Actions of the braid group, and new algebraic proofs of results of Dehornoy and Larue.

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To Martin Dunwoody on the occasion of his 70th birthday.

Abstract

This article surveys many standard results about the braid group, with emphasis on simplifying the usual algebraic proofs.

We use van der Waerden's trick to illuminate the Artin-Magnus proof of the classic presentation of the braid group considered as the *algebraic* mapping-class group of a disc with punctures.

We give a simple, new proof of the σ_1-trichotomy for the braid group, and, hence, recover the Dehornoy right-ordering of the braid group.

We give three proofs of the Birman-Hilden theorem concerning the fidelity of braid-group actions on free products of finite cyclic groups, and discuss the consequences derived by Perron-Vannier and the connections with Artin groups and the Wada representations.

The first, very direct, proof, is due to Crisp-Paris and uses the σ_1-trichotomy and the Larue-Shpilrain technique. The second proof arises by studying ends of free groups, and gives interesting extra information. The third proof arises from Larue's study of polygonal curves in discs with punctures, and gives extremely detailed information.

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1 General Notation

Let $\mathbb{N}$ denote the set of finite cardinals, $\{0, 1, 2, \ldots\}$.

Throughout, we fix an element n of $\mathbb{N}$.

Let G be a multiplicative group.

For elements a, b of G, we write $\overline{a} := a^{-1}$, $a^b := \overline{b}ab$, $[a] := \{a^g \mid g \in G\}$, the conjugacy class of a in G, and $a^{nb} := \overline{b}a^n b$. We let $\operatorname{Aut} G$ denote the group of all automorphisms of G, acting on G on the right with exponent notation.

For two subsets A, B of a set X, the complement of $A \cap B$ in A will be denoted by $A - B$ (and not by $A \setminus B$ since we let $G \backslash Y$ denote the set of G-orbits of a left G-set Y).

An *ordering* of a set will mean a *total* ordering for the set. An *ordered* set is a set endowed with a specific ordering.

We will make frequent use of sequences, usually with vector notation. We shall use the language of sequences to introduce indexed symbols and to realize free monoids. Formally, we define a *sequence* as a set endowed with a specified listing of its elements. Thus a sequence has an underlying set; with vector notation, the coordinates are the elements of (the underlying set of) the sequence. For two sequences A, B, their concatenation will be denoted $A \vee B$. By a sequence A *in* a given set X, we mean a sequence endowed with a specified map of sets $A \to X$; to avoid extra notation, we shall use the same symbol to denote an element of A and its image in X even when the map is not injective. We often treat A as an element in the free monoid on X with concatenation as binary operation, and then the elements of A are its atomic factors.

Let i, $j \in \mathbb{Z}$.

We write $[i{\uparrow}j] := \begin{cases} (i, i+1, \ldots, j-1, j) \in \mathbb{Z}^{j-i+1} & \text{if } i \leq j, \\ () \in \mathbb{Z}^0 & \text{if } i > j. \end{cases}$

Also, $[i{\uparrow}\infty[\ := (i, i+1, i+2, \ldots)$. We define $[j{\downarrow}i]$ to be the reverse of the sequence $[i{\uparrow}j]$, $(j, j-1, \ldots, i+1, i)$.

Let v be a symbol.

For each $k \in \mathbb{Z}$, we let v_k denote the ordered pair (v, k).

We let $v_{[i\uparrow j]} := \begin{cases} (v_i, v_{i+1}, \cdots, v_{j-1}, v_j) & \text{if } i \leq j, \\ () & \text{if } i > j. \end{cases}$

Also, $v_{[i\uparrow\infty[} := (v_i, v_{i+1}, v_{i+2}, \ldots)$. We define $v_{[j\downarrow i]}$ to be the reverse of the sequence $v_{[i\uparrow j]}$.

Now suppose that $v_{[i\uparrow j]}$ is a sequence *in* the multiplicative group G, that is, there is specified a map of sets $v_{[i\uparrow j]} \to G$, and we treat the elements of $v_{[i\uparrow j]}$ as elements of G (possibly with repetitions). We let

$$\Pi v_{[i\uparrow j]} := \begin{cases} v_i v_{i+1} \cdots v_{j-1} v_j \in G & \text{if } i \leq j, \\ 1 \in G & \text{if } i > j. \end{cases}$$

$$\Pi v_{[j\downarrow i]} := \begin{cases} v_j v_{j-1} \cdots v_{i+1} v_i \in G & \text{if } j \geq i, \\ 1 \in G & \text{if } j < i. \end{cases}$$

2 Outline

Recall that $n \in \mathbb{N}$.

Let $\Sigma_{0,1,n} := \langle z_1, t_{[1\uparrow n]} \mid z_1 \Pi t_{[1\uparrow n]} = 1 \rangle$. Here, z and t are symbols, and $\Sigma_{0,1,n}$ is presented as a one-relator group with generating sequence $(z_1, t_1, \ldots, t_n) =$

$(z_1) \vee t_{[1\uparrow n]}$. In particular, $t_{[1\uparrow n]} \vee \bar{t}_{[1\uparrow n]}$ is a sequence in $\Sigma_{0,1,n}$, and we see that $\Sigma_{0,1,n}$ is freely generated by $t_{[1\uparrow n]}$.

Let $\mathrm{Out}^+_{0,1,n}$ denote the subgroup of $\mathrm{Aut}\,\Sigma_{0,1,n}$ consisting of all the automorphisms of $\Sigma_{0,1,n}$ which respect the sets $\{z_1\}$ and $\{[t_i]\}_{i\in[1\uparrow n]}$. Let $\mathrm{Out}_{0,1,0}$ denote a group of order two, and, for $n \geq 1$, let $\mathrm{Out}_{0,1,n}$ denote the group of all automorphisms of $\Sigma_{0,1,n}$ which respect the sets $\{z_1, \bar{z}_1\}$ and $\{[t_i] \cup [\bar{t}_i]\}_{i\in[1\uparrow n]}$. Then $\mathrm{Out}^+_{0,1,n}$ is a subgroup of index two in $\mathrm{Out}_{0,1,n}$. We call $\mathrm{Out}_{0,1,n}$ the algebraic mapping-class group of the surface of genus 0 with 1 boundary component and n punctures; see [18] for background on algebraic mapping-class groups.

Frequently, $\mathrm{Out}^+_{0,1,n}$ will be denoted $\mathcal{B}_n$ and called the *n-string braid group*. (The similar symbol B_n denotes a Coxeter diagram.)

In Section 3, we define a sequence $\sigma_{[1\uparrow(n-1)]}$ in $\mathrm{Out}^+_{0,1,n}$, we review Artin's 1925 proof that $\sigma_{[1\uparrow(n-1)]}$ generates $\mathrm{Out}^+_{0,1,n}$, and we present related results that we shall apply in subsequent sections. In Section 4, we recall the definition of Artin groups, specifically $\mathrm{Artin}\langle A_n\rangle$, $\mathrm{Artin}\langle B_n\rangle$ and $\mathrm{Artin}\langle D_n\rangle$. In Section 5, we verify Artin's 1925 result that $\mathrm{Out}^+_{0,1,n} \simeq \mathrm{Artin}\langle A_{n-1}\rangle$, by combining Magnus' 1934 proof, Manfredini's observation that $\mathrm{Out}^+_{0,1,(n-1)\perp 1} \simeq \mathrm{Artin}\langle B_{n-1}\rangle$, and the van der Waerden trick, to condense the calculations involved.

In Section 6, we use results of Section 4 to recover the celebrated σ_1-trichotomy and the Dehornoy right-ordering of $\mathcal{B}_n$. This free-group-action approach represents a substantial simplification over previous arguments. Let us emphasize that we verify directly that $\mathrm{Out}^+_{0,1,n}$ satisfies the σ_1-trichotomy, which is the reverse of the route taken by Larue [22], where the σ_1-trichotomy for $\mathrm{Artin}\langle A_{n-1}\rangle$ is used to verify that $\mathrm{Artin}\langle A_{n-1}\rangle$ acts faithfully on $\Sigma_{0,1,n}$.

In Section 7, we review the action of $\mathcal{B}_n$ on the set of ends of $\Sigma_{0,1,n}$. We recall the argument of Thurston [29] that yields the Dehornoy right-ordering of $\mathcal{B}_n$, but not the σ_1-trichotomy. By analysing further, we obtain new results about the $\mathcal{B}_n$-orbit of t_1 in $\Sigma_{0,1,n}$.

In Section 8, for each $m \geq 2$, we introduce $\mathrm{Out}_{0,1,n^{(m)}}$, the algebraic mapping-class group of the disc with n C_m-points. We recall the Larue-Shpilrain-type proof by Crisp-Paris of the Birman-Hilden result that the natural map from $\mathrm{Out}_{0,1,n}$ to $\mathrm{Out}_{0,1,n^{(m)}}$ is injective. We then modify an argument of Steve Humphries to show that there is a natural identification $\mathrm{Out}_{0,1,n^{(m)}} = \mathrm{Out}_{0,1,n}$. The results previously obtained in Section 7 then provide additional information in this context.

In Section 9, we review some applications by Perron-Vannier [27] of the above Birman-Hilden result, and discuss connections with the actions given by Wada [31] and studied by Shpilrain [30] and Crisp-Paris [10], [11].

Following a kind suggestion of Patrick Dehornoy, we studied the analysis of the $\mathcal{B}_n$-orbit of t_1 in $\Sigma_{0,1,n}$ given by David Larue [21]. Larue's approach is combinatorial and uses polygonal curves in the punctured disc. By combining Larue's approach with Whitehead's use of graphs, we were able to simplify Larue's main arguments; we record our combinatorial approach in an appendix.

We also show how Larue's results imply the results we had obtained in Section 7 by studying ends.

3 Artin's generators of $\mathcal{B}_n$

In this section we recall Artin's generating sequence $\sigma_{[1\uparrow(n-1)]}$ of $\mathcal{B}_n$.

Let us first fix more notation related to $\Sigma_{0,1,n} = \langle z_1, t_{[1\uparrow n]} \mid z_1\Pi t_{[1\uparrow n]} = 1\rangle$ and $\mathcal{B}_n \leq \operatorname{Aut}\Sigma_{0,1,n}$.

3.1 Notation. Let $m \in \mathbb{N}$. Consider an element w of $\Sigma_{0,1,n}$ and a sequence $a_{[1\uparrow m]}$ in $t_{[1\uparrow n]} \vee \bar{t}_{[1\uparrow n]}$. We also view $a_{[1\uparrow m]}$ as a sequence in $\Sigma_{0,1,n}$.

If $\Pi a_{[1\uparrow m]} = w$ in $\Sigma_{0,1,n}$, we say that $a_{[1\uparrow m]}$ is a monoid *expression* for w, in $t_{[1\uparrow n]} \vee \bar{t}_{[1\uparrow n]}$, of *length* m. We say that $a_{[1\uparrow m]}$ is *reduced* if, for all $j \in [1\uparrow(n-1)]$, $a_{j+1} \neq \bar{a}_j$ in $t_{[1\uparrow n]} \vee \bar{t}_{[1\uparrow n]}$. Each element of $\Sigma_{0,1,n}$ has a unique reduced expression, called the *normal form*.

Suppose that $a_{[1\uparrow m]}$ is the normal form for w. We define the *length* of w to be $|w| := m$. The set of elements of $\Sigma_{0,1,n}$ whose normal forms have $a_{[1\uparrow m]}$ as an initial segment is denoted $(w\star)$; and, the set of elements of $\Sigma_{0,1,n}$ whose normal forms have $a_{[1\uparrow m]}$ as a terminal segment is denoted $(\star w)$. The elements of $(w\star)$ are said to *begin with* w, and the elements of $(\star w)$ are said to *end with* w.

Let Sym_n denote the group of permutations of (the set underlying) $[1\uparrow n]$, acting with exponent notation.

Let $\phi \in \mathcal{B}_n$. There exists a unique permutation $\pi \in \operatorname{Sym}_n$, and a unique sequence $w_{[0\uparrow n+1]}$ in $\Sigma_{0,1,n}$ such that $w_0 = 1$ and $w_{n+1} = 1$, and, for each $i \in [1\uparrow n]$, $w_i \notin (t_{i^\pi}\star) \cup (\bar{t}_{i^\pi}\star)$ and

$$t_i^\phi = t_{i^\pi}^{w_i}.$$

For each $i \in [0\uparrow n]$, let $u_i = w_i\overline{w}_{i+1}$. If $j \in [i\uparrow n]$, then $\Pi u_{[i\uparrow j]} = w_i\overline{w}_{j+1}$. In particular, $\Pi u_{[i\uparrow n]} = w_i$. We define $\pi(\phi) := \pi$, $w_i(\phi) := w_i$ for $i \in [0\uparrow n+1]$, and $u_i(\phi) := u_i$ for $i \in [0\uparrow n]$. We write $\|\phi\| := \sum\limits_{i\in[1\uparrow n]} |t_i^\phi| = n + 2\sum\limits_{i\in[1\uparrow n]} |w_i(\phi)|$.

Let $\sigma_{[1\uparrow(n-1)]}$ be the sequence in $\mathcal{B}_n$ defined as follows: for all $i \in [1\uparrow(n-1)]$ and all $k \in [1\uparrow n]$, $t_k^{\sigma_i} = \begin{cases} t_k & \text{if } k \in [1\uparrow(i-1)] \vee [(i+2)\uparrow n], \\ t_{i+1} & \text{if } k = i, \\ t_i^{t_{i+1}} & \text{if } k = i+1. \end{cases}$

In the literature, σ_i is sometimes represented in $2 \times n$-matrix notation, for example, in the format

$$\sigma_i = \begin{pmatrix} t_1 & \dots & t_{i-1} & t_i & t_{i+1} & t_{i+2} & \dots & t_n \\ t_1 & \dots & t_{i-1} & t_{i+1} & t_i^{t_{i+1}} & t_{i+2} & \dots & t_n \end{pmatrix}.$$

We find it convenient to avoid dots and we say that σ_i and $\overline{\sigma}_i$ are *determined by* the expressions

$$\begin{array}{cccc} \overset{k\in[1\uparrow(i-1)]}{\overline{}} & & & \overset{k\in[(i+1)\uparrow n]}{\overline{}} \\ (t_k & t_i & t_{i+1} & t_k)^{\sigma_i} \\ =(t_k & t_{i+1} & t_i^{t_{i+1}} & t_k), \end{array} \quad\text{and}\quad \begin{array}{cccc} \overset{k\in[1\uparrow(i-1)]}{\overline{}} & & & \overset{k\in[(i+1)\uparrow n]}{\overline{}} \\ (t_k & t_i & t_{i+1} & t_k)^{\overline{\sigma}_i} \\ =(t_k & t_{i+1}^{\overline{t}_i} & t_i & t_k). \end{array} \quad \square$$

We shall apply the following result in different situations.

3.2 Lemma (Artin [3])**.** *Let $\phi \in \mathcal{B}_n$. Let $\pi = \pi(\phi)$ and, for each $i \in [0\uparrow n]$, let $u_i = u_i(\phi)$.*

(i). *Suppose that there exists some $i \in [1\uparrow(n-1)]$ such that $u_i \in (\star\overline{t}_{(i+1)^\pi})$. Then $\|\sigma_i\phi\| \le \|\phi\| - 2$. Moreover, for each $j \in [1\uparrow i]$, $t_j^{\sigma_i\phi}$ and t_j^ϕ both begin with the same element of $t_{[1\uparrow n]} \vee \overline{t}_{[1\uparrow n]}$.*

(ii). *Suppose that there exists some $i \in [1\uparrow(n-1)]$ such that $u_i \in (\overline{t}_{i^\pi}\star)$. Then $\|\overline{\sigma}_i\phi\| \le \|\phi\| - 2$. Moreover, for each $j \in [1\uparrow(i-1)]$, $t_j^{\overline{\sigma}_i\phi}$ and t_j^ϕ both begin with the same element of $t_{[1\uparrow n]} \vee \overline{t}_{[1\uparrow n]}$.*

(iii). *Suppose that, for each $i \in [1\uparrow(n-1)]$, $u_i \notin (\overline{t}_{i^\pi}\star) \cup (\star\overline{t}_{(i+1)^\pi})$. Then $\phi = 1$.*

Proof. (i). There exists some $v \in \Sigma_{0,1,n} - (\star t_{(i+1)^\pi})$ such that $u_i = v\overline{t}_{(i+1)^\pi}$. Since $w_i(\phi) = u_i w_{i+1}(\phi)$, we have

$$w_i(\phi) = v\overline{t}_{(i+1)^\pi} w_{i+1}(\phi). \tag{3.2.1}$$

Since $v \notin (\star t_{(i+1)^\pi})$ and $w_{i+1}(\phi) \notin (t_{(i+1)^\pi}\star)$, there is no cancellation in the expression $t_{i^\pi}^{v\overline{t}_{(i+1)^\pi} w_{i+1}(\phi)}$ for t_i^ϕ; hence

$$t_i^\phi \in (\overline{w}_{i+1}(\phi) t_{(i+1)^\pi}\star) \text{ and } |t_i^\phi| = 1 + 2|v| + 2 + 2|w_{i+1}(\phi)|. \tag{3.2.2}$$

For all $j \in [1\uparrow(i-1)] \vee [(i+2)\uparrow n]$, $t_j^{\sigma_i\phi} = t_j^\phi$; hence, $t_j^{\sigma_i\phi}$ has the same first letter as t_j^ϕ, and, $|t_j^{\sigma_i\phi}| = |t_j^\phi|$.

Since $t_i^{\sigma_i\phi} = t_{i+1}^\phi \in (\overline{w}_{i+1}(\phi) t_{(i+1)^\pi}\star)$, we see, from (3.2.2), that $t_i^{\sigma_i\phi}$ has the same first letter as t_i^ϕ. Also, $|t_i^{\sigma_i\phi}| = |t_{i+1}^\phi|$.

By (3.2.1), $w_i(\phi)\overline{w}_{i+1}(\phi)t_{(i+1)^\pi} = v$; hence

$$t_{i+1}^{\sigma_i\phi} = (t_i^{t_{i+1}})^\phi = (t_{i^\pi}^{w_i(\phi)})^{(t_{(i+1)^\pi}^{w_{i+1}(\phi)})} = t_{i^\pi}^{v w_{i+1}(\phi)}.$$

Hence, $|t_{i+1}^{\sigma_i\phi}| \le 1 + 2|v| + 2|w_{i+1}(\phi)| \overset{(3.2.2)}{=} |t_i^\phi| - 2$.

It now follows that $\|\sigma_i\phi\| \le \|\phi\| - 2$, and (i) is proved.

(ii). There exists some $v \in \Sigma_{0,1,n} - (t_{i^\pi}\star)$ such that $u_i = \overline{t}_{i^\pi}v$. Since $w_{i+1}(\phi) = \overline{u}_i w_i(\phi)$, we have

$$w_{i+1}(\phi) = \overline{v} t_{i^\pi} w_i(\phi). \tag{3.2.3}$$

Since $\overline{v} \notin (\star\bar{t}_{i^\pi})$ and $w_i(\phi) \notin (\bar{t}_{i^\pi}\star)$, there is no cancellation in the expression $t_{(i+1)^\pi}^{\overline{v}t_{i^\pi}w_i(\phi)}$ for t_{i+1}^ϕ; hence

$$|t_{i+1}^\phi| = 1 + 2|\overline{v}| + 2 + 2|w_i(\phi)|. \tag{3.2.4}$$

For all $j \in [1\uparrow(i-1)] \vee [(i+2)\uparrow n]$, $t_j^{\overline{\sigma}_i\phi} = t_j^\phi$; hence, $t_j^{\overline{\sigma}_i\phi}$ has the same first letter as t_j^ϕ, and, $|t_j^{\overline{\sigma}_i\phi}| = |t_j^\phi|$.

Since $t_{i+1}^{\overline{\sigma}_i\phi} = t_i^\phi$, we see that $|t_{i+1}^{\overline{\sigma}_i\phi}| = |t_i^\phi|$.

By (3.2.3), $w_{i+1}(\phi)\overline{w}_i(\phi)\bar{t}_{i^\pi} = \overline{v}$; hence

$$t_i^{\overline{\sigma}_i\phi} = (t_{i+1}^{\bar{t}_i})^\phi = (t_{(i+1)^\pi}^{w_{i+1}(\phi)})^{(\bar{t}_{i^\pi}^{w_i(\phi)})} = t_{i^\pi}^{\overline{v}w_i(\phi)}.$$

Hence, $|t_i^{\overline{\sigma}_i\phi}| \le 1 + 2|\overline{v}| + 2|w_i(\phi)| \overset{(3.2.4)}{=} |t_{i+1}^\phi| - 2$.

It now follows that $\|\overline{\sigma}_i\phi\| \le \|\phi\| - 2$, and (ii) is proved.

(iii). Since $u_0 = \overline{w}_1(\phi) \notin (\star\bar{t}_{1^\pi})$ and $u_n = w_n(\phi) \notin (\bar{t}_{n^\pi}\star)$, we see that there is no cancellation anywhere in the expression $u_0 \prod\limits_{i\in[1\uparrow n]} (t_{i^\pi}u_i)$. Hence,

$$|u_0 \prod_{i\in[1\uparrow n]} (t_{i^\pi}u_i)| = \sum_{i\in[0\uparrow n]} |u_i| + n, \text{ that is, } \sum_{i\in[0\uparrow n]} |u_i| = |u_0 \prod_{i\in[1\uparrow n]} (t_{i^\pi}u_i)| - n.$$

Recall that $u_0 \prod\limits_{i\in[1\uparrow n]} (t_{i^\pi}u_i) = \prod\limits_{i\in[1\uparrow n]} (t_{i^\pi}^{w_i(\phi)}) = (\prod\limits_{i\in[1\uparrow n]} t_i)^\phi = \prod\limits_{i\in[1\uparrow n]} t_i$. Hence

$$|u_0 \prod_{i\in[1\uparrow n]} (t_{i^\pi}u_i)| = n \text{ and } \sum_{i\in[0\uparrow n]} |u_i| = n - n = 0.$$

Hence, all the elements of $u_{[0\uparrow n]}$ are trivial.

For each $i \in [0\uparrow(n+1)]$, $w_i = \Pi u_{[i\uparrow n]}$; hence, all the elements of $w_{[1\uparrow n]}$ are trivial. Also, $\prod\limits_{i\in[1\uparrow n]} t_{i^\pi} = u_0 \prod\limits_{i\in[1\uparrow n]} (t_{i^\pi}u_i) = \prod\limits_{i\in[1\uparrow n]} t_i$. Hence π is trivial. Thus $\phi = 1$. □

The following is then immediate.

3.3 Proposition (Artin [3])**.** *For each $\phi \in \mathcal{B}_n$, either $\phi = 1$, or there exists some $\sigma_i^\epsilon \in \sigma_{[1\uparrow(n-1)]} \vee \overline{\sigma}_{[1\uparrow(n-1)]}$ such that $\|\sigma_i^\epsilon\phi\| \le \|\phi\| - 2$. Hence, $\langle\sigma_{[1\uparrow(n-1)]}\rangle = \mathcal{B}_n$.* □

3.4 Remarks. If $w \in \Sigma_{0,1,n}$ has odd length, then w^{σ_i} has odd length, and $|w^{\sigma_i}| \le 2|w| + 1$, with equality being achieved only if every odd letter of w equals t_{i+1} or $\bar{t}_{i+1}$. Similar statements hold with $\overline{\sigma}_i$ in place of σ_i.

Let $\phi \in \mathcal{B}_n$ and let $|\phi|$ denote the minimum length of a monoid expression for ϕ in $\sigma_{[1\uparrow(n-1)]} \vee \overline{\sigma}_{[1\uparrow(n-1)]}$. Thus, $|t_i^\phi| \le 2^{|\phi|+1} - 1$. Hence, $\|\phi\| \le n2^{|\phi|+1} - n$. Proposition 3.3 gives an algorithm which yields a monoid expression for ϕ in $\sigma_{[1\uparrow(n-1)]} \vee \overline{\sigma}_{[1\uparrow(n-1)]}$ of length at most $\frac{\|\phi\|-n}{2}$, and we have now seen that $\frac{\|\phi\|-n}{2} \le \frac{n2^{|\phi|+1}-2n}{2} = n2^{|\phi|} - n$. □

4 Coxeter diagrams and Artin groups

4.1 Definition. A *Coxeter diagram* X consists of a set V together with a function $V \times V \to \mathbb{N} \cup \{\infty\}$, $(x,y) \mapsto m_{x,y}$, such that, for all x, $y \in V$, $m_{x,x} = 0$ and $m_{x,y} = m_{y,x}$. The elements of V are called the *vertices* of X, and, for all x, $y \in V$, we say that $m_{x,y}$ is the *number of edges joining* x *and* y; thus we can represent X diagrammatically. We then define the *Artin group* of X, denoted $\mathrm{Artin}\langle X\rangle$, to be the group presented with generating set V and relations saying that, for all x, $y \in V$,

$$\begin{array}{rcll} xy &=& yx & \text{if } m_{x,y}=0,\\ xyx &=& yxy & \text{if } m_{x,y}=1,\\ xyxy &=& yxyx & \text{if } m_{x,y}=2,\\ \text{etc.} &&& \end{array}$$

Notice that if $m_{x,y} = \infty$, then no relation is imposed. Notice also that if V is empty, then $\mathrm{Artin}\langle X\rangle$ is the trivial group. □

4.2 Notation. (i). Let A_n denote the Coxeter diagram

$$a_1 \text{ --- } a_2 \text{ --- } \cdots \text{ --- } a_{n-1} \text{ --- } a_n.$$

It is understood that A_0 is empty. We define A_{-1} to be empty also.

Thus, in A_n, the vertex set is $a_{[1\uparrow n]}$, and, for i, $j \in [1\uparrow n]$, the number of edges joining a_i to a_j is $\begin{cases} 1 & \text{if } |i-j| = 1,\\ 0 & \text{if } |i-j| \neq 1.\end{cases}$

Hence, $\mathrm{Artin}\langle A_n\rangle$ has a presentation with generating set $a_{[1\uparrow n]}$ and relations saying that, for i, $j \in [1\uparrow n]$,

$$\begin{array}{rcll} a_ia_j &=& a_ja_i & \text{if } |i-j|\neq 1,\\ a_ia_ja_i &=& a_ja_ia_j & \text{if } |i-j| = 1.\end{array}$$

(ii). Let B_n denote the Coxeter diagram

$$b_1 \text{ --- } b_2 \text{ --- } \cdots \text{ --- } b_{n-1} \text{ === } b_n.$$

Here, the vertex set is $b_{[1\uparrow n]}$, and, for i, $j \in [1\uparrow n]$, the number of edges joining b_i to b_j is $\begin{cases} 2 & \text{if } \{i,j\} = \{n-1,n\},\\ 1 & \text{if } |i-j| = 1 \text{ and } \{i,j\} \neq \{n-1,n\},\\ 0 & \text{if } |i-j| \neq 1.\end{cases}$

(iii). For $n \geq 2$, let D_n denote the Coxeter diagram

$$\begin{array}{ccccccccccccc} &&&&&&&& d_n &&\\ &&&&&&&& | &&\\ d_1 & \text{---} & d_2 & \text{---} & \cdots & \text{---} & d_{n-3} & \text{---} & d_{n-2} & \text{---} & d_{n-1}. \end{array}$$

Here, the vertex set is $d_{[1\uparrow n]}$, and, for i, $j \in [1\uparrow n]$, the number of edges joining d_i to d_j is

$$\begin{cases} 1 & \text{if } \{i,j\} \in \{\{1,2\},\{2,3\},\ldots,\{n-2,n-1\},\{n-2,n\}\},\\ 0 & \text{otherwise.}\end{cases}$$

□

5 Artin's presentation of $\mathcal{B}_n$

In this section, we verify Artin's result that there exists an isomorphism $\gamma_n\colon \operatorname{Artin}\langle A_{n-1}\rangle \to \mathcal{B}_n$ determined by $\underset{=(\sigma_i)}{\overset{\underline{i\in[1\uparrow(n-1)]}}{(a_i)^{\gamma_n}}}$. We express this result by writing $\mathcal{B}_n = \operatorname{Artin}\langle\, \sigma_1 \,\text{—}\, \sigma_2 \,\text{—}\, \cdots \,\text{—}\, \sigma_{n-1}\rangle$.

5.1 Proposition. *There exists a homomorphism $\gamma_n\colon \operatorname{Artin}\langle A_{n-1}\rangle \to \mathcal{B}_n$ determined by $\underset{=(\sigma_i)}{\overset{\underline{i\in[1\uparrow(n-1)]}}{(a_i)^{\gamma_n}}}$, and γ_n is surjective.*

Proof. (a). Suppose that $1 \le i \le i+2 \le j \le n-1$. We have the following.

$$\begin{array}{lcccccc}
\overset{\underline{k\in[1\uparrow(i-1)]}}{(t_k} & t_i & t_{i+1} & \overset{\underline{k\in[(i+2)\uparrow(j-1)]}}{t_k} & t_j & t_{j+1} & \overset{\underline{k\in[(j+2)\uparrow n]}}{t_k)^{\sigma_i\sigma_j}} \\
=(t_k & t_{i+1} & t_i^{t_{i+1}} & t_k & t_j & t_{j+1} & t_k)^{\sigma_j} \\
=(t_k & t_{i+1} & t_i^{t_{i+1}} & t_k & t_{j+1} & t_j^{t_{j+1}} & t_k) \\
=(t_k & t_i & t_{i+1} & t_k & t_{j+1} & t_j^{t_{j+1}} & t_k)^{\sigma_i} \\
=(t_k & t_i & t_{i+1} & t_k & t_j & t_{j+1} & t_k)^{\sigma_j\sigma_i}.
\end{array}$$

(b). Suppose that $1 \le i \le n-2$. We have the following.

$$\begin{array}{lccccc}
\overset{\underline{k\in[1\uparrow(i-1)]}}{(t_k} & t_i & t_{i+1} & t_{i+2} & \overset{\underline{k\in[(i+3)\uparrow n]}}{t_k)^{\sigma_i\sigma_{i+1}\sigma_i}} \\
=(t_k & t_{i+1} & t_i^{t_{i+1}} & t_{i+2} & t_k)^{\sigma_{i+1}\sigma_i} \\
=(t_k & t_{i+2} & t_i^{t_{i+2}} & t_{i+1}^{t_{i+2}} & t_k)^{\sigma_i} \\
=(t_k & t_{i+2} & t_{i+1}^{t_{i+2}} & t_i^{t_{i+1}t_{i+2}} & t_k) \\
=(t_k & t_{i+1} & t_{i+2} & t_i^{t_{i+1}t_{i+2}} & t_k)^{\sigma_{i+1}} \\
=(t_k & t_i & t_{i+2} & t_{i+1}^{t_{i+2}} & t_k)^{\sigma_i\sigma_{i+1}} \\
=(t_k & t_i & t_{i+1} & t_{i+2} & t_k)^{\sigma_{i+1}\sigma_i\sigma_{i+1}}
\end{array}$$

By (a) and (b), there exists a homomorphism $\gamma_n\colon \operatorname{Artin}\langle A_{n-1}\rangle \to \mathcal{B}_n$ determined by $\underset{=(\sigma_i)}{\overset{\underline{i\in[1\uparrow(n-1)]}}{(a_i)^{\gamma_n}}}$. By Proposition 3.3, $\langle\sigma_{[1\uparrow(n-1)]}\rangle = \mathcal{B}_n$, and, hence, γ_n is surjective. □

In the remainder of this section, we shall use induction on n to show that the surjective homomorphism $\gamma_n\colon \operatorname{Artin}\langle A_{n-1}\rangle \to \mathcal{B}_n$ of Proposition 5.1 is an isomorphism. Notice that γ_n endows $\operatorname{Artin}\langle A_{n-1}\rangle$ with a canonical action on $\Sigma_{0,1,n}$.

The following is precisely [25, Proposition 1] and, also, [10, Proposition 2.1(2)].

5.2 Lemma (Manfredini [25]). *If $n \geq 1$, then*

$$\mathrm{Artin}\langle A_{n-1}\rangle \ltimes \Sigma_{0,1,n} = \mathrm{Artin}\langle\, a_1 \;\text{—}\; a_2 \;\text{—}\; \cdots \;\text{—}\; a_{n-1} \;=\!=\; \overline{t}_n \,\rangle \simeq \mathrm{Artin}\langle B_n\rangle.$$

Proof. For $n = 1$, the result is clear.

For $n = 2$, we have the following.

$$\begin{aligned}
&\mathrm{Artin}\langle A_1\rangle \ltimes \Sigma_{0,1,2} = \langle a_1, t_{[1\uparrow 2]} \mid t_1^{a_1} = t_2,\ t_2^{a_1} = \overline{t}_2 t_1 t_2\rangle \\
&= \langle a_1, t_2 \mid t_2^{a_1} = \overline{t}_2 t_2^{\overline{a}_1} t_2\rangle = \langle a_1, t_2 \mid (\overline{a}_1 t_2)(a_1) = (\overline{t}_2 a_1)(t_2 \overline{a}_1 t_2)\rangle \\
&= \langle a_1, t_2 \mid (a_1)(\overline{t}_2 a_1 \overline{t}_2) = (\overline{t}_2 a_1)(\overline{t}_2 a_1)\rangle = \mathrm{Artin}\langle\, a_1 \;=\!=\; \overline{t}_2 \,\rangle.
\end{aligned}$$

From the case $n = 2$, we see that there exists a homomorphism

$\mu\colon \mathrm{Artin}\langle\, B_n \,\rangle \to \mathrm{Artin}\langle A_{n-1}\,\rangle \ltimes \Sigma_{0,1,n}$ determined by $\begin{array}{c} \overline{\scriptstyle i\in[1\uparrow(n-1)]} \\ (b_i \\ = (a_i \end{array} \begin{array}{c} \\ b_n)^{\mu} \\ \overline{t}_n) \end{array}$.

For each $k \in [1\uparrow n]$, let t_k denote the element $\overline{b}_n^{\,\overline{\Pi \overline{b}_{[n-1\downarrow k]}}}$ of $\mathrm{Artin}\langle B_n\rangle$. For each $i \in [1\uparrow(n-1)]$ and $k \in [1\uparrow n]$, let us formally define

$$\mathsf{t}_k^{\overline{\sigma}_i} := \begin{cases} \mathsf{t}_k & \text{if } k \in [1\uparrow i-1] \vee [i+2\uparrow n], \\ \mathsf{t}_{i+1}^{\overline{\mathsf{t}}_i} & \text{if } k = i, \\ \mathsf{t}_i & \text{if } k = i+1. \end{cases}$$

We shall see that $\mathsf{t}_k^{\overline{b}_i} = \mathsf{t}_k^{\overline{\sigma}_i}$; this immediately implies that there exists a homomorphism $\overline{\mu}\colon \mathrm{Artin}\langle A_{n-1}\,\rangle \ltimes \Sigma_{0,1,n} \to \mathrm{Artin}\langle B_n\,\rangle$ determined by $\begin{array}{c} \overline{\scriptstyle i\in[1\uparrow(n-1)]} \\ (a_i \\ = (b_i \end{array} \begin{array}{c} \overline{\scriptstyle k\in[1\uparrow n]} \\ \mathsf{t}_k)^{\overline{\mu}} \\ \mathsf{t}_k) \end{array}$
which is then clearly inverse to μ, and the result will be proved.

For each $m \in [n\downarrow 1]$, we shall show, by decreasing induction on m, that, for each $k \in [n\downarrow m]$ and each $i \in [(n-1)\downarrow m]$, $\mathsf{t}_k^{\overline{b}_i} = \mathsf{t}_k^{\overline{\sigma}_i}$. For $m = n$, this is trivial, and, for $m = n-1$, it follows from the case $n = 2$. Suppose that $m \in [(n-2)\downarrow 1]$.

(a). For each $k \in [n\downarrow(m+1)]$ and each $i \in [(n-1)\downarrow(m+1)]$, $\mathsf{t}_k^{\overline{b}_i} = \mathsf{t}_k^{\overline{\sigma}_i}$, by hypothesis.

(b). For each $k \in [n\downarrow(m+2)]$, $\mathsf{t}_k \in \langle b_{[n\downarrow(m+2)]}\rangle$ and, hence, $\mathsf{t}_k^{\overline{b}_m} = \mathsf{t}_k = \mathsf{t}_k^{\overline{\sigma}_m}$.

(c). $\mathsf{t}_{m+1}^{\overline{b}_m} = \overline{b}_n^{(\Pi \overline{b}_{[(n-1)\downarrow(m+1)]})\overline{b}_m} = \mathsf{t}_m = \mathsf{t}_{m+1}^{\overline{\sigma}_m}$.

(d). For each $i \in [(n-1)\downarrow(m+2)]$, $\mathsf{t}_m^{\overline{b}_i} \overset{\text{(c)}}{=} \mathsf{t}_{m+1}^{\overline{b}_m \overline{b}_i} = \mathsf{t}_{m+1}^{\overline{b}_i \overline{b}_m} \overset{\text{(a)}}{=} \mathsf{t}_{m+1}^{\overline{b}_m} \overset{\text{(c)}}{=} \mathsf{t}_m = \mathsf{t}_m^{\overline{\sigma}_i}$.

(e). $\mathsf{t}_m^{\overline{b}_{m+1}} \overset{\text{(c)}}{=} \mathsf{t}_{m+1}^{\overline{b}_m \overline{b}_{m+1}} \overset{\text{(a)}}{=} \mathsf{t}_{m+2}^{\overline{b}_{m+1} \overline{b}_m \overline{b}_{m+1}} = \mathsf{t}_{m+2}^{\overline{b}_m \overline{b}_{m+1} \overline{b}_m} \overset{\text{(b)}}{=} \mathsf{t}_{m+2}^{\overline{b}_{m+1} \overline{b}_m} \overset{\text{(a)}}{=} \mathsf{t}_{m+1}^{\overline{b}_m} \overset{\text{(c)}}{=} \mathsf{t}_m = \mathsf{t}_m^{\overline{\sigma}_{m+1}}$.

(f). $\mathsf{t}_m^{\overline{b}_m} = \mathsf{t}_m^{b_{m+1} b_m \overline{b}_{m+1} \overline{b}_m \overline{b}_{m+1}} \overset{\text{(e)}}{=} \mathsf{t}_m^{b_m \overline{b}_{m+1} \overline{b}_m \overline{b}_{m+1}} \overset{\text{(c)}}{=} \mathsf{t}_{m+1}^{\overline{b}_{m+1} \overline{b}_m \overline{b}_{m+1}}$
$\overset{\text{(a)}}{=} (\mathsf{t}_{m+1} \mathsf{t}_{m+2} \overline{\mathsf{t}}_{m+1})^{\overline{b}_m \overline{b}_{m+1}} \overset{\text{(c),(b),(c)}}{=} (\mathsf{t}_m \mathsf{t}_{m+2} \overline{\mathsf{t}}_m)^{\overline{b}_{m+1}} \overset{\text{(e),(a),(e)}}{=} \mathsf{t}_m \mathsf{t}_{m+1} \overline{\mathsf{t}}_m = \mathsf{t}_m^{\overline{\sigma}_m}$.

Now the result follows by induction. □

We write $\mathrm{Stab}(\mathrm{Artin}\langle A_n\rangle;[t_{n+1}])$ to denote the $\mathrm{Artin}\langle A_n\rangle$-stabilizer of the conjugacy class $[t_{n+1}]$ under the $\mathrm{Artin}\langle A_n\rangle$-action on $\Sigma_{0,1,n+1}$. The Reidemeister-Schreier rewriting technique automatically gives a useful presentation of $\mathrm{Stab}(\mathrm{Artin}\langle A_n\rangle;[t_{n+1}])$ but the resulting exposition is tedious. Once the presentation has been found, we can verify it directly using the van der Waerden trick, as in the following proof.

5.3 Theorem (Magnus [24])**.** *Let $n \geq 1$.*

(i). *There exists a homomorphism* $\phi_n\colon \mathrm{Artin}\langle A_{n-1}\rangle \ltimes \Sigma_{0,1,n} \to \mathrm{Artin}\langle A_n\rangle$ *determined by*
$$\begin{array}{rll} & \overset{i\in[1\uparrow(n-1)]}{\overline{(a_i}} & t_n\,)^{\phi_n} \\ = & (a_i & \overline{a}_n^{\,2}). \end{array}$$

(ii). *ϕ_n is injective.*

(iii). *For each $i \in [1\uparrow n]$, $t_i^{\phi_n} = \overline{a}_i^{\,2\Pi a_{[(i+1)\uparrow n]}}$ in* $\mathrm{Artin}\langle A_n\rangle$.

(iv). *The image of ϕ_n is* $\mathrm{Stab}(\mathrm{Artin}\langle A_n\rangle;[t_{n+1}])$.

Proof. Let us write $G = \mathrm{Artin}\langle A_n\rangle$ and $H = \mathrm{Artin}\langle A_{n-1}\rangle \ltimes \Sigma_{0,1,n}$.

In G,

$$(a_{n-1}a_n^2a_{n-1})^{a_n}=(\overline{a}_na_{n-1}a_n)(a_na_{n-1}a_n)=(a_{n-1}a_n\overline{a}_{n-1})(a_{n-1}a_na_{n-1})=a_{n-1}a_n^2a_{n-1},$$

and, hence, $a_{n-1}a_n^2a_{n-1}a_n^2 = a_n^2a_{n-1}a_n^2a_{n-1}$. By Lemma 5.2, $H \simeq \mathrm{Artin}\langle B_n\rangle$, and we see that there exist a homomorphism $\phi_n\colon H \to G$ determined by
$$\begin{array}{rll} & \overset{i\in[1\uparrow(n-1)]}{\overline{(a_i}} & \overline{t}_n\,)^{\phi_n} \\ = & (a_i & a_n^{\,2}) \end{array}$$
and (i) is proved.

Let v be a symbol and let $H \times v_{[1\uparrow(n+1)]}$ denote a free left H-set with left H-transversal $v_{[1\uparrow(n+1)]}$. We construct a right G-action on $H \times v_{[1\uparrow(n+1)]}$ such that $H \times v_{[1\uparrow(n+1)]}$ becomes an (H,G)-bi-set. For each $i \in [1\uparrow n]$, the element a_i of G acts on the right on $H \times v_{[1\uparrow(n+1)]}$ as the left H-map that is determined by the following.

$$\begin{array}{rlllll} & \overset{k\in[1\uparrow(i-1)]}{\overline{(\quad v_k}} & v_i & v_{i+1} & \overset{k\in[(i+2)\uparrow(n+1)]}{\overline{v_k)a_i}} \\ = & (a_{i-1}v_k & v_{i+1} & \overline{t}_iv_i & a_iv_k). \end{array}$$

We now verify that the relations of G are respected.

(a). Suppose that $1 \leq i < i+2 \leq j \leq n$. We have the following.

$$\begin{array}{rlllllll} & \overset{k\in[1\uparrow(i-1)]}{\overline{(\qquad v_k}} & v_i & v_{i+1} & \overset{k\in[(i+2)\uparrow(j-1)]}{\overline{v_k}} & v_j & v_{j+1} & \overset{k\in[(j+2)\uparrow(n+1)]}{\overline{v_k)a_ia_j}} \\ = & (\quad a_{i-1}v_k & v_{i+1} & \overline{t}_iv_i & a_iv_k & a_iv_j & a_iv_{j+1} & a_iv_k)a_j \\ = & (a_{i-1}a_{j-1}v_k & a_{j-1}v_{i+1} & \overline{t}_ia_{j-1}v_i & a_ia_{j-1}v_k & a_iv_{j+1} & a_i\overline{t}_jv_j & a_ia_jv_k) \\ = & (a_{j-1}a_{i-1}v_k & a_{j-1}v_{i+1} & a_{j-1}\overline{t}_iv_i & a_{j-1}a_iv_k & a_iv_{j+1} & \overline{t}_ja_iv_j & a_ja_iv_k) \\ = & (\quad a_{j-1}v_k & a_{j-1}v_i & a_{j-1}v_{i+1} & a_{j-1}v_k & v_{j+1} & \overline{t}_jv_j & a_jv_k)a_i \\ = & (\qquad v_k & v_i & v_{i+1} & v_k & v_j & v_{j+1} & v_k)a_ja_i. \end{array}$$

(b). Suppose that $1 \le i \le n-1$. We have the following.

$$
\begin{array}{rrrrrl}
 & \underline{\scriptstyle k\in[1\uparrow(i-1)]} & & & & \underline{\scriptstyle k\in[(i+3)\uparrow(n+1)]} \\
(& v_k & v_i & v_{i+1} & v_{i+2} & v_k)a_i a_{i+1} a_i \\
=(& a_{i-1}v_k & v_{i+1} & \overline{t}_i v_i & a_i v_{i+2} & a_i v_k)a_{i+1}a_i \\
=(& a_{i-1}a_i v_k & v_{i+2} & \overline{t}_i a_i v_i & a_i \overline{t}_{i+1} v_{i+1} & a_i a_{i+1} v_k)a_i \\
=(& a_{i-1}a_i a_{i-1} v_k & a_i v_{i+2} & \overline{t}_i a_i v_{i+1} & a_i \overline{t}_{i+1}\overline{t}_i v_i & a_i a_{i+1} a_i v_k) \\
=(& a_i a_{i-1} a_i v_k & a_i v_{i+2} & a_i \overline{t}_{i+1} v_{i+1} & \overline{t}_{i+1}\overline{t}_i a_i v_i & a_{i+1} a_i a_{i+1} v_k) \\
=(& a_i a_{i-1} v_k & a_i v_{i+1} & a_i v_{i+2} & \overline{t}_{i+1}\overline{t}_i v_i & a_{i+1} a_i v_k)a_{i+1} \\
=(& a_i v_k & a_i v_i & v_{i+2} & \overline{t}_{i+1} v_{i+1} & a_{i+1} v_k)a_i a_{i+1} \\
=(& v_k & v_i & v_{i+1} & v_{i+2} & v_k)a_{i+1}a_i a_{i+1}.
\end{array}
$$

By (a) and (b), the relations of G are respected. Hence, we have a right G-action on $H \times v_{[1\uparrow(n+1)]}$ by left H-maps.

Notice that $v_{n+1}\overline{t}_n^{\,\phi_n} = v_{n+1}a_n^2 = \overline{t}_n v_n a_n = \overline{t}_n v_{n+1}$, and, for each $i \in [1\uparrow(n–1)]$, $v_{n+1}a_i^{\phi_n} = v_{n+1}a_i = a_i v_{n+1}$. It follows that, for each $h \in H$, $v_{n+1}h^{\phi_n} = hv_{n+1}$. Hence, ϕ_n is injective. This proves (ii).

Recall that $G = \text{Artin}\langle A_n\rangle$.

Let $i \in [1\uparrow n]$.

We shall show by decreasing induction on i that

$$a_n^{\Pi\overline{a}_{[(n-1)\downarrow i]}} = a_i^{\Pi a_{[(i+1)\uparrow n]}}. \tag{5.3.1}$$

If $i = n$, then (5.3.1) holds. Now suppose that $i \ge 2$, and that (5.3.1) holds. Conjugating (5.3.1) by $\overline{a}_{i-1}$ yields

$$a_n^{\Pi\overline{a}_{[(n-1)\downarrow(i-1)]}} = a_i^{(\Pi a_{[(i+1)\uparrow n]})\overline{a}_{i-1}} = a_i^{\overline{a}_{i-1}\Pi a_{[(i+1)\uparrow n]}} = a_{i-1}^{a_i\Pi a_{[(i+1)\uparrow n]}} = a_{i-1}^{\Pi a_{[i\uparrow n]}}.$$

By induction, (5.3.1) holds.

Now $\overline{t}_i^{\,\phi_n} = (\overline{t}_n^{\,\Pi\overline{a}_{[(n-1)\downarrow i]}})^{\phi_n} = a_n^{2\Pi\overline{a}_{[(n-1)\downarrow i]}} \overset{(5.3.1)}{=} a_i^{2\Pi a_{[(i+1)\uparrow n]}}$. This proves (iii). Also, $\overline{t}_i^{\,\phi_n}\Pi\overline{a}_{[n\downarrow i]} = (\Pi\overline{a}_{[n\downarrow(i+1)]})a_i$.

If $k \in [1\uparrow(i-1)]$, then

$$a_i^{\Pi a_{[k\uparrow n]}} = a_i^{\Pi a_{[k\uparrow(i-2)]}\Pi a_{[(i-1)\uparrow i]}\Pi a_{[(i+1)\uparrow n]}} = a_i^{\Pi a_{[(i-1)\uparrow i]}\Pi a_{[(i+1)\uparrow n]}} = a_{i-1}^{\Pi a_{[(i+1)\uparrow n]}} = a_{i-1}.$$

Hence, $a_{i-1}\Pi\overline{a}_{[n\downarrow k]} = (\Pi\overline{a}_{[n\downarrow k]})a_i$.

Let ψ_n denote the map of sets

$$\psi_n \colon H \times v_{[1\uparrow n+1]} \to G, \quad hv_k \mapsto h^{\phi_n}\Pi\overline{a}_{[n\downarrow k]} \text{ for all } hv_k = (h, v_k) \in H \times v_{[1\uparrow(n+1)]}.$$

Hence, for each $h \in H$, we have the following, in G.

$$
\begin{array}{rlllll}
 & & \underline{\scriptstyle k\in[1\uparrow(i-1)]} & & & \underline{\scriptstyle k\in[(i+2)\uparrow(n+1)]} \\
 & (h\ \ (& v_k & v_i & v_{i+1} & v_k\ \ ))^{\psi_n}a_i \\
= & (h^{\phi_n}(& \Pi\overline{a}_{[n\downarrow k]} & \Pi\overline{a}_{[n\downarrow i]} & \Pi\overline{a}_{[n\downarrow(i+1)]} & \Pi\overline{a}_{[n\downarrow k]}))a_i \\
= & (h^{\phi_n}(& a_{i-1}\Pi\overline{a}_{[n\downarrow k]} & \Pi\overline{a}_{[n\downarrow(i+1)]} & \overline{t}_i^{\,\phi_n}\Pi\overline{a}_{[n\downarrow i]} & a_i\Pi\overline{a}_{[n\downarrow k]})) \\
= & (h\ \ (& a_{i-1}v_k & v_{i+1} & \overline{t}_i v_i & a_i v_k\ \ ))^{\psi_n} \\
= & (h\ \ (& v_k & v_i & v_{i+1} & v_k\ \ )a_i)^{\psi_n}.
\end{array}
$$

This proves that ψ_n is a map of right G-sets, and, hence, ψ_n must be surjective. Thus, $G = \bigcup\limits_{k\in[1\uparrow(n+1)]} H^{\phi_n} v_k^{\psi_n}$, and, hence, the index of H^{ϕ_n} in G is at most $n+1$.

Consider the action of G on the set of conjugacy classes $\{[t_k]\}_{k\in[1\uparrow(n+1)]}$ in $\Sigma_{0,1,n+1}$. For any $i \in [1\uparrow n]$, a_i acts as the transposition $([t_i],[t_{i+1}])$. In particular, the index of $\mathrm{Stab}(G;[t_{n+1}])$ in G is $n+1$. Also, the elements of $a_{[1\uparrow(n-1)]} \vee (a_n^2)$ fix $[t_{n+1}]$, and, hence, $H^{\phi_n} \le \mathrm{Stab}(G;[t_{n+1}])$. By comparing indices in G, we see that $H^{\phi_n} = \mathrm{Stab}(G;[t_{n+1}])$. This proves (iv). □

5.4 Theorem (Artin)**.** $\mathcal{B}_n = \mathrm{Artin}\langle \sigma_1 — \sigma_2 — \cdots — \sigma_{n-1}\rangle$.

Proof. This is trivial for $n \le 1$.

Hence, we may assume that $n \ge 1$ and that the homomorphism $\gamma_n\colon \mathrm{Artin}\langle A_{n-1}\rangle \to \mathcal{B}_n$, of Proposition 5.1, determined by $\underset{=(\sigma_i)}{\overset{i\in[1\uparrow(n-1)]}{(a_i)^{\gamma_n}}}$ is an isomorphism. By induction, it remains to show that the surjective homomorphism $\gamma_{n+1}\colon \mathrm{Artin}\langle A_n\rangle \to \mathcal{B}_{n+1}$ is injective.

Consider an element w of the kernel of γ_{n+1}. In particular, w fixes t_{n+1} in the $\mathrm{Artin}\langle A_n\rangle$-action on $\Sigma_{0,1,n+1}$. By Theorem 5.3(iv), w lies in the image of the homomorphism $\phi_n\colon \mathrm{Artin}\langle A_{n-1}\rangle \ltimes \Sigma_{0,1,n} \to \mathrm{Artin}\langle A_n\rangle$ determined by $\underset{=(a_i \quad \overline{a}_n^{\,2})}{\overset{i\in[1\uparrow(n-1)]}{(a_i \quad t_n)^{\phi_n}}}$, and there is a resulting factorization of the form $w = w_1(a_{[1\uparrow(n-1)]})w_2(t_{[1\uparrow n]}^{\phi_n})$. Now,

$$\text{(5.4.1)} \quad \text{in } \mathrm{Artin}\langle A_n\rangle \ltimes \Sigma_{0,1,n+1}, \quad t_{n+1} = t_{n+1}^{w} = t_{n+1}^{w_1(a_{[1\uparrow n-1]})w_2(t_{[1\uparrow n]}^{\phi_n})} = t_{n+1}^{w_2(t_{[1\uparrow n]}^{\phi_n})}.$$

Consider the homomorphism $\phi_{n+1}\colon \mathrm{Artin}\langle A_n\rangle \ltimes \Sigma_{0,1,n+1} \to \mathrm{Artin}\langle A_{n+1}\rangle$ determined by $\underset{=(a_i \quad \overline{a}_{n+1}^{\,2})}{\overset{i\in[1\uparrow n]}{(a_i \quad t_{n+1})^{\phi_{n+1}}}}$. Let $i \in [1\uparrow n]$. By Theorem 5.3(iii),

$$\begin{aligned}
(t_i^{\phi_n})^{\phi_{n+1}a_{n+1}} &= (\overline{a}_i^{\,2\Pi a_{[(i+1)\uparrow n]}})^{\phi_{n+1}a_{n+1}} = (\overline{a}_i^{\,2\Pi a_{[(i+1)\uparrow n]}})^{a_{n+1}} \\
&= (\overline{a}_i^{\,2\Pi a_{[(i+1)\uparrow(n+1)]}}) = (t_i)^{\phi_{n+1}}, \\
(t_{n+1})^{\phi_{n+1}a_{n+1}} &= (\overline{a}_{n+1}^{\,2})^{a_{n+1}} = \overline{a}_{n+1}^{\,2} = (t_{n+1})^{\phi_{n+1}}.
\end{aligned}$$

In particular, the two sequences $t_{[1\uparrow n]}^{\phi_n} \vee (t_{n+1})$ and $t_{[1\uparrow n+1]}$ (in $\mathrm{Artin}\langle A_n\rangle \ltimes \Sigma_{0,1,n+1}$) become conjugate (in $\mathrm{Artin}\langle A_{n+1}\rangle$) under ϕ_{n+1}. By Theorem 5.3(ii), ϕ_{n+1} is injective. Since $t_{[1\uparrow(n+1)]}$ freely generates the free subgroup $\Sigma_{0,1,n+1}$ of $\mathrm{Artin}\langle A_n\rangle \ltimes \Sigma_{0,1,n+1}$, we see that $t_{[1\uparrow n]}^{\phi_n} \vee (t_{n+1})$ also freely generates a free subgroup of $\mathrm{Artin}\langle A_n\rangle \ltimes \Sigma_{0,1,n+1}$. From (5.4.1), we see that w_2 must be trivial.

Hence, $w = w_1(a_{[1\uparrow(n-1)]})$ in $\text{Artin}\langle A_n\rangle$. By the induction hypothesis, $w_1(a_{[1\uparrow(n-1)]}) = 1$ in $\text{Artin}\langle A_{n-1}\rangle$. Hence $w = 1$ in $\text{Artin}\langle A_n\rangle$.

Now the result holds by induction. □

Combining Lemma 5.2, Theorem 5.3 and Theorem 5.4, we have the following.

5.5 Corollary (Artin-Magnus-Manfredini)**.** *If* $n \geq 2$*, then*

$$\begin{aligned}
\mathcal{B}_n &= \text{Artin}\langle \sigma_1 \text{ —— } \sigma_2 \text{ —— } \cdots \text{ —— } \sigma_{n-2} \text{ —— } \sigma_{n-1}\rangle \simeq \text{Artin}\langle A_{n-1}\rangle,\\
\text{Stab}(\mathcal{B}_n;[t_n]) &= \text{Artin}\langle \sigma_1 \text{ —— } \sigma_2 \text{ —— } \cdots \text{ —— } \sigma_{n-2} \text{ ══ } \sigma_{n-1}^2\rangle \simeq \text{Artin}\langle B_{n-1}\rangle,\\
\mathcal{B}_{n-1} \ltimes \Sigma_{0,1,n-1} &= \text{Artin}\langle \sigma_1 \text{ —— } \sigma_2 \text{ —— } \cdots \text{ —— } \sigma_{n-2} \text{ ══ } \overline{t}_{n-1}\rangle \simeq \text{Artin}\langle B_{n-1}\rangle.
\end{aligned}$$

□

5.6 Historical Remarks. In 1925, Artin [3] found the above presentation of $\mathcal{B}_n$ by an intuitive topological argument; later [4], he indicated that there were difficulties that could be corrected. In 1934, Magnus [24] gave an algebraic proof that the relations suffice. In 1945, Markov [26] gave a similar algebraic proof. In 1947, Bohnenblust [7] gave a similar algebraic proof; in 1948, Chow [8] simplified the latter proof. All these algebraic proofs of the sufficiency of the relations involve the Reidemeister-Schreier rewriting process for the subgroup of index n.

Larue [22] gave a new algebraic proof of the sufficiency of the relations, by using the σ_1-trichotomy [14] for braid groups. We shall proceed in the opposite direction. Proofs of the σ_1-trichotomy for $\text{Artin}\langle A_{n-1}\rangle$ have tended to be more difficult than proofs that $\text{Out}^+_{0,1,n} = \text{Artin}\langle A_{n-1}\rangle$, and we shall now see that Artin's generation argument easily gives the σ_1-trichotomy for $\text{Out}^+_{0,1,n}$. □

6 Three trichotomies

6.1 Definitions. Let $\phi \in \mathcal{B}_n$.

We say that ϕ is σ_1*-neutral* if ϕ lies in the subgroup of $\mathcal{B}_n$ generated by $\sigma_{[2\uparrow(n-1)]}$. This holds automatically if $n \leq 1$.

We say that ϕ is σ_1*-positive* if $n \geq 2$ and ϕ has a monoid expression in $\sigma_{[1\uparrow(n-1)]} \vee \overline{\sigma}_{[2\uparrow(n-1)]}$ such that at least one term of the expression is σ_1. We say that ϕ is σ*-positive* if $n \geq 2$ and, for some $i \in [1\uparrow(n-1)]$, ϕ has a monoid expression in $\sigma_{[i\uparrow(n-1)]} \vee \overline{\sigma}_{[(i+1)\uparrow(n-1)]}$ such that at least one term of the expression is σ_i.

We say that ϕ is σ_1*-negative* if $\overline{\phi}$ is σ_1-positive, that is, $n \geq 2$ and ϕ has a monoid expression in $\sigma_{[2\uparrow(n-1)]} \vee \overline{\sigma}_{[1\uparrow(n-1)]}$ such that at least one term of the expression is $\overline{\sigma}_1$.

If ϕ satisfies exactly one of the properties of being σ_1-neutral, σ_1-positive σ_1-negative, we say that ϕ *satisfies the* σ_1*-trichotomy.*

If every element of $\mathcal{B}_n$ satisfies the σ_1-trichotomy, then we say that $\mathcal{B}_n$ *satisfies the* σ_1*-trichotomy.* □

6.2 Historical Remarks. View Artin$\langle A_n\rangle$ as a subgroup of Artin$\langle A_{n+1}\rangle$ in a natural way, and let Artin$\langle A_\infty\rangle$ denote the union of the resulting chain; thus Artin$\langle A_\infty\rangle = \langle a_{[1\uparrow\infty[}\rangle$. Dehornoy [14, Theorem 6] gave a one-sided ordering of Artin$\langle A_\infty\rangle$; the positive semigroup for this ordering is the set of 'a-positive' elements of Artin$\langle A_\infty\rangle$.

Let $\phi \in \mathcal{B}_n$. By replacing ϕ with $\overline{\phi}$ if necessary, we can apply Dehornoy's result to deduce that there exists some $n' \geq n$ such that ϕ is σ-negative in $\mathcal{B}_{n'}$, or $\phi = 1$. Larue [21] showed that this implies that $t_1^\phi \in (t_1\star)$ and that this in turn implies that ϕ has a monoid expression in $\sigma_{[2\uparrow(n-1)]} \vee \overline{\sigma}_{[1\uparrow(n-1)]}$, of length at most $|\phi| + \frac{1}{4}n^2 3^{|\phi|}$. Thus, $\mathcal{B}_n$ satisfies the σ_1-trichotomy. Larue's work is surveyed in [16, Chapter 5]. Fenn-Greene-Rolfsen-Rourke-Wiest [19] gave a direct topological proof of the σ_1-trichotomy for $\mathcal{B}_n$ without being aware of Larue's work and without applying Dehornoy's result. Their work is surveyed in [16, Chapter 6].

We shall give elementary direct proofs of the foregoing results and replace Larue's bound $|\phi| + \frac{1}{4}n^2 3^{|\phi|}$ with the much smaller bound $n2^{|\phi|} - n$. Larue's proof contains much interesting information that we shall rework in the Appendix. □

Part (iii) of the following is new.

6.3 Lemma. *Let $n \geq 1$ and let ϕ be an element of $\mathcal{B}_n$ such that $t_1^\phi \in (t_1\star)$. Let $\pi = \pi(\phi)$ and, for each $i \in [1\uparrow n]$, let $u_i = u_i(\phi)$.*

(i). *Suppose that there exists some $i \in [1\uparrow(n-1)]$ such that $u_i \in (\star\overline{t}_{(i+1)^\pi})$. Then $\|\sigma_i\phi\| \leq \|\phi\| - 2$ and $t_1^{\sigma_i\phi} \in (t_1\star)$. Moreover, if $t_1^\phi = t_1$, then $i \in [2\uparrow(n-1)]$.*

(ii). *Suppose that there exists some $i \in [2\uparrow(n-1)]$ such that $u_i \in (\overline{t}_{i^\pi}\star)$. Then $\|\overline{\sigma}_i\phi\| \leq \|\phi\| - 2$ and $t_1^{\overline{\sigma}_i\phi} \in (t_1\star)$.*

(iii). *Suppose that, for each $i \in [1\uparrow(n-1)]$, $u_i \notin (\star\overline{t}_{(i+1)^\pi})$ and, for each $i \in [2\uparrow(n-1)]$, $u_i \notin (\overline{t}_{i^\pi}\star)$. Then $\phi = 1$.*

Proof. For each $i \in [0\uparrow(n+1)]$, let $w_i = w_i(\phi)$.

(i). The first conclusion follows from Artin's Lemma 3.2(i). Notice that, if $t_1^\phi = t_1$, then $w_1 = 1$ and $u_1 = \overline{w}_2 \notin (\star\overline{t}_{2^\pi})$.

(ii) follows from Lemma 3.2(ii).

(iii). Recall that $u_0 \prod\limits_{i\in[1\uparrow n]}(t_{i^\pi}u_i) = \prod\limits_{i\in[1\uparrow n]}(t_{i^\pi}^{w_i}) = (\prod\limits_{i\in[1\uparrow n]} t_i)^\phi = \prod\limits_{i\in[1\uparrow n]} t_i$. Hence, $u_0 t_{1^\pi} u_1 \prod\limits_{i\in[2\uparrow n]}(t_{i^\pi}u_i) = t_1 \prod\limits_{i\in[2\uparrow n]} t_i$, and, hence,

$$\text{(6.3.1)} \qquad |u_1 \prod_{i\in[2\uparrow n]}(t_{i^\pi}u_i)| = |\overline{t}_{1^\pi}\overline{u}_0 t_1 \prod_{i\in[2\uparrow n]} t_i| \leq |\overline{t}_{1^\pi}\overline{u}_0 t_1| + n - 1.$$

Since $u_n = w_n \notin (\bar{t}_{n^\pi}\star)$, the hypotheses imply that there is no cancellation anywhere in the expression $u_1 \prod\limits_{i\in[2\uparrow n]} (t_{i^\pi} u_i)$. Hence,

$$\text{(6.3.2)} \qquad \sum_{i\in[1\uparrow n]} |u_i| + n - 1 = |u_1 \prod_{i\in[2\uparrow n]} (t_{i^\pi} u_i)| \overset{(6.3.1)}{\leq} |\bar{t}_{1^\pi}\overline{u}_0 t_1| + n - 1.$$

Since $t_{1^\pi}^{\overline{u}_0} = t_{1^\pi}^{w_1} = t_1^\phi \in (t_1\star)$, we see that $u_0 t_{1^\pi} \in (t_1\star)$, and

$$\text{(6.3.3)} \qquad |\bar{t}_1 u_0 t_{1^\pi}| = -1 + |u_0 t_{1^\pi}| \leq -1 + |u_0| + 1 = |u_0|.$$

Since $\prod u_{[0\uparrow n]} = w_0 \overline{w}_{n+1} = 1$, we see that

$$\text{(6.3.4)} \qquad \prod u_{[1\uparrow n]} = \overline{u}_0 = w_1 \notin (\bar{t}_{1^\pi}\star).$$

Now, $\sum\limits_{i\in[1\uparrow n]} |u_i| \overset{(6.3.2)}{\leq} |\bar{t}_{1^\pi}\overline{u}_0 t_1| \overset{(6.3.3)}{\leq} |\overline{u}_0| \overset{(6.3.4)}{=} |\prod u_{[1\uparrow n]}|$. Therefore, there is no cancellation in $\prod u_{[1\uparrow n]}$, and, by (6.3.4), $u_1 \notin (\bar{t}_{1^\pi}\star)$. By Lemma 3.2(iii), $\phi = 1$. □

As in Remarks 3.4, we deduce the following from Lemma 6.3 by induction on $\|\phi\|$.

6.4 Corollary (Larue [21]). *Let $n \geq 1$ and let $\phi \in \mathcal{B}_n$.*

(i). *If $t_1^\phi \in (t_1\star)$, then ϕ has a monoid expression in $\sigma_{[2\uparrow(n-1)]} \vee \overline{\sigma}_{[1\uparrow(n-1)]}$ of length at most $\frac{\|\phi\|-n}{2} \leq n2^{|\phi|} - n$. In particular, ϕ is σ_1-negative or σ_1-neutral.*

(ii). *ϕ is σ_1-neutral if and only if $t_1^\phi = t_1$.* □

6.5 Notation. For each $i \in [1\uparrow(n-1)]$, let σ_i' and σ_i'' be the automorphisms of $\Sigma_{0,1,n}$ determined by

$$\begin{array}{llll} & \overset{k\in[1\uparrow i]}{\overline{}} & & \overset{k\in[(i+2)\uparrow n]}{\overline{}} \\ & (t_k & t_{i+1} & t_k)^{\sigma_i'} \\ = & (t_k & t_{i+1}^{t_i} & t_k), \end{array} \qquad \begin{array}{lllll} & \overset{k\in[1\uparrow(i-1)]}{\overline{}} & & & \overset{k\in[(i+1)\uparrow n]}{\overline{}} \\ & (t_k & t_i & t_{i+1} & t_k)^{\sigma_i''} \\ = & (t_k & t_{i+1} & t_i & t_k). \end{array}$$

Then $\sigma_i = \sigma_i'\sigma_i''$. Any normal form in $t_{[1\uparrow n]}$ factorizes into an alternating product with factors which are normal forms of non-trivial elements of $\langle t_{[i\uparrow(i+1)]}\rangle$ alternating with factors which are normal forms of non-trivial elements of $\langle t_{[1\uparrow(i-1)]\vee[(i+2)\uparrow n]}\rangle$. On $\langle t_{[i\uparrow(i+1)]}\rangle$, σ_i' acts as conjugation by t_i, while σ_i'' interchanges the two free generators. On $\langle t_{[1\uparrow(i-1)]\vee[(i+2)\uparrow n]}\rangle$, σ_i' and σ_i'' act as the identity map. □

The next result gives three trichotomies, called (a), (b) and (c), which hold for elements of $\mathcal{B}_n$. Attribution is not sharply defined, but it is reasonable to attribute (b) to Dehornoy [14], and (c) to Larue [21].

6.6 Theorem (Dehornoy-Larue [14], [21]). *Let $n \geq 1$, let $\phi \in \mathcal{B}_n$, and consider the following nine assertions.*

(a1). $t_1^{\phi} = t_1$. (a2). $t_1^{\phi} \in (t_1\star) - \{t_1\}$. (a3). $t_1^{\phi} \notin (t_1\star)$.

(b1). *ϕ is σ_1-neutral.* (b2). *ϕ is σ_1-negative.* (b3). *ϕ is σ_1-positive.*

(c1). $(t_1\star)^{\phi} = (t_1\star)$ (c2). $(t_1\star)^{\phi} \subset (t_1\star)$. (c3). $(t_1\star)^{\phi} \supset (t_1\star)$.

Then the following column-equivalences hold:

(a1) $\Leftrightarrow$ (b1) $\Leftrightarrow$ (c1); (a2) $\Leftrightarrow$ (b2) $\Leftrightarrow$ (c2); (a3) $\Leftrightarrow$ (b3) $\Leftrightarrow$ (c3).

Hence, exactly one of (b1), (b2), (b3), *holds; that is, ϕ satisfies the σ_1-trichotomy. Hence, $\mathcal{B}_n$ satisfies the σ_1-trichotomy.*

Proof. (a1) $\Leftrightarrow$ (b1) by Corollary 6.4(ii). We shall use (a1) and (b1) interchangeably in the remainder of the proof.

(b1) $\Rightarrow$ (c1). If ϕ is σ_1-neutral, then so is $\overline{\phi}$. It follows that $(t_1\star)^{\phi} \subseteq (t_1\star)$ and $(t_1\star)^{\overline{\phi}} \subseteq (t_1\star)$. Thus, $(t_1\star)^{\phi} = (t_1\star)$.

(a2) $\Rightarrow$ (b2). If (a2) holds, then Corollary 6.4(i) shows that (b1) or (b2) holds. Since (a1) fails, (b1) fails. Thus (b2) holds.

(b2) $\Rightarrow$ (c2). Using Notation 6.5, we see that

$$(t_1\star)^{\overline{\sigma}_1} = (t_1\star)^{\overline{\sigma}_1''\overline{\sigma}_1'} = (t_2\star)^{\overline{\sigma}_1'} \subseteq (t_1t_2\star) \subset (t_1\star).$$

Since the composition of injective self-maps of $(t_1\star)$ can be bijective only if all the factors are bijective, we see that (b2) $\Rightarrow$ (c2).

(a3) $\Rightarrow$ (b3). We translate into algebra the crucial reflection argument of [16, Corollary 5.2.4].

Suppose that (a3) holds.

With Notation 3.1, let $w_1 = w_1(\phi)$ and $\pi = \pi(\phi)$. Then $\overline{w}_1 t_{1^\pi} w_1 = t_1^{\phi} \notin (t_1\star)$. It follows that $\overline{w}_1 t_{1^\pi} \notin (t_1\star)$. Hence, $\overline{w}_1\,\overline{t}_{1^\pi} \notin (t_1\star)$. Hence, $\overline{t}_1^{\phi} = \overline{w}_1\,\overline{t}_{1^\pi} w_1 \notin (t_1\star) \cup \{1\}$. On conjugating by t_1, we see that $\overline{t}_1^{\phi t_1} \in (\overline{t}_1\star)$.

Let ζ be the automorphism of $\Sigma_{0,1,n}$ determined by $\Big(\underset{k\in[1\uparrow n]}{t_k}\Big)^{\zeta} = \Big(\overline{t}_k^{\,\Pi\overline{t}_{[(k-1)\downarrow 1]}}\Big)$.

For each $k \in [1\uparrow n]$, $(\Pi t_{[1\uparrow k]})^{\zeta} = \Pi\overline{t}_{[k\downarrow 1]}$. It follows that $\zeta^2 = 1$. Notice that ζ belongs to $\mathrm{Out}_{0,1,n}^{-} := \mathrm{Out}_{0,1,n} - \mathrm{Out}_{0,1,n}^{+}$. Also, $\Big(t_1 \;\; \underset{k\in[2\uparrow n]}{t_k}\Big)^{\overline{t}_1\zeta} = \Big(\overline{t}_1 \;\; \overline{t}_k^{\,\Pi\overline{t}_{[(k-1)\downarrow 2]}}\Big)$. Hence,

$$t_1^{\phi^{\zeta}} = t_1^{\zeta\phi\zeta} = \overline{t}_1^{\phi t_1 \overline{t}_1 \zeta} \in (\overline{t}_1\star)^{\overline{t}_1\zeta} \subseteq (t_1\star).$$

By Corollary 6.4(i), ϕ^{ζ} has a monoid expression in $\sigma_{[2\uparrow(n-1)]} \vee \overline{\sigma}_{[1\uparrow(n-1)]}$. It is not difficult to check that, for each $i \in [1\uparrow(n-1)]$, $\sigma_i^{\zeta} = \overline{\sigma}_i$ in $\mathrm{Out}_{0,1,n}$. Hence

$\phi^{\zeta^2}(=\phi)$ has a monoid expression in $\sigma^{\zeta}_{[2\uparrow(n-1)]} \vee \overline{\sigma}^{\zeta}_{[1\uparrow(n-1)]}(=\overline{\sigma}_{[2\uparrow(n-1)]} \vee \sigma_{[1\uparrow(n-1)]})$. Hence, (b3) or (b1) holds. Since (a3) holds, (a1) fails, and (b1) fails. Thus (b3) holds.

(b3) $\Rightarrow$ (c3). If ϕ is σ_1-positive, then $\overline{\phi}$ is σ_1-negative, and, by (b2) $\Rightarrow$ (c2), $(t_1\star)^{\overline{\phi}} \subset (t_1\star)$ and, hence, $(t_1\star) \subset (t_1\star)^{\phi}$.

(c1) $\Rightarrow$ (a1). Suppose that (a1) fails. Then (a2) or (a3) holds. Hence (c2) or (c3) holds. Hence (c1) fails.

(c2) $\Rightarrow$ (a2) and (c3) $\Rightarrow$ (a3) are proved similarly.

Thus the desired equivalences hold.

Since exactly one of (a1), (a2), (a3) holds, exactly one of (b1), (b2), (b3) holds. □

Recall the definition of σ-positive from Definitions 6.1.

6.7 Theorem (Dehornoy [14]). *For each $\phi \in \mathcal{B}_n$ exactly one of the following holds: $\phi = 1$; ϕ is σ-positive; ϕ is σ-negative. The set of σ-positive elements of $\mathcal{B}_n$ is the positive cone of a right-ordering of $\mathcal{B}_n$, called* the Dehornoy right-ordering of $\mathcal{B}_n$.

Proof. Suppose that $\phi \neq 1$.

Let i be the largest element of $[1\uparrow(n-1)]$ such that $\phi \in \langle \sigma_{[i\uparrow(n-1)]}\rangle$. The natural subscript-shifting isomorphism from $\langle t_{[i\uparrow n]}\rangle$ to $\Sigma_{0,1,n-i+1}$ induces an isomorphism from $\langle \sigma_{[i\uparrow(n-1)]}\rangle$ to $\mathcal{B}_{n-i+1}$. Notice that ϕ is mapped to an element of $\mathcal{B}_{n-i+1}$ which is not σ_1-neutral; by Theorem 6.6, this image is σ_1-positive or σ_1-negative but not both. Hence exactly one of ϕ, $\overline{\phi}$ is σ-positive.

It is easy to see that the product of two σ-positive elements of $\mathcal{B}_n$ is σ-positive.

Hence the set of σ-positive elements of $\mathcal{B}_n$ is the positive cone for a right-ordering of $\mathcal{B}_n$. □

7 Ends, right-orderings and squarefreeness

7.1 Review. An *end* of $\Sigma_{0,1,n}$ is a sequence $a_{[1\uparrow\infty[}$ in $t_{[1\uparrow n]} \vee \overline{t}_{[1\uparrow n]}$ such that, for each $i \in [1\uparrow\infty[$, $a_{i+1} \neq \overline{a}_i$. We represent $a_{[1\uparrow\infty[}$ as a formal right-infinite reduced product, $a_1a_2\cdots$ or $\Pi a_{[1\uparrow\infty[}$.

We denote the set of ends of $\Sigma_{0,1,n}$ by $\mathfrak{E}(\Sigma_{0,1,n})$, or simply by $\mathfrak{E}$ if there is no risk of confusion.

An element of $\Sigma_{0,1,n} \cup \mathfrak{E}(\Sigma_{0,1,n})$ is said to be *squarefree* if, in its reduced expression, no two consecutive terms are equal; for example: $(t_1t_2)^\infty$ is a squarefree end; $t_1t_2t_2t_3$ is non-squarefree.

For each $w \in \Sigma_{0,1,n}$, we define the *shadow* of w in $\mathfrak{E}$ to be

$$(w\blacktriangleleft) := \{a_{[1\uparrow\infty[} \in \mathfrak{E} \mid \ \Pi a_{[1\uparrow|w|]} = w\}.$$

Thus, for example, $(1\blacktriangleleft) = \mathfrak{E}$.

We now give $\mathfrak{E}$ an ordering, $<$, as follows. For each $w \in \Sigma_{0,1,n}$, we assign an ordering, $<$, to a partition of $(w\blacktriangleleft)$ into $2n$ or $2n-1$ subsets, depending as $w = 1$ or $w \neq 1$, as follows. We set

$$(t_1\blacktriangleleft) < (\bar{t}_1\blacktriangleleft) < (t_2\blacktriangleleft) < (\bar{t}_2\blacktriangleleft) < \cdots < (t_n\blacktriangleleft) < (\bar{t}_n\blacktriangleleft).$$

If $i \in [1{\uparrow}n]$ and $w \in (\star\bar{t}_i)$, then we set

$$\begin{aligned}(w\bar{t}_i\blacktriangleleft) &< (wt_{i+1}\blacktriangleleft) < (w\bar{t}_{i+1}\blacktriangleleft) < (wt_{i+2}\blacktriangleleft) < (w\bar{t}_{i+2}\blacktriangleleft) < \cdots \\ &\cdots < (wt_n\blacktriangleleft) < (w\bar{t}_n\blacktriangleleft) < (wt_1\blacktriangleleft) < (w\bar{t}_1\blacktriangleleft) < (wt_2\blacktriangleleft) < \cdots \\ &\cdots < (wt_{i-1}\blacktriangleleft) < (w\bar{t}_{i-1}\blacktriangleleft).\end{aligned}$$

If $i \in [1{\uparrow}n]$ and $w \in (\star t_i)$, then we set

$$\begin{aligned}(wt_{i+1}\blacktriangleleft) &< (w\bar{t}_{i+1}\blacktriangleleft) < (wt_{i+2}\blacktriangleleft) < (w\bar{t}_{i+2}\blacktriangleleft) < \cdots \\ &\cdots < (wt_n\blacktriangleleft) < (w\bar{t}_n\blacktriangleleft) < (wt_1\blacktriangleleft) < (w\bar{t}_1\blacktriangleleft) < (wt_2\blacktriangleleft) < \cdots \\ &\cdots < (wt_{i-1}\blacktriangleleft) < (w\bar{t}_{i-1}\blacktriangleleft) < (wt_i\blacktriangleleft).\end{aligned}$$

Hence, for each $w \in \Sigma_{0,1,n}$, we have an ordering $<$ of a partition of $(w\blacktriangleleft)$ into $2n$ or $2n-1$ subsets.

If $a_{[1\uparrow\infty[}$ and $b_{[1\uparrow\infty[}$ are two different ends, then there exists $i \in \mathbb{N}$ such that $a_{[1\uparrow i]} = b_{[1\uparrow i]}$ and $a_{i+1} \neq b_{i+1}$. Let $w = \Pi a_{[1\uparrow i]} = \Pi b_{[1\uparrow i]}$ in $\Sigma_{0,1,n}$. Then $a_{[1\uparrow\infty[}$ and $b_{[1\uparrow\infty[}$ lie in $(w\blacktriangleleft)$, but lie in different elements of the partition of $(w\blacktriangleleft)$ into $2n$ or $2n-1$ subsets. We then order $a_{[1\uparrow\infty[}$ and $b_{[1\uparrow\infty[}$ using the order of the elements of the partition of $(w\blacktriangleleft)$ that they belong to. This completes the definition of the ordering $<$ of $\mathfrak{E}$.

We remark that the smallest element of $\mathfrak{E}$ is $\bar{z}_1^\infty = (\Pi t_{[1\uparrow n]})^\infty$ and the largest element of $\mathfrak{E}$ is $z_1^\infty = (\Pi \bar{t}_{[n\downarrow 1]})^\infty$. □

7.2 Review. By work of Nielsen-Thurston [9], [29], there is an order-preserving action of $\mathcal{B}_n$ on $(\mathfrak{E}(\Sigma_{0,1,n}), \leq)$; we shall give an elementary version of this result.

We assume that $n \geq 2$, and we first define the action of σ_1 on $\mathfrak{E}$.

Consider any $\mathfrak{e} \in \mathfrak{E}$. There is then a unique factorization $\mathfrak{e} = \Pi w_{[1\uparrow i]}$ or $\mathfrak{e} = \Pi w_{[1\uparrow\infty[}$, where, in the former case, $w_{[1\uparrow(i-1)]}$ is a finite sequence of non-trivial group elements, and w_i is an end, and, in the latter case, $w_{[1\uparrow\infty[}$ is an infinite sequence of non-trivial group elements, and in both cases, the w_j alternate between elements of $\langle t_{[1\uparrow 2]}\rangle \cup \mathfrak{E}(\langle t_{[1\uparrow 2]}\rangle)$, and elements of $\langle t_{[3\uparrow n]}\rangle \cup \mathfrak{E}(\langle t_{[3\uparrow n]}\rangle)$. We shall express this factorization as $\mathfrak{e} = [w_1][w_2]\cdots$.

Recall, from Notation 6.5, that we have the factorization $\sigma_1 = \sigma_1'\sigma_1''$. On $\langle t_{[1\uparrow 2]}\rangle \cup \mathfrak{E}(\langle t_{[1\uparrow 2]}\rangle)$, σ_1' acts as conjugation by t_1, while σ_1'' interchanges the two free generators. On $\langle t_{[3\uparrow n]}\rangle \cup \mathfrak{E}(\langle t_{[3\uparrow n]}\rangle)$, σ_1' and σ_1'' act as the identity map. This completes the description of the action of σ_1', σ_1'' and σ_1 on $\mathfrak{E}$.

It is not difficult to show that, for any ends $a_{[1\uparrow\infty[}$ and $b_{[1\uparrow\infty[}$, if $(a_{[1\uparrow\infty[})^{\sigma_1} = b_{[1\uparrow\infty[}$, then for all $i, j \in \mathbb{N}$, if $j \geq 2i$, then $(\Pi a_{[1\uparrow j]})^{\sigma_1} \in (\Pi b_{[1\uparrow i]}\star)$. Thus, $(a_{[1\uparrow\infty[})^{\sigma_1} = \lim\limits_{j\to\infty}((\Pi a_{[1\uparrow j]})^{\sigma_1})$.

It is clear that σ_1', σ_1'' and, hence, σ_1 act bijectively on $\mathfrak{E}$. Hence we have the action of $\overline{\sigma}_1$ on $\mathfrak{E}$. It is then not difficult to verify that we have an action of $\mathcal{B}_n$ on $\mathfrak{E}$.

We next show that σ_1 respects the ordering of $\mathfrak{E}$. We do this by considering all the ways that two ends can be compared, and the resulting effect of σ_1' and σ_1. We represent the information in tables. In all of the following, we understand that t_1a, $\bar t_1 b$, t_2c, and $\bar t_2 d$ are reduced expressions for elements of $\langle t_{[1\uparrow 2]}\rangle \cup \mathfrak{E}(\langle t_{[1\uparrow 2]}\rangle)$, and $b \neq 1$. Since a does not begin with $\bar t_1$, $a^{\sigma_1''}t_2$ begins with t_1 or $\bar t_1$ or t_2. We make the convention that $\Sigma_{0,1,n}$ acts trivially on the right on $\mathfrak{E}$.

$(\cdots][wt_1\blacktriangleleft)$	$(\cdots][wt_1\blacktriangleleft)^{\sigma_1'}$	$(\cdots][wt_1\blacktriangleleft)^{\sigma_1}$
$\cdots][wt_1\ \ t_2c][\cdots$	$\cdots][(\bar t_1 w)t_1\ \ t_2(ct_1)][\cdots$	$\cdots][(\bar t_2 w^{\sigma_1''})t_2\ \ t_1(c^{\sigma_1''}t_2)][\cdots$
$\cdots][wt_1\ \ \bar t_2 d][\cdots$	$\cdots][(\bar t_1 w)t_1\ \ \bar t_2(dt_1)][\cdots$	$\cdots][(\bar t_2 w^{\sigma_1''})t_2\ \ \bar t_1(d^{\sigma_1''}t_2)][\cdots$
$\cdots][wt_1][t_3\uparrow\bar t_n\cdots$	$\cdots][(\bar t_1 w)t_1\ \ t_1][t_3\uparrow\bar t_n\cdots$	$\cdots][(\bar t_2 w^{\sigma_1''})t_2\ \ t_2][t_3\uparrow\bar t_n\cdots$
$\cdots][wt_1\ \ t_1a][\cdots$	$\cdots][(\bar t_1 w)t_1\ \ t_1(at_1)][\cdots$	$\cdots][(\bar t_2 w^{\sigma_1''})t_2\ \ t_2(a^{\sigma_1''}t_2)][\cdots$

Here, the case $w = 1$ does not present any problems.

$(\cdots][w\bar t_1\blacktriangleleft)$	$(\cdots][w\bar t_1\blacktriangleleft)^{\sigma_1'}$	$(\cdots][w\bar t_1\blacktriangleleft)^{\sigma_1}$
$\cdots][w\bar t_1\ \ \bar t_1 b][\cdots$	$\cdots][(\bar t_1 w)\ \ \bar t_1\bar t_1(bt_1)][\cdots$	$\cdots][(\bar t_2 w^{\sigma_1''})\ \ \bar t_2\bar t_2(b^{\sigma_1''}t_2)][\cdots$
$\cdots][w\bar t_1\ \ \bar t_1][t_3\uparrow\bar t_n\cdots$	$\cdots][(\bar t_1 w)\ \ \bar t_1][t_3\uparrow\bar t_n\cdots$	$\cdots][(\bar t_2 w^{\sigma_1''})\ \ \bar t_2][t_3\uparrow\bar t_n\cdots$
$\cdots][w\bar t_1\ \ t_2c][\cdots$	$\cdots][(\bar t_1 w)\ \ \bar t_1 t_2(ct_1)][\cdots$	$\cdots][(\bar t_2 w^{\sigma_1''})\ \ \bar t_2 t_1(c^{\sigma_1''}t_2)][\cdots$
$\cdots][w\bar t_1\ \ \bar t_2 d][\cdots$	$\cdots][(\bar t_1 w)\ \ \bar t_1\bar t_2(dt_1)][\cdots$	$\cdots][(\bar t_2 w^{\sigma_1''})\ \ \bar t_2\bar t_1(d^{\sigma_1''}t_2)][\cdots$
$\cdots][w\bar t_1][t_3\uparrow\bar t_n\cdots$	$\cdots][(\bar t_1 w)][t_3\uparrow\bar t_n\cdots$	$\cdots][(\bar t_2 w^{\sigma_1''})][t_3\uparrow\bar t_n\cdots$

Here, w does not end with t_1, and, hence, $(\bar t_2 w^{\sigma_1''})$ ends with t_1, $\bar t_1$ or $\bar t_2$.

$(\cdots][wt_2\blacktriangleleft)$	$(\cdots][wt_2\blacktriangleleft)^{\sigma_1'}$	$(\cdots][wt_2\blacktriangleleft)^{\sigma_1}$
$\cdots][wt_2][t_3\uparrow\bar t_n\cdots$	$\cdots][(\bar t_1 w)t_2\ \ t_1][t_3\uparrow\bar t_n\cdots$	$\cdots][(\bar t_2 w^{\sigma_1''})t_1\ \ t_2][t_3\uparrow\bar t_n\cdots$
$\cdots][wt_2\ \ t_1a][\cdots$	$\cdots][(\bar t_1 w)t_2\ \ t_1(at_1)][\cdots$	$\cdots][(\bar t_2 w^{\sigma_1''})t_1\ \ t_2(a^{\sigma_1''}t_2)][\cdots$
$\cdots][wt_2\ \ \bar t_1 b][\cdots$	$\cdots][(\bar t_1 w)t_2\ \ \bar t_1(bt_1)][\cdots$	$\cdots][(\bar t_2 w^{\sigma_1''})t_1\ \ \bar t_2(b^{\sigma_1''}t_2)][\cdots$
$\cdots][wt_2\ \ \bar t_1][t_3\uparrow\bar t_n\cdots$	$\cdots][(\bar t_1 w)t_2][t_3\uparrow\bar t_n\cdots$	$\cdots][(\bar t_2 w^{\sigma_1''})t_1][t_3\uparrow\bar t_n\cdots$
$\cdots][wt_2\ \ t_2c][\cdots$	$\cdots][(\bar t_1 w)t_2\ \ t_2(ct_1)][\cdots$	$\cdots][(\bar t_2 w^{\sigma_1''})t_1\ \ t_1(c^{\sigma_1''}t_2)][\cdots$

$(\cdots][w\bar t_2\blacktriangleleft)$	$(\cdots][w\bar t_2\blacktriangleleft)^{\sigma_1'}$	$(\cdots][w\bar t_2\blacktriangleleft)^{\sigma_1}$
$\cdots][w\bar t_2\ \ \bar t_2 d][\cdots$	$\cdots][(\bar t_1 w)\bar t_2\ \ \bar t_2(dt_1)][\cdots$	$\cdots][(\bar t_2 w^{\sigma_1''})\bar t_1\ \ \bar t_1(d^{\sigma_1''}t_2)][\cdots$
$\cdots][w\bar t_2][t_3\uparrow\bar t_n\cdots$	$\cdots][(\bar t_1 w)\bar t_2\ \ t_1][t_3\uparrow\bar t_n\cdots$	$\cdots][(\bar t_2 w^{\sigma_1''})\bar t_1\ \ t_2][t_3\uparrow\bar t_n\cdots$
$\cdots][w\bar t_2\ \ t_1a][\cdots$	$\cdots][(\bar t_1 w)\bar t_2\ \ t_1(at_1)][\cdots$	$\cdots][(\bar t_2 w^{\sigma_1''})\bar t_1\ \ t_2(a^{\sigma_1''}t_2)][\cdots$
$\cdots][w\bar t_2\ \ \bar t_1 b][\cdots$	$\cdots][(\bar t_1 w)\bar t_2\ \ \bar t_1(bt_1)][\cdots$	$\cdots][(\bar t_2 w^{\sigma_1''})\bar t_1\ \ \bar t_2(b^{\sigma_1''}t_2)][\cdots$
$\cdots][w\bar t_2\ \ \bar t_1][t_3\uparrow\bar t_n\cdots$	$\cdots][(\bar t_1 w)\bar t_2][t_3\uparrow\bar t_n\cdots$	$\cdots][(\bar t_2 w^{\sigma_1''})\bar t_1][t_3\uparrow\bar t_n\cdots$

$(\cdots t_3◀)$	$(\cdots t_3◀)^{\sigma_1'}$	$(\cdots t_3◀)^{\sigma_1}$
$\cdots t_3\ \ t_4\!\uparrow\!\bar{t}_n\cdots$	$\cdots t_3\ \ t_4\!\uparrow\!\bar{t}_n\cdots$	$\cdots t_3\ \ t_4\!\uparrow\!\bar{t}_n\cdots$
$\cdots t_3][t_1a][\cdots$	$\cdots t_3][(at_1)][\cdots$	$\cdots t_3][(a^{\sigma_1''}t_2)][\cdots$
$\cdots t_3][\bar{t}_1b][\cdots$	$\cdots t_3][\bar{t}_1\bar{t}_1(bt_1)][\cdots$	$\cdots t_3][\bar{t}_2\bar{t}_2(b^{\sigma_1''}t_2)][\cdots$
$\cdots t_3][\bar{t}_1][t_3\!\uparrow\!\bar{t}_n\cdots$	$\cdots t_3][\bar{t}_1][t_3\!\uparrow\!\bar{t}_n\cdots$	$\cdots t_3][\bar{t}_2][t_3\!\uparrow\!\bar{t}_n\cdots$
$\cdots t_3][t_2c][\cdots$	$\cdots t_3][\bar{t}_1t_2(ct_1)][\cdots$	$\cdots t_3][\bar{t}_2t_1(c^{\sigma_1''}t_2)][\cdots$
$\cdots t_3][\bar{t}_2d][\cdots$	$\cdots t_3][\bar{t}_1\bar{t}_2(dt_1)][\cdots$	$\cdots t_3][\bar{t}_2\bar{t}_1(d^{\sigma_1''}t_2)][\cdots$
$\cdots t_3\ \ t_3\cdots$	$\cdots t_3\ \ t_3\cdots$	$\cdots t_3\ \ t_3\cdots$

The remaining tables are clearly of the same form as the last one. Thus we have proved that the action of σ_1 respects the ordering of $\mathfrak{E}$. It follows that the action of $\overline{\sigma}_1$ respects the ordering of $\mathfrak{E}$. Similarly, the action of $\sigma_{[2\uparrow(n-1)]}\vee\overline{\sigma}_{[2\uparrow(n-1)]}$ respects the ordering of $\mathfrak{E}$. Hence $\mathcal{B}_n$ acts on $(\mathfrak{E},\le)$. □

7.3 Remarks (Thurston [29])**.** The (right) action of $\mathcal{B}_n$ on $(\mathfrak{E},\le)$ gives rise to many right orderings of $\mathcal{B}_n$.

Let us use the left-to-right lexicographic ordering on $(\mathfrak{E}^n,\le)$, and consider the $\mathcal{B}_n$-orbit of $t^\infty_{[1\uparrow n]} := (t_i^\infty)_{i\in[1\uparrow n]}$. It is not difficult to show that the $\mathcal{B}_n$-stabilizer of $t^\infty_{[1\uparrow n]}$ is trivial. Thus we have an injective map

$$\mathcal{B}_n\to\mathfrak{E}^n,\qquad \phi\mapsto t^{\infty\phi}_{[1\uparrow n]} := ((t_i^\infty)^\phi)_{i\in[1\uparrow n]}.$$

Let $\le$ denote the ordering of $\mathcal{B}_n$ induced by pullback from $\mathfrak{E}^n$. Clearly $\le$ is a right-ordering of $\mathcal{B}_n$.

If $n\ge 2$ and $\phi\in\mathcal{B}_n$ is σ_1-negative, then, as in the proof of Theorem 6.6(b2)⇒(c2), we have $(t_1◀)^\phi\subset(t_1◀)$. Since $\max(t_1◀)=t_1^\infty$ and ϕ respects the ordering, we see that $(t_1^\infty)^\phi<t_1^\infty$. Hence $\phi<1$ and $1<\overline{\phi}$. Similar arguments with $(t_i◀)$, $i\in[2\uparrow n]$, show that, if $\phi\in\mathcal{B}_n$ is σ-positive (resp. σ-negative), then $1<\phi$ (resp. $1>\phi$). Hence the right-ordering of $\mathcal{B}_n$ obtained from $(t^\infty_{[1\uparrow n]})^{\mathcal{B}_n}\subseteq(\mathfrak{E}^n,\le)$ coincides with the Dehornoy right-ordering. □

The following will be useful in the study of squarefreeness.

7.4 Lemma. *Let $n\ge 1$, let $i\in[1\uparrow n]$, and let $w\in\Sigma_{0,1,n}-(\star t_i)-(\star\bar{t}_i)$. Then, in $(\mathfrak{E}(\Sigma_{0,1,n}),\le)$, the following hold:*

(i). $wt_i\overline{w}((\Pi t_{[1\uparrow n]})^\infty)\le wt_i((\Pi t_{[i\uparrow n]\vee[1\uparrow i-1]})^\infty)=\min(wt_it_i◀)$;

(ii). $\min(wt_it_i◀)<\max(w\bar{t}_i\bar{t}_i◀)$;

(iii). $\max(w\bar{t}_i\bar{t}_i◀)=w\bar{t}_i((\Pi\bar{t}_{[i\downarrow 1]\vee[n\downarrow i+1]})^\infty)\le w\bar{t}_i\overline{w}((\Pi\bar{t}_{[n\downarrow 1]})^\infty)$;

(iv). $(wt_it_i◀)\cup(w\bar{t}_i\bar{t}_i◀)\subseteq[(wt_i\overline{w}((\Pi t_{[1\uparrow n]})^\infty))\uparrow(w\bar{t}_i\overline{w}((\Pi\bar{t}_{[n\downarrow 1]})^\infty))]$.

(v). *If $n\ge 3$, then one of the following holds:*

(a). $t_1((\Pi\bar{t}_{[n\downarrow 1]})^\infty)\ <\ wt_i\overline{w}((\Pi t_{[1\uparrow n]})^\infty)$;
(b). $t_1((\Pi\bar{t}_{[n\downarrow 1]})^\infty)\ >\ w\bar{t}_i\overline{w}((\Pi\bar{t}_{[n\downarrow 1]})^\infty)$;

and, hence, $t_1((\Pi\bar{t}_{[n\downarrow 1]})^\infty) \notin [(wt_i\overline{w}((\Pi t_{[1\uparrow n]})^\infty))\uparrow(w\bar{t}_i\overline{w}((\Pi\bar{t}_{[n\downarrow 1]})^\infty))]$*, that is,* $t_1(z_1^\infty) \notin [(wt_i\overline{w}(\overline{z}_1^\infty))\uparrow(w\bar{t}_i\overline{w}(z_1^\infty))]$

Proof. Recall that:

$$(t_1◀) < (\bar{t}_1◀) < (t_2◀) < \cdots < (t_n◀) < (\bar{t}_n◀),$$
$$(t_it_{i+1}◀) < (t_i\bar{t}_{i+1}◀) < \cdots < (t_i\bar{t}_n◀) < (t_it_1◀) < \cdots < (t_i\bar{t}_{i-1}◀) < (t_it_i◀),$$
$$(\bar{t}_i\bar{t}_i◀) < (\bar{t}_it_{i+1}◀) < \cdots < (\bar{t}_i\bar{t}_n◀) < (\bar{t}_it_1◀) < \cdots < (\bar{t}_it_{i-1}◀) < (\bar{t}_i\bar{t}_{i-1}◀).$$

(i). It is straightforward to see that $wt_i((\Pi t_{[i\uparrow n]\vee[1\uparrow i-1]})^\infty) = \min(wt_it_i◀)$.

Let x denote the element of $t_{[1\uparrow n]} \vee \bar{t}_{[1\uparrow n]}$ such that $\overline{w}((\Pi t_{[1\uparrow n]})^\infty) \in (x◀)$; notice that $x \neq \bar{t}_i$.

If $x \neq t_i$, then $(wt_ix◀) < (wt_it_i◀)$, and we have

$$wt_i\overline{w}((\Pi t_{[1\uparrow n]})^\infty) \in (wt_ix◀) < (wt_it_i◀) \ni \min(wt_it_i◀).$$

If $x = t_i$, then $\overline{w}$ is completely cancelled in $\overline{w}((\Pi t_{[1\uparrow n]})^\infty)$, and, moreover,

$$wt_i\overline{w}((\Pi t_{[1\uparrow n]})^\infty) = wt_i((\Pi t_{[i\uparrow n]}\Pi t_{[1\uparrow i-1]})^\infty) = \min(wt_it_i◀).$$

Thus, (i) holds.

(ii) is clear.

(iii). It is straightforward to see that $w\bar{t}_i((\Pi\bar{t}_{[i\downarrow 1]\vee[n\downarrow i+1]})^\infty) = \max(w\bar{t}_i\bar{t}_i◀)$.

Let x denote the element of $t_{[1\uparrow n]} \vee \bar{t}_{[1\uparrow n]}$ such that $\overline{w}((\Pi\bar{t}_{[n\downarrow 1]})^\infty) \in (x◀)$; notice that $x \neq t_i$.

If $x \neq \bar{t}_i$, then $(w\bar{t}_i\bar{t}_i◀) < (w\bar{t}_ix◀)$, and we have

$$\max(w\bar{t}_i\bar{t}_i◀) \in (w\bar{t}_i\bar{t}_i◀) < (w\bar{t}_ix◀) \ni w\bar{t}_i\overline{w}((\Pi\bar{t}_{[n\downarrow 1]})^\infty).$$

If $x = \bar{t}_i$, then $\overline{w}$ is completely cancelled in $\overline{w}(\Pi\bar{t}_{[n\downarrow 1]})^\infty$, and, moreover,

$$w\bar{t}_i\overline{w}((\Pi\bar{t}_{[n\downarrow 1]})^\infty) = w\bar{t}_i((\Pi\bar{t}_{[i\downarrow 1]\vee[n\downarrow i+1]})^\infty) = \max(w\bar{t}_i\bar{t}_i◀).$$

Thus, (iii) holds.

(iv) follows from (i)-(iii).

(v). It is not difficult to see that

$$wt_i\overline{w}((\Pi t_{[1\uparrow n]})^\infty) \in (wt_i◀) \quad \text{and} \quad w\bar{t}_i\overline{w}((\Pi\bar{t}_{[n\downarrow 1]})^\infty) \in (w\bar{t}_i◀).$$

Case 1. $w = 1$.

Here, $t_1((\Pi\bar{t}_{[n\downarrow 1]})^\infty) \in (t_1\bar{t}_n◀) < (t_it_1◀) \ni t_i((\Pi t_{[1\uparrow n]})^\infty) = wt_i\overline{w}((\Pi t_{[1\uparrow n]})^\infty)$, and (a) holds.

Case 2. $w \notin (t_1\star) \cup \{1\}$.

Here, $t_1((\Pi\bar{t}_{[n\downarrow 1]})^\infty) \in (t_1◀) < (w◀) \ni wt_i\overline{w}((\Pi t_{[1\uparrow n]})^\infty)$, and (a) holds.

Case 3. $w \in (t_1t_1\star)$.

Here, $t_1((\Pi\bar{t}_{[n\downarrow 1]})^\infty) \in (t_1\bar{t}_n◀) < (t_1t_1◀) \ni wt_i\overline{w}((\Pi t_{[1\uparrow n]})^\infty)$, and (a) holds.

Case 4. $w \in (t_1\star) - (t_1t_1\star)$.

Here, $w\bar{t}_i\overline{w}((\Pi\bar{t}_{[n\downarrow 1]})^\infty) \in (w\bar{t}_i◀) \subseteq (t_1◀) - (t_1t_1◀)$. Hence,

$$w\bar{t}_i\overline{w}((\Pi\bar{t}_{[n\downarrow 1]})^\infty) \le \max((t_1◀) - (t_1t_1◀)) = \max(t_1\bar{t}_n◀) = t_1((\Pi\bar{t}_{[n\downarrow 1]})^\infty).$$

To prove that (b) holds, it remains to show that

$$w\bar{t}_i\overline{w}((\Pi\bar{t}_{[n\downarrow 1]})^\infty) \ \neq \ t_1((\Pi\bar{t}_{[n\downarrow 1]})^\infty),$$

that is, $\bar{t}_1 w\bar{t}_i\overline{w}((\Pi\bar{t}_{[n\downarrow 1]})^\infty) \ \neq \ (\Pi\bar{t}_{[n\downarrow 1]})^\infty$, that is, $\bar{t}_1 w\bar{t}_i\overline{w} \notin \langle \Pi\bar{t}_{[n\downarrow 1]}\rangle$. We can write $w = t_1 u$ where $u \notin (\bar{t}_1\star)$. Then $\bar{t}_1 w\bar{t}_i\overline{w} = u\bar{t}_i\overline{u}\bar{t}_1$, in normal form. Thus it suffices to show that $u\bar{t}_i\overline{u}\bar{t}_1 \notin \langle \Pi\bar{t}_{[n\downarrow 1]}\rangle$.

If $u = 1$, then $u\bar{t}_i\overline{u}\bar{t}_1 = \bar{t}_i\bar{t}_1 \notin \langle \Pi\bar{t}_{[n\downarrow 1]}\rangle$, since $n \geq 3$.

If $u \neq 1$, then $u\bar{t}_i\overline{u}\bar{t}_1 \notin \langle \Pi\bar{t}_{[n\downarrow 1]}\rangle$, since $u\bar{t}_i\overline{u}\bar{t}_1$ does not lie in the submonoid of $\Sigma_{0,1,n}$ generated by $t_{[1\uparrow n]}$, nor in the submonoid generated by $\bar{t}_{[1\uparrow n]}$.

In both subcases, (b) holds.

In all four cases, (v) holds. □

The following appeared as [5, Lema 2.2.17].

7.5 Theorem. *If $n \geq 1$ then, for each $\phi \in \mathcal{B}_n$, $t_1^\phi((\Pi\bar{t}_{[n\downarrow 1]})^\infty)$ is a squarefree end.*

Proof. This is clear if $n = 1$.

For $n = 2$, $\mathcal{B}_2 = \langle \sigma_1 \rangle$, and

$$t_1^{\mathcal{B}_2} = \{t_1^{\sigma_1^{2m}}, t_1^{\sigma_1^{1+2m}} \mid m \in \mathbb{Z}\} = \{t_1^{(t_1t_2)^m}, t_2^{(t_1t_2)^m} \mid m \in \mathbb{Z}\}.$$

Thus, every element of $t_1^{\mathcal{B}_2}$ is squarefree and does not end in $\bar{t}_2$. Hence, every end in $t_1^{\mathcal{B}_2}((\Pi\bar{t}_{[n\downarrow 1]})^\infty)$ is squarefree.

Thus, we may assume that $n \geq 3$.

Recall that $z_1 = \Pi\bar{t}_{[n\uparrow 1]}$ and $\overline{z}_1 = \Pi t_{[1\uparrow n]}$. Let $\cup[t]_{[1\uparrow n]}$ denote $\bigcup\limits_{i\in[1\uparrow n]} [t_i]$. By Lemma 7.4(v), $t_1(z_1^\infty)$ does not lie in

$$\bigcup_{x\in\cup[t]_{[1\uparrow n]}} [(x(\overline{z}_1^\infty))\uparrow(\overline{x}(z_1^\infty))] \, (= \bigcup_{i=1}^{n} \bigcup_{w\in\Sigma_{0,1,n}-(\star t_i)-(\star\bar{t}_i)} [(wt_i\overline{w}(\overline{z}_1^\infty))\uparrow(w\bar{t}_i\overline{w}(z_1^\infty))]).$$

Notice that ϕ permutes the elements of each of the following sets:

$$\cup[t]_{[1\uparrow n]}; \quad \{\overline{z}_1^\infty\}; \quad \{z_1^\infty\}; \quad \text{and} \quad \bigcup_{x\in\cup[t]_{[1\uparrow n]}} [x(\overline{z}_1^\infty), \, \overline{x}(z_1^\infty)].$$

Hence $(t_1(z_1^\infty))^\phi$ does not lie in $\bigcup\limits_{x\in\cup[t]_{[1\uparrow n]}} [x(\overline{z}_1^\infty), \, \overline{x}(z_1^\infty)]$. By Lemma 7.4(iv),

$$\bigcup_{x\in\cup[t]_{[1\uparrow n]}} [x(\overline{z}_1^\infty), \, \overline{x}(z_1^\infty)] \quad \supseteq \quad \bigcup_{i=1}^{n} \bigcup_{w\in\Sigma_{0,1,n}-(\star t_i)-(\star\bar{t}_i)} ((wt_it_i\blacktriangleleft) \cup (w\bar{t}_i\bar{t}_i\blacktriangleleft)).$$

Hence, $(t_1(z_1^\infty))^\phi$ does not lie in the latter set either, and, hence, $(t_1(z_1^\infty))^\phi$ is a squarefree end. Since $(t_1(z_1^\infty))^\phi = t_1^\phi(z_1^\infty)$, the desired result holds. □

We now obtain new information about the $\mathcal{B}_n$-orbit of t_1 in $\Sigma_{0,1,n}$.

7.6 Corollary. *Let* $n \geq 1$*, let* $\phi \in \mathcal{B}_n$*, and let* $k \in [1{\uparrow}n]$.

(i). t_1^{ϕ} *is squarefree.*

(ii). $t_1^{\phi} \notin ((\Pi\overline{t}_{[n\downarrow(k+1)]})t_k\star) - \{t_k^{\Pi t_{[(k+1)\uparrow n]}}\}$.

(iii). $t_1^{\phi} \notin ((\Pi t_{[1\uparrow(k-1)]})\overline{t}_k\star)$.

Proof. Recall from Notation 3.1 that we write $t_1^{\phi} = t_{1^{\pi(\phi)}}^{w_1(\phi)}$. Let $\pi = \pi(\phi)$ and $w_1 = w_1(\phi)$.

It is not difficult to see that

$$t_1^{\phi}(z_1^{\infty}) = \overline{w}_1 t_{1^{\pi}} w_1((\Pi\overline{t}_{[n\downarrow 1]})^{\infty}) \ \ \in \ \ (\overline{w}_1\blacktriangleleft).$$

By Theorem 7.5, $t_1^{\phi}(z_1^{\infty})$ is a squarefree end. Hence, $\overline{w}_1$ is squarefree, and $w_1 \notin (\star\overline{t}_k\Pi t_{[(k+1)\downarrow n]})$.

Since $\overline{w}_1$ is squarefree, t_1^{ϕ} is also squarefree. Hence (i) holds.

Also, $w_1 \notin (\star\overline{t}_k\Pi t_{[(k+1)\uparrow n]})$ implies that $\overline{w}_1 \notin ((\Pi\overline{t}_{[n\downarrow(k+1)]})t_k\star)$ and, hence, $t_1^{\phi} \notin ((\Pi\overline{t}_{[n\downarrow(k+1)]})t_k\star) - \{t_k^{\Pi t_{[k+1\uparrow n]}}\}$ and, also, $\overline{t}_1^{\phi} \notin ((\Pi\overline{t}_{[n\downarrow(k+1)]})t_k\star)$. In particular, (ii) holds.

Let ξ be the automorphism of $\Sigma_{0,1,n}$ determined by $\overline{\begin{matrix} j\in[1\uparrow n] \\ (t_j)^{\xi} \\ = (\overline{t}_{n+1-j}) \end{matrix}}$. Then $\xi^2 = 1$ and $\xi \in \mathrm{Out}_{0,1,n}^{-} := \mathrm{Out}_{0,1,n} - \mathrm{Out}_{0,1,n}^{+}$. Also,

$$t_n^{\phi\xi} = t_n^{\xi\phi\xi} = \overline{t}_1^{\phi\xi} \notin ((\Pi\overline{t}_{[n\downarrow(k+1)]})t_k\star)^{\xi} = ((\Pi t_{[1\uparrow(n-k)]})\overline{t}_{n+1-k}\star).$$

It follows that $t_n^{\mathcal{B}_n^{\xi}} \cap ((\Pi t_{[1\uparrow(n-k)]})\overline{t}_{n+1-k}\star) = \emptyset$. Since $\mathcal{B}_n^{\xi} = \mathcal{B}_n$ and $t_n^{\mathcal{B}_n} = t_1^{\mathcal{B}_n}$, we see that $t_1^{\phi} \notin ((\Pi t_{[1\uparrow(n-k)]})\overline{t}_{n+1-k}\star)$. Now replacing k with $n+1-k$ gives (iii). □

In Remark IV.3, we shall give a second proof of Corollary 7.6 using Larue-Whitehead diagrams.

8 Actions on free products of cyclic groups

8.1 Notation. Throughout this section, we assume that $n \geq 1$ and we fix a positive integer N.

Let $p_{[1\uparrow N]}$ be a partition of n, that is, $p_{[1\uparrow N]}$ is a sequence in $[1{\uparrow}\infty[$ such that $p_1 + \cdots + p_N = n$.

Let $m_{[1\uparrow N]}$ be a sequence in $\mathbb{N} - \{1\}$.

We let $\Sigma_{0,1,p_1^{(m_1)}\perp p_2^{(m_2)}\perp\cdots\perp p_N^{(m_N)}}$ denote the group with presentation

$$\langle z, \tau_{[1\uparrow n]} \mid z\Pi\tau_{[1\uparrow n]}, \{\tau_{j+\sum p_{[1\uparrow i-1]}}^{m_i}\}_{i\in[1\uparrow N], j\in[1\uparrow p_i]}\rangle.$$

Thus, $\Sigma_{0,1,p_1^{(m_1)}\perp p_2^{(m_2)}\perp\cdots\perp p_N^{(m_N)}}$ is isomorphic to a free product of cyclic groups, $C_{m_1}^{*p_1} * C_{m_2}^{*p_2} * \cdots * C_{m_N}^{*p_N}$, where C_0 is interpreted as C_∞, and $p_i^{(0)}$ is also written p_i.

We let $\mathrm{Out}^+_{0,1,p_1^{(m_1)}\perp p_2^{(m_2)}\perp\cdots\perp p_N^{(m_N)}}$ denote the group of all automorphisms of $\Sigma_{0,1,p_1^{(m_1)}\perp p_2^{(m_2)}\perp\cdots\perp p_N^{(m_N)}}$ which respect $\{z\}$ and $\{[\tau_i]\}_{i\in[(p_1+\ldots+p_{j-1}+1)\uparrow(p_1+\ldots+p_j)]}$ for each $j\in[1\uparrow N]$.

We let $\mathrm{Out}_{0,1,p_1^{(m_1)}\perp p_2^{(m_2)}\perp\cdots\perp p_N^{(m_N)}}$ denote the group of all automorphisms of $\Sigma_{0,1,p_1^{(m_1)}\perp p_2^{(m_2)}\perp\cdots\perp p_N^{(m_N)}}$ which respect $\{z,\overline{z}\}$ and

$$\{[\tau_i]\cup[\overline{\tau}_i]\}_{i\in[(p_1+\ldots+p_{j-1}+1)\uparrow(p_1+\ldots+p_j)]}$$

for each $j\in[1\uparrow N]$.

In the case where all the m_i are 0, we get groups denoted $\mathrm{Out}_{0,1,p_1\perp p_2\perp\cdots\perp p_N}$ and $\mathrm{Out}^+_{0,1,p_1\perp p_2\perp\cdots\perp p_N}$. Notice that $\mathrm{Out}_{0,1,p_1\perp p_2\perp\cdots\perp p_N}$ is the subgroup of $\mathrm{Out}_{0,1,n}$ consisting of those elements such that the permutation in Sym_n, arising from the permutation of $\{[t_i]\cup[\overline{t}_i]\}_{i\in[1\uparrow n]}$, lies in the natural image of

$$\mathrm{Sym}_{p_1}\times\mathrm{Sym}_{p_2}\times\cdots\times\mathrm{Sym}_{p_N}$$

in Sym_n.

There are natural maps

$$\mathrm{Out}_{0,1,p_1\perp p_2\perp\cdots\perp p_N}\to\mathrm{Out}_{0,1,p_1^{(m_1)}\perp p_2^{(m_2)}\perp\cdots\perp p_N^{(m_N)}},\tag{8.1.1}$$

$$\mathrm{Out}^+_{0,1,p_1\perp p_2\perp\cdots\perp p_N}\to\mathrm{Out}^+_{0,1,p_1^{(m_1)}\perp p_0^{(m_2)}\perp\cdots\perp p_N^{(m_N)}}.\tag{8.1.2}$$

Since (8.1.2) is of index two in (8.1.1), we see that (8.1.1) is injective, surjective or bijective, if and only if (8.1.2) has the same property. □

For topological reasons, we suspect that (8.1.1) and (8.1.2) are isomorphisms. In this section, we shall prove that this holds in the case where all the m_i are equal, which includes the case $N=1$. We begin by proving that (8.1.1) and (8.1.2) are injective, which seems to be new.

8.2 Theorem. *With* Notation 8.1*, the maps*

$$\mathrm{Out}_{0,1,p_1\perp p_2\perp\cdots\perp p_N}\to\mathrm{Out}_{0,1,p_1^{(m_1)}\perp p_2^{(m_2)}\perp\cdots\perp p_N^{(m_N)}},\tag{8.1.1}$$

$$\mathrm{Out}^+_{0,1,p_1\perp p_2\perp\cdots\perp p_N}\to\mathrm{Out}^+_{0,1,p_1^{(m_1)}\perp p_2^{(m_2)}\perp\cdots\perp p_N^{(m_N)}}\tag{8.1.2}$$

are injective.

Proof. Suppose that ϕ is an element of the kernel of (8.1.1) or (8.1.2). Clearly, $\phi\in\mathrm{Out}^+_{0,1,n}$. Also $t^\phi_{[1\uparrow n]}$ and $t_{[1\uparrow n]}$ both have the same image in $\Sigma_{0,1,p_1^{(m_1)}\perp p_2^{(m_2)}\perp\cdots\perp p_N^{(m_N)}}$. By Theorem 7.5, $t^\phi_{[1\uparrow n]}$ is a sequence of squarefree elements of $\Sigma_{0,1,n}$, and, hence, they have the same normal form in $\Sigma_{0,1,n}$ and in $\Sigma_{0,1,p_1^{(m_1)}\perp p_2^{(m_2)}\perp\cdots\perp p_N^{(m_N)}}$. Hence $t^\phi_{[1\uparrow n]}=t_{[1\uparrow n]}$, as sequences in $\Sigma_{0,1,n}$. Thus $\phi=1$, and the result is proved. □

8.3 Historical Remarks. Let us now restrict to the classic case where $N = 1$. Here, for an integer $m \geq 2$, we are considering the action of $\mathrm{Out}_{0,1,n}$ on C_m^{*n}, and it induces maps

$$\mathrm{Out}_{0,1,n} \to \mathrm{Out}_{0,1,n^{(m)}}, \tag{8.3.1}$$

$$\mathrm{Out}^+_{0,1,n} \to \mathrm{Out}^+_{0,1,n^{(m)}}. \tag{8.3.2}$$

Theorem 8.2 shows that these maps are injective. Birman-Hilden [6, Theorem 7] gave a topological proof that (8.3.2) is injective, thus answering a question of Magnus. Crisp-Paris [11] gave an elegant algebraic proof of the injectivity of (8.3.2) using the σ_1-trichotomy and the technique of Larue [22] and Shpilrain [30]. Here is the essence of their proof.

Suppose that ϕ is a non-trivial element of $\mathcal{B}_n = \mathrm{Out}^+_{0,1,n}$. We will show that ϕ acts non-trivially on $\Sigma_{0,1,n^{(m)}}$.

We may assume that $n \geq 3$. By Theorem 6.7, we may replace ϕ with $\overline{\phi}$ if necessary, and assume that ϕ is σ-negative. Thus there exists some $i \in [1{\uparrow}(n-1)]$ such that ϕ has a monoid expression in $\sigma_{[(i+1)\uparrow(n-1)]} \vee \overline{\sigma}_{[i\uparrow(n-1)]}$, and $\overline{\sigma}_i$ appears at least once in the expression.

Let $(\tau_i^* \star)$ denote the set of elements of $\Sigma_{0,1,n^{(m)}}$ whose free-product normal form begins with an element of $\langle \tau_i \rangle - \{1\}$. With Notation 6.5,

$$(\tau_i^*\star)^{\overline{\sigma}_i} = (\tau_i^*\star)^{\overline{\sigma}''_i\overline{\sigma}'_i} = (\tau_{i+1}^*\star)^{\overline{\sigma}'_i} \subseteq \tau_i(\tau_{i+1}^*\star) \overset{(n>2)}{\subset} (\tau_i^*\star).$$

Because the elements of $\sigma_{[(i+1)\uparrow(n-1)]} \vee \overline{\sigma}_{[i\uparrow(n-1)]}$ act as injective self-maps on $(\tau_i^*\star)$, it follows that $(\tau_i^*\star)^\phi \subset (\tau_i^*\star)$, and, hence, ϕ acts non-trivially on $\Sigma_{0,1,n^{(m)}}$, as desired. □

Let us now verify the surjectivity of the maps (8.3.1) and (8.3.2). The case where $m = 2$ is due to Stephen Humphries [2, Lemma 2.1.7].

8.4 Notation. Let $m, n \in \mathbb{N}$ with $n \geq 1$ and $m \geq 2$. Let $\lfloor \frac{m}{2} \rfloor$ denote the greatest integer not exceeding $\frac{m}{2}$. Then $[0{\uparrow}\lfloor\frac{m}{2}\rfloor] \vee [(-1){\downarrow}(-\lfloor\frac{m-1}{2}\rfloor)]$ is a sequence of representatives for the integers modulo m. For $\tau^k \in \langle \tau \mid \tau^m = 1 \rangle$, we define $|\tau^k|$ by

$$\begin{array}{cc} \underline{k\in[0\uparrow\lfloor\frac{m}{2}\rfloor]} & \underline{k\in[(-1)\downarrow(-\lfloor\frac{m-1}{2}\rfloor)]} \\ (\ |\tau^k| & |\tau^k|\ ) \\ =(\ 2k & -2k-1) \end{array}$$

and we then extend $|-|$ to all of $\Sigma_{0,1,n^{(m)}}$ additively on normal forms for the free product C_m^{*n}.

Let $\phi \in \mathrm{Out}^+_{0,1,n^{(m)}}$. There exists a unique permutation $\pi \in \mathrm{Sym}_n$, and a unique sequence $w_{[0\uparrow(n+1)]}$ in $\Sigma_{0,1,n^{(m)}}$ such that $w_0 = 1$ and $w_{n+1} = 1$, and, for each $i \in [1{\uparrow}n]$, $w_i \notin (\tau_{i^\pi}^*\star)$ and $\tau_i^\phi = \tau_{i^\pi}^{w_i}$. For each $i \in [0{\uparrow}n]$, let $u_i = w_i\overline{w}_{i+1}$. We define $\pi(\phi) := \pi$, $w_i(\phi) := w_i$, $i \in [0{\uparrow}(n+1)]$, and $u_i(\phi) := u_i$, $i \in [0{\uparrow}n]$. We write $\|\phi\| := n + 2 \sum\limits_{i\in[1\uparrow n]} |w_i(\phi)|$. □

The following is similar to Artin's Lemma 3.2.

8.5 Lemma. *Let $n \geq 1$, $m \geq 2$ and let $\phi \in \mathrm{Out}_{0,1,n^{(m)}}$. Let $\pi = \pi(\phi)$. For each $i \in [0{\uparrow}n]$, let $u_i = u_i(\phi)$. For each $i \in [1{\uparrow}n]$, let a_i, b_i denote the elements of $[0{\uparrow}(m-1)]$ determined by the following:*

there exists some $u_i' \in \Sigma_{0,1,n^{(m)}} - (\star\tau_{i^\pi}^*)$ such that $u_{i-1} = u_i'\tau_{i^\pi}^{a_i}$;

there exists some $u_i'' \in \Sigma_{0,1,n^{(m)}} - (\tau_{i^\pi}^*\star)$ such that $u_i = \tau_{i^\pi}^{b_i}u_i''$.

In particular, $a_1 = b_n = 0$.

(i). *Let $i \in [2{\uparrow}n]$. If $a_i \in [\lfloor\frac{m}{2}\rfloor{\uparrow}(m-1)]$, then $\|\sigma_{i-1}\phi\| < \|\phi\|$.*

(ii). *Let $i \in [1{\uparrow}(n-1)]$. If $b_i \in [\lfloor\frac{m+1}{2}\rfloor{\uparrow}(m-1)]$, then $\|\overline{\sigma}_i\phi\| < \|\phi\|$.*

(iii). *If $\phi \neq 1$, there exists some $\sigma_i^\epsilon \in \sigma_{[1\uparrow(n-1)]} \vee \overline{\sigma}_{[1\uparrow(n-1)]}$ such that $\|\sigma_i^\epsilon\phi\| < \|\phi\|$.*

Proof. (i). Let $a = a_i$. There exists some $v \in \Sigma_{0,1,n^{(m)}} - (\star\tau_{i^\pi}^*)$ such that $u_{i-1} = v\tau_{i^\pi}^a$. Since $w_{i-1}(\phi) = u_{i-1}w_i(\phi)$, we have

$$w_{i-1}(\phi) = v\tau_{i^\pi}^a w_i(\phi); \qquad (8.5.1)$$

since $w_i(\phi) \notin (\tau_{i^\pi}^*\star)$ and $v \notin (\star\tau_{i^\pi}^*)$, $v\tau_{i^\pi}^a w_i(\phi)$ is a free-product normal form for $w_{i-1}(\phi)$.

Claim. $|\tau_{i^\pi}^{a+1}| < |\tau_{i^\pi}^a|$.

Proof of claim. If $a' \in [(\lfloor\frac{m}{2}\rfloor + 1){\uparrow}(m-1)]$, then $a' - m \in [(-\lfloor\frac{m-1}{2}\rfloor){\uparrow}(-1)]$, and, hence,

$$|\tau_{i^\pi}^{a'}| = |\tau_{i^\pi}^{a'-m}| = -2(a'-m) - 1 = 2m - 2a' - 1.$$

Therefore, if $a \in [\lfloor\frac{m}{2}\rfloor{\uparrow}(m-2)]$, $|\tau_{i^\pi}^{a+1}| = 2m - 2(a+1) - 1 = 2m - 2a - 3$.
Thus, $|\tau_{i^\pi}^{a+1}| < |\tau_{i^\pi}^a|$ if $a \in [(\lfloor\frac{m}{2}\rfloor + 1){\uparrow}(m-2)]$.
For $a = \lfloor\frac{m}{2}\rfloor$, $a \geq \frac{m-1}{2}$, and $|\tau_{i^\pi}^a| = 2a > 2m - 2a - 3 = |\tau_{i^\pi}^{a+1}|$.
For $a = m-1$, $|\tau_{i^\pi}^a| = 1$ and $|\tau_{i^\pi}^{a+1}| = 0$. This proves the claim. □

Thus, $|w_{i-1}(\phi)| = |v| + |\tau_{i^\pi}^a| + |w_i(\phi)| > |v| + |\tau_{i^\pi}^{a+1}| + |w_i(\phi)|$.
By (8.5.1), $w_{i-1}(\phi)\overline{w}_i(\phi)\tau_{i^\pi} = v\tau_{i^\pi}^{a+1}$; hence

$$\tau_i^{\sigma_{i-1}\phi} = (\tau_{i-1}^{\tau_i})^\phi = (\tau_{(i-1)^\pi}^{w_{i-1}(\phi)})^{(\tau_{i^\pi}^{w_i(\phi)})} = \tau_{(i-1)^\pi}^{v\tau_{i^\pi}^{a+1}w_i(\phi)}.$$

Hence, $|w_i(\sigma_{i-1}\phi)| = |v\tau_{i^\pi}^{a+1}w_i(\phi)| \leq |v| + |\tau_{i^\pi}^{a+1}| + |w_i(\phi)| < |w_{i-1}(\phi)|$.

For each $j \in [1{\uparrow}(i-2)] \vee [(i+1){\uparrow}n]$, $\tau_j^{\sigma_{i-1}\phi} = \tau_j^\phi$, and, hence, $|w_j(\sigma_{i-1}\phi)| = |w_j(\phi)|$.

Also, $\tau_{i-1}^{\sigma_{i-1}\phi} = \tau_i^\phi$; in particular, $|w_{i-1}(\sigma_{i-1}\phi)| = |w_i(\phi)|$.

It now follows that $\|\sigma_{i-1}\phi\| < \|\phi\|$.

(ii). Let $b = b_i$. There exists some $v \in \Sigma_{0,1,n^{(m)}} - (\tau^*_{i^\pi}\star)$ such that $u_i = \tau^b_{i^\pi} v$. Since $w_{i+1}(\phi) = \overline{u}_i w_i(\phi)$, we have

$$w_{i+1}(\phi) = \overline{v}\ \overline{\tau}^b_{i^\pi} w_i(\phi). \tag{8.5.2}$$

Since $w_i(\phi) \notin (\langle\tau_{i^\pi}\rangle\star)$ and $\overline{v} \notin (\star\langle\tau_{i^\pi}\rangle)$, $\overline{v}\ \overline{\tau}^b_{i^\pi} w_i(\phi)$ is a free-product normal form for $w_{i+1}(\phi)$. Hence, $|w_{i+1}(\phi)| = |\overline{v}| + |\overline{\tau}^b_{i^\pi}| + |w_i(\phi)|$.

Claim. $|\overline{\tau}^{b+1}_{i^\pi}| < |\overline{\tau}^b_{i^\pi}|$.

Proof of claim. Suppose that $b' \in [\lfloor\frac{m+1}{2}\rfloor\uparrow m]$. Then $m - b' \in [\lfloor\frac{m}{2}\rfloor\downarrow 0]$, and, hence,

$$|\overline{\tau}^{b'}_{i^\pi}| = |\tau^{m-b'}_{i^\pi}| = 2(m-b) = 2m - 2b'.$$

Since $b \in [\lfloor\frac{m+1}{2}\rfloor\uparrow(m-1)]$,

$$|\overline{\tau}^{b+1}_{i^\pi}| = 2m - 2(b+1) = 2m - 2b - 2 < |\overline{\tau}^b_{i^\pi}|.$$

This proves the claim. □

Hence $|w_{i+1}(\phi)| > |\overline{v}| + |\overline{\tau}^{b+1}_{i^\pi}| + |w_i(\phi)|$.

For all $j \in [1\uparrow(i-1)] \vee [(i+2)\uparrow n]$, $\tau_j^{\overline{\sigma}_i\phi} = \tau_j^\phi$; hence, $|w_j(\overline{\sigma}_i\phi)| = |w_j(\phi)|$.

Since $\tau_{i+1}^{\overline{\sigma}_i\phi} = \tau_i^\phi$, we see that $|w_{i+1}(\overline{\sigma}_i\phi)| = |w_i(\phi)|$.

By (8.5.2), $w_{i+1}(\phi)\overline{w}_i(\phi)\overline{\tau}_{i^\pi} = \overline{v}\ \overline{\tau}^{b+1}_{i^\pi}$; hence

$$\tau_i^{\overline{\sigma}_i\phi} = (\tau_{i+1}^{\overline{\tau}_i})^\phi = (\tau_{(i+1)^\pi}^{w_{i+1}(\phi)})^{(\overline{\tau}_{i^\pi}^{w_i(\phi)})} = \tau_{i^\pi}^{\overline{v}\ \overline{\tau}^{b+1}_{i^\pi} w_i(\phi)}.$$

Hence, $|w_i(\overline{\sigma}_i\phi)| = |\overline{v}\ \overline{\tau}^{b+1}_{i^\pi} w_i(\phi)| \leq |\overline{v}| + |\overline{\tau}^{b+1}_{i^\pi}| + |w_i(\phi)| < |w_{i+1}(\phi)|$.

It now follows that $\|\overline{\sigma}_i\phi\| < \|\phi\|$, and (ii) is proved.

(iii). If $\phi \neq 1$, we choose a distinguished element of $[1\uparrow n]$ as follows.

If, for some $i \in [1\uparrow n]$, $\tau_{i^\pi}^{a_i+1+b_i} = 1$, we take any such i to be our distinguished element of $[1\uparrow n]$.

Consider then the case where, for all $i \in [1\uparrow n]$, $\tau_{i^\pi}^{a_i+1+b_i} \neq 1$. Thus, there is no further cancellation in $\Pi\tau^\phi_{[1\uparrow n]}$. Since ϕ fixes $\Pi\tau_{[1\uparrow n]}$, it is not difficult to see that, for all $i \in [1\uparrow n]$, $\tau_{i^\pi}^{a_i+1+b_i} = \tau_i$. Since $\phi \neq 1$, it is then not difficult to show that there exists some $i \in [1\uparrow n]$ such that $(a_i, b_i) \neq (0,0)$. We take any such i to be our distinguished element of $[1\uparrow n]$.

In each case, let i denote our distinguished element of $[1\uparrow n]$.

Notice that $(a_i, b_i) \neq (0,0)$ and that $\tau_{i^\pi}^{a_i+1+b_i} \in \{1, \tau_{i^\pi}\}$.

Hence, $a_i + 1 + b_i \in \{m, m+1\}$, and, hence, $b_i \in \{m - a_i - 1, m - a_i\}$.

Case 1. $a_i \in [\lfloor\frac{m}{2}\rfloor\uparrow(m-1)]$.

Here, $i \in [2\uparrow n]$ and, by (i), $\|\sigma_{i-1}\phi\| < \|\phi\|$.

Case 2. $a_i \in [0\uparrow\lfloor\frac{m-2}{2}\rfloor]$

Here, $m - a_i - 1 \in [(m-1)\downarrow\lfloor\frac{m+1}{2}\rfloor]$, and, hence, $b_i \in [\lfloor\frac{m+1}{2}\rfloor\uparrow(m-1)]$. Here, $i \in [1\uparrow(n-1)]$ and, by (ii), $\|\overline{\sigma}_i\phi\| < \|\phi\|$. □

8.6 Theorem. *Let $n \geq 1$, $m \geq 2$. The natural map* $\mathrm{Out}^+_{0,1,n} \to \mathrm{Out}^+_{0,1,n^{(m)}}$ *is an isomorphism, and, hence, the natural map* $\mathrm{Out}_{0,1,n} \to \mathrm{Out}_{0,1,n^{(m)}}$ *is an isomorphism.*

With Notation 8.1, *the maps* $\mathrm{Out}_{0,1,p_1\perp p_2\perp\cdots\perp p_N} \to \mathrm{Out}_{0,1,p_1^{(m)}\perp p_2^{(m)}\perp\cdots\perp p_N^{(m)}}$, *and* $\mathrm{Out}^+_{0,1,p_1\perp p_2\perp\cdots\perp p_N} \to \mathrm{Out}^+_{0,1,p_1^{(m)}\perp p_2^{(m)}\perp\cdots\perp p_N^{(m)}}$ *are isomorphisms.* □

The following is essentially an algebraic translation of a part of a topological argument in [27, Section 3].

8.7 Proposition. *With* Notation 8.1, *in* $\Sigma_{0,1,p_1^{(m_1)}\perp p_2^{(m_2)}\perp\cdots\perp p_N^{(m_N)}}$ *let H be any subgroup of finite index, and now in* $\mathrm{Out}_{0,1,p_1^{(m_1)}\perp p_2^{(m_2)}\perp\cdots\perp p_N^{(m_N)}}$ *let A be any subgroup consisting of automorphisms which map H to itself. Then, either the induced map $A \to \mathrm{Aut}\, H$ is injective or $(n, N, m_1) = (2, 1, 2)$.*

Proof. Suppose that $\phi \in \mathrm{Out}_{0,1,p_1^{(m_1)}\perp p_2^{(m_2)}\perp\cdots\perp p_N^{(m_N)}}$, and that ϕ acts as the identity on H. We shall show that $\phi = 1$ or $(n, N, m_1) = (2, 1, 2)$.

Let $G = \Sigma_{0,1,p_1^{(m_1)}\perp p_2^{(m_2)}\perp\cdots\perp p_N^{(m_N)}}$.

For any $g \in G$, right multiplication by g permutes the elements of the finite set $H\backslash G$, so there exists some positive integer k such that g^k acts trivially on $H\backslash G$. In particular, $Hg^k = H$ and, hence, $g^k \in H$.

Hence, there exists some positive integer k such that $(\Pi\tau_{[1\uparrow n]})^k \in H$. Now $(\Pi\tau_{[1\uparrow n]})^\phi = (\Pi\tau_{[1\uparrow n]})^\epsilon$ for some $\epsilon \in \{1, -1\}$, and, hence,

$$(\Pi\tau_{[1\uparrow n]})^k = (\Pi\tau_{[1\uparrow n]})^{k\phi} = (\Pi\tau_{[1\uparrow n]})^{\phi k} = (\Pi\tau_{[1\uparrow n]})^{\epsilon k} = (\Pi\tau_{[1\uparrow n]})^{k\epsilon}.$$

Since $\Pi\tau_{[1\uparrow n]}$ has infinite order in G, we see that $\epsilon = 1$. Thus ϕ fixes $\Pi\tau_{[1\uparrow n]}$.

Consider any $i \in [1\uparrow n]$. Since $(\Pi\tau_{[1\uparrow n]})^{\tau_i} \in G$, there exists some positive integer k such that $(\Pi\tau_{[1\uparrow n]})^{\tau_i k} \in H$. Hence,

$$(\Pi\tau_{[1\uparrow n]})^{k\tau_i} = (\Pi\tau_{[1\uparrow n]})^{\tau_i k} = (\Pi\tau_{[1\uparrow n]})^{\tau_i k\phi} = (\Pi\tau_{[1\uparrow n]})^{k\phi\tau_i^\phi} = (\Pi\tau_{[1\uparrow n]})^{k\tau_i^\phi}.$$

Hence $\tau_i^\phi\overline{\tau}_i$ commutes with $(\Pi\tau_{[1\uparrow n]})^k$. A straightforward normal-form argument shows that $\tau_i^\phi\overline{\tau}_i \in \langle\Pi\tau_{[1\uparrow n]}\rangle$.

Hence there exists an integer j such that $\tau_i^\phi = (\Pi\tau_{[1\uparrow n]})^j\tau_i$. Since τ_i^ϕ is a conjugate of $\tau_{i^{\pi(\phi)}}$, the cyclically-reduced form of $(\Pi\tau_{[1,n]})^j\tau_i$ is $\tau_{i^{\pi(\phi)}}$. Either $j = 0$, or there must be cyclic cancellation, and a straightforward analysis then shows that $(n, N, m_1) = (2, 1, 2)$. Since i was arbitrary, this completes the proof. □

9 The $\mathcal{B}_{n+1}$-group Φ_n

9.1 Notation. Recall that $\Sigma_{0,1,(n+1)^{(2)}} = C_2^{*(n+1)} = \langle\tau_{[1\uparrow(n+1)]} \mid \tau^2_{[1\uparrow n+1]} = 1\rangle$. By Theorem 8.6, $\mathcal{B}_{n+1} = \mathrm{Out}^+_{0,1,n+1} = \mathrm{Out}^+_{0,1,(n+1)^{(2)}}$. We define Φ_n to be the subgroup of $\Sigma_{0,1,(n+1)^{(2)}}$ consisting of the elements which have even exponent sum in the τ_i. It is not difficult to see that Φ_n is a free group of rank n, and that there is induced a map from $\mathrm{Out}_{0,1,(n+1)^{(2)}}$ to $\mathrm{Aut}\,\Phi_n$. Hence Φ_n has a $\mathcal{B}_{n+1}$-action; we say that Φ_n is a $\mathcal{B}_{n+1}$-group, and that Φ_n is a $\mathcal{B}_{n+1}$-subgroup of $\Sigma_{0,1,(n+1)^{(2)}}$.

Proposition 8.7 shows that, if $n \neq 1$, then the map from $\mathrm{Out}_{0,1,(n+1)^{(2)}}$ to $\mathrm{Aut}\,\Phi_n$ is injective, and we say that the $\mathcal{B}_{n+1}$-action is *faithful*, and that Φ_n is a *faithful* $\mathcal{B}_{n+1}$-group. □

Over the course of this section, we shall choose various free generating sets of Φ_n to obtain interesting actions. In the next two examples, we identify $\Sigma_{g,1,0}$ with Φ_{2g} and identify $\Sigma_{g,2,0}$ with Φ_{2g+1}.

9.2 Example. Now that algebraic proofs of the requisite theorems are known to us, let us review [18, Example 15.6] which was an algebraic approximation of results in [27, Section 3].

Let $g \in \mathbb{N}$. Let

$$\Sigma_{g,1,0} := \langle x_{[1\uparrow g]}, y_{[1\uparrow g]}, z_1 \mid (\Pi_{i\in[1\uparrow g]}[x_i, y_i])z_1 = 1\rangle,$$

where the commutator $[x, y]$ of group elements x, y is $\overline{x}\,\overline{y}xy$. Let $\mathrm{Out}^{+}_{g,1,0}$ denote the group of all automorphisms of $\Sigma_{g,1,0}$ which fix z_1. Then $\Sigma_{g,1,0}$ is free of rank $2g$, freely generated by $x_{[1\uparrow g]} \vee y_{[1\uparrow g]}$, and $\mathrm{Out}^{+}_{g,1,0}$ is the group of all automorphisms of $\Sigma_{g,1,0}$ which fix $\Pi_{i\in[1\uparrow g]}[x_i, y_i]$.

We now recall some Dehn-twist elements of $\mathrm{Out}^{+}_{g,1,0}$ from Definitions 3.10 and Remarks 5.1 of [18].

For each $i \in [1\uparrow g]$, we define α_i, $\beta_i \in \mathrm{Out}^{+}_{g,1,0}$ by

$$\begin{array}{rcccc} & \overline{\scriptstyle k\in[1\uparrow(i-1)]} & & & \overline{\scriptstyle k\in[(i+1)\uparrow g]} \\ & (x_k \quad y_k & x_i & y_i & x_k \quad y_k)^{\alpha_i} \\ = & (x_k \quad y_k & \overline{y}_i x_i & y_i & x_k \quad y_k), \end{array} \quad \text{and} \quad \begin{array}{rcccc} & \overline{\scriptstyle k\in[1\uparrow(i-1)]} & & & \overline{\scriptstyle k\in[(i+1)\uparrow g]} \\ & (x_k \quad y_k & x_i & y_i & x_k \quad y_k)^{\beta_i} \\ = & (x_k \quad y_k & x_i & x_i y_i & x_k \quad y_k). \end{array}$$

For each $i \in [1\uparrow(g-1)]$, write $f_i = y_i\overline{x}_{i+1}\overline{y}_{i+1}x_{i+1}$ and define $\gamma_i \in \mathrm{Out}^{+}_{g,1,0}$ by

$$\begin{array}{rcccccc} & \overline{\scriptstyle k\in[1\uparrow(i-1)]} & & & & & \overline{\scriptstyle k\in[(i+2)\uparrow g]} \\ & (x_k \quad y_k & x_i & y_i & x_{i+1} & y_{i+1} & x_k \quad y_k)^{\gamma_i} \\ = & (x_k \quad y_k & \overline{f}_i x_i & y_i^{f_i} & x_{i+1}f_i & y_{i+1} & x_k \quad y_k). \end{array}$$

Let us identify $\Sigma_{g,1,0}$ with Φ_{2g} via

$$\begin{array}{rccc} & \multicolumn{2}{c}{\overline{\qquad\qquad\scriptstyle k\in[1\uparrow g]\qquad\qquad}} & \\ & (x_k & y_k & z_1)^{\Sigma_{g,1,0}\xrightarrow{\sim}\Phi_{2g}} \\ = & (\Pi\tau_{[(2k+1)\downarrow(2k)]} & \tau_{2k+1}\Pi\tau_{[1\uparrow(2k+1)]} & z_1^2). \end{array}$$

Notice that $[x_k, y_k] = \overline{x}_k\overline{y}_k x_k y_k$ is then identified with

$$(\Pi\tau_{[(2k)\uparrow(2k+1)]})(\Pi\tau_{[(2k+1)\downarrow 1]})\tau_{2k+1}(\Pi\tau_{[(2k+1)\downarrow(2k)]})\tau_{2k+1}(\Pi\tau_{[1\uparrow(2k+1)]})$$

which equals $(\Pi\tau_{[(2k-1)\downarrow 1]})(\Pi\tau_{[(2k)\uparrow(2k+1)]})(\Pi\tau_{[1\uparrow(2k+1)]})$. It follows that $\prod\limits_{k\in[1\uparrow g]}[x_k, y_k]$ is identified with $(\Pi\tau_{[1\uparrow(2g+1)]})^2$.

This corresponds to the surface of genus g with one boundary component arising as a two-sheeted branched cover of a sphere with one boundary component and $2g+1$ double points. Then $\mathcal{B}_{2g+1} = \mathrm{Out}^+_{0,1,2g+1} = \mathrm{Out}^+_{0,1,(2g+1)^{(2)}}$ becomes embedded in $\mathrm{Out}^+_{g,1,0}$ via the homomorphism represented as

$$\begin{pmatrix} \sigma_1 & \sigma_2 & \sigma_3 & \sigma_4 & \sigma_5 & \cdots & \sigma_{2g-2} & \sigma_{2g-1} & \sigma_{2g} \\ \alpha_1 & \beta_1 & \gamma_1 & \beta_2 & \gamma_2 & \cdots & \beta_{g-1} & \gamma_{g-1} & \beta_g \end{pmatrix}. \qquad \square$$

Clearly, in the preceding example, the subgroup $\mathcal{B}_{2g}$ of $\mathcal{B}_{2g+1}$ is also embedded in $\mathrm{Out}_{g,1,0}$, but it is more natural to remove from the surface a handle containing the boundary component (a sphere with three boundary components, a 'pair of pants'), and embed $\mathcal{B}_{2g}$ in $\mathrm{Out}_{g-1,2,0}$, as follows.

9.3 Example. Now that algebraic proofs of the requisite theorems are known to us, let us review [18, Example 15.7] which was an algebraic approximation of results in [27, Section 3].

Let $g \in \mathbb{N}$. Let

$$\Sigma_{g,2,0} := \langle x_{[1\uparrow g]}, y_{[1\uparrow g]}, z_{[1\uparrow 2]} \mid (\prod_{i\in[1\uparrow g]}[x_i, y_i])\,\Pi\, z_{[1\uparrow 2]} = 1\rangle.$$

Recall that $[x,y] := \overline{x}\,\overline{y}xy$. Then $\Sigma_{g,2,0}$ is free of rank $2g+1$ with free generating sequence $x_{[1\uparrow g]} \vee y_{[1\uparrow g]} \vee (z_1)$ and distinguished element z_2 such that $\overline{z}_2 = (\prod\limits_{i\in[1\uparrow g]}[x_i, y_i])z_1$. Let $\mathrm{Out}^+_{g,1\perp 1,0}$ denote the group of all automorphisms of $\Sigma_{g,2,0} * \langle e_1 \mid \quad \rangle$ which map $\Sigma_{g,2,0}$ to itself, and fix $z_1^{e_1}$ and z_2. It can be shown that $\mathrm{Out}^+_{g,1\perp 1,0}$ acts faithfully on the subset $\Sigma_{g,2,0} \cup \Sigma_{g,2,0}e_1$ of $\Sigma_{g,2,0} * \langle e_1 \mid \quad \rangle$.

Here, e_1 represents an arc from the base-point of one boundary component, to the base-point of the other boundary component. Karen Vogtmann calls such an arc a 'tether joining the basepoint to the second boundary component'. For any surface-with-boundaries, A'Campo [1, Section 4, Remarque 6], [27, p.232] identifies basepoints of all the boundary components, which makes tethers into loops, to obtain a topological quotient space whose (free) fundamental group is (faithfully) acted on by the mapping-class group of the surface-with-boundaries.

We now recall some Dehn-twist elements of $\mathrm{Out}^+_{g,1\perp 1,0}$ from Definitions 3.10 and Remarks 5.1 of [18].

For each $i \in [1\uparrow g]$, we define α_i, $\beta_i \in \mathrm{Out}^+_{g,1\perp 1,0}$ by

$$\begin{aligned} &(\overbrace{x_k \quad y_k}^{k\in[1\uparrow(i-1)]} \quad x_i \quad y_i \quad \overbrace{x_k \quad y_k}^{k\in[(i+1)\uparrow g]} \quad z_1 \quad e_1)^{\alpha_i} \\ = &(x_k \quad y_k \quad \overline{y}_i x_i \quad y_i \quad x_k \quad y_k \quad z_1 \quad e_1), \end{aligned}$$

$$\begin{array}{rcccccccc} & \overline{\scriptstyle k\in[1\uparrow(i-1)]} & & & & \overline{\scriptstyle k\in[(i+1)\uparrow g]} & & & \\ & (x_k & y_k & x_i & y_i & x_k & y_k & z_1 & e_1)^{\beta_i} \\ = & (x_k & y_k & x_i & x_iy_i & x_k & y_k & z_1 & e_1). \end{array}$$

For $i \in [1\uparrow(g-1)]$, write $f_i = y_i\overline{x}_{i+1}\overline{y}_{i+1}x_{i+1}$ and define $\gamma_i \in \mathrm{Out}^+_{g,1\perp1,0}$ by

$$\begin{array}{rcccccccccc} & \overline{\scriptstyle k\in[1\uparrow(i-1)]} & & & & & & \overline{\scriptstyle k\in[(i+2)\uparrow g]} & & & \\ & (x_k & y_k & x_i & y_i & x_{i+1} & y_{i+1} & x_k & y_k & z_1 & e_1)^{\gamma_i} \\ = & (x_k & y_k & \overline{f}_ix_i & y_i^{f_i} & x_{i+1}f_i & y_{i+1} & x_k & y_k & z_1 & e_1), \end{array}$$

and write $f_g = y_gz_1$ and define $\gamma_g \in \mathrm{Out}^+_{g,1\perp1,0}$ by

$$\begin{array}{rccccccc} & \overline{\scriptstyle k\in[1\uparrow(i-1)]} & & & & & \\ & (x_k & y_k & x_g & y_g & z_1 & e_1)^{\gamma_g} \\ = & (x_k & y_k & \overline{f}_gx_g & y_g^{f_g} & z_1^{f_g} & \overline{f}_ge_1). \end{array}$$

Let us identify $\Sigma_{g,2,0}$ with Φ_{2g+1} and $\Sigma_{g,2,0} \cup \Sigma_{g,2,0}e_1$ with $\Sigma_{0,1,(2g+2)^{(2)}}$ via the map $\Sigma_{g,2,0} * \langle e_1\rangle \to \Sigma_{0,1,(2g+2)^{(2)}}$ determined by

$$\begin{array}{rccccc} & \overline{\qquad\qquad\scriptstyle k\in[1\uparrow g]\qquad\qquad} & & & & \\ & (\quad x_k & y_k & z_1 & e_1 & z_2)^{\Sigma_{g,2,0}*\langle e_1\rangle\to\Sigma_{0,1,(2g+2)^{(2)}}} \\ = & (\Pi\tau_{[(2k+1)\downarrow(2k)]} & \tau_{2k+1}\Pi\tau_{[1\uparrow(2k+1)]} & z_1^{\tau_{2g+2}} & \tau_{2g+2} & z_1). \end{array}$$

This corresponds to the surface of genus g with two boundary components arising as a two-sheeted branched cover of a sphere with one boundary component and $2g+2$ double points. Now $\mathcal{B}_{2g+2} = \mathrm{Out}^+_{0,1,2g+2} = \mathrm{Out}^+_{0,1,(2g+2)^{(2)}}$ is embedded in $\mathrm{Out}^+_{g,1\perp1,0}$ via a homomorphism represented as

$$\begin{pmatrix} \sigma_1 & \sigma_2 & \sigma_3 & \sigma_4 & \sigma_5 & \cdots & \sigma_{2g-2} & \sigma_{2g-1} & \sigma_{2g} & \sigma_{2g+1} \\ \alpha_1 & \beta_1 & \gamma_1 & \beta_2 & \gamma_2 & \cdots & \beta_{g-1} & \gamma_{g-1} & \beta_g & \gamma_g \end{pmatrix}.$$

For $g \geq 1$, Proposition 8.7 shows that this is an embedding. In the case where $g = 0$, the interpretation of the notation is as follows: σ_1 is mapped to γ_0 which fixes z_1 and sends e_1 to $\overline{z}_1e_1$. □

Clearly, in the preceding example, the subgroup $\mathcal{B}_{2g+1}$ of $\mathcal{B}_{2g+2}$ is also embedded in $\mathrm{Out}^+_{g,1\perp1,0}$, but it is more natural to remove from the surface a disc containing the two boundary components (a sphere with three boundary components), and embed $\mathcal{B}_{2g+1}$ in $\mathrm{Out}^+_{g,1,0}$, as in Example 9.2.

We next discuss the Perron-Vannier isomorphism $\mathcal{B}_{n+1} \ltimes \Phi_n \simeq \mathrm{Artin}\langle D_{n+1}\rangle$ for $n \geq 1$. The following was shown to us by Mladen Bestvina.

9.4 Lemma. *Let $n \geq 2$. Then,* Artin$\langle D_n\rangle$ *has a unique automorphism υ of order two which fixes $d_{[1\uparrow(n-2)]}$ and interchanges d_{n-1} and d_n. The semidirect product* Artin$\langle D_n\rangle \rtimes \langle\upsilon\rangle$ *has presentation*

$$\mathrm{Artin}\langle\, d_1 \text{ --- } d_2 \text{ --- } \cdots \text{ --- } d_{n-3} \text{ --- } d_{n-2} \text{ --- } d_{n-1} = \upsilon \mid \upsilon^2 = 1\,\rangle.$$

Proof. Notice that

$$\begin{aligned}&\langle d_{n-1}, d_n, \upsilon \mid \upsilon^2 = 1, d_{n-1}^{\upsilon} = d_n, d_{n-1}d_n = d_n d_{n-1}\rangle \\ &= \langle d_{n-1}, \upsilon \mid \upsilon^2 = 1, d_{n-1}d_{n-1}^{\upsilon} = d_{n-1}^{\upsilon}d_{n-1}\rangle = \mathrm{Artin}\langle\ d_{n-1} =\!= \upsilon \mid \upsilon^2 = 1\ \rangle.\end{aligned}$$

The result now follows easily. □

Part of the following appears in [27] and [10].

9.5 Theorem (Perron-Vannier [27]). *Let $n \geq 2$.*

(i). $$\begin{aligned}\mathcal{B}_n \ltimes \Phi_{n-1} &= \mathrm{Artin}\langle \sigma_1 \text{ — } \sigma_2 \text{ — } \cdots \text{ — } \sigma_{n-3} \text{ — } \begin{matrix}\sigma_{n-1}\tau_n\tau_{n-1} \\ | \\ \sigma_{n-2}\end{matrix} \text{ — } \sigma_{n-1}\rangle \\ &\simeq \mathrm{Artin}\langle D_n\rangle.\end{aligned}$$

(ii). *$\mathcal{B}_n \ltimes \Phi_{n-1}$ has a unique automorphism υ of order two which fixes $\sigma_{[1\uparrow(n-2)]}$ and interchanges σ_{n-1} and $\sigma_{n-1}\tau_n\tau_{n-1}$.*

(iii). $$\begin{aligned}&(\mathcal{B}_n \ltimes \Phi_{n-1}) \rtimes \langle \upsilon\rangle \\ &= \mathrm{Artin}\langle \sigma_1 \text{ — } \sigma_2 \text{ — } \cdots \text{ — } \sigma_{n-3} \text{ — } \sigma_{n-2} \text{ — } \sigma_{n-1} =\!= \upsilon \mid \upsilon^2 = 1\rangle.\end{aligned}$$

Proof. By Corollary 5.5, we have a presentation

$$\mathcal{B}_n \ltimes \Sigma_{0,1,n} \quad = \quad \mathrm{Artin}\langle \sigma_1 \text{ — } \cdots \text{ — } \sigma_{n-1} =\!= \bar{t}_n\rangle.$$

If we impose the relation $t_n^2 = 1$, we transform $\mathcal{B}_n \ltimes \Sigma_{0,1,n}$ into $\mathcal{B}_n \ltimes \Sigma_{0,1,n^{(2)}}$, and we have

$$\begin{aligned}\mathcal{B}_n \ltimes \Sigma_{0,1,n^{(2)}} &= \mathrm{Artin}\langle \sigma_1 \text{ — } \cdots \text{ — } \sigma_{n-1} =\!= \tau_n \mid \tau_n^2 = 1\rangle \\ &= \mathrm{Artin}\langle\ \sigma_1 \text{ — } \cdots \text{ — } \sigma_{n-3} \text{ — } \begin{matrix}\sigma_{n-1}^{\tau_n} \\ | \\ \sigma_{n-2}\end{matrix} \text{ — } \sigma_{n-1}\ \rangle \rtimes \langle \tau_n \mid \tau_n^2 = 1\rangle,\end{aligned}$$

by Lemma 9.4. This group has a retraction to $\langle \tau_n \mid \tau_n^2 = 1\rangle$ with kernel the normal subgroup generated by $\sigma_{[1\uparrow(n-1)]}$. This normal subgroup contains $\sigma_i^{\tau_{i+1}} = \sigma_i\tau_{i+1}\tau_i$ for all $i \in [1\uparrow(n-1)]$, and we see that this normal subgroup is

$$\mathcal{B}_n \ltimes \Phi_{n-1} \quad = \quad \mathrm{Artin}\langle\ \sigma_1 \text{ — } \cdots \text{ — } \sigma_{n-3} \text{ — } \begin{matrix}\sigma_{n-1}^{\tau_n} \\ | \\ \sigma_{n-2}\end{matrix} \text{ — } \sigma_{n-1}\ \rangle;$$

this agrees with the desired presentation. □

9.6 Remarks. Corollary 5.5 says that, for $n \geq 1$, we can go down by index $n+1$ from $\mathrm{Artin}\langle A_n\rangle$ by squaring the last generator, and arrive at $\mathrm{Artin}\langle B_n\rangle \simeq \mathrm{Artin}\langle A_{n-1}\rangle \ltimes \Sigma_{0,1,n}$.

Theorem 9.5 says that, for $n \geq 2$, we can kill the square of the new last generator, go down by index 2, and arrive at $\mathrm{Artin}\langle D_n\rangle \simeq \mathrm{Artin}\langle A_{n-1}\rangle \ltimes \Phi_{n-1}$. □

We now review some other free generating sets of Φ_n which appear in the literature.

9.7 Examples. Recall Notation 9.1. In particular, the $\mathcal{B}_{n+1}$-action on Φ_n is faithful if $n \neq 1$.

(1). For each $k \in [1{\uparrow}n]$, set $x_k = \tau_k\tau_{k+1}$ in Φ_n. Then $x_{[1\uparrow n]}$ is a free generating set for Φ_n, and, for each $i \in [1{\uparrow}n]$, the action of σ_i on Φ_n is determined by

$$\begin{array}{rccccc} & \underline{\scriptstyle k\in[1\uparrow(i-2)]} & & & & \underline{\scriptstyle k\in[(i+2)\uparrow n]} \\ & (x_k & x_{i-1} & x_i & x_{i+1} & x_k)^{\sigma_i} \\ = & (x_k & x_{i-1}x_i & x_i & \overline{x}_i x_{i+1} & x_k), \end{array}$$

interpreted appropriately for $i = 1$ and $i = n$.

(2). For each $k \in [1{\uparrow}n]$, set $x_k = \tau_{n+1}\tau_k$ in Φ_n, Then $x_{[1\uparrow n]}$ is a free generating set for Φ_n, and, for each $i \in [1{\uparrow}(n-1)]$, σ_i acts on $x_{[1\uparrow n]}$ as follows.

$$\begin{array}{rcccc} & \underline{\scriptstyle k\in[1\uparrow(i-1)]} & & & \underline{\scriptstyle k\in[(i+2)\uparrow n]} \\ & (x_k & x_i & x_{i+1} & x_k)^{\sigma_i} \\ = & (x_k & x_{i+1} & x_{i+1}\overline{x}_i x_{i+1} & x_k). \end{array} \qquad \begin{array}{rcc} & \underline{\scriptstyle k\in[1\uparrow(n-1)]} & \\ & (\ \ x_k & x_n)^{\sigma_n} \\ = & (x_{n-1}x_k & x_n). \end{array}$$

(3). We next consider a free generating set indicated by the proof of [11, Proposition A.1(2)].

For each $k \in [1{\uparrow}n]$, set $x_k = (\tau_{n+1}^{\Pi\tau_{[n\downarrow 1]}}\tau_k)^{\Pi\tau_{[k\uparrow(n+1)]}}$ in Φ_n. Then $x_{[1\uparrow n]}$ is a free generating set for Φ_n, and, for each $i \in [1{\uparrow}(n-1)]$, σ_i acts on $x_{[1\uparrow n]}$ as follows.

$$\begin{array}{rcccc} & \underline{\scriptstyle k\in[1\uparrow(i-1)]} & & & \underline{\scriptstyle k\in[(i+2)\uparrow n]} \\ & (x_k & x_i & x_{i+1} & x_k)^{\sigma_i} \\ = & (x_k & x_i\Pi\overline{x}_{[(i+1)\downarrow i]} & (\Pi x_{[i\uparrow(i+1)]})x_{i+1} & x_k). \end{array}$$

Let $w = (\Pi x^2_{[1\uparrow(n-1)]})x_n$; then σ_n acts as follows.

$$\begin{array}{rcc} & \underline{\scriptstyle k\in[1\uparrow(n-1)]} & \\ & (\qquad x_k & x_n \qquad)^{\sigma_n} \\ = & (w^{(-1)^k\Pi x_{[1\uparrow(k-1)]}}x_k & w^{(-1)^n\Pi x_{[1\uparrow(n-1)]}}x_n w). \end{array}$$

□

9.8 Historical Remarks. Let us view $\mathcal{B}_n$ as a subgroup of $\mathcal{B}_{n+1}$ by suppressing σ_n. Then the $\mathcal{B}_{n+1}$-group Φ_n becomes a faithful $\mathcal{B}_n$-group, even if $n = 1$.

Wada [31] defined various left actions of $\mathcal{B}_n$ on a free group of rank n. All but four of the actions are obviously non-faithful, and two of the remaining four are obviously equivalent up to changing the free generating set, leaving three actions to be studied for faithfulness. Shpilrain [30] ingeniously used the σ_1-trichotomy to prove that these three are all faithful. Crisp-Paris [11, Proposition A.1(2)] showed that the second and third of these three faithful actions are equivalent up to changing the free generating set. In fact, they correspond to Examples 9.7(2), (3), above, with σ_n suppressed, where our actions on the right are the inversions of their actions on the left. Thus, the *second* and *third* of the faithful Wada actions of $\mathcal{B}_n$ are both obtained by choosing suitable free generating sets of the Perron-Vannier $\mathcal{B}_{n+1}$-group Φ_n and suppressing σ_n. Hence, Shpilrain [30] had given the first algebraic proof that $\mathcal{B}_n$ acts faithfully on Φ_n; this includes the information that $\mathcal{B}_n$ acts faithfully on the overgroup $\Sigma_{0,1,(n+1)^{(2)}}$, and on the free factor thereof $\Sigma_{0,1,n^{(2)}}$.

Sakuma [28] observed that the third Wada action of $\mathcal{B}_n$ on $\langle x_{[1\uparrow n]} \mid \quad \rangle$ induces an action of $\mathcal{B}_n$ on $\langle x_{[1\uparrow n]} \mid x^2_{[1\uparrow n]} \rangle$ which, when pre-composed with the inversion-of-the-generators automorphism, agrees with the Artin action of $\mathcal{B}_n$ on $\Sigma_{0,1,n^{(2)}}$. Since the latter is faithful by the Birman-Hilden Theorem [6, Theorem 7], the third Wada action is faithful.

Shpilrain [30], unaware of Sakuma's article, repeats the observation that the third Wada action of $\mathcal{B}_n$ on $\langle x_{[1\uparrow n]} \mid \quad \rangle$ induces an action of $\mathcal{B}_n$ on $\langle x_{[1\uparrow n]} \mid x^2_{[1\uparrow n]} \rangle$ and notes that it does not agree with the Artin action of $\mathcal{B}_n$ on $\Sigma_{0,1,n^{(2)}}$. It seems to be tacitly understood in his discussion that the second Wada action of $\mathcal{B}_n$ on $\langle x_{[1\uparrow n]} \mid \quad \rangle$ induces an action of $\mathcal{B}_n$ on $\langle x_{[1\uparrow n]} \mid x^2_{[1\uparrow n]} \rangle$ which clearly agrees with the Artin action of $\mathcal{B}_n$ on $\Sigma_{0,1,n^{(2)}}$, and then, by the Birman-Hilden Theorem, the second Wada action is faithful.

The *first* faithful Wada action is constructed by choosing a non-zero integer m, and, for each $i \in [1\uparrow(n-1)]$, letting σ_i act on $\langle x_{[1\uparrow n]} \mid \quad \rangle$ by

$$\begin{array}{rcccc} & \underline{k\in[1\uparrow(i-1)]} & & & \underline{k\in[(i+2)\uparrow n]} \\ & (x_k & x_i & x_{i+1} & x_k)^{\sigma_i} \\ = & (x_k & x_{i+1} & x_i^{x_{i+1}^m} & x_k). \end{array}$$

Edward Formanek has pointed out that $x^m_{[1,n]}$ freely generates a faithful $\mathcal{B}_n$-subgroup of $\langle x_{[1,n]} \mid \quad \rangle$, where faithfulness can be seen from the fact that the $\mathcal{B}_n$-action is the standard Artin action with respect to this free generating set. This argument gives a transparent proof that this action is faithful. □

Appendix. Larue-Whitehead diagrams

In this appendix, we rework ideas from Chapter 2 and Appendix A of Larue's thesis [21], using combinatorial arguments to obtain a description of the $\mathcal{B}_n$-orbit of t_1. A topological treatment of similar ideas was given by

Fenn-Greene-Rolfsen-Rourke-Wiest [19], and it was arrived at independently of Larue's work; see [16, Chapters 5, 6].

I Self-homeomorphisms

This section is purely motivational. We shall briefly indicate the mapping-class viewpoint of the braid group, and the Jordan-curve nature of the Whitehead graphs of the elements in the $\mathcal{B}_n$-orbit of t_1 if $n \geq 1$.

Let $\mathbb{C}$ denote the complex plane, and $\widehat{\mathbb{C}}$ the Riemann sphere, or projective complex line, $\mathbb{C} \cup \{\infty\}$. For each $z \in \mathbb{C}$ and each non-negative real number r, let $\mathbf{D}(z, r)$ denote the closed disc in $\mathbb{C}$ with centre z and radius r, and let $\mathbf{D}^\circ(z, r)$ denote the interior of $\mathbf{D}(z, r)$.

Let $S_{0,1,n}$ denote the surface formed by deleting from a sphere one open disc and n points. We shall think of the discs and points as being distinguished rather than deleted; for example, it is then meaningful to speak of the self-homeomorphisms of $S_{0,1,n}$ as permuting the points. We take as our model of $S_{0,1,n}$ the sphere $\widehat{\mathbb{C}}$ having $[1{\uparrow}n]$ as its set of distinguished points, and $\mathbf{D}^\circ(0, \frac{1}{2})$ as its distinguished open disc. We are particularly interested in the set $[0{\uparrow}n]$, and, in our diagrams, we shall indicate these points by drawing small discs around them.

For each $k \in [0{\uparrow}n]$, we have a distinguished oriented tether, or arc,

$$\{k + r\mathbf{i} \mid r \text{ is } \infty \text{ or real, with } r \text{ decreasing from } \infty \text{ to } 0\},$$

joining ∞ to k. We label the right flank of this oriented arc t_k, and label the left flank $\overline{t}_k$; we then cut $\widehat{\mathbb{C}}$ open along these arcs and obtain a $(2n + 2)$-gon, with clockwise boundary label $\prod\limits_{k \in [0{\uparrow}n]} (t_k \overline{t}_k)$; see Fig. I.1.4. We shall use t_0 and z_1 interchangeably in this section. Performing the boundary identifications then gives back $\widehat{\mathbb{C}}$.

The self-homeomorphism λ of $\mathbf{D}(0, 1)$ given by $\lambda(re^{\mathbf{i}\theta}) := re^{\mathbf{i}(\theta - 2\pi r)}$ fixes the boundary of $\mathbf{D}(0, 1)$ and interchanges $\frac{1}{2}$ and $-\frac{1}{2}$; see Fig. I.1.1. For each

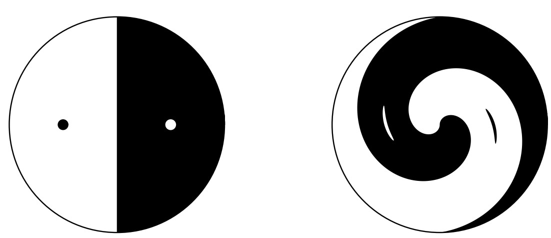

Figure I.1.1: The map $\lambda\colon \mathbf{D}(0, 1) \to \mathbf{D}(0, 1)$, $re^{\mathbf{i}\theta} \mapsto re^{\mathbf{i}(\theta - 2\pi r)}$.

$i \in [1{\uparrow}(n - 1)]$, let ϕ_i denote the self-homeomorphism of $\widehat{\mathbb{C}}$ which acts as the identity map on $\widehat{\mathbb{C}} - \mathbf{D}(i+\frac{1}{2},1)$,

and by $z \mapsto \lambda(z-i-\frac{1}{2})+i+\frac{1}{2}$ on $\mathbf{D}(i+\frac{1}{2},1)$.

Then $\phi_{[1\uparrow(n-1)]}$ generates a group $\langle\phi_{[1\uparrow(n-1)]}\rangle$ of self-homeomorphisms of $\widehat{\mathbb{C}}$ which sheds light on the $\mathcal{B}_n$-orbit of t_1. To describe the induced action of $\langle\phi_{[1\uparrow(n-1)]}\rangle$ on the fundamental group of $S_{0,1,n}$, we first give $\widehat{\mathbb{C}}$ a CW-structure by specifying a graph $S^{(1)}_{0,1,n}$ embedded in $\widehat{\mathbb{C}}$.

For each $k \in [(-1)\uparrow n]$, we have vertices $w_k := k+\frac{1}{2}-\mathbf{i}$ and $v_k := k+\frac{1}{2}+\mathbf{i}$, and (in $\mathbb{C}$) an oriented straight edge f_k joining w_k to v_k. For each $k\in[0\uparrow n]$, we have an oriented straight edge e_k joining w_{k-1} to w_k, and an oriented straight edge d_k joining v_{k-1} to v_k. This completes the description of the graph $S^{(1)}_{0,1,n}$. Each distinguished point $k\in[0\uparrow n]$ is the midpoint of the rectangle in $\mathbb{C}$ cut out by the path $f_{k-1}d_k\overline{f}_k\overline{e}_k$. For $n=3$, $S^{(1)}_{0,1,3}$ can be seen in Fig. I.1.2.

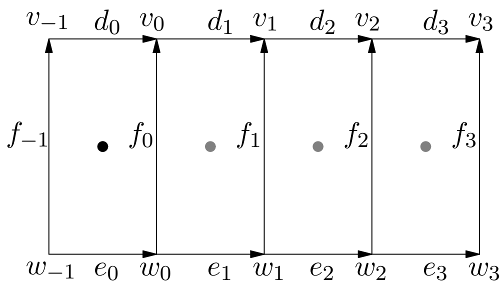

Figure I.1.2: $S_{0,1,3}$.

Let $\langle S^{(1)}_{0,1,n} \mid \ \rangle$ denote the (free) fundamental groupoid of $S^{(1)}_{0,1,n}$, and let $\langle S^{(1)}_{0,1,n} \mid \ \rangle(w_{-1},w_{-1})$ denote the (free) fundamental group of $S^{(1)}_{0,1,n}$ at w_{-1}. The subgraph of $S^{(1)}_{0,1,n}$ spanned by $e_{[0\uparrow n]}\vee f_{[(-1)\uparrow n]}$ is a maximal subtree of $S^{(1)}_{0,1,n}$, and $d_{[0\uparrow n]}$ then determines a free generating set $t_{[0\uparrow n]}$ of $\langle S^{(1)}_{0,1,n} \mid \ \rangle(w_{-1},w_{-1})$; explicitly, for each $k\in[0\uparrow n]$, $t_k = \Pi e_{[0\uparrow(k-1)]}f_{k-1}d_k\overline{f}_k\Pi\overline{e}_{[k\downarrow 0]}$.

The path $f_{-1}\Pi d_{[0\uparrow n]}\overline{f}_n\Pi\overline{e}_{[n\downarrow 0]}$ cuts out a rectangle in $\mathbb{C}$; the complementary region in $\widehat{\mathbb{C}}$ together with the graph $S^{(1)}_{0,1,n}$ is then a retract of $\widehat{\mathbb{C}}-[0\uparrow n]$. Let $\sim$ denote homotopy for closed paths at w_{-1} in $\widehat{\mathbb{C}}-[0\uparrow n]$. We can identify the fundamental groupoid of $S_{0,1,n}$ with $\langle S^{(1)}_{0,1,n} \mid f_{-1}\Pi d_{[0\uparrow n]}\overline{f}_n\Pi\overline{e}_{[n\downarrow 0]}\sim w_{-1}\rangle$. We then identify $\Sigma_{0,1,n}$ with the fundamental group of $S_{0,1,n}$ at w_{-1},

$$\begin{aligned}\Sigma_{0,1,n} &= \langle S^{(1)}_{0,1,n} \mid f_{-1}\Pi d_{[0\uparrow n]}\overline{f}_n\Pi\overline{e}_{[n\downarrow 0]}\sim w_{-1}\rangle(w_{-1},w_{-1})\\ &= \langle t_{[0\uparrow n]} \mid \Pi t_{[0\uparrow n]} = 1\rangle.\end{aligned}$$

Consider the action of ϕ_1 on the graph $S^{(1)}_{0,1,n}$. For $n=3$, the result can be

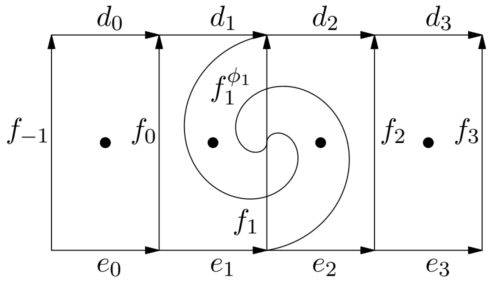

Figure I.1.3: $S^{(1)}_{0,1,3}$ and its image under ϕ_1.

seen in Fig. I.1.3. The crucial point is that $f_1^{\phi_1} \sim e_2 f_2 \overline{d}_2 \overline{f}_1 \overline{e}_1 f_0 d_1$, and all the other elements of $S^{(1)}_{0,1,3}$ are fixed by ϕ_1; this makes the action quite simple algebraically. Then, $\overline{f}_1^{\,\phi_1} \sim \overline{d}_1 \overline{f}_0 e_1 f_1 d_2 \overline{f}_2 \overline{e}_2$, and, for the free generator $t_1 = e_0 f_0 d_1 \overline{f}_1 \Pi \overline{e}_{[1\downarrow 0]}$, we have

$$t_1^{\phi_1} \sim e_0 f_0 d_1 (\overline{d}_1 \overline{f}_0 e_1 f_1 d_2 \overline{f}_2 \overline{e}_2) \Pi \overline{e}_{[1\downarrow 0]} \sim \Pi e_{[0\uparrow 1]} f_1 d_2 \overline{f}_2 \Pi \overline{e}_{[2\downarrow 0]} = t_2.$$

Similarly, for this element, t_2, we have

$$\begin{aligned} t_2^{\phi_1} &\sim \Pi e_{[0\uparrow 1]} (e_2 f_2 \overline{d}_2 \overline{f}_1 \overline{e}_1 f_0 d_1) d_2 \overline{f}_2 \Pi \overline{e}_{[2\downarrow 0]} \\ &\sim \Pi e_{[0\uparrow 2]} f_2 \overline{d}_2 \overline{f}_1 \overline{e}_1 f_0 \Pi d_{[1\uparrow 2]} \overline{f}_2 \Pi \overline{e}_{[2\downarrow 0]} \sim \overline{t}_2 t_1 t_2, \end{aligned}$$

where the latter homotopy can be seen directly by collapsing the elements of $e_{[0\uparrow 2]} \vee f_{[0\uparrow 2]}$, which lie in the maximal subtree. Thus, we see that ϕ_1 acts on $\Sigma_{0,1,n}$ as the automorphism σ_1. It follows that the action of any given element of $\mathcal{B}_n$ on $\Sigma_{0,1,n}$ is induced by some self-homeomorphism $\phi \in \langle \phi_{[1\uparrow(n-1)]} \rangle$. The interesting feature now is that ϕ carries the oriented Jordan curve $f_{-1} \Pi d_{[0\uparrow 1]} \overline{f}_1 \Pi \overline{e}_{[1\downarrow 0]}$ ($\sim t_0 t_1$) to an oriented Jordan curve $f_{-1} \Pi d_{[0\uparrow 1]} \overline{f}_1^{\,\phi} \Pi \overline{e}_{[1\downarrow 0]}$ ($\sim (t_0 t_1)^{\phi} \sim t_0 t_1^{\phi}$).

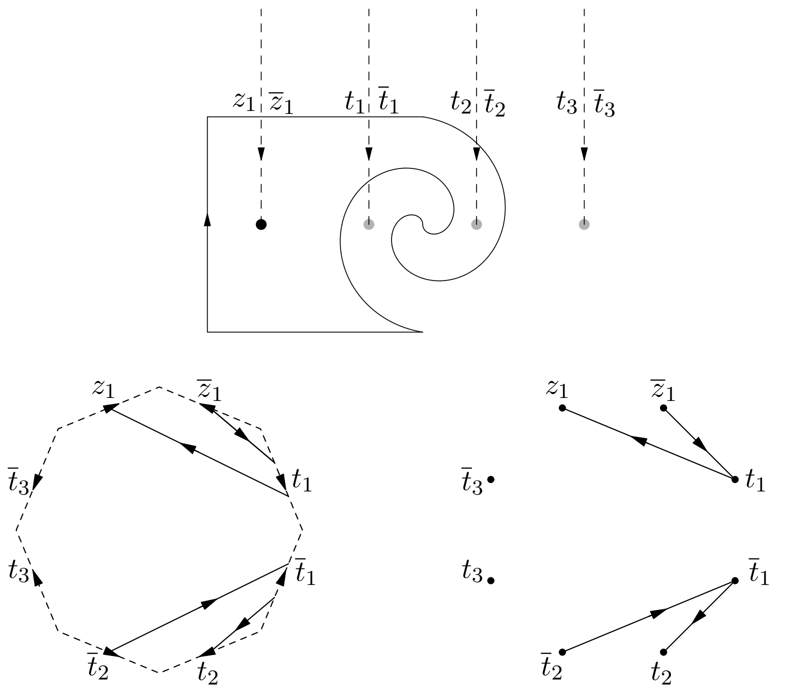

Figure I.1.4: Jordan curves for $z_1 t_1^{\overline{\phi}_1}$ and a Whitehead graph for $t_1^{\overline{\sigma}_1} = t_1 t_2 \overline{t}_1$.

Recall that $\widehat{\mathbb{C}}$ is obtained by edge identification from the $(2n+2)$-gon with clockwise boundary label $\prod\limits_{i\in[0\uparrow n]} (t_i \overline{t}_i)$. The Jordan curve $f_{-1}\Pi d_{[0\uparrow 1]}\overline{f}_1^{\phi}\Pi\overline{e}_{[1\downarrow 0]}$ has as its preimage, in the $(2n+2)$-gon, the union of a family of disjoint oriented arcs. These arcs can be used to reconstruct t_1^{ϕ}, since the Jordan curve cyclically reads off $t_0 t_1^{\phi}$ from its meetings with the labelled oriented tethers; notice that the set of tethers is now dual to the set of generators $t_{[0\uparrow n]}$. The purpose of this appendix is to define and study a combinatorial representation of the family of arcs, and recover Larue's characterization of the elements of $t_1^{\mathcal{B}_n}$.

Although it will not be used in our arguments, let us mention the fact that, on collapsing the interior of each labelled edge of the $(2n+2)$-gon to a labelled vertex, each oriented arc in the family becomes an oriented edge, and we recover the (directed, multi-edge, non-cyclic) Whitehead graph of t_1^{ϕ}; see Fig. I.1.4.

II Nested sets

We now introduce some formal definitions that will allow us to associate a combinatorial Jordan curve to each element of $t_1^{\mathcal{B}_n}$.

II.1 Definitions. Let $(A, \leq)$ be a finite ordered set, and let $m \in \mathbb{N}$.

Let N denote the number of elements of A. Then A is order-isomorphic to $[1\uparrow N]$ in a unique way, and we assign to A the induced metric, denoted d_A. Thus $d_A(a_1, a_2) = 1$ if and only if $a_1 \neq a_2$ and no element of A lies strictly between a_1 and a_2.

Let a_1, a_2, b_1, b_2 be elements of A. We say that $\{a_1, b_1\}$ is *nested with* $\{a_2, b_2\}$ (for $(A, \le)$) if a_1, a_2, b_1, b_2 are distinct elements of A, and either both of, or neither of, a_2 and b_2 lie between a_1 and b_1 in $(A, \le)$. It is not difficult to see that, in this event, $\{a_2, b_2\}$ is nested with $\{a_1, b_1\}$.

Let $a_{[1\uparrow m]}$ and $b_{[1\uparrow m]}$ be sequences in A.

We say that $a_{[1\uparrow m]}$ is a sequence *without repetitions* if $a_i \ne a_j$ for all $i \ne j$ in $[1\uparrow m]$.

We say that $a_{[1\uparrow m]}$ is an *ascending* sequence (in $(A, \le)$) if $a_1 \le a_2 \le \cdots \le a_m$ in $(A, \le)$.

We say that $\{\{a_i, b_i\}\}_{i\in[1\uparrow m]}$ is *nested* (for $(A, \le)$) if, for all $i \ne j$ in $[1\uparrow m]$, $\{a_i, b_i\}$ is nested with $\{a_j, b_j\}$ for $(A, \le)$.

We let Sym_m act on A^m, on the left, by ${}^{\pi}(a_{[1\uparrow m]}) := a_{[1\uparrow m]^{\overline{\pi}}}$. For example, ${}^{(1,2,3)}(a_1, a_2, a_3) = (a_3, a_1, a_2)$, and, hence, ${}^{(1,2,3)}(a, b, c) = (c, a, b)$. The *ascending rearrangement* of $a_{[1\uparrow m]}$ is the unique ascending sequence in $(A, \le)$ that lies in the Sym_m-orbit of $a_{[1\uparrow m]}$.

Let $a_{[1\uparrow(2m)]}$ be a sequence in A.

A permutation $\pi \in \mathrm{Sym}_{2m}$ is said to *embed* $a_{[1\uparrow(2m)]}$ *in a plane* if ${}^{\pi}a_{[1\uparrow(2m)]}$ is ascending for $(A, \le)$, and both $\{\{2i-1, 2i\}^{\pi}\}_{i\in[1\uparrow m]}$ and $\{\{2i, 2i+1\}^{\pi}\}_{i\in[1\uparrow(m-1)]}$ are nested in $(\mathbb{N}, \le)$. We call $\{\{2i-1, 2i\}^{\pi}\}_{i\in[1\uparrow m]}$ the *odd-even pairing*, and call $\{\{2i, 2i+1\}^{\pi}\}_{i\in[1\uparrow(m-1)]}$ the *even-odd pairing*.

We say that $a_{[1\uparrow(2m)]}$ is a *planar* sequence (in $(A, \le)$) if there exists some $\pi \in \mathrm{Sym}_{2m}$ which embeds $a_{[1\uparrow(2m)]}$ in a plane. (If no two consecutive terms of $a_{[1\uparrow(2m)]}$ are equal, π is then unique, but we shall not need this fact.) There is then an associated diagram in $\mathbb{C}$ formed as follows. We assign, to each point $i \in [1\uparrow(2m)] \subset \mathbb{C}$ the label $a_{i^{\overline{\pi}}}$; notice that this means that the label of i^{π} is a_i. For each $i \in [1\uparrow m]$, we join $(2i-1)^{\pi}$ (labelled a_{2i-1}) to $(2i)^{\pi}$ (labelled a_{2i}) by an oriented semi-circle in the upper half-plane. For each $i \in [1\uparrow(m-1)]$, we join $(2i)^{\pi}$ (labelled a_{2i}) to $(2i+1)^{\pi}$ (labelled a_{2i+1}) by an oriented semi-circle in the lower half-plane. These oriented semi-circles link up to form an oriented arc which traces out the sequence $a_{[1\uparrow(2m)]}$, and the nesting property means that the arc has no self-crossings. □

II.2 Example. Suppose that $a_{[1\uparrow 8]} = (\overline{z}_1, t_1, \overline{t}_1, t_2, \overline{t}_2, \overline{t}_1, t_1, z_1)$ is a sequence in some ordered set $(A, \le)$, and that the ascending rearrangement of $a_{[1\uparrow 8]}$ is $(\overline{z}_1, t_1, t_1, \overline{t}_1, \overline{t}_1, t_2, \overline{t}_2, z_1)$.

The permutation $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 1 & 2 & 5 & 6 & 7 & 4 & 3 & 8 \end{pmatrix} = (3,5,7)(4,6)$ embeds $a_{[1\uparrow 8]}$ in a plane since both $\{\{1,2\},\{5,6\},\{7,4\},\{3,8\}\}$ and $\{\{2,5\},\{6,7\},\{4,3\}\}$ are nested in $(\mathbb{N}, \le)$, and ${}^{(3,5,7)(4,6)}(\overline{z}_1, t_1, \overline{t}_1, t_2, \overline{t}_2, \overline{t}_1, t_1, z_1) = (\overline{z}_1, t_1, t_1, \overline{t}_1, \overline{t}_1, t_2, \overline{t}_2, z_1)$. The associated diagram can be seen in Fig. II.2.1.

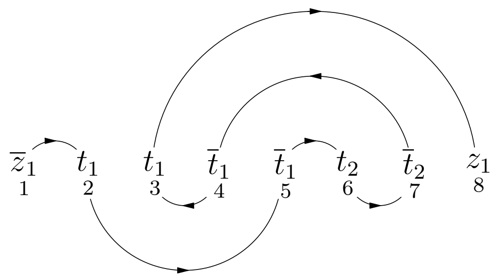

Figure II.2.1: $(\overline{z}_1, t_1, \overline{t}_1, t_2, \overline{t}_2, \overline{t}_1, t_1, z_1)$. □

Let us record two results which will be useful later.

II.3 Lemma. *Let* $(A, \leq)$ *be an ordered set, let* $m \in \mathbb{N}$, *and let* $a_{[1\uparrow(2m)]}$ *be a sequence in* A.

Then $a_{[1\uparrow(2m)]}$ *is planar for* $(A, \leq)$ *if and only if there exists an ordered set* $(B, \leq)$ *with* $|B| = 2m$, *and a sequence* $b_{[1\uparrow(2m)]}$ *in* B, *without repetitions, and an ordered-set map* $B \to A$, $b \mapsto \mathrm{label}(b)$, *such that* $\mathrm{label}(b_{[1\uparrow(2m)]}) = a_{[1\uparrow(2m)]}$, *and* $\{\{b_{2i}, b_{2i+1}\}\}_{i\in[1\uparrow(m-1)]}$ *and* $\{\{b_{2i-1}, b_{2i}\}\}_{i\in[1\uparrow m]}$ *are nested for* $(B, \leq)$.

Proof. Suppose first that $a_{[1\uparrow(2m)]}$ is planar for $(A, \leq)$, and let π be an element of Sym_{2m} that embeds $a_{[1\uparrow(2m)]}$ in a plane. We take B to be $[1\uparrow(2m)]$ with the usual ordering. For each $i \in [1\uparrow(2m)]$, let $\mathrm{label}(i) = a_{i^{\overline{\pi}}}$ and let $b_i = i^\pi$; thus, $\mathrm{label}(b_i) = \mathrm{label}(i^\pi) = a_i$. All the conditions are satisfied.

Conversely, if B exists, we can identify B with $[1\uparrow(2m)]$ with the usual ordering, in a unique way. Then the map $i \mapsto b_i$ is an element π of Sym_{2m} that embeds $a_{[1\uparrow(2m)]}$ in a plane. □

II.4 Lemma. *Let* $(A, \leq)$ *be an ordered set, and let* m *be a positive integer. Let* $c_{[1\uparrow m]}$ *and* $\overline{c}_{[1\uparrow m]}$ *be sequences without repetitions in* $(A, \leq)$ *such that* $\{\{c_i, \overline{c}_i\}\}_{i\in[1\uparrow m]}$ *is nested, and* $\max(c_{[1\uparrow m]}) < \min(\overline{c}_{[1\uparrow m]})$. *If* $c_{[1\uparrow m]}$ *is ascending in* $(A, \leq)$, *then* $\overline{c}_{[m\downarrow 1]}$ *is also ascending in* $(A, \leq)$.

Proof. We argue by induction on m. If $m = 1$, the conclusion is trivial. Now, assume that $m \geq 2$ and that the implication holds with $m - 1$ in place of m. We see that $c_1 < c_2 \leq \max(c_{[1\uparrow m]}) < \min(\overline{c}_{[1\uparrow m]}) \leq \overline{c}_1$. Since $\{c_1, \overline{c}_1\}$ is nested with $\{c_2, \overline{c}_2\}$, we also see that $c_1 < \overline{c}_2 < \overline{c}_1$. By the induction hypothesis, $\overline{c}_{[m\downarrow 2]}$ is ascending, and hence $\overline{c}_{[m\downarrow 1]}$ is ascending. The result is proved. □

III Planar elements of $\Sigma_{0,1,n}$

III.1 Definitions. Let A denote $(z_1, \overline{z}_1) \vee t_{[1\uparrow n]} \vee \overline{t}_{[1\uparrow n]}$, the usual monoid-generating sequence of $\Sigma_{0,1,n}$. We form the ordered set $(A, \leq)$ with

$$\overline{z}_1 < t_1 < \overline{t}_1 < \cdots < t_n < \overline{t}_n < z_1.$$

We remark that, for $n \neq 1$, the ordering on A is reminiscent of the ordering of the ends of $\Sigma_{0,1,n}$ in Section 7. We emphasize that, even if $n = 1$, $z_1 \neq \overline{t}_1$ in A.

Let $m \in \mathbb{N}$. Consider a sequence $a_{[1\uparrow m]}$ in $t_{[1\uparrow n]} \vee \overline{t}_{[1\uparrow n]}$, and let $w = \Pi a_{[1\uparrow m]} \in \Sigma_{0,1,n}$; thus $a_{[1\uparrow m]}$ is an expression for w. We define the *Whitehead expansion* of $a_{[1\uparrow m]}$ to be the sequence

$$(\overline{z}_1) \vee (\bigvee_{i\in[1\uparrow m]}(a_i, \overline{a}_i)) \vee (z_1) = (\overline{z}_1, a_1, \overline{a}_1, a_2, \overline{a}_2, \ldots, a_m, \overline{a}_m, z_1)$$

in A. We say that $a_{[1\uparrow m]}$ is a *planar* expression for w if the Whitehead expansion of $a_{[1\uparrow m]}$ is planar for $(A, \leq)$. If the unique reduced expression for w is a planar expression for w, then we say that w is a *planar* element of $\Sigma_{0,1,n}$. □

III.2 Examples. (i). $t_1\overline{t}_2\overline{t}_1$ is planar, since the Whitehead expansion of the reduced expression is $(\overline{z}_1, t_1, \overline{t}_1, t_2, \overline{t}_2, \overline{t}_1, t_1, z_1)$, which is planar for $(A, \leq)$, by Example II.2; in a sense, Fig. II.2.1 reflects Fig. I.1.4. We call Fig. II.2.1 *the Larue-Whitehead diagram* of $t_1\overline{t}_2\overline{t}_1$.

(ii). $t_1\overline{t}_2$ is not planar; there is only one permutation to consider.

(iii). t_1^2 is not planar; there are four permutations to consider.

(iv). $t_3^{t_1\overline{t}_2\overline{t}_1}$ is planar, while $t_3^{t_1t_2\overline{t}_1}$ is not planar, and these two group elements have the same Whitehead graph. □

III.3 Proposition. *Let $w \in \Sigma_{0,1,n}$. If there exists some planar expression for w, then* (*the reduced expression for*) *w is planar.*

Proof. Suppose that $a_{[1\uparrow m]}$ is a planar expression for w, as in Definitions III.1.

By Lemma II.3, there exists an ordered set $(B, \leq)$ with $|B| = 2m + 2$, and a planar sequence $b_{[1\uparrow(2m+2)]}$ in $(B, \leq)$, without repetitions, and an order-respecting labelling $B \to A$, $b \mapsto \text{label}(b)$, such that $\text{label}(b_{[1\uparrow(2m+2)]})$ is the Whitehead expansion of $a_{[1\uparrow m]}$.

Suppose that the given planar expression $a_{[1\uparrow m]}$ is not reduced. We shall find a shorter planar expression for w.

There exists some $j \in [1\uparrow(m-1)]$ such that $a_{j+1} = \overline{a}_j$ in $t_{[1\uparrow n]} \vee \overline{t}_{[1\uparrow n]}$, and we may suppose that we have chosen this j in such a way that $d_B(b_{2j+1}, b_{2j+2})$ has the minimum possible value. Notice that $\text{label}(b_{[2j\uparrow 2j+3]}) = (a_j, \overline{a}_j, \overline{a}_j, a_j)$.

Clearly, $w = \Pi a_{[1\uparrow(j-1)]}\Pi a_{[(j+1)\uparrow m]}$, and

$$\text{label}(b_{[1\uparrow(2j-1)]\vee[(2j+4)\uparrow(2m+2)]}) = (\overline{z}_1)\vee(\bigvee_{i\in[1\uparrow(j-1)]}(a_i, \overline{a}_i))\vee(\bigvee_{i\in[(j+2)\uparrow m]}(a_i, \overline{a}_i))\vee(z_1).$$

It suffices to show that $b_{[1\uparrow(2j-1)]\vee[(2j+4)\uparrow(2m+2)]}$ is planar for $(B, \leq)$.

Claim. $d_B(b_{2j}, b_{2j+3}) = 1$.

Proof of claim. Consider any $k \in [1\uparrow(2m-1)]$ such that b_k lies between b_{2j} and b_{2j+3}.

Let η denote $(-1)^k$.

Since label(b_{2j}) = label$(b_{2j+3}) = a_j$, we see that label$(b_k) = a_j$. Hence label$(b_{k+\eta}) = \overline{a}_j$ = label(b_{2j+1}) = label(b_{2j+2}).

Either $a_j < \overline{a}_j$ or $a_j > \overline{a}_j$ in $(A, \leq)$. Hence,

$$\text{either } \max\{b_{2j}, b_k, b_{2j+3}\} < \min\{b_{2j+1}, b_{k+\eta}, b_{2j+2}\} \text{ in } (B, \leq),$$

$$\text{or } \min\{b_{2j}, b_k, b_{2j+3}\} > \max\{b_{2j+1}, b_{k+\eta}, b_{2j+2}\} \text{ in } (B, \leq),$$

respectively.

Since $\{\{b_{2j}, b_{2j+1}\}, \{b_{2j+2}, b_{2j+3}\}, \{b_k, b_{k+\eta}\}\}$ is nested, and b_k lies between b_{2j} and b_{2j+3}, we see, from Lemma II.4, that $b_{k+\eta}$ lies between b_{2j+1} and b_{2j+2}.

Since $\{b_{2j+1}, b_{2j+2}\}$ is nested with $\{b_{k+\eta}, b_{k+2\eta}\}$ and $b_{k+\eta}$ lies between b_{2j+1} and b_{2j+2}, we see that $b_{k+2\eta}$ lies between b_{2j+1} and b_{2j+2}. Hence,

$$d_B(b_{k+2\eta}, b_{k+\eta}) \leq d_B(b_{2j+1}, b_{2j+2}),$$

with equality holding only if $\{b_{k+2\eta}, b_{k+\eta}\} = \{b_{2j+1}, b_{2j+2}\}$. Also, label$(b_{k+2\eta}) = \overline{a}_j$, and, hence, label$(b_{k+3\eta}) = a_j$. Thus

$$\text{label}(b_k, b_{k+\eta}, b_{k+2\eta}, b_{k+3\eta}) = (a_j, \overline{a}_j, \overline{a}_j, a_j).$$

By the minimality of $d_B(b_{2j+1}, b_{2j+2})$, we see that $k = 2j$ or $k = 2j+3$. This proves the claim. □

Now consider the passage from $b_{[1\uparrow(2m+2)]}$ to $b_{[1\uparrow(2j-1)]} \vee b_{[(2j+4)\uparrow(2m+2)]}$.

For the odd-even pairing, we pass from $\{\{b_{2i-1}, b_{2i}\}\}_{i\in[1\uparrow(m+1)]}$ to

$$\{\{b_{2i-1}, b_{2i}\}\}_{i\in[1\uparrow(j-1)]\vee[(j+3)\uparrow(m+1)]} \cup \{\{b_{2j-1}, b_{2j+4}\}\}.$$

Thus, we remove $\{\{b_{2j-1}, b_{2j}\}$, $\{b_{2j+1}, b_{2j+2}\}$, $\{b_{2j+3}, b_{2j+4}\}\}$, and we add only $\{\{b_{2j-1}, b_{2j+4}\}\}$. To see that, for all $k \in [1\uparrow(j-1)] \vee [(j+3)\uparrow(m+1)]$, $\{b_{2k-1}, b_{2k}\}$ is nested with $\{b_{2j-1}, b_{2j+4}\}$, we note the following:

$$\begin{aligned}
&(b_{2j-1} \text{ lies between } b_{2k-1} \text{ and } b_{2k})\\
&\qquad \Leftrightarrow (b_{2j} \text{ lies between } b_{2k-1} \text{ and } b_{2k})\\
&\qquad\qquad \text{since } \{b_{2j-1}, b_{2j}\} \text{ is nested with } \{b_{2k-1}, b_{2k}\}\\
&\qquad \Leftrightarrow (b_{2j+3} \text{ lies between } b_{2k-1} \text{ and } b_{2k})\\
&\qquad\qquad \text{since } d_B(b_{2j}, b_{2j+3}) = 1\\
&\qquad \Leftrightarrow (b_{2j+4} \text{ lies between } b_{2k-1} \text{ and } b_{2k})\\
&\qquad\qquad \text{since } \{b_{2j+3}, b_{2j+4}\} \text{ is nested with } \{b_{2k-1}, b_{2k}\}.
\end{aligned}$$

For the even-odd pairing, we pass from $\{\{b_{2i}, b_{2i+1}\}\}_{i\in[1\uparrow m]}$ to

$$\{\{b_{2i}, b_{2i+1}\}\}_{i\in[1\uparrow(j-1)]\vee[(j+2)\uparrow m]}.$$

Thus, we remove $\{\{b_{2j}, b_{2j+1}\}, \{b_{2j+2}, b_{2j+3}\}\}$, and we add nothing. Hence this remains nested.

This completes the proof. □

At the end of the next section, we shall see that the following generalizes Corollary 7.6.

III.4 Proposition. *Let w be a planar element of $\Sigma_{0,1,n}$, and let $k \in [1{\uparrow}n]$.*

(i). *w is squarefree.*

(ii). $w \notin ((\Pi \bar{t}_{[n\downarrow(k+1)]})t_k\star) - \{t_k^{\Pi t_{[(k+1)\uparrow n]}}\}$.

(iii). $w \notin ((\Pi t_{[1\uparrow(k-1)]})\bar{t}_k\star)$.

Proof. Suppose that $a_{[1\uparrow m]}$ is the reduced planar expression for w, as in Definitions III.1. By Lemma II.3, there exists an ordered set $(B, \leq)$ with $|B| = 2m+2$, and a planar sequence $b_{[1\uparrow(2m+2)]}$ in $(B, \leq)$, without repetitions, and an order-respecting labelling $B \to A$, $b \mapsto \text{label}(b)$, such that

$$\text{label}(b_{[1\uparrow(2m+2)]}) = (\bar{z}_1) \vee (\bigvee_{i\in[1\uparrow m]} (a_i, \bar{a}_i)) \vee (z_1).$$

(i). Suppose that w is not squarefree; hence, for some $i \in [1{\uparrow}m]$ and some $\ell \in [1{\uparrow}(m-1)]$, $a_{[\ell\uparrow(\ell+1)]}$ is (t_i, t_i) or $(\bar{t}_i, \bar{t}_i)$. Hence $\text{label}(b_{[(2\ell)\uparrow(2\ell+3)]})$ is $(t_i, \bar{t}_i, t_i, \bar{t}_i)$ or $(\bar{t}_i, t_i, \bar{t}_i, t_i)$.

Let m_i be the number of elements of B with label t_i. Let $c_{[1\uparrow m_i]}$ be the ascending sequence in $(B, \leq)$ which is the interval of elements labelled t_i. For each $k \in [1{\uparrow}m_i]$, let $\bar{c}_k$ denote the element of B such that $\{c_k, \bar{c}_k\}$ is an element of the even-to-odd pairing for $b_{[1\uparrow(2m+2)]}$. By the definition of the Whitehead expansion, the label of $\bar{c}_k$ is $\bar{t}_i$. By Lemma II.4, $\bar{c}_{[m_i\downarrow 1]}$ is the ascending sequence in $(B, \leq)$ which is the interval of elements of B labelled $\bar{t}_i$.

By hypothesis, there exists $\ell \in [1{\uparrow}(m-1)]$ such that $\{b_{2l+1}, b_{2l+2}\} = \{\bar{c}_j, c_k\}$ for some $j, k \in [1{\uparrow}m_i]$. Let us choose ℓ so that $j + k$ is as large as possible. We claim that $k = m_i$. Suppose not; then $c_k < c_{k+1} < \bar{c}_j$. Consider the $d \in B$ such that $\{c_{k+1}, d\}$ lies in the odd-even pairing for $b_{[1\uparrow(2m+2)]}$. Then $d \in \,]c_k{\uparrow}\bar{c}_j[$. Hence $\text{label}(d)$ is t_i or $\bar{t}_i$. Since $a_{[1\uparrow m]}$ is reduced, $\text{label}(d) \neq t_i$. Hence $\text{label}(d) = \bar{t}_i$. Thus, $d = \bar{c}_{j'}$ for some $j' \in [m_i{\downarrow}(j+1)]$. This contradicts the maximality of $k + j$. Hence $k = m_i$, as claimed. Similarly, $j = m_i$. Thus $\{c_{m_i}, \bar{c}_{m_i}\}$ lies in both the even-odd pairing and the odd-even pairing. This gives a closed-curve component within an arc which joins $\bar{z}_1$ to z_1. Hence, we have a contradiction.

(ii). Suppose that $w \in ((\Pi \bar{t}_{[n\downarrow(k+1)]})t_k\star)$.

Then $a_{[1\uparrow(n-k+1)]} = \bar{t}_{[n\downarrow(k+1)]} \vee (t_k)$ and

$$\text{label}(b_{[1\uparrow(2n-2k+3)]}) = (\bar{z}_1, \bar{t}_n, t_n, \bar{t}_{n-1}, t_{n-1}, \ldots, \bar{t}_{k+1}, t_{k+1}, t_k, \bar{t}_k).$$

Since $\text{label}(b_{[(2n-2k+1)\uparrow(2n-2k+3)]}) = (t_{k+1}, t_k, \bar{t}_k)$, we see that, in $(B, \leq)$,

$$b_{2n-2k+2} < b_{2n-2k+3} < b_{2n-2k+1} \text{ with labels } t_k,\ \bar{t}_k,\ t_{k+1}.$$

Since $\{b_{2n-2k+1}, b_{2n-2k+2}\}$ and $\{b_{2n-2k+3}, b_{2n-2k+4}\}$ are nested in $(B, \leq)$, we see that $b_{2n-2k+4} \in \,]b_{2n-2k+2}{\uparrow}b_{2n-2k+1}[$. In particular, $\text{label}(b_{2n-2k+4}) \in \{t_k, \bar{t}_k, t_{k+1}\}$

and label($b_{2n-2k+4}$) $= a_{n-k+2}$. Since $a_{[1\uparrow m]}$ is reduced, $a_{n-k+2} \neq \bar{t}_k$. By (i), $a_{n-k+2} \neq t_k$. Hence $a_{n-k+2} = t_{k+1}$ and the nesting is

$$b_{2n-2k+2} < b_{2n-2k+3} < b_{2n-2k+4} < b_{2n-2k+1} \text{ with labels } t_k, \bar{t}_k, t_{k+1}, t_{k+1}.$$

Using the last inequality and Lemma II.4, we see that

$$b_{2n-2k+4} < b_{2n-2k+1} < b_{2n-2k} < b_{2n-2k+5} \text{ with labels } t_{k+1}, t_{k+1}, \bar{t}_{k+1}, \bar{t}_{k+1}.$$

Now label($b_{[1\uparrow(2n-2k+7)]}$) is

$$(\bar{z}_1, \bar{t}_n, t_n, \bar{t}_{n-1}, t_{n-1}, \ldots, \bar{t}_{k+2}, t_{k+2}, \bar{t}_{k+1}, t_{k+1}, t_k, \bar{t}_k, t_{k+1}, \bar{t}_{k+1}, a_{n-k+3}, \bar{a}_{n-k+3})$$

Notice that

$$b_{2n-2k} < b_{2n-2k+5} < b_{2n-2k-1} \text{ with labels } t_{k+1}, \bar{t}_{k+1}, t_{k+2}.$$

Also $\{b_{2n-2k+5}, b_{2n-2k+6}\}$ is nested with $\{b_{2n-2k}, b_{2n-2k-1}\}$. Hence label($b_{2n-2k+6}$) lies in $\{t_{k+1}, \bar{t}_{k+1}, t_{k+2}\}$, and label($b_{2n-2k+6}$) $= a_{n-k+3}$. Since $a_{[1\uparrow m]}$ is reduced, $a_{n-k+3} \neq \bar{t}_{k+1}$. By (i), $a_{n-k+3} \neq t_{k+1}$. Hence $a_{n-k+3} = t_{k+2}$ and the nesting is

$$b_{2n-2k} < b_{2n-2k+5} < b_{2n-2k+6} < b_{2n-2k-1} \text{ with labels } t_{k+1}, \bar{t}_{k+1}, t_{k+2}, t_{k+2}.$$

Using the last inequality and Lemma II.4, we see that

$$b_{2n-2k+6} < b_{2n-2k-1} < b_{2n-2k-2} < b_{2n-2k+7} \text{ with labels } t_{k+2}, t_{k+2}, \bar{t}_{k+2}, \bar{t}_{k+2}.$$

By repeating the argument in the last paragraph, we eventually find that $w = t_k^{\Pi t_{[(k+1)\uparrow n]}}$.

(iii). Suppose that $w \in (\Pi t_{[1\uparrow k-1]} \bar{t}_k \star)$.

Then $a_{[1\uparrow k]} = t_{[1\uparrow(k-1)]} \vee (\bar{t}_k)$,

$$\text{label}(b_{[1\uparrow(2k+1)]}) = (\bar{z}_1, t_1, \bar{t}_1, t_2, \bar{t}_2, \ldots, t_{k-1}, \bar{t}_{k-1}, \bar{t}_k, t_k),$$

and by an argument similar to that given in (ii), we find that this is impossible. □

IV $\mathcal{B}_n$ permutes the planar elements of $\Sigma_{0,1,n}$

IV.1 Proposition. *Let $w \in \Sigma_{0,1,n}$ and let $i \in [1\uparrow(n-1)]$. If w is planar, then w^{σ_i} is planar.*

Proof. Suppose that $r_{[1\uparrow m]}$ is any planar expression for w, as in Definitions III.1.

In applying σ_i to $(\bar{z}_1) \vee (\bigvee_{i\in[1\uparrow m]}(r_i, \bar{r}_i)) \vee (z_1)$ we replace

each $(t_i, \bar{t}_i)$ with $(t_{i+1}, \bar{t}_{i+1})$,

each $(\bar{t}_i, t_i)$ with $(\bar{t}_{i+1}, t_{i+1})$,

each $(t_{i+1}, \bar{t}_{i+1})$ with $(\bar{t}_{i+1}, t_{i+1}, t_i, \bar{t}_i, t_{i+1}, \bar{t}_{i+1})$,

each $(\bar{t}_{i+1}, t_{i+1})$ with $(\bar{t}_{i+1}, t_{i+1}, \bar{t}_i, t_i, t_{i+1}, \bar{t}_{i+1})$.
We will not perform any cancellations in the resulting sequence.

By Lemma II.3, there exists an ordered set $(B, \leq)$ with $|B| = 2m + 2$, and a planar sequence $p_{[1\uparrow(2m+2)]}$ in $(B, \leq)$, without repetitions, and an order-respecting labelling $B \to A$, $b \mapsto \text{label}(b)$, such that

$$\text{label}(p_{[1\uparrow(2m+2)]}) = (\bar{z}_1) \vee (\textstyle\bigvee_{i\in[1\uparrow m]}(r_i, \bar{r}_i)) \vee (z_1).$$

Let m_i denote the number of elements of B with label t_i, and let m_{i+1} denote the number of elements of B with label t_{i+1}. To begin, we have to add $4m_{i+1}$ elements to B, and we have to specify the ordering on the expanded set.

Let $c_{[1\uparrow m_i]}$ denote the ascending sequence of those elements of B which have label t_i. Let $\bar{c}_{[m_i\downarrow 1]}$ denote the ascending sequence of those elements of B which have label $\bar{t}_i$. Let $d_{[1\uparrow m_{i+1}]}$ denote the ascending sequence of those elements of B which have label t_{i+1}. Let $\bar{d}_{[m_{i+1}\downarrow 1]}$ denote the ascending sequence of those elements of B which have label $\bar{t}_{i+1}$. We then have an interval in B

$$[c_1{\uparrow}\bar{d}_1] = c_{[1\uparrow m_i]} \vee \bar{c}_{[m_i\downarrow 1]} \vee d_{[1\uparrow m_{i+1}]} \vee \bar{d}_{[m_{i+1}\downarrow 1]}.$$

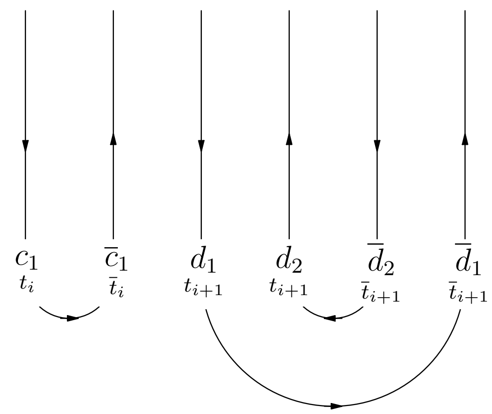

Figure IV.1.1: $c_{[1\uparrow 1]} \vee \bar{c}_{[1\downarrow 1]} \vee d_{[1\uparrow 2]} \vee \bar{d}_{[2\downarrow 1]}$

We create an interval of $4m_{i+1}$ new elements

$$[a_1{\uparrow}\bar{b}_1] = a_{[1\uparrow m_{i+1}]} \vee \bar{a}_{[m_{i+1}\downarrow 1]} \vee b_{[1\uparrow m_{i+1}]} \vee \bar{b}_{[m_{i+1}\downarrow 1]}$$

and expand B by inserting this interval $[a_1{\uparrow}\bar{b}_1]$ just before the interval $[c_1{\uparrow}\bar{d}_1]$. We then have a new ordered set B' with $2m + 2 + 4m_{i+1}$ elements.

We now specify the labelling $B' \to A$. On $c_{[1\uparrow m_i]}$, we change the labels from t_i to t_{i+1}. On $\bar{c}_{[m_i\downarrow 1]}$, we change the labels from $\bar{t}_i$ to $\bar{t}_{i+1}$. On $d_{[1\uparrow m_{i+1}]}$, we change the labels from t_{i+1} to $\bar{t}_{i+1}$. On $\bar{d}_{[m_{i+1}\downarrow 1]}$, we keep the same labels, $\bar{t}_{i+1}$. On $B - [c_1{\uparrow}\bar{d}_1]$, we keep the same labels. We give all the elements of $a_{[1\uparrow m_{i+1}]}$ the label t_i; all the elements of $\bar{a}_{[m_{i+1}\downarrow 1]}$ the label $\bar{t}_i$; and all the elements of

$b_{[1\uparrow m_{i+1}]} \vee \overline{b}_{[m_{i+1}\downarrow 1]}$ the label t_{i+1}. The labelling clearly respects the orderings of B' and A.

It follows from Lemma II.4 that

$$\{\{p_{2k}, p_{2k+1}\}\}_{k\in[1\uparrow m]} \quad \supseteq \quad \{\{c_i, \overline{c}_i\}\}_{i\in[1\uparrow m_i]} \cup \{\{d_j, \overline{d}_j\}\}_{j\in[1\uparrow m_{i+1}]}.$$

Let $q_{[1\uparrow(2m+4m_{i+1})]}$ be the sequence in B' obtained from $p_{[1\uparrow(2m+2)]}$ as follows. For each $j \in [1\uparrow m_{i+1}]$, there exists a unique $k \in [1\uparrow m]$ such that $\{p_{2k}, p_{2k+1}\} = \{d_j, \overline{d}_j\}$. If $(p_{2k}, p_{2k+1}) = (d_j, \overline{d}_j)$ in $p_{[1\uparrow(2m+2)]}$, then it is to be expanded to $(d_j, \overline{b}_j, a_j, \overline{a}_j, b_j, \overline{d}_j)$ in $q_{[1\uparrow(2m+4m_{i+1})]}$. If $(p_{2k}, p_{2k+1}) = (\overline{d}_j, d_j)$ in $p_{[1\uparrow(2m+2)]}$, then it is to be expanded to $(\overline{d}_j, b_j, \overline{a}_j, a_j, \overline{b}_j, d_j)$ in $q_{[1\uparrow(2m+4m_{i+1})]}$. This completes the definition of $q_{[1\uparrow(2m+4m_{i+1})]}$.

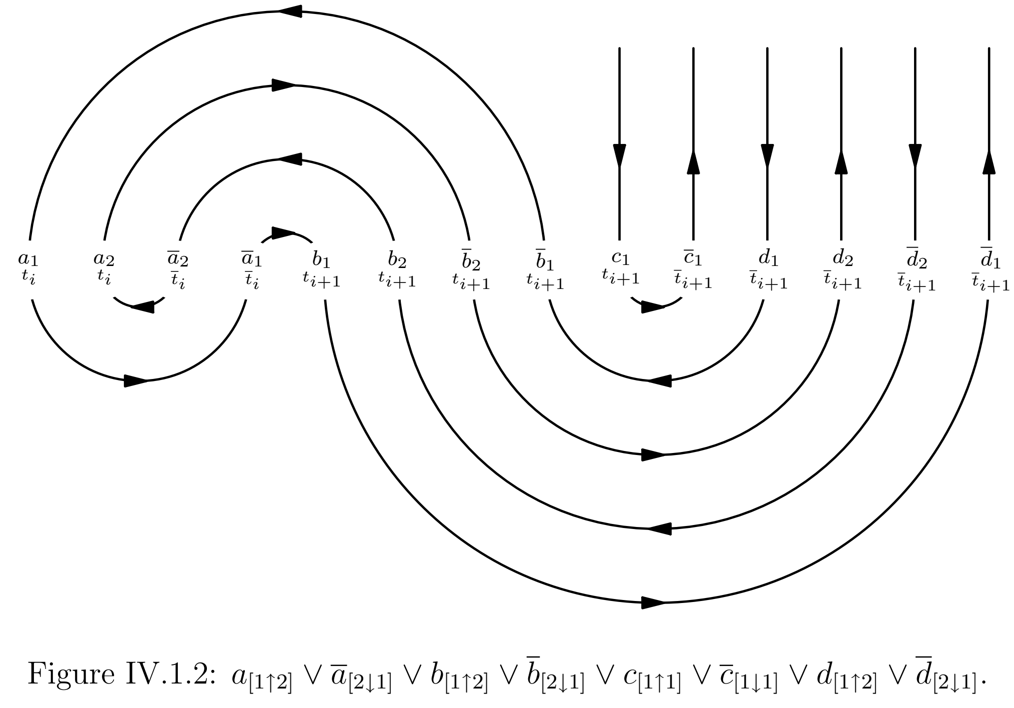

Figure IV.1.2: $a_{[1\uparrow 2]} \vee \overline{a}_{[2\downarrow 1]} \vee b_{[1\uparrow 2]} \vee \overline{b}_{[2\downarrow 1]} \vee c_{[1\uparrow 1]} \vee \overline{c}_{[1\downarrow 1]} \vee d_{[1\uparrow 2]} \vee \overline{d}_{[2\downarrow 1]}$.

In passing from $\{\{p_{2k-1}, p_{2k}\}\}_{k\in[1\uparrow m+1]}$ to $\{\{q_{2k-1}, q_{2k}\}\}_{k\in[1\uparrow(m+2m_{i+1})]}$, we add $\{\{\overline{a}_j, b_j\}, \{a_j, \overline{b}_j\}\}_{j\in[1\uparrow m_{i+1}]}$. In B', for each $j \in [1\uparrow m_{i+1}]$,

$[\overline{a}_j\uparrow b_j] = \overline{a}_{[j\downarrow 1]} \vee b_{[1\uparrow j]}$
has induced odd-even pairing $\{\{\overline{a}_k, b_k\}\}_{k\in[1\uparrow j]}$,

$[a_j\uparrow \overline{b}_j] = a_{[j\uparrow m_{i+1}]} \vee \overline{a}_{[m_{i+1}\downarrow 1]} \vee b_{[1\uparrow m_{i+1}]} \vee \overline{b}_{[m_{i+1}\downarrow j]}$
has induced odd-even pairing $\{\{\overline{a}_k, b_k\}, \{a_k, \overline{b}_k\}\}_{k\in[j\uparrow m_{i+1}]}$.

Both types of intervals are closed under the odd-even pairing; this shows that $\{\{q_{2k-1}, q_{2k}\}\}_{k\in[1\uparrow(m+2m_{i+1})]}$ is nested.

In passing from $\{\{p_{2k}, p_{2k+1}\}\}_{k\in[1\uparrow m]}$ to $\{\{q_{2k}, q_{2k+1}\}\}_{k\in[1\uparrow(m+2m_{i+1}-1)]}$, we delete $\{\{d_j, \overline{d}_j\}\}_{j\in[1\uparrow m_{i+1}]}$, and add $\{\{d_j, \overline{b}_j\}, \{a_j, \overline{a}_j\}, \{b_j, \overline{d}_j\}\}_{j\in[1\uparrow m_{i+1}]}$. In B', for each $j \in [1\uparrow m_{i+1}]$,

$$\begin{aligned}
&[a_j\uparrow\overline{a}_j] = a_{[1\uparrow j]} \vee \overline{a}_{[j\downarrow 1]} \text{ has induced even-odd pairing } \{\{a_k, \overline{a}_k\}\}_{k\in[1\uparrow j]},\\
&[\overline{b}_j\uparrow d_j] = \overline{b}_{[j\downarrow 1]} \vee c_{[1\uparrow r]} \vee \overline{c}_{[r\downarrow 1]} \vee d_{[1\uparrow j]} \text{ has induced even-odd pairing}\\
&\qquad\quad \{\{\overline{b}_k, d_k\}\}_{k\in[1\uparrow j]} \cup \{\{c_i, \overline{c}_i\}\}_{i\in[1\uparrow m_i]},\\
&[b_j\uparrow\overline{d}_j] = b_{[j\uparrow m_{i+1}]} \vee \overline{b}_{[m_{i+1}\downarrow 1]} \vee c_{[1\uparrow m_i]} \vee \overline{c}_{[m_i\downarrow 1]} \vee d_{[1\uparrow m_{i+1}]} \vee \overline{d}_{[m_{i+1}\downarrow j]}\\
&\qquad\quad \text{has induced even-odd pairing}\\
&\qquad\quad \{\{\overline{b}_k, d_k\}\}_{k\in[1\uparrow m_{i+1}]} \cup \{\{b_k, \overline{d}_k\}\}_{k\in[j\uparrow m_{i+1}]} \cup \{\{c_i, \overline{c}_i\}\}_{i\in[1\uparrow m_i]}.
\end{aligned}$$

All three types of intervals are closed under the even-odd pairing; this shows that $\{\{q_{2k}, q_{2k+1}\}\}_{k\in[1\uparrow(m+2m_{i+1}-1)]}$ is nested. □

A similar argument shows that $\overline{\sigma}_i$ carries planar elements to planar elements.

IV.2 Theorem. *The group $\mathcal{B}_n$ acts on the set of planar elements of $\Sigma_{0,1,n}$, and, hence, if $n \geq 1$, every element of $t_1^{\mathcal{B}_n}$ is planar.*

IV.3 Remark. By combining Theorem IV.2 and Proposition III.4, we get another proof of Corollary 7.6. □

V The $\mathcal{B}_n$-orbits of the planar elements of $\Sigma_{0,1,n}$

In this section we rework [21, Lemma 2.3.12] and in this case our argument is longer than Larue's. The object is to show that the number of $\mathcal{B}_n$-orbits in the set of all planar elements of $\Sigma_{0,1,n}$ is $n+1$, and that $\{\Pi t_{[1\uparrow k]}\}_{k\in[0\uparrow n]}$ is a complete set of representatives.

V.1 Lemma. *Let i, j be elements of $[1\uparrow n]$ such that $j \leq i-1$, let $\phi = \Pi\sigma_{[j\uparrow(i-1)]}$, and let w be a planar element of $\Sigma_{0,1,n}$.*

(i) *If $w \in ((\Pi t_{[1\uparrow i]})t_j\star)$, then $|w^\phi| < |w|$.*

(ii) *If $w \in ((\Pi t_{[1\uparrow i]})\overline{t}_j\star)$, then $|w^\phi| < |w|$.*

Proof. It is straightforward to show that ϕ acts as

$$\begin{array}{cccccc}
 & \underline{\scriptstyle k\in[1\uparrow(j-1)]} & & \underline{\scriptstyle k\in[(j+1)\uparrow i]} & \underline{\scriptstyle k\in[(i+1)\uparrow n]} \\
 & (t_k & t_j & t_k & t_k)^\phi \\
= & (t_k & t_i & t_{k-1}^{t_i} & t_k).
\end{array}$$

(i). Suppose that $w \in ((\Pi t_{[1\uparrow i]})t_j\star)$.

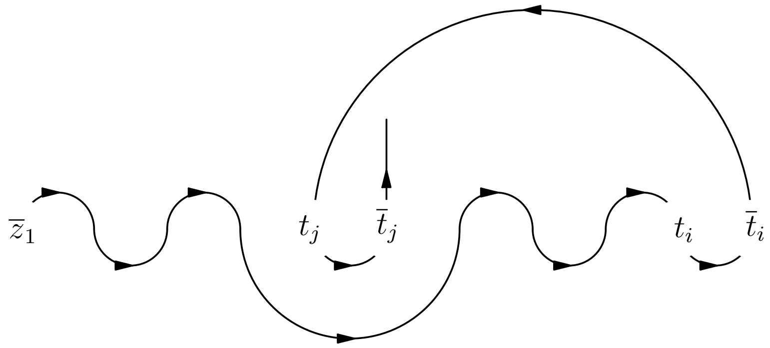

Figure V.1.1: $w \in ((\Pi t_{[1\uparrow i]})t_j\star)$, $j \leq i-1$.

Since $t_i t_j$ is a subword of w and w is planar, every letter occurring in w that belongs to $t_{[j\uparrow i]} \vee \overline{t}_{[j\uparrow i]}$ belongs to a (reduced) subword of w of the form $av\overline{b}$, where $a, b \in \{\overline{t}_i, t_j\}$ and $v \in \langle t_{[j\uparrow i]}\rangle$. Since, moreover, w begins with $\Pi t_{[1\uparrow i]}$, it can be shown that it is not possible to have $a = \overline{t}_i$ or $b = \overline{t}_i$. Thus $a = b = t_j$. Here, $|(av\overline{b})^{\phi}| = |avb| - 2$.

We factor w into syllables consisting of all such subwords together with the individual remaining letters, all of which lie in $t_{[1\uparrow(j-1)]\vee[(i+1)\uparrow n]} \vee \overline{t}_{[1\uparrow(j-1)]\vee[(i+1)\uparrow n]}$, and all of which are mapped to single letters by ϕ.

Since t_j occurs in w, we see that $|w^{\phi}| < |w|$.

(ii). Suppose that $w \in ((\Pi t_{[1\uparrow i]})\overline{t}_j\star)$.

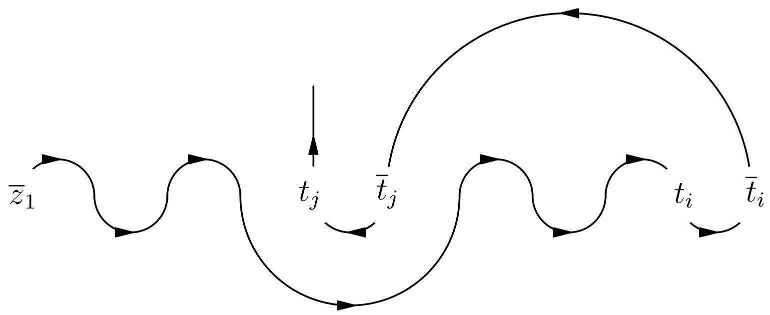

Figure V.1.2: $w \in ((\Pi t_{[1\uparrow i]})\overline{t}_j\star)$, $j \leq i-1$.

Since $t_i\overline{t}_j$ is a subword of w and w is planar, every letter occurring in w that belongs to $t_{[(j+1)\uparrow i]} \vee \overline{t}_{[(j+1)\uparrow i]}$ belongs to a (reduced) subword of w of the form $av\overline{b}$, where $a, b \in \{t_j, \overline{t}_i\}$ and $v \in \langle t_{[j+1\uparrow i]}\rangle$. Since, moreover, w begins with $\Pi t_{[1\uparrow i]}$, it can be shown that it is not possible to have $a = \overline{t}_i$ or $b = \overline{t}_i$. Thus $a = b = t_j$. Here, $|(av\overline{b})^{\phi}| = |avb| - 2$.

We factor w into syllables consisting of all such subwords together with the individual remaining letters, all of which lie in $t_{[1\uparrow j]\vee[(i+1)\uparrow n]} \vee \overline{t}_{[1\uparrow j]\vee[(i+1)\uparrow n]}$, and all of which are mapped to single letters by ϕ.

Since t_i occurs in w, it is then clear that $|w^{\phi}| \leq |w| - 2$. □

V.2 Lemma. *Let i, j be elements of $[1\uparrow n]$ such that $j \geq i+2$, let $\phi = \Pi\overline{\sigma}_{[(j-1)\downarrow(i+1)]}$, and let w be a planar element of $\Sigma_{0,1,n}$.*

(i) *If* $w \in ((\Pi t_{[1\uparrow i]})t_j\star)$, *then* $|w^\phi| \le |w|$, *and, moreover, if* $|w^\phi| = |w|$ *then* $w^\phi \in (\Pi t_{[1\uparrow i+1]}\star)$.

(ii) *If* $w \in ((\Pi t_{[1\uparrow i]})\overline{t}_j\star)$, *then* $|w^\phi| < |w|$.

Proof. It is straightforward to show that ϕ acts as

$$\begin{array}{rcccc} & \underline{\scriptstyle k\in[1\uparrow i]} & \underline{\scriptstyle k\in[(i+1)\uparrow(j-1)]} & & \underline{\scriptstyle k\in[(j+1)\uparrow n]} \\ & (t_k & t_k & t_j & t_k)^\phi \\ = & (t_k & t_{k+1}^{\overline{t}_{i+1}} & t_{i+1} & t_k). \end{array}$$

(i). Suppose that $w \in ((\Pi t_{[1\uparrow i]})t_j\star)$.

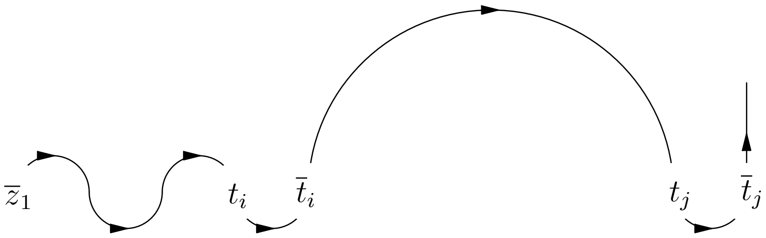

Figure V.2.1: $w \in ((\Pi t_{[1\uparrow i]})t_j\star)$, $j \ge i+2$.

Since $t_i t_j$ is a subword of w, every letter occurring in w that belongs to $t_{[(i+1)\uparrow(j-1)]} \vee \overline{t}_{[(i+1)\uparrow(j-1)]}$ belongs to a (reduced) subword of w of the form $av\overline{b}$, where $a, b \in \{t_i, \overline{t}_j\}$ and $v \in \langle t_{[(i+1)\uparrow(j-1)]}\rangle$. Since, moreover, w begins with $\Pi t_{[1\uparrow i]}$, it can be shown that it is not possible to have $a = t_i$ or $b = t_i$. Thus $a = b = \overline{t}_j$. Here, $|(av\overline{b})^\phi| = |av\overline{b}| - 2$.

We factor w into syllables consisting of all such subwords together with the individual remaining letters, all of which lie in $t_{[1\uparrow i]\vee[j\uparrow n]} \vee \overline{t}_{[1\uparrow i]\vee[j\uparrow n]}$, and all of which are mapped to single letters by ϕ.

It is then clear that $|w^\phi| \le |w|$.

Moreover, if $|w^\phi| = |w|$, then $w \in \langle t_{[1\uparrow i]\vee[j\uparrow n]}\rangle$, and $w^\phi \in (\Pi t_{[1\uparrow(i+1)]}\star)$.

(ii). Suppose that $w \in ((\Pi t_{[1\uparrow i]})\overline{t}_j\star)$.

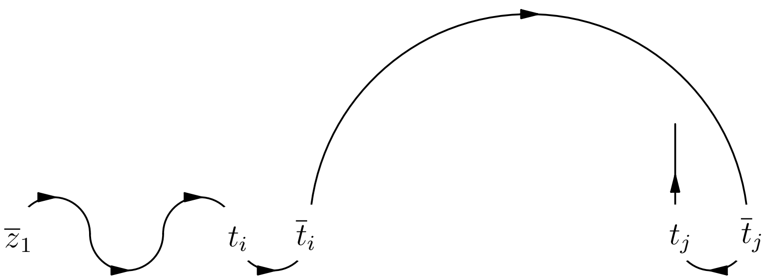

Figure V.2.2: $w \in ((\Pi t_{[1\uparrow i]})\overline{t}_j\star)$, $j \ge i+2$.

Since $t_i\overline{t}_j$ is a subword of w and w is planar, every letter occurring in w that belongs to $t_{[(i+1)\uparrow j]} \vee \overline{t}_{[(i+1)\uparrow j]}$ belongs to a (reduced) subword of w of the form $av\overline{b}$, where $a, b \in \{t_i, \overline{t}_j\}$ and $v \in \langle t_{[(i+1)\uparrow j]}\rangle$. Since, moreover, w begins with

$\Pi t_{[1\uparrow i]}$, it can be shown that it is not possible to have $a = t_i$ or $b = t_i$. Thus $a = b = \bar{t}_j$. Here, $|(av\bar{b})^\phi| = |av\bar{b}| - 2$.

We factor w into syllables consisting of all such subwords together with the individual remaining letters, all of which lie in $t_{[1\uparrow i]\vee[(j+1)\uparrow n]} \vee \bar{t}_{[1\uparrow i]\vee[(j+1)\uparrow n]}$, and all of which are mapped to single letters by ϕ.

Since $\bar{t}_j$ occurs in w, it is then clear that $|w^\phi| \le |w| - 2$. □

V.3 Theorem (Larue)**.** *The set* $\{\Pi t_{[1\uparrow k]}\}_{k\in[0\uparrow n]}$ *is a complete set of representatives of the* $\mathcal{B}_n$*-orbits in the set of all planar elements of* $\Sigma_{0,1,n}$*.*

Proof. Let w be a planar element of $\Sigma_{0,1,n}$. We wish to show that there exists some $k \in [0\uparrow n]$ such that $\Pi t_{[1\uparrow k]} \in w^{\mathcal{B}_n}$.

Let i be the largest integer such that $w \in (\Pi t_{[1\uparrow i]}\star)$.

We may assume that, for all $v \in w^{\mathcal{B}_n}$, $|v| \ge |w|$, and if $|v| = |w|$, then $v \notin (\Pi t_{[1\uparrow i+1]}\star)$.

By Lemma V.1, for all $j \in [1\uparrow(i-1)]$, $w \notin ((\Pi t_{[1\uparrow i]})t_j\star) \cup ((\Pi t_{[1\uparrow i]})\bar{t}_j\star)$.

By Proposition III.4(i), $w \notin ((\Pi t_{[1\uparrow i]})t_i\star)$.

By the maximality of i, $w \notin ((\Pi t_{[1\uparrow i]})t_{i+1}\star)$.

By Proposition III.4(iii), $w \notin ((\Pi t_{[1\uparrow i]})\bar{t}_{i+1}\star)$.

By Lemma V.2, for all $j \in [(i+2)\uparrow n]$, $w \notin ((\Pi t_{[1\uparrow i]})t_j\star) \cup ((\Pi t_{[1\uparrow i]})\bar{t}_j\star)$.

Hence, $w = \Pi t_{[1\uparrow i]}$, as desired. □

V.4 Remarks. (i). Let w be a planar element of $\Sigma_{0,1,n}$.

Lemmas V.1 and V.2 give an effective procedure for finding $\phi \in \mathcal{B}_n$ first to minimize $|w^\phi|$, and then to obtain the form $w^\phi = \Pi t_{[1\uparrow k]}$ for some $k \in [0\uparrow n]$.

(ii). Let $n \ge 1$ and let $w \in \Sigma_{0,1,n}$.

Theorem V.3 shows that w lies in the $\mathcal{B}_n$-orbit of t_1 if and only if the cyclically-reduced form of w lies in $t_{[1\uparrow n]}$ and w is planar. Moreover, in this event, Lemmas V.1 and V.2 effectively produce a $\phi \in \mathcal{B}_n$ such that $w^\phi = t_1$.

(iii). There is an algorithm which, for any $k \in [1\uparrow n]$, and any sequence $w_{[1\uparrow k]}$ in $\Sigma_{0,1,n}$, decides if there exists some $\phi \in \mathcal{B}_n$ such that $w^\phi_{[1\uparrow k]} = t_{[1\uparrow k]}$, and effectively finds such a ϕ, by using (ii) to convert w_1 to t_1 if possible, and then restricting to $\langle \sigma_{[2\uparrow(n-1)]}\rangle$.

This algorithm for $\mathcal{B}_n$ is simpler than the Whitehead algorithm for the larger group $\operatorname{Aut}\Sigma_{0,1,n}$, essentially because the information carried by planarity is more detailed then the information carried by the Whitehead graph used in the Whitehead algorithm. Enric Ventura has pointed out to us that Whitehead's algorithm has the power to decide whether *any* pair of *conjugacy classes* in $\Sigma_{0,1,n}$ lie in the same $\mathcal{B}_n$-orbit or not; see, for example, [23, Proposition I.4.21]. □

Let us conclude by emphasizing (ii).

V.5 Theorem (Larue)**.** *Let* $n \ge 1$ *and let* $w \in \Sigma_{0,1,n}$*. Then* w *lies in the* $\mathcal{B}_n$*-orbit of* t_1 *if and only if the cyclically-reduced form of* w *lies in* $t_{[1\uparrow n]}$ *and* w *is planar.* □

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