2p+6)-(6ip^2-6ip-12p^2-6ip+6i+12p))\mathbf{x}(i)$$
$$= \sum_{i\in[3\uparrow p]} (6ip^2-18pi-12p^2+36p-6ip^2+6ip+12p^2+6ip-6i-12p)\mathbf{x}(i)$$

$$= \sum_{i\in[3\uparrow p]} 6(-ip + 4p - i)\mathbf{x}(i)$$

which is positive since (II.3.b) holds. □

II.9 Lemma. *Let* Notation II.1 *hold. Then at least one of the following holds:*

(II.9.a) *For all* $\mathbf{y} \in \Delta$, $\min(\text{bound}_2(\mathbf{y}), \text{bound}_3(\mathbf{y})) \leqslant 3\kappa\,\text{s}(\mathbf{x})s$.

(II.9.b) $p \geqslant 5$ *and their exists some point* $\mathbf{y}_{(3,p)} \in \Delta$ *with the following properties:*

- (i) *for all* $j \in [4\uparrow(p-1)]$, $\mathbf{y}_{(3,p)}(j) = 0$,
- (ii) $\text{bound}_2(\mathbf{y}_{(3,p)}) = \text{bound}_3(\mathbf{y}_{(3,p)})$,
- (iii) *for all* $\mathbf{y} \in \Delta$, $\min(\text{bound}_2(\mathbf{y}), \text{bound}_3(\mathbf{y})) \leqslant \text{bound}_2(\mathbf{y}_{(3,p)}) = \text{bound}_3(\mathbf{y}_{(3,p)})$.

Proof. By Lemma II.3, we may assume that (II.3.b) holds.

Consider any $j \in [4\uparrow p]$.

It follows from Lemmas II.6 and II.7 that in travelling along the oriented line segment $[\mathbf{y}_3, \mathbf{y}_j]$ in Δ, bound_2 starts strictly below bound_3 and ends above bound_3. Hence bound_2 and bound_3 agree at a unique point $\mathbf{y}_{(3,j)}$ of $[\mathbf{y}_3, \mathbf{y}_j]$. Notice that $\min(\text{bound}_2, \text{bound}_3)$ agrees with bound_2 on $[\mathbf{y}_3, \mathbf{y}_{(3,j)}]$ and with bound_3 on $[\mathbf{y}_{(3,j)}, \mathbf{y}_j]$.

It follows from Lemma II.4 that bound_2 increases in travelling along $[\mathbf{y}_3, \mathbf{y}_j]$, and hence $\min(\text{bound}_2, \text{bound}_3)$ increases in travelling along $[\mathbf{y}_3, \mathbf{y}_{(3,j)}]$.

It follows from Lemma II.5 that bound_3 decreases in travelling along $[\mathbf{y}_3, \mathbf{y}_j]$, and hence $\min(\text{bound}_2, \text{bound}_3)$ decreases in travelling along $[\mathbf{y}_{(3,j)}, \mathbf{y}_j]$.

Hence, on $[\mathbf{y}_3, \mathbf{y}_j]$, $\min(\text{bound}_2, \text{bound}_3)$ achieves a maximum value at $\mathbf{y}_{(3,j)}$.

By linearity, there exists some $j_0 \in [4\uparrow p]$ such that the maximum value achieved by $\min(\text{bound}_2, \text{bound}_3)$ on all of Δ is $\text{bound}_2(\mathbf{y}_{(3,j_0)}) = \text{bound}_3(\mathbf{y}_{(3,j_0)})$.

If $j_0 = p$, then (II.9.b) holds.

Thus we may assume that $j_0 \in [4\uparrow(p-1)]$.

It follows from Lemma II.4 that $\text{bound}_2(\mathbf{y}_{j_0}) \leqslant \text{bound}_2(\mathbf{y}_{p-1})$, and it follows from Lemma II.8 that $\text{bound}_2(\mathbf{y}_{p-1}) \leqslant 3\kappa\,\text{s}(\mathbf{x})s$. Thus

$$\text{bound}_2(\mathbf{y}_{(3,j_0)}) \leqslant \text{bound}_2(\mathbf{y}_{j_0}) \leqslant \text{bound}_2(\mathbf{y}_{p-1}) \leqslant 3\kappa\,\text{s}(\mathbf{x})s,$$

and (II.9.a) holds. □

III Proof of the inequality

III.1 Theorem. *Let* Notation I.1 *hold. Then, for all* $\mathbf{x}, \mathbf{y} \in [0,\infty[^{[3\uparrow p]}$,

$$\min(\text{bound}_2(\mathbf{x},\mathbf{y}),\ \text{bound}_3(\mathbf{x},\mathbf{y})) \leqslant 3\kappa\,\text{s}(\mathbf{x})\,\text{s}(\mathbf{y}).$$

Proof. We may assume that $\mathrm{s}(\mathbf{x}) > 0$ and $\mathrm{s}(\mathbf{y}) > 0$.

If $(\mathbf{x}, \mathbf{y})$ is a counter-example then, by Lemma II.9, $p \geqslant 5$ and we can fix $\mathbf{x}$ and $\mathrm{s}(\mathbf{y})$ and alter $\mathbf{y}$ to arrange that, for all $j \in [4{\uparrow}(p-1)]$, $\mathbf{y}(j) = 0$, and $(\mathbf{x}, \mathbf{y})$ is still a counter-example. By the left-right dual of Lemma II.9, we can then fix $\mathbf{y}$ and $\mathrm{s}(\mathbf{x})$ and alter $\mathbf{x}$ to arrange that, for all $i \in [4{\uparrow}(p-1)]$, $\mathbf{x}(i) = 0$ and $\mathrm{bound}_2(\mathbf{x}, \mathbf{y}) = \mathrm{bound}_3(\mathbf{x}, \mathbf{y})$, and $(\mathbf{x}, \mathbf{y})$ is still a counter-example.

Thus we are left with four variables $\mathbf{x}(3)$, $\mathbf{x}(p)$, $\mathbf{y}(3)$, $\mathbf{y}(p)$ in $[0, \infty[$ subject to $\mathrm{bound}_2(\mathbf{x}, \mathbf{y}) = \mathrm{bound}_3(\mathbf{x}, \mathbf{y})$, where

$$\mathrm{bound}_2(\mathbf{x}, \mathbf{y}) = 6\mathbf{x}(3)\mathbf{y}(3) + 6p\mathbf{x}(3)\mathbf{y}(p) + 6p\mathbf{x}(p)\mathbf{y}(3) + 6p(p-2)\mathbf{x}(p)\mathbf{y}(p),$$
$$\mathrm{bound}_3(\mathbf{x}, \mathbf{y}) = 9\mathbf{x}(3)\mathbf{y}(3) + 3p\mathbf{x}(3)\mathbf{y}(p) + 3p\mathbf{x}(p)\mathbf{y}(3) + p(5p-12)\mathbf{x}(p)\mathbf{y}(p).$$

Taking the difference we find that

$$\text{(III.1.1)} \qquad p^2\mathbf{x}(p)\mathbf{y}(p) + 3p\mathbf{x}(3)\mathbf{y}(p) + 3p\mathbf{x}(p)\mathbf{y}(3) - 3\mathbf{x}(3)\mathbf{y}(3) = 0.$$

Also, $\mathrm{s}(\mathbf{x}) = \mathbf{x}(3) + (p-2)\mathbf{x}(p) > 0$ and $\mathrm{s}(\mathbf{y}) = \mathbf{y}(3) + (p-2)\mathbf{y}(p) > 0$.

Define

$$X(\mathbf{x}) := 2(2p^2 - 6p + 3)\tfrac{\mathbf{x}(p)}{\mathrm{s}(\mathbf{x})} - 3(p-1), \qquad Y(\mathbf{y}) := 2(2p^2 - 6p + 3)\tfrac{\mathbf{y}(p)}{\mathrm{s}(\mathbf{y})} - 3(p-1).$$

It is not difficult to show that $X(\mathbf{x})\,\mathrm{s}(\mathbf{x}) = p(p-3)\mathbf{x}(p) - 3(p-1)\mathbf{x}(3)$.

Straightforward calculations show that

$$\begin{aligned}
&(3p^2 - X(\mathbf{x})Y(\mathbf{y}))\,\mathrm{s}(\mathbf{x})\,\mathrm{s}(\mathbf{y}) = 3p^2\,\mathrm{s}(\mathbf{x})\,\mathrm{s}(\mathbf{y}) - X(\mathbf{x})\,\mathrm{s}(\mathbf{x})Y(\mathbf{y})\,\mathrm{s}(\mathbf{y})\\
&= 3p^2(\mathbf{x}(3) + (p-2)\mathbf{x}(p))(\mathbf{y}(3) + (p-2)\mathbf{y}(p))\\
&\qquad - (p(p-3)\mathbf{x}(p) - 3(p-1)\mathbf{x}(3))(p(p-3)\mathbf{y}(p) - 3(p-1)\mathbf{y}(3))\\
&= (2p^4 - 6p^3 + 3p^2)\mathbf{x}(p)\mathbf{y}(p) + (6p^3 - 18p^2 + 9p)(\mathbf{x}(3)\mathbf{y}(p) + \mathbf{x}(p)\mathbf{y}(3))\\
&\qquad + (-6p^2 + 18p - 9)\mathbf{x}(3)\mathbf{y}(3).\\
&= (2p^2 - 6p + 3)(p^2\mathbf{x}(p)\mathbf{y}(p) + 3p\mathbf{x}(3)\mathbf{y}(p) + 3p\mathbf{x}(p)\mathbf{y}(3) - 3\mathbf{x}(3)\mathbf{y}(3))\\
&= 0 \text{ by (III.1.1).}
\end{aligned}$$

It follows that $X(\mathbf{x})Y(\mathbf{y}) = 3p^2$.

Since $(X(\mathbf{x}) + Y(\mathbf{y}))^2 = (X(\mathbf{x}) - Y(\mathbf{y}))^2 + 4X(\mathbf{x})Y(\mathbf{y}) \geqslant 4X(\mathbf{x})Y(\mathbf{y}) = 12p^2$, we see that $|X(\mathbf{x}) + Y(\mathbf{y})| \geqslant 2p\sqrt{3}$.

Since $(p-2)\mathbf{x}(p) \leqslant \mathrm{s}(\mathbf{x})$, we see that $\frac{\mathbf{x}(p)}{\mathrm{s}(\mathbf{x})} \leqslant \frac{1}{p-2}$, and $X(\mathbf{x}) \leqslant \frac{p(p-3)}{p-2} \leqslant \frac{3p}{2}$. It follows that $X(\mathbf{x}) + Y(\mathbf{y}) \leqslant 3p < 2p\sqrt{3}$. Hence, $X(\mathbf{x}) + Y(\mathbf{y}) \leqslant -2p\sqrt{3}$.

Further straightforward calculations show that

$$\begin{aligned}
&3\,\mathrm{s}(\mathbf{x})\,\mathrm{s}(\mathbf{y})(-(p-2)X(\mathbf{x})Y(\mathbf{y}) + \left(p^2-3p\right)(X(\mathbf{x})+Y(\mathbf{y})) + 8p^4 - 33p^3 + 36p^2 - 9p) \\
&= 6p(p-2)\left(2p^2-6p+3\right)^2 \mathbf{x}(p)\mathbf{y}(p) \\
&\qquad + 6p\left(2p^2-6p+3\right)^2 (\mathbf{x}(3)\mathbf{y}(p) + \mathbf{x}(p)\mathbf{y}(3)) + 6\left(2p^2-6p+3\right)^2 \mathbf{x}(3)\mathbf{y}(3). \\
&= \left(2p^2-6p+3\right)^2 \mathrm{bound}_2(\mathbf{x},\mathbf{y}).
\end{aligned}$$

Hence,

$$\begin{aligned}
&\left(2p^2-6p+3\right)^2 \tfrac{\mathrm{bound}_2(\mathbf{x},\mathbf{y})}{3\,\mathrm{s}(\mathbf{x})\,\mathrm{s}(\mathbf{y})} \\
&= -(p-2)XY + \left(p^2-3p\right)(X+Y) + 8p^4 - 33p^3 + 36p^2 - 9p \\
&\leqslant -(p-2)(3p^2) + \left(p^2-3p\right)(-2p\sqrt{3}) + 8p^4 - 33p^3 + 36p^2 - 9p \\
&= 8p^4 - 2p^3\sqrt{3} - 36p^3 + 6p^2\sqrt{3} + 42p^2 - 9p.
\end{aligned}$$

Thus

$$\begin{aligned}
\tfrac{\mathrm{bound}_2(\mathbf{x},\mathbf{y})}{3\,\mathrm{s}(\mathbf{x})\,\mathrm{s}(\mathbf{y})} &\leqslant \tfrac{8p^4-2p^3\sqrt{3}-36p^3+6p^2\sqrt{3}+42p^2-9p}{(2p^2-6p+3)^2} = \tfrac{\frac{1}{2}p(4p-6+3\sqrt{3})(2p-3-\sqrt{3})^2}{\frac{1}{4}(2p-3+\sqrt{3})^2(2p-3-\sqrt{3})^2} \\
&= \tfrac{2p(4p-6+3\sqrt{3})}{(2p-3+\sqrt{3})^2} = \kappa,
\end{aligned}$$

expressed in factors that are linear in p. This completes the argument. □

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References

[1] Warren Dicks and M. J. Dunwoody, *Groups acting on graphs*, Cambridge Stud. Adv. Math. **17**, CUP, Cambridge, 1989. Errata at `http://mat.uab.cat/~dicks/DDerr.html`

[2] Warren Dicks and S. V. Ivanov, *On the intersection of free subgroups in free products of groups*, Math. Proc. Cambridge Phil. Soc. **144** (2008), 511–534.

[3] David J. Grynkiewicz, *On extending Pollard's Theorem for t-representable sums*, Israel J. Math. **177** 2010, 413–439.

[4] S. V. Ivanov, *On the intersection of finitely generated subgroups in free products of groups*, Int. J. Algebra Comp. **9** (1999), 521–528.

[5] S. V. Ivanov, *Intersecting free subgroups in free products of groups*, Int. J. Algebra Comp. **11** (2001), 281–290.

[6] W. Lück, *L^2-invariants: theory and applications to geometry and K-theory*, Ergeb. Math. Grenzgeb. (3) **44**, Springer-Verlag, Berlin, 2002.

[7] Hanna Neumann, *On intersections of finitely generated subgroups of free groups*, Publ. Math. Debrecen **4** (1956), 186–189.

[8] Hanna Neumann, *On intersections of finitely generated subgroups of free groups. Addendum*, Publ. Math. Debrecen **5** (1957), 128.

[9] W. D. Neumann, *On intersections of finitely generated subgroups of free groups*, pp. 161–170 in *Groups - Canberra 1989* (ed. L. G. Kovács), Lecture Notes Math. **1456**, Springer-Verlag, Berlin, 1990.

[10] J. M. Pollard, *A generalisation of the theorem of Cauchy and Davenport*, J. London Math. Soc. **8** (1974), 460–462.

[11] M. Sykiotis, *On subgroups of finite complexity in groups acting on trees*, J. Pure Appl. Algebra **200** (2005), 1–23.

[12] G. Tardos, *On the intersection of subgroups of a free group*, Invent. Math. **108** (1992), 29–36.

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