Retracts of vertex sets of trees and the almost stability theorem

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Abstract

Let G be a group, let T be an (oriented) G-tree with finite edge stabilizers, and let VT denote the vertex set of T. We show that, for each G-retract V' of the G-set VT, there exists a G-tree whose edge stabilizers are finite and whose vertex set is V'. This fact leads to various new consequences of the almost stability theorem.

We also give an example of a group G, a G-tree T and a G-retract V' of VT such that no G-tree has vertex set V'.

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1 Outline

Throughout the article, let G be a group, and let $\mathbb{N}$ denote the set of finite cardinals, $\{0, 1, 2, \ldots\}$. All our G-actions will be on the left.

The following extends Definitions II.1.1 of [3] (where A is assumed to have trivial G-action).

1.1 Definition. Let E and A be G-sets.

Let (E, A) denote the set of all functions from E to A. An element v of (E, A) has the form $v\colon E \to A$, $e \mapsto v(e)$. There is a natural G-action on (E, A) such that $(gv)(e) := g(v(g^{-1}e))$ for all $v \in (E, A)$, $g \in G$, $e \in E$.

Two elements v and w of (E, A) are said to be *almost equal* if the set

$$\{e \in E \mid v(e) \neq w(e)\}$$

is finite. Almost equality is an equivalence relation; the equivalence classes are called the *almost equality classes in* (E, A).

A subset V of (E, A) is said to be *G-stable* if V is closed under the G-action. In general, a G-stable subset is the same as a G-subset. $\square$

In this article, we wish to strengthen the following result.

1.2 The almost stability theorem [3, Theorem III.8.5]. *If E is a G-set with finite stabilizers, and A is a nonempty set with trivial G-action, and V is a G-stable almost equality class in the G-set (E, A), then there exists a G-tree with finite edge stabilizers and vertex set V.* $\square$

In the light of Bass-Serre theory, the almost stability theorem can be thought of as a broad generalization of Stallings' ends theorem; see [3, Theorem III.2.1].

Let us now recall the notion of a G-retract of a G-set. The following alters Definition III.1.1 of [3] slightly.

1.3 Definition. A *G-retract* U of a G-set V is a G-subset of V with the property that, for each $w \in V - U$, there exists $u \in U$ such that $G_w \leq G_u$, or, equivalently, with the property that there exists a G-map, called a *G-retraction*, from V to U which is the identity on U. □

Chapter IV of [3] collects together a wide variety of consequences of the almost stability theorem 1.2. In some of these applications, the conclusions assert that certain naturally arising G-sets are G-retracts of vertex sets of G-trees with finite edge stabilizers. This leads to the question of whether or not the class of vertex sets of G-trees with finite edge stabilizers is closed under taking G-retracts. We are now able to answer this in the affirmative; in Section 4 below, we prove that any G-retract of the vertex set of a G-tree with finite edge stabilizers is itself the vertex set of a G-tree with finite edge stabilizers.

In Section 5, we record the resulting generalizations of the almost stability theorem and the applications which are affected. In the most classic example, if G has cohomological dimension one, and $\omega\mathbb{Z}G$ is the augmentation ideal of the group ring $\mathbb{Z}G$, one can deduce that G acts freely on a tree whose vertex set is the G-set $1 + \omega\mathbb{Z}G$, and, hence, G is a free group; this is a slightly more detailed version of a theorem of Stallings and Swan.

In Section 6, we record an even more general form of the almost stability theorem in which the G-action on A need not be trivial.

In Section 7, we construct a group G and a G-retract of a vertex set of a G-tree (with infinite edge stabilizers) that is not itself the vertex set of a G-tree.

2 Operations on trees

Throughout this section we will be working with the following.

2.1 Hypotheses. Let $T = (T, V, E, \iota, \tau)$ be a G-tree, as in [3, Definition I.2.3].

We write $VT = V$ and $ET = E$, and we view the underlying G-set of T as the disjoint union of V and E, written $T = V \vee E$. Here $\iota\colon E \to V$ is the *initial vertex* map and $\tau\colon E \to V$ is the *terminal vertex* map. □

We first consider a simple form of retraction, which amplifies Definitions III.7.1 of [3]. Recall that a vertex v of a tree is called a *sink* if every edge of the tree is oriented towards v.

2.2 The compressing lemma. *Suppose that Hypotheses* 2.1 *hold.*

Let E' be a G-subset of E such that each component of the subforest $T - E'$ of T has a (unique) sink. Let V' denote the set of sinks of the components of $T - E'$.

Let $i\colon E' \to E$ denote the inclusion map, and let $\phi\colon V \to V'$ denote the G-retraction which assigns, to each $v \in V$, the sink of that component of $T - E'$ which contains v.

Then the G-graph $T' = (T', V', E', \phi \circ \iota \circ i, \phi \circ \tau \circ i)$ is a G-tree.

Let $E'' = E - E'$ and let $V'' = V - V'$. Then $T - E'$ is the G-subforest of T with vertex set V and edge set E''. For each $v \in V$, $\phi(v)$ is reached in T by starting at v and travelling as far as possible along edges in E'' respecting the orientation. The initial vertex map $\iota\colon E \to V$ induces a bijective map $E'' \to V''$.

We say that T' is obtained from T by *compressing the closures of the elements of E'' to their terminal vertices* or by *compressing the components of $T - E'$ to their sinks.*

In applications, we usually first G-equivariantly reorient T and then, in the resulting tree, compress a G-set of closed edges to their terminal vertices; we then call the combined procedure a *G-equivariant compressing operation.*

Proof of Lemma 2.2. The map ϕ induces a surjective G-map $T \to T'$ in which the fibres are the components of $T - E'$. It follows that T' is a G-tree. □

We now recall the sliding operation of Rips-Sela [8, p. 59] as generalized by Forester [7, Section 3.6]; see also the Type 1 operation of [6, p. 146]. We find it convenient to express the result and the proof in the notation of [3].

2.3 The sliding lemma. *Suppose that Hypotheses* 2.1 *hold.*

Let e and f be elements of E.

Suppose that $\tau e = \iota f$, $G_e \le G_f$, and $Gf \cap Ge = \emptyset$.

Let $\tau'\colon E \to V$ denote the map given by

$$e' \mapsto \tau'(e') := \begin{cases} \tau(e') & \text{if } e' \in E - Ge, \\ \tau(gf) & \text{if } e' = ge \text{ for some } g \in G, \end{cases}$$

for all $e' \in E$.

Then the G-graph $T' = (T', V, E, \iota, \tau')$ is a G-tree.

Here, we say that T' is obtained from T by *G-equivariantly sliding τe along f from ιf to τf.*

In applications, we usually first G-equivariantly reorient Ge, or Gf, or both, or neither, and then, in the resulting tree, G-equivariantly slide τe along f from ιf to τf, and then reorient back again. We then call the combined procedure a *G-equivariant sliding operation.*

Proof of Lemma 2.3. It is clear that T' is a G-graph.

Let X be the G-graph obtained from T by deleting the two edge orbits $Ge \cup Gf$, and then inserting one new vertex orbit Gv and three new edge orbits $Ge' \cup Gf_1 \cup Gf_2$, with $G_{e'} = G_e$, $G_v = G_{f_1} = G_{f_2} = G_f$, and setting

$$\iota(e') := \iota(e), \quad \iota(f_1) := \iota(f) = \tau(e), \quad \iota(f_2) := \tau(e') := \tau(f_1) := v, \quad \tau(f_2) := \tau(f).$$

Thus we are G-equivariantly subdividing f into f_1 and f_2 by adding v, and then sliding τe along f_1 from ιf_1 to $\tau f_1 = v$.

Then T is recovered from X by G-equivariantly compressing the closure of f_1 to $\iota(f_1)$, and renaming f_2 as f, e' as e. Thus X maps onto T with fibres which are trees. It follows that X is a tree; see [3, Proposition III.3.3].

Also T' is recovered from X by G-equivariantly compressing the closure of f_2 to $\tau(f_2)$, and renaming f_1 as f, e' as e. By the compressing lemma 2.2, T' is a tree. □

3 Filtrations

Throughout this section we will be working with the following.

3.1 Hypotheses. Let $T = (T, V, E, \iota, \tau)$ be a G-tree, let U be a G-retract of the G-set V, and let $W = V - U$. □

3.2 Conventions. We shall use interval notation for ordinals; for example, if κ is an ordinal, then $[0, \kappa)$ denotes the set of all ordinals α such that $\alpha < \kappa$.

If we have an ordinal κ and a specified map from a set X to $[0, \kappa)$, then we will understand that the following notation applies. Denoting the image of each $x \in X$ by $\mathrm{height}(x) \in [0, \kappa)$, we write, for each $\alpha \in [0, \kappa)$ and each $\beta \in [0, \kappa]$,

$$X[\alpha] := \{x \in X \mid \mathrm{height}(x) = \alpha\} \quad \text{and} \quad X[0, \beta) := \{x \in X \mid \mathrm{height}(x) < \beta\}. \ \square$$

3.3 Definitions. Suppose that Hypotheses 3.1 hold.

Let $P(T)$ denote the set of paths in T, as in Definitions I.2.3 of [3]. Thus, for each $p \in P(T)$, we have the *initial vertex* of p, denoted ιp, the *terminal vertex* of p, denoted τp, the *set of edges which occur in* p, denoted $E(p) \subseteq E$, the *length* of p, denoted $\text{length}(p) \in \mathbb{N}$, and the *$G$-stabilizer* of p, denoted $G_p \leq G$.

Let κ be an ordinal and let

$$(3.3.1) \qquad T \to [0, \kappa), \quad x \mapsto \text{height}(x),$$

be a map. Since T is nonempty, κ must be nonzero. As a set, $T = V \cup E$. Thus, for each $\alpha \in [0, \kappa)$, we have $T[\alpha]$, $E[\alpha]$ and $V[\alpha]$, and, for each $\beta \in [0, \kappa]$, we have $T[0, \beta)$, $E[0, \beta)$ and $V[0, \beta)$.

For each $w \in W$, we then define

$$\begin{aligned} P_T(w) := \{p \in P(T) \mid \iota p = w,\, G_p = G_w,\, \text{height}(\tau p) < \text{height}(w),& \\ \text{height}(E(p)) \subseteq \{\text{height}(w), \text{height}(w) + 1\}\}&. \end{aligned}$$

We say that (3.3.1) is a *U-filtration of T* if all of the following hold:

(3.3.2) for each $\beta \in [0, \kappa]$, $T[0, \beta)$ is a G-subforest of T;

(3.3.3) $T[0] = U$;

(3.3.4) for each $\alpha \in [1, \kappa)$, $T[\alpha]$ is a G-finite G-subset of T; and,

(3.3.5) for each $w \in W$, $P_T(w)$ is nonempty. □

3.4 Lemma. *If Hypotheses* 3.1 *hold, then there exists a U-filtration of T.*

Proof. We shall recursively construct a family $(E[\alpha] \mid \alpha \in [0, \kappa))$ of G-subsets of E, for some nonzero ordinal κ.

We take $E[0] = \emptyset$.

Suppose that γ is a nonzero ordinal, and that we have a family $(E[\alpha] \mid \alpha \in [0, \gamma))$ of G-subsets of E.

For each $\beta \in [0, \gamma]$, we define

$$E[0, \beta) := \bigcup_{\alpha \in [0, \beta)} E[\alpha] \quad \text{and} \quad V[0, \beta) := \begin{cases} \emptyset & \text{if } \beta = 0, \\ U \cup \iota(E[0, \beta)) \cup \tau(E[0, \beta)) & \text{if } \beta > 0. \end{cases}$$

For each $\alpha \in [0, \gamma)$, we define $V[\alpha] := V[0, \alpha + 1) - V[0, \alpha)$. Thus

$$V[0, \beta) = \bigcup_{\alpha \in [0, \beta)} V[\alpha].$$

If $E[0, \gamma) = E$, we take $\kappa = \gamma$ and the construction terminates.

Now suppose that $E[0, \gamma) \subset E$. We shall explain how to choose $E[\gamma]$.

If γ is a limit ordinal or 1, we take $E[\gamma]$ to be an arbitrary single G-orbit in $E - E[0, \gamma)$.

If γ is a successor ordinal greater than 1 then there is a unique $\alpha \in [1, \gamma)$ such that $\gamma = \alpha + 1$, and we want to construct $E[\alpha + 1]$. Notice that $V[0, \alpha)$ is a G-retract of V because $V[0, \alpha)$ contains U. Thus we can G-equivariantly specify, for each $w \in V[\alpha]$, a T-geodesic $p = p(w)$ from w to an element $v = v(w) \in V[0, \alpha)$ fixed by G_w. Since G_w fixes both ends of p, G_w fixes p. Hence we may assume that v is the first, and hence only, vertex of p that lies in $V[0, \alpha)$. Clearly G_p fixes w. Thus $G_w = G_p$. Let $P_{\alpha+1}$ denote the set of edges which occur in the $p(w)$, as w

ranges over $V[\alpha]$. Then $P_{\alpha+1} \subseteq E - E[0,\alpha)$, since each element of $E[0,\alpha)$ has *both* vertices in $V[0,\alpha)$. If $P_{\alpha+1} \subseteq E[\alpha]$, we choose $E[\alpha+1]$ to be an arbitrary single G-orbit in $E - E[0,\alpha+1)$. If $P_{\alpha+1} \not\subseteq E[\alpha]$, we take $E[\alpha+1] = P_{\alpha+1} - E[\alpha]$. This completes the description of the recursive construction.

We now verify that we have a U-filtration of T.

It can be seen that, for each ordinal γ such that $(E[\alpha] \mid \alpha \in [0,\gamma))$ is defined, the $E[\alpha]$, $\alpha \in [1,\gamma)$, are pairwise disjoint, nonempty, G-subsets of E. Hence the cardinal of γ is at most one more than the cardinal of E. Therefore the construction terminates at some stage. This implies that there exists a nonzero ordinal κ such that $E[0,\kappa) = E$. Also $V[0,\kappa) = V$, and $(V[\alpha] \mid \alpha \in [0,\kappa))$ gives a partition of V. Thus we have an implicit map $T \to [0,\kappa)$ and we denote it by $x \mapsto \text{height}(x)$.

Clearly (3.3.2), (3.3.3) and (3.3.5) hold.

If $\alpha \in [1,\kappa)$ and $E[\alpha]$ is G-finite, then either $E[0,\alpha+1) = E$ or $V[\alpha]$, $P_{\alpha+1}$ and $E[\alpha+1]$ are G-finite. It follows, by transfinite induction, that $E[\alpha]$ and $V[\alpha]$ are G-finite for all $\alpha \in [1,\kappa)$. Thus (3.3.4) holds. □

4 The main result

Let us introduce a technical concept which generalizes that of a finite subgroup.

4.1 Definitions. A subgroup H of G is said to be *G-conjugate incomparable* if, for each $g \in G$, $H^g \subseteq H$ (if and) only if $H^g = H$. This clearly holds if H is finite.

We say that a G-set X *has G-conjugate-incomparable stabilizers* if, for each $x \in X$, the G-stabilizer G_x is a G-conjugate-incomparable subgroup, that is, for each $g \in G$, $G_x \subseteq G_{gx}$ (if and) only if $G_x = G_{gx}$. □

Throughout this section we will be working with the following.

4.2 Hypotheses. Let $T = (T, V, E, \iota, \tau)$ be a G-tree, let U be a G-retract of the G-set V, and let $W = V - U$.

Suppose that the G-set W has G-conjugate-incomparable stabilizers.

Let κ be an ordinal and let

(4.2.1) $\quad \text{height}\colon V \cup E \to [0,\kappa), \quad x \mapsto \text{height}(x),$

be a U-filtration of T. □

4.3 Definitions. Suppose that Hypotheses 4.2 hold.

Let $w \in W$. Define $d_T(w) := \min\{\text{length}(p) \mid p \in P_T(w)\}$. Then $d_T(w)$ is a positive integer and

(4.3.1) $\quad d_T(gw) = d_T(w)$ for all $g \in G$.

For v_0, v_1 in V, we say that v_1 is *lower than* v_0 if one of the following holds:

(4.3.2) $\quad \text{height}(v_0) > \text{height}(v_1)$;

(4.3.3) $\quad \text{height}(v_0) = \text{height}(v_1) > 0$ and $G_{v_0} < G_{v_1}$; or,

(4.3.4) $\quad \text{height}(v_0) = \text{height}(v_1) > 0$ and $G_{v_0} = G_{v_1}$ and $d_T(v_0) > d_T(v_1)$.

An edge e of T is said to be *problematic* if it joins vertices v_0, v_1 such that $\text{height}(e) = \text{height}(v_1) = \text{height}(v_0) + 1$. Notice that $\text{height}(e)$ is a successor ordinal and that v_0 is lower than v_1.

For each $v_0 \in W$, there exists a path

(4.3.5) $\quad v_0, e_1^{\epsilon_1}, v_1, e_2^{\epsilon_2}, v_2, \ldots, e_d^{\epsilon_d}, v_d$ in $P_T(v_0)$ such that $d = d_T(v_0)$.

Here height$(v_1) \le$ height$(v_0) + 1$. We say that v_0 is a *problematic* vertex of T if there exists a path as in (4.3.5) such that height$(v_1) =$ height$(v_0) + 1$. In this event height$(e_1) =$ height(v_1) and e_1 is a problematic edge of T. □

4.4 Lemma. *If Hypotheses* 4.2 *hold, then applying some transfinite sequence of G-equivariant sliding operations to T yields a G-tree $T' = (T', V, E, \iota', \tau')$ such that* (4.2.1) *is also a U-filtration of T' and T' has no problematic vertices.*

Proof. We shall construct a family of trees

$$(T_\beta = (T_\beta, V, E, \iota_\beta, \tau_\beta) \mid \beta \in [0, \kappa])$$

such that, for each $\beta \in [0, \kappa]$, (4.2.1) is a U-filtration of T_β, and T_β has no problematic vertices in $V[0, \beta)$.

We take $T_0 = T$.

For each successor ordinal $\beta = \alpha + 1 \in [0, \kappa)$, $T_{\alpha+1}$ will be obtained from T_α by altering, if necessary, ι_α and τ_α on $E[\alpha + 1]$, as described below.

For each limit ordinal $\beta \in [0, \kappa]$, we let ι_β be given on $E[\alpha]$ by ι_α, for each $\alpha \in [0, \beta)$, and similarly for τ_β.

Suppose then that $\beta = \alpha+1 \in [0, \kappa)$, that we have a tree $T_\alpha = (T_\alpha, V, E, \iota_\alpha, \tau_\alpha)$, and that (4.2.1) is a U-filtration of T_α, and that T_α has no problematic vertices in $V[0, \alpha)$.

We now describe a crucial *problem-reducing procedure* that can be applied in the case where there exists some $v_0 \in V[\alpha]$ which is a problematic vertex of T_α.

Let $d = d_{T_\alpha}(v_0)$. Thus, there exists a path

$$v_0, e_1^{\epsilon_1}, v_1, e_2^{\epsilon_2}, v_2, \ldots, e_d^{\epsilon_d}, v_d$$

in $P_{T_\alpha}(v_0)$ such that $v_1 \in V[\alpha+1]$. Hence, $e_1 \in E[\alpha+1]$. Without loss of generality, let us assume that $\epsilon_1 = -1$.

There exists a least $i \in [2, d]$ such that $v_i \in V[0, \alpha + 1)$. Then

$$\{v_1, \ldots, v_{i-1}\} \subseteq V[\alpha + 1] \quad \text{and, hence,} \quad \{e_1, \ldots, e_i\} \subseteq E[\alpha + 1].$$

We claim that $Ge_1 \cap \bigcup_{j=2}^{i} Ge_j = \emptyset$. Suppose this fails. Then $e_1 \in \bigcup_{j=2}^{i} Ge_j$. Here, $v_0 \in \bigcup_{j=1}^{i} Gv_j$. Since $v_0 \in V[\alpha]$ and $\bigcup_{j=1}^{i-1} Gv_j \subseteq V[\alpha + 1]$ we see that $v_0 \in Gv_i$. Hence $v_i \in V[\alpha]$ and, by (4.3.1), $d_{T_\alpha}(v_i) = d_{T_\alpha}(v_0) = d$. But $G_{v_0} = G_p \subseteq G_{v_i}$. Since G_{v_0} is a G-conjugate-incomparable subgroup, $G_{v_0} = G_{v_i}$. It follows that

$$v_i, e_{i+1}^{\epsilon_{i+1}}, v_{i+1}, \ldots, e_d^{\epsilon_d}, v_d$$

lies in $P_{T_\alpha}(v_i)$. Hence $d_{T_\alpha}(v_i) \le d - i$, which is a contradiction. This proves the claim.

By Lemma 2.3, we can G-equivariantly slide ιe_1 along $e_2^{\epsilon_2}$ from v_1 to v_2, and then G-equivariantly slide ιe_1 along $e_3^{\epsilon_3}$ from v_2 to v_3, and so on, up to v_i. We then get a new G-tree $T_{\alpha,1} = (T_{\alpha,1}, V, E, \iota_{\alpha,1}, \tau_{\alpha,1})$ by G-equivariantly sliding ιe_1 along our path from v_1 to v_i.

Let e_1' denote e_1 viewed as an edge of $T_{\alpha,1}$. Wherever v_1, e_1, v_0 occurs in a path in T_α, it can be replaced with the sequence

$$v_1, e_2^{\epsilon_2}, v_2, \ldots, v_{i-1}, e_i^{\epsilon_i}, v_i, e_1', v_0$$

to obtain a path in $T_{\alpha,1}$. It is important to note that all the edges involved here lie in $E[\alpha + 1]$. In terms of the free groupoid on $E[\alpha + 1]$, $e_1 = e_2^{\epsilon_2} e_3^{\epsilon_3} \cdots e_i^{\epsilon_i} e_1'$, and we are performing the change-of-basis which replaces e_1 with e_1'.

It is easy to see that (3.3.2)–(3.3.5) then hold for $T_{\alpha,1}$. Thus (4.2.1) is a U-filtration of $T_{\alpha,1}$. Notice that $T_{\alpha,1}$, like T_α, has no problematic vertices in $V[0,\alpha)$. We have reduced the number of G-orbits of problematic edges in $E[\alpha+1]$.

This completes the description of a problem-reducing procedure.

Since $E[\alpha+1]$ is G-finite by (3.3.4), on repeating problem-reducing procedures as often as possible, we find some $m \in \mathbb{N}$, and a sequence

$$T_\alpha = T_{\alpha,0},\, T_{\alpha,1},\, \ldots,\, T_{\alpha,m},$$

such that $T_{\alpha,m}$ has no problematic vertices in $V[0,\alpha) \cup V[\alpha] = V[0,\alpha+1)$. We define $T_{\alpha+1} = (T_{\alpha+1}, V, E, \iota_{\alpha+1}, \tau_{\alpha+1})$ to be $T_{\alpha,m}$. Notice that $\iota_{\alpha+1}$ agrees with ι_α on $E - E[\alpha+1]$, and similarly for $\tau_{\alpha+1}$.

Continuing this procedure transfinitely, we arrive at a tree T_κ which has no problematic vertices. □

4.5 Lemma. *If Hypotheses* 4.2 *hold and T has no problematic vertices, then applying some G-equivariant compressing operation on T yields a G-tree with vertex set U.*

Proof. We claim that any sequence in V is finite if each term is lower than all its predecessors.

Let $\alpha \in [0,\kappa)$.

If v_0, v_1 are elements of the same G-orbit of $V[\alpha]$, then v_1 is not lower than v_0, that is, (4.3.2)–(4.3.4) all fail; this follows from (4.3.1) and the fact that $V[\alpha]$ has G-conjugate-incomparable stabilizers.

Thus, if $n \in \mathbb{N}$ and $v_1, v_2, \ldots, v_n$ is a sequence in $V[\alpha]$ such that each term is lower than all its predecessors, then $Gv_1, Gv_2, \ldots, Gv_n$ are pairwise disjoint, and n is at most the number of G-orbits in $V[\alpha]$. It follows that any sequence in $V[\alpha]$ is finite if each term is lower than all its predecessors. The claim now follows.

Let us G-equivariantly reorient T so that, for each edge e, ιe is not lower than τe.

Let $v_0 \in W$. Let us G-equivariantly choose a path

$$v_0, e_1^{\epsilon_1}, v_1, e_2^{\epsilon_2}, v_2, \ldots, e_d^{\epsilon_d}, v_d$$

in $P_T(v_0)$ such that $d = d_T(v_0)$. Then we call e_1 the *distinguished edge* associated to v_0, and v_1 the *distinguished neighbour* of v_0.

Let E'' denote the set of distinguished edges chosen in this way.

Let us consider the above path for v_0. From Definitions 4.3, we see that, since T has no problematic vertices, $\text{height}(v_0) \geq \text{height}(v_1)$. We claim that v_1 is lower than v_0. The claim is clear if $\text{height}(v_0) > \text{height}(v_1)$ (in which case, $d = 1$), and we may assume that $\text{height}(v_0) = \text{height}(v_1)$ (> 0). Again, the claim is clear if $G_{v_0} < G_{v_1}$, and we may assume that $G_{v_0} = G_{v_1}$. Here G_{v_1} fixes p, and the path

$$v_1, e_2^{\epsilon_2}, v_2, \ldots, e_d^{\epsilon_d}, v_d$$

shows that $d_T(v_1) \leq d-1 < d = d_T(v_0)$, and the claim is proved. Hence $\epsilon_1 = 1$.

Thus ι induces a bijection $E'' \to W$.

Moreover, in travelling along the distinguished edge e_1 respecting the orientation, from v_0 to its distinguished neighbour v_1, we move to a lower vertex.

Thus, starting at any element v of V, after travelling a finite number of steps along distinguished edges respecting the orientation, we arrive at a vertex, denoted $\phi(v)$, with no distinguished neighbours, that is, $\phi(v) \in U$.

By Lemma 2.2, compressing the closures of the distinguished edges to their terminal vertices gives a G-tree with vertex set U and edge set $E - E''$. □

We now come to our main result. In Section 7, we will see that the G-conjugate-incomparability hypotheses cannot be omitted.

4.6 Theorem. *Let T be a G-tree, and let U be a G-retract of the G-set VT. Suppose that the G-set ET has G-conjugate-incomparable stabilizers, or, more generally, that the G-set $VT - U$ has G-conjugate-incomparable stabilizers.*

Then applying to T some transfinite sequence of G-equivariant sliding operations followed by some G-equivariant compressing operation yields a G-tree T' such that $VT' = U$.

Here ET' is a G-subset of ET, and there exists a G-set isomorphism

$$ET - ET' \simeq VT - VT' = VT - U.$$

Proof. For each $w \in VT - U$, there exists $u \in U$ such that $G_w \leq G_u$. If e denotes the first edge in the T-geodesic from w to u, then $G_e = G_w$. Thus, if E has G-conjugate-incomparable stabilizers, then the same holds for $VT - U$.

By Lemma 3.4, we may assume that Hypotheses 4.2 hold. By Lemma 4.4, we may assume that T itself has no problematic vertices. Applying Lemma 4.5, we obtain the result; the final assertion follows from the compression lemma 2.2. □

We record the special case of Theorem 4.6 that is of interest to us.

4.7 The retraction lemma. *Let T be a G-tree whose edge stabilizers are finite, and let U be any G-retract of the G-set VT. Then there exists a G-tree whose edge stabilizers are finite and whose vertex set is the G-set U.* □

5 The almost stability theorem and applications

We now combine the almost stability theorem 1.2 and the retraction lemma 4.7.

5.1 Theorem. *Let E and A be G-sets such that E has finite stabilizers and A is nonempty and has trivial G-action. If V is a G-retract of a G-stable almost equality class in (E, A), then there exists a G-tree whose edge stabilizers are finite and whose vertex set is the G-set V.*

Proof. Let $\widetilde{V}$ be a G-stable almost equality class in (E, A) which contains V as a G-retract. By the almost stability theorem 1.2, there exists a G-tree whose edge stabilizers are finite and whose vertex set is $\widetilde{V}$. By the retraction lemma 4.7, there exists a G-tree whose edge stabilizers are finite and whose vertex set is V. □

We now recall Definitions IV.2.1 and IV.2.2 of [3].

5.2 Definitions. Let M be a *G-module*, that is, an additive abelian group which is also a G-set such that G acts as group automorphisms on M. Thus a G-module is simply a left module over the integral group ring $\mathbb{Z}G$.

If $d\colon G \to M$ is a *derivation*, that is, a map such that $d(xy) = d(x) + xd(y)$ for all x, $y \in G$, then M_d denotes the set M endowed with the G-action

$$G \times M \to M, \quad (g, m) \mapsto g \cdot m := gm + d(g) \quad \text{for all } g \in G \text{ and all } m \in M.$$

It is straightforward to show that M_d is a G-set. This construction has made other appearances in the literature; see [1, Remarque 4.a.5].

We say that M is an *induced* G-module if there exists an abelian group A such that M is isomorphic, as G-module, to $AG := \mathbb{Z}G \otimes_{\mathbb{Z}} A$.

We say that M is a *G-projective* G-module if M is isomorphic, as G-module, to a direct summand of an induced G-module. □

5.3 Example. If R is any ring and P is a projective left RG-module, then there exists a free left R-module F such that P is isomorphic, as RG-module, to an RG-summand of

$$RG \otimes_R F = \mathbb{Z}G \otimes_{\mathbb{Z}} R \otimes_R F = \mathbb{Z}G \otimes_{\mathbb{Z}} F = FG.$$

Hence P is G-projective. □

The following generalizes Theorem IV.2.5 and Corollary IV.2.8 of [3].

5.4 Theorem. *If P is a G-projective G-module, and $d\colon G \to P$ is a derivation, then there exists a G-tree whose edge stabilizers are finite and whose vertex set is the G-set P_d.*

Proof. There exists an abelian group A such that P is isomorphic to a G-summand of AG. We view P as a G-submodule of AG. There exists an additive G-retraction $\pi\colon AG \to P$.

We view AG as the almost equality class of (G, A) which contains the zero map. Thus AG is a G-submodule of (G, A), and we have a derivation

$$d : G \to P \subseteq AG \subseteq (G, A).$$

By a classic result of Hochschild's, there exists $v \in (G, A)$ such that, for all $g \in G$, $d(g) = gv - v$. For example, we can take $v\colon x \mapsto -(d(x))(x)$, for all $x \in G$. See the proof of Proposition IV.2.3 in [3].

Let $U = v + P$ and $V = v + AG$. Then $U \subseteq V \subseteq (G, A)$, and V is the almost equality class which contains v. Also, U and V are G-stable, since, for each $g \in G$, $gv = v + d(g) \in v + P \subseteq v + AG$. The map

$$V \to U, \quad v + m \mapsto v + \pi(m), \text{ for all } m \in AG,$$

is a G-retraction, since, for all $m \in AG$,

$$\begin{aligned} g(v + m) = v + gm + d(g) \quad \mapsto \quad v + \pi(gm + d(g)) &= v + g\pi(m) + d(g) \\ &= g(v + \pi(m)). \end{aligned}$$

By Theorem 5.1, there exists a G-tree whose edge stabilizers are finite and whose vertex set is the G-set U.

The bijective map $P \to U$, $p \mapsto v + p$, is an isomorphism of G-sets $P_d \xrightarrow{\sim} U$. Now the result follows. □

5.5 Remark. Notice that, in Theorem 5.4, the stabilizer of a vertex $p \in P_d$ is precisely the kernel of the derivation

$$d + \operatorname{ad} p\colon G \to P, \quad g \mapsto d(g) + gp - p = (g - 1)(v + p). \qquad \square$$

The following generalizes Corollary IV.2.10 of [3] and is used in the proof of Lemma 5.16 of [5].

5.6 Corollary. *Let M be a G-module, let P be a G-projective G-submodule of M, and let v be an element of M. If the subset $v+P$ of M is G-stable, then there exists a G-tree whose edge stabilizers are finite and whose vertex set is the G-set $v + P$.*

Proof. The inner derivation $\operatorname{ad} v\colon G \to M$ restricts to a derivation $d\colon G \to P$, $g \mapsto gv - v \in P \subseteq M$, for all $g \in G$. The bijective map $P \to v+P$, $p \mapsto v+p$, is then an isomorphism of G-sets $P_d \xrightarrow{\sim} v+P$. Now the result follows from Theorem 5.4. □

5.7 Example. Let R be a nonzero associative ring, and let ωRG be the augmentation ideal of the group ring RG.

Notice that, in the (left) G-set RG, the G-subset $RG - \{0\}$ has finite stabilizers. The coset $1 + \omega RG$ lies in $RG - \{0\}$ and is G-stable. Hence $1 + \omega RG$ is a G-set with finite stabilizers.

If ωRG is projective as left RG-module, then, by Corollary 5.6, there exists a G-tree T with $VT = 1 + \omega RG$; hence T has finite stabilizers. This sheds some light on the main step in the characterization of groups of cohomological dimension at most one over R. See, for example, [3, Theorem IV.3.13]. □

6 A more general almost stability theorem

We next want to generalize Theorem 5.1.

The following is similar to Lemma 2.2 of [4], and the proof is straightforward.

6.1 Lemma. *Let E and A be G-sets such that, for each $e \in E$, G_e acts trivially on A.*

Let $\bar{A}$ denote the G-set with the same underlying set as A but with trivial G-action.

Let E_0 be a G-transversal in E.

For each $\phi \in (E, A)$, let $\widehat{\phi} \in (E, \bar{A})$ be defined by $\widehat{\phi}(ge) = g^{-1} \cdot \phi(ge)$ for all $(g, e) \in G \times E_0$, where $\cdot$ denotes the G action on A.

For each $\psi \in (E, \bar{A})$, let $\widetilde{\psi} \in (E, A)$ be defined by $\widetilde{\psi}(ge) = g \cdot \psi(ge)$ for all $(g, e) \in G \times E_0$.

Then

$$(E, A) \to (E, \bar{A}), \quad \phi \mapsto \widehat{\phi}, \quad \text{and} \quad (E, \bar{A}) \to (E, A), \quad \psi \mapsto \widetilde{\psi},$$

are mutually inverse isomorphisms of G-sets which preserve almost equality between functions. □

Combined, Lemma 6.1 and Theorem 5.1 give the most general form that we know of the almost stability theorem.

6.2 Theorem. *Let E and A be G-sets such that A is nonempty and, for each $e \in E$, G_e is finite and acts trivially on A. If V is a G-retract of a G-stable almost equality class in (E, A), then there exists a G-tree whose edge stabilizers are finite and whose vertex set is the G-set V.* □

For each $e \in E$, if G_e is trivial, then G_e is finite and acts trivially on A. It was this case that was useful in [4].

7 An example

In this section, we shall give an example of a group G and a retract of a vertex set of a G-tree that is not the vertex set of any G-tree.

7.1 Hypotheses. Let $Y = (Y, \overline{V}, \overline{E}, \overline{\iota}, \overline{\tau})$ be the graph given as follows:

$$\overline{V} = \{\overline{u}, \overline{w}\}, \quad \overline{E} = \{\overline{e}, \overline{f}\}, \quad \overline{\iota}(\overline{e}) = \overline{u}, \quad \overline{\tau}(\overline{e}) = \overline{\iota}(\overline{f}) = \overline{\tau}(\overline{f}) = \overline{w}.$$

Let $Y_0 := (Y_0, \overline{V}, \{\overline{e}\}, \overline{\iota}, \overline{\tau})$ be the unique maximal subtree of Y.

We now use the notation of Definitions I.3.1 of [3] to define a graph of groups $(G(-), Y)$ as follows. Let the vertex groups be given by

$$G(\overline{u}) = \langle x, y \mid \quad \rangle, \qquad\qquad G(\overline{w}) = \langle x', y' \mid \quad \rangle.$$

Let the edge groups be given by

$$G(\overline{e}) = \langle x^4,\, xyx,\, y^4 \mid \quad \rangle, \qquad\qquad G(\overline{f}) = \langle x',\, y' \mid \quad \rangle,$$

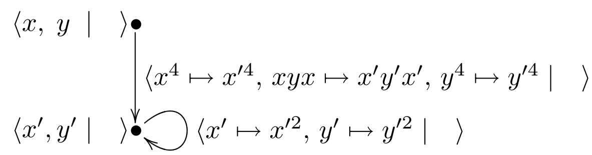
where we have

$$\begin{aligned} G(\overline{e}) &= \langle x^4,\, xyx,\, y^4 \mid \quad \rangle \;\le\; \langle x,\, y \mid \quad \rangle = G(\overline{u}) = G(\overline{\iota}\,\overline{e}),\\ G(\overline{f}) &= \langle x',\, y' \mid \quad \rangle = G(\overline{w}) = G(\overline{\iota}\,\overline{f}). \end{aligned}$$

Finally, let the edge-group monomorphisms be given by

$$\begin{array}{rlcl} t_{\overline{e}}\colon G(\overline{e}) = & \langle x^4,\, xyx,\, y^4 \mid \quad \rangle & \to & \langle x',\, y' \mid \quad \rangle = G(\overline{w}) = G(\overline{\tau}\,\overline{e}),\\ & (x^4, xyx, y^4) & \mapsto & (x'^4, x'y'x', y'^4),\\ t_{\overline{f}}\colon G(\overline{f}) = & \langle x', y' \mid \quad \rangle & \to & \langle x',\, y' \mid \quad \rangle = G(\overline{w}) = G(\overline{\tau}\,\overline{f}),\\ & (x', y') & \mapsto & (x'^2, y'^2). \end{array}$$

Using notation whose interpretation we hope is clear, we represent the resulting graph of groups as follows.



Let G be the fundamental group of the graph of groups, $\pi(G(-), Y, Y_0)$, as in Definitions I.3.4 of [3]. We shall write t for the element of G that realizes the edge-group monomorphism $t_{\overline{f}}\colon G(\overline{f}) \to G(\overline{w})$; thus

(7.1.1) $\quad G = \langle x, y, x', y', t \mid x^4 = x'^4,\, xyx = x'y'x',\, y^4 = y'^4,\, x'^t = x'^2,\, y'^t = y'^2 \rangle,$

where x'^t denotes $t^{-1}x't$. Here $G(\overline{u})$ and $G(\overline{w})$ are subgroups of G; see [3, Corollary I.7.5].

Let $T = (T, V, E, \iota, \tau)$ be the Bass-Serre tree $T(G(-), Y, Y_0)$, as in Notation I.7.1 of [3]. Thus, using $\vee$ to denote disjoint union, we can write

$$V = Gu \vee Gw, \quad G_u = \langle x,\, y \rangle, \quad G_w = \langle x',\, y' \rangle,$$

$$E = Ge \vee Gf, \quad G_e = \langle x^4,\, xyx,\, y^4 \rangle, \quad G_f = \langle x',\, y' \rangle,$$

$$\iota(e) = u, \quad \tau(e) = w, \quad \iota(f) = w, \quad \tau(f) = tw.$$

By Bass-Serre Theory, T is a G-tree; see [3, Theorem I.7.6].

For any subset S of T, let S^{xyx} denote $\{s \in S \mid (xyx)s = s\}$. □

We shall see that Gu is a retract of a vertex set of a G-tree, but is not itself the vertex set of a G-tree.

7.2 Lemma. *Suppose that Hypotheses* 7.1 *hold. In particular, in* T, $V = Gu \vee Gw$, $E = Ge \vee Gf$, $\iota(e) = u$, $\tau(e) = w$, $\iota(f) = w$, *and* $\tau(f) = tw$.

(i) *In* G, $x' = x^{4t^{-2}}$ *and* $y' = y^{4t^{-2}}$.

(ii) $G = \langle x, y, t \mid x^{4t} = x^8,\, y^{4t} = y^8,\, x^{t^2} y^{t^2} x^{t^2} = x^4 y^4 x^4 \rangle$.

(iii) *In* T, $G_u = \langle x,\, y \rangle$, $G_w = \langle x^4,\, y^4 \rangle^{t^{-2}}$, $G_e = \langle x^4,\, xyx,\, y^4 \rangle$, $G_f = \langle x^4,\, y^4 \rangle^{t^{-2}}$.

(iv) *Gu is a G-retract of V.*

Proof. (i). Now $x'^{t^2} = x'^{2t} = x'^4 = x^4$. Thus $x' = x^{4t^{-2}}$. Similarly, $y' = y^{4t^{-2}}$.

(ii). By (7.1.1),

$$\begin{aligned} G &= \langle x, y, x', y', t \mid x^4 = x'^4,\ xyx = x'y'x',\ y^4 = y'^4,\ x'^t = x'^2,\ y'^t = y'^2 \rangle \\ &= \langle x, y, x', y', t \mid x^4 = x'^4,\ xyx = x'y'x',\ y^4 = y'^4, \\ &\qquad\qquad x'^t = x'^2,\ y'^t = y'^2,\ x' = x^{4t^{-2}},\ y' = y^{4t^{-2}} \rangle \\ &= \langle x, y, t \mid x^4 = x^{16t^{-2}},\ xyx = x^{4t^{-2}} y^{4t^{-2}} x^{4t^{-2}},\ y^4 = y^{16t^{-2}}, \\ &\qquad x^{4t^{-1}} = x^{8t^{-2}},\ y^{4t^{-1}} = y^{8t^{-2}} \rangle \\ &= \langle x, y, t \mid x^{4t^2} = x^{16},\ x^{t^2} y^{t^2} x^{t^2} = x^4 y^4 x^4,\ y^{4t^2} = y^{16}, \\ &\qquad x^{4t} = x^8, \qquad\qquad\qquad\qquad y^{4t} = y^8 \rangle \\ &= \langle x, y, t \mid x^{4t} = x^8,\ x^{t^2} y^{t^2} x^{t^2} = x^4 y^4 x^4,\ y^{4t} = y^8 \rangle. \end{aligned}$$

(iii). $G_f = G_w = \langle x',\, y' \rangle = \langle x^4,\, y^4 \rangle^{t^{-2}}$.

(iv). We have $G_w = \langle x^4,\, y^4 \rangle^{t^{-2}} \le \langle x,\, y \rangle^{t^{-2}} = G_u^{t^{-2}} = G_{t^2 u}$. Thus Gu is a G-retract of $Gu \vee Gw = V$. □

It remains to show that Gu is not the vertex set of any G-tree. We shall use a sequence of technical lemmas.

It is straightforward to prove the following, using Lemma 7.2(ii).

7.3 Lemma. *Suppose that Hypotheses* 7.1 *hold, and let* $n \in \mathbb{N}$.

(i) *In* G, $(xyx)^{t^{n+2}} = (x^4)^{2^n} (y^4)^{2^n} (x^4)^{2^n}$.

(ii) *If* $n \neq 1$, *then, in* G, $(xyx)^{t^n} = x^{2^n} y^{2^n} x^{2^n}$. □

The next result concerns the free group of rank two.

7.4 Lemma. *Suppose that Hypotheses* 7.1 *hold, let* $n \in \mathbb{N}$, *and let* $g \in G_u$. *In particular,* $G_u = \langle x, y \mid \quad \rangle$.

(i) *If* $x^{2^n} y^{2^n} x^{2^n} \in \langle x^2, y^2 \rangle^g$, *then* $n \neq 0$ *and* $g \in \langle x^2, y^2 \rangle$.

(ii) *If* $x^{2^n} y^{2^n} x^{2^n} \in \langle x^4, xyx, y^4 \rangle^g$, *then* $n \neq 1$ *and* $g \in \langle x^4, xyx, y^4 \rangle$.

Proof. Let $T_u = X(G_u, \{x, y\})$, the Cayley graph of G_u with respect to $\{x, y\}$, as in [3, Definitions I.2.1]. Each (oriented) edge of T_u is labelled x or y.

Let H be any subgroup of G_u; we have in mind the cases $H = \langle x^2, y^2 \rangle$ and $H = \langle x^4, xyx, y^4 \rangle$.

Let $w = x^{2^n} y^{2^n} x^{2^n} \in G_u$.

Let $X := H \backslash T_u$, let $Y := \langle w \rangle \backslash T_u$, and let $Z := G_u \backslash T_u$.

The pullback of the two natural maps $X \to Z$, $Y \to Z$ provides detailed information about all nontrivial subgroups of G_u of the form $\langle w \rangle \cap H^g$; see [2, p. 380]. However, this pullback can be rather cumbersome and we do not require detailed information. For our purposes, special considerations will suffice, as follows.

Define $g^{-1}X := (H^g) \backslash T_u$.

There is a graph isomorphism $X \simeq g^{-1}X$, $Hx \leftrightarrow H^g g^{-1} x$.

The fundamental group of X with basepoint $H1$, $\pi(X, H1)$, is naturally isomorphic to H, with the elements of H being read off closed paths based at $H1$.

Similarly, H^g is naturally isomorphic to $\pi(g^{-1}X, H^g 1)$, and this in turn is naturally isomorphic to $\pi(X, Hg)$ via the graph isomorphism $g^{-1}X \simeq X$.

Suppose that w lies in H^g. Then w can be read off a closed path in X based at Hg. Since w is a cyclically reduced word, the closed path is cyclically reduced. The smallest subgraph of X which contains all the cyclically reduced closed paths in X is called the *core* of X, denoted core(X). It follows that the vertex Hg lies in core(X), and that we can start at Hg, read w and stay inside core(X).

(i) Suppose that $H = \langle x^2, y^2\rangle$.

Here core(X) has vertex set $\{H1,\ Hx,\ Hy\}$ and labelled-edge set

$$\{(H1, x, Hx), (Hx, x, Hx^2), (H1, y, Hy), (Hy, y, Hy^2)\}$$

with $Hx^2 = Hy^2 = H1$.

We note that Hxy and Hyx are outside core(X).

Notice that $(Hy)x = Hyx$. This lies outside core(X). Thus, $Hg \neq Hy$, since Hgw can be read in core(X). Hence, $Hg \in \{H1, Hx\}$.

Notice that $(H1)(xy) = Hxy$ and $(Hx)(xyx) = Hyx$. These lie outside core(X). Thus $n \neq 0$. Hence, $x^{2^n} \in H$.

Notice that $(Hx)(x^{2^n}y) = Hxy$ lies outside core(X). Thus $Hg \neq Hx$. Hence, $Hg = H1$, that is, $g \in H$.

This proves (i).

(ii). Suppose that $H = \langle x^4, xyx, y^4\rangle$.

Here core(X) has vertex set

$$\{H1\} \cup \{Hx^i,\ Hy^i \mid 1 \leq i \leq 3\}.$$

and labelled-edge set

$$\{(Hx^i, x, Hx^{i+1}),\ (Hy^i, y, Hy^{i+1}) \mid 0 \leq i \leq 3\} \cup \{(Hx, y, Hxy)\},$$

with $Hx^4 = Hy^4 = H1$ and $Hxy = Hx^3$.

We note that $Hxy^2 = Hx^3y$, Hx^2y, Hyx, Hy^2x, and Hy^3x all lie outside core(X).

Consider any j with $1 \leq j \leq 3$. Notice that $(Hy^j)(x) = Hy^jx$. This lies outside core(X). It follows that $Hg \neq Hy^j$. Hence $Hg = Hx^i$ for some i with $0 \leq i \leq 3$.

Notice that $(Hx)(xy) = Hx^2y$, $(Hx^2)(xy) = Hx^3y$, and $(Hx^3)(xyx) = Hyx$. These all lie outside core(X). Thus, if $n = 0$, then $Hg = H1$.

Notice that $(H1)(x^2y) = Hx^2y$, $(Hx)(x^2y) = Hx^3y$, $(Hx^2)(x^2y^2x) = Hy^2x$, and $(Hx^3)(x^2y^2) = Hxy^2$. These all lie outside core(X). Thus $n \neq 1$.

Now suppose that $n \geq 2$. Thus $x^{2^n} = (x^4)^{2^{n-2}} \in H$.

Notice that $(Hx)(x^{2^n}y^2) = Hxy^2$, $(Hx^2)(x^{2^n}y) = Hx^2y$, and $(Hx^3)(x^{2^n}y) = Hx^3y$. These all lie outside core(X). Thus $Hg = H1$.

This proves (ii). □

7.5 Lemma. *Suppose that Hypotheses* 7.1 *hold and let* $n \in \mathbb{N}$.

(i) $(t^nG_ue)^{xyx} = \{t^ne\}$ *if* $n \neq 1$.

(ii) $(t^nG_we)^{xyx} = \begin{cases} \{t^ne\} & \text{if } n \neq 1, \\ \emptyset & \text{if } n = 1. \end{cases}$

(iii) $(t^nG_wt^{-1}f)^{xyx} = \begin{cases} \{t^{n-1}f\} & \text{if } n \neq 0, \\ \emptyset & \text{if } n = 0. \end{cases}$

(iv) $(t^nG_wf)^{xyx} = \{t^nf\}$.

Proof. (i). Let $g \in G_u = \langle x, y\rangle$.

Suppose that $n \neq 1$ and that $(xyx)t^nge = t^nge$. Then $(xyx)^{t^ng} \in G_e$. By Lemma 7.3(ii),

$$(x^{2^n}y^{2^n}x^{2^n})^g \in G_e = \langle x^4,\ xyx,\ y^4\rangle.$$

By Lemma 7.4(ii), $g \in \langle x^4, xyx, y^4\rangle = G_e$. Hence $t^n ge = t^n e$. It is now easy to see that (i) holds.

(ii). Let $h \in G_w = \langle x^4, y^4\rangle^{t^{-2}}$. Let $g = h^{t^2} \in \langle x^4, y^4\rangle$.

Suppose that $(xyx)t^n he = t^n he$. Then $(xyx)t^{n+2}gt^{-2}e = t^{n+2}gt^{-2}e$, and $(xyx)^{t^{n+2}gt^{-2}} \in G_e$. By Lemma 7.3(i),

$$((x^4)^{2^n}(y^4)^{2^n}(x^4)^{2^n})^g \in G_e^{t^2} = \langle x^4, xyx, y^4\rangle^{t^2} = \langle (x^4)^4, (x^4)(y^4)(x^4), (y^4)^4\rangle.$$

By Lemma 7.4(ii) with x^4 , y^4 in place of x, y, we see that $n \neq 1$ and

$$g \in \langle (x^4)^4, (x^4)(y^4)(x^4), (y^4)^4\rangle = G_e^{t^2}.$$

Hence $h \in G_e$ and $t^n he = t^n e$. It is now clear that (ii) holds.

(iii). Let $h \in G_w = \langle x^4, y^4\rangle^{t^{-2}}$. Let $g = h^{t^2} \in \langle x^4, y^4\rangle$.

Suppose that $(xyx)t^n ht^{-1}f = t^n ht^{-1}f$. Then $(xyx)t^{n+2}gt^{-3}f = t^{n+2}gt^{-3}f$, and $(xyx)^{t^{n+2}gt^{-3}} \in G_f$. By Lemma 7.3(i),

$$((x^4)^{2^n}(y^4)^{2^n}(x^4)^{2^n})^g \in G_f^{t^3} = \langle x^4, y^4\rangle^t = \langle (x^4)^2, (y^4)^2\rangle.$$

By Lemma 7.4(i), with x^4 , y^4 in place of x, y, we see that $n \neq 0$ and

$$g \in \langle (x^4)^2, (y^4)^2\rangle = G_f^{t^3}.$$

Hence $h^{t^{-1}} \in G_f$ and $t^n ht^{-1}f = t^{n-1}f$. It is now clear that (iii) holds.

(iv). By Lemma 7.3(i), $(xyx)^{t^n} \in \langle x^4, y^4\rangle^{t^{-2}} = G_f = G_w$. □

7.6 Lemma. *Suppose that Hypotheses* 7.1 *hold. Then*

$$V^{xyx} \quad = \quad \{t^n u \mid n \in \mathbb{N} - \{1\}\} \quad \cup \quad \{t^n w \mid n \in \mathbb{N}\}.$$

Proof. Let $n \in \mathbb{N}$.

From [3, Definitions I.3.4], we obtain the following.

$$\iota^{-1}(t^n u) = t^n G_u e, \qquad \tau^{-1}(t^n u) = \emptyset,$$
$$\iota^{-1}(t^n w) = t^n G_w f, \qquad \tau^{-1}(t^n w) = t^n G_w e \ \cup\ t^n G_w t^{-1} f.$$

By Lemma 7.5(ii), (iii) and (iv), the edges of T^{xyx} incident to w are e and f, the edges of T^{xyx} incident to tw are f and tf, and, for $n \geq 2$, the edges of T^{xyx} incident to $t^n w$ are $t^n e$, $t^{n-1}f$ and $t^n f$.

Hence, in T^{xyx}, the neighbours of w are u and tw, the neighbours of tw are w and $t^2 w$, and, for $n \geq 2$, the neighbours of $t^n w$ are $t^n u$, $t^{n-1}w$ and $t^{n+1}w$.

By Lemma 7.5(i), if $n \neq 1$, then the unique edge of T^{xyx} incident to $t^n u$ is $t^n e$, and hence the unique neighbour of $t^n u$ in T^{xyx} is $t^n w$.

The result now follows. □

7.7 Lemma. *Suppose that Hypotheses* 7.1 *hold. There exists no G-tree with vertex set Gu.*

Proof. Suppose that there exists a G-tree T' with $VT' = Gu$. We will derive a contradiction.

Let L denote the subtree of T with vertex set $\langle t\rangle w$ and edge set $\langle t\rangle f$. Then L is homeomorphic to $\mathbb{R}$ and t acts on L by translation. In particular, $\langle t\rangle$ acts freely on VT. Hence, $\langle t\rangle$ acts freely on $VT' \subseteq VT$. As in [3, Proposition I.4.11], there exists a subtree L' of T' homeomorphic to $\mathbb{R}$ on which t acts by translation.

Let v' denote the vertex of L' closest to u in T'. It is well known, and easy to prove, that the T'-geodesic from u to t^2u, denoted $T'[u, t^2u]$, is the concatenation of the four T'-geodesics $T'[u, v']$, $T'[v', tv']$, $T'[tv', t^2v']$, and $T'[t^2v', t^2u]$.

By Lemma 7.6, and the fact that $\langle t\rangle$ acts freely on VT',

$$VT'^{xyx} = (Gu)^{xyx} = \{t^n u \mid n \in \mathbb{N} - \{1\}\} = \{t^n u \mid n \in \mathbb{N}\} - \{tu\}. \tag{7.7.1}$$

By (7.7.1), or by direct calculation, xyx fixes u, moves tu, and fixes t^2u. Thus, xyx fixes $T'[u, t^2u]$, and, hence, xyx fixes v', fixes tv', and fixes t^2v'.

In particular, $tu \neq tv'$, hence $u \neq v'$, that is, $u \notin L'$.

Since xyx fixes v', we see, by (7.7.1), that $v' = t^n u$ for some $n \in \mathbb{N} - \{1\}$. Hence $u = t^{-n}v' \in t^{-n}L' = L'$. This is a contradiction. □

We now have the desired example.

7.8 Theorem. *There exists a group G and a G-set U such that U is a G-retract of the vertex set of some G-tree but U is not the vertex set of any G-tree.* □

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