

HOW DO DIFFERENCE BODIES IN COMPLEX VECTOR SPACES LOOK LIKE? A GEOMETRICAL APPROACH.

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ABSTRACT. We investigate geometrical properties and inequalities satisfied by the complex difference body, in the sense of studying which of the classical ones for the difference body have an analog in the complex framework. Among others we give an equivalent expression for the support function of the complex difference body and prove that, unlike the classical case, the dimension of the complex difference body depends on the position of the body with respect to the complex structure of the vector space. We use spherical harmonics to characterize the bodies for which the complex difference body is a ball, we prove that it is a polytope if and only if the two bodies involved in the construction are polytopes and provide several inequalities for classical magnitudes of the complex difference body, as volume, quermassintegrals and diameter, in terms of the corresponding ones for the involved bodies.

1. INTRODUCTION

Let V denote a real vector space of dimension n and $\mathcal{K}(V)$ the space of compact convex sets, shortly, convex bodies, in V , endowed with the Hausdorff metric. The n -dimensional volume of a set $M \subseteq \mathbb{R}^n$, i.e., its n -dimensional Lebesgue measure, is denoted by $\text{Vol}(M)$ (or $\text{Vol}_n(M)$ if the distinction of the dimension is useful). For K, L convex bodies, the vectorial or Minkowski sum $K + L$ is also a convex body. It is well known that the Minkowski sum in $\mathcal{K}(V)$ satisfies the cancellation law and behaves nicely with dilations by positive scalars. The *support function* of $K \in \mathcal{K}(V)$ in the direction $\xi \in V^*$ is defined as $h(K, \xi) = \max\{\xi(x) : x \in K\}$. When V is identified with \mathbb{R}^n and endowed with the euclidean inner product $\langle \cdot, \cdot \rangle$, we denote by B_n , the euclidean unit ball and by S^{n-1} its topological boundary.

Given a convex body K , its image under reflection in the origin is denoted by $-K$ and the set

$$DK = K + (-K) =: K - K$$

is a centrally symmetric convex body called the *difference body* of K .

The difference body, seen as an operator $D : \mathcal{K}(V) \rightarrow \mathcal{K}(V)$ enjoys many important properties. For instance, it is a *Minkowski valuation*, namely, it satisfies $D(K \cup L) + D(K \cap L) = D(K) + D(L)$ whenever $K, L, K \cup L \in \mathcal{K}(V)$, and it is *Minkowski additive*, since $D(K + L) = D(K) + D(L)$. Furthermore, it is *continuous* (with respect to the Hausdorff topology), *translation invariant* (i.e., $D(K + x) = D(K)$ for every $x \in V$) and *SL(V, ℝ)-covariant* (i.e., $Z(gK) = gZ(K)$, for every $g \in \text{SL}(V, \mathbb{R})$).

The so-called complex difference body has naturally appeared in the framework of the Minkowski valuation theory. Let W be a complex vector space of complex dimension $m \geq 1$, $\alpha \in \mathbb{C}$ and $K \in \mathcal{K}(W)$. We denote by αK the convex body obtained under the action of

the $m \times m$ complex diagonal matrix $\begin{pmatrix} \alpha & \dots & 0 \\ & \ddots & \\ 0 & \dots & \alpha \end{pmatrix}$ on K . For a convex body $C \subset \mathbb{C}$, we

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denote by $d\mathcal{S}(C, \cdot)$ the area measure of the convex body C (see, for instance, [31, Section 4.3] for a complete description of the area measure of a convex body).

Definition 1.1. *Let W be a complex vector space of complex dimension $m \geq 1$ and C a convex body in \mathbb{C} . The complex difference body operator $D_C : \mathcal{K}(W) \rightarrow \mathcal{K}(W)$ with respect to C is defined by*

$$(1) \quad h(D_C K, \xi) = \int_{S^1} h(\alpha K, \xi) d\mathcal{S}(C, \alpha), \quad \xi \in W^*.$$

By simplicity, we will usually refer to complex difference body operator with respect to C only by *complex difference body operator*. We notice that Equation (1) can also be written as

$$(2) \quad D_C K = \int_{S^1} \alpha K d\mathcal{S}(C, \alpha).$$

We will also consider the operator $D : \mathcal{K}(\mathbb{C}) \times \mathcal{K}(W) \rightarrow \mathcal{K}(W)$, defined by $D(C, K) = D_C K$, which will be again called complex difference body operator.

The following result characterizes the complex difference body operator for dimension $m \geq 3$.

Theorem A ([2]). *Let W be a complex vector space of complex dimension $m \geq 3$. A map $Z : \mathcal{K}(W) \rightarrow \mathcal{K}(W)$ is a continuous translation invariant and $\text{SL}(W, \mathbb{C})$ -covariant Minkowski valuation if and only if there exists a unique planar convex body C with Steiner point $s(C) = 0$ such that $Z = D_C$.*

The motivation to study the previous classification result was given by Ludwig's characterization theorem for the difference body (and projection body) operator. More precisely, from [23], it follows that in a real vector space of dimension $n \geq 2$ the difference body is, up to positive constants, the unique continuous Minkowski valuation which is translation invariant and $\text{SL}(V, \mathbb{R})$ -covariant (see also [22]). In [1, 2, 3], the complex analog of Ludwig's result was studied, obtaining, in particular, Theorem A.

We remark that the "usual" difference body of a convex body K , i.e., DK , is indeed obtained as a particular case of $D_C K$, namely for $C = [-i/2, i/2]$. (Here, and along the paper, we identify $\mathbb{R}^2 \equiv \mathbb{C}$.) In fact, the unit normal vectors to C are $(1, 0)$ and $(-1, 0)$, which correspond to 1 and -1 in \mathbb{C} . Thus, the surface area measure of C is $\mathcal{S}(C, \cdot) = \delta_1 + \delta_{-1}$ and the statement follows.

In view of the previous remark and the natural definition of the complex difference body operator in a complex setting, we aim to shed light on the geometry of this newly defined operator. To achieve this objective, we study geometrical properties and inequalities satisfied by the complex difference body, comparing them with the well-known properties and inequalities of the real difference body, which we recall in the next lines.

Let $K \in \mathcal{K}(V)$ and V be endowed with the euclidean inner product. It is easy to see that for every $u \in S^{n-1}$, $h(DK, u) = w(K, u)$, where $w(K, u) := h(K, u) + h(K, -u)$ denotes the *width of K in the direction u* . As a consequence, DK is a ball if and only if K has constant width and DK is homothetic to K if and only if K is centrally symmetric. We will denote centrally symmetric convex bodies by symmetric convex bodies.

The classical Brunn-Minkowski inequality yields that

$$(3) \quad 2^n \text{Vol}_n(K) \leq \text{Vol}_n(DK)$$

with equality precisely if K is centrally symmetric. An upper bound for $\text{Vol}_n(DK)$ is given by the so called *Rogers-Shephard inequality*, which establishes that

$$(4) \quad \text{Vol}_n(DK) \leq \binom{2n}{n} \text{Vol}_n(K)$$

with equality precisely if K is a simplex (see [27] and [31, Section 7.3]).

To study similar facts for the complex difference body operator, we have to remark, first of all, that the dimension of the complex difference body of K depends on the position of $K \in \mathcal{K}(W)$ as a subset of W , that is, the dimension of $D_C K$ is not rotation-invariant, except for the case in which C is a segment. The following result gives the precise dimension of $D_C K$.

Theorem 1.2. *Let $K \in \mathcal{K}(W)$ be a convex body of dimension l contained in an l -dimensional subspace E . Let $a := \dim_{\mathbb{C}}(E \cap JE)$. Then, for any $C \in \mathcal{K}(\mathbb{C})$ with interior points, it holds*

$$\dim D_C K = 2(l - a).$$

Here J denotes a complex structure of the complex vector space W .

This implies that a Rogers-Shephard's type inequality cannot hold for the complex difference body. However, we provide sharp (lower) bounds for the volume and other quermassintegrals of $D_C K$ in terms of the quermassintegrals of both, K and C in Section 3.

The classical difference body of K is a symmetric convex body. In the complex case, this is no longer true, in general. Indeed, the image of D_C is contained in the space of symmetric convex bodies if and only if C itself is a symmetric one (see Corollary 7.7). In turn, in Lemma 3.3 we will prove that for any $C \in \mathcal{K}(\mathbb{C})$ with $\text{length}(C) = 1$, the width of $D_C K$ does not decrease whereas the diameter does not increase, which allows us to say, roughly, that the complex difference body of K with respect to C is closer to a ball than K , for any C .

As mentioned above, DK is a ball if and only if K is a convex body of constant width. It is also well known that DK is a polytope if and only if K is also a polytope. We address these questions and similar ones for the complex difference body proving, for instance, the following result.

Theorem 1.3. *Let $C \in \mathcal{K}(\mathbb{C})$ and $K \in \mathcal{K}(W)$. Then, $D_C K$ is a polytope if and only if both C and K are polytopes.*

As a consequence we obtain the following

Corollary 1.4. *The map $D: \mathcal{K}(\mathbb{C}) \times \mathcal{K}(W) \rightarrow \mathcal{K}(W)$ is neither injective nor surjective.*

This is the analog to the well known fact that the classical difference body operator is neither injective nor surjective on $\mathcal{K}(V)$. We further study the range of injectivity of D_C for a fixed C .

We characterize the pairs (C, K) for which $D_C K$ is a ball. In order to do so, we calculate the harmonic expansion of the support function of $D_C K$, which is given in terms of the Fourier coefficients of the harmonic expansion of the support functions of C , $c_j(h_C)$, and K , $\pi_{j,l}(h_K)$, for $j, l \geq 0$ (see Sections 4 and 5 for precise definitions and details).

Corollary 1.5. *Let $C \in \mathcal{K}(\mathbb{C})$ and $K \in \mathcal{K}(W)$. Then, $D_C K$ is a ball if and only if for every $(j, l) \in \mathbb{N}^2 \setminus \{(0, 0)\}$, either $c_{j-l}(h_C) = 0$ or $\pi_{j,l}(h_K) = 0$.*

The previous corollary implies that if C is a symmetric convex body with $c_{2j} \neq 0$ for every $j \in \mathbb{Z}$, then $D_C K$ is a ball if and only if K is of constant width, recovering the classical case. The same ideas allow us to characterize when $D_C K$ is of constant width, symmetric or S^1 -invariant (see Corollaries 7.3, 7.6 and 7.7).

A convex body $K \subset W$ is said to be S^1 -invariant if $\alpha K = K$ for every $\alpha \in S^1$. The S^1 -invariant convex bodies are also called R_θ -invariant or equilibrated bodies. This class of bodies is well-adapted to the complex structure of the ambient space (see, for instance, [18, 19, 28]) and shall be used along the paper. Related results concerning convex bodies or valuations in a complex vector space can be found in [4, 6, 7, 18, 19, 28, 33].

The paper is organized as follows. In Section 2 we give some equivalent descriptions of the complex difference body, and prove some auxiliary results as well as Theorems 1.2 and 1.3. Section 3 is devoted to prove several inequalities involving the complex difference body and magnitudes such as quermassintegrals, circumradius and diameter. In Section 4 we

introduce the necessary notation and results from the theory of spherical harmonics, which will be used later on. In Section 5 we prove that the extension of the complex difference body operator to the space of continuous functions on the sphere is a multiplier transformation (in the context of spherical harmonics). As a consequence of this, we provide the coefficients of the harmonic expansion of the support function of the complex difference body. Section 6 concerns the study of the fixed points of the complex difference body operator and the relation of the latter to some particular classes of convex bodies, such as universal convex bodies or M -classes. In the last section we deduce Corollaries 1.3 and 1.4, as well as some other related properties of $D_C K$.

2. FUNCTIONAL PROPERTIES OF D_C

In this section we study some functional properties of the complex difference body. From now on, V denotes always a real vector space of dimension n and W a complex vector space of complex dimension m . We will often identify W with a real vector space of dimension $2m$. If a basis of W is fixed, then we use the following map for the identification

$$(5) \quad (w_1, \dots, w_m) = (w_{11} + iw_{12}, \dots, w_{m1} + iw_{m2}) \mapsto (w_{11}, w_{12}, \dots, w_{m1}, w_{m2}).$$

By V^* and W^* we denote the dual vector spaces of V and W , respectively.

First we deal with equivalent constructions or descriptions of the complex difference body. The next proposition provides an equivalent expression for the operator D_C (cf. [3]).

Proposition 2.1. *Let $\mu : \mathcal{K}(\mathbb{C}) \rightarrow \mathbb{R}$ be a continuous, translation invariant, monotone valuation homogeneous of degree 1. Then the operator $Z : \mathcal{K}(W) \rightarrow \mathcal{K}(W)$ defined by*

$$h(ZK, \xi) = \mu(\xi(K)), \quad K \in \mathcal{K}(W), \xi \in W^*$$

is a continuous, translation invariant, $\mathrm{SL}(W, \mathbb{C})$ -covariant Minkowski valuation. $\xi(K)$ is defined as $\xi(K) := \{z \in \mathbb{C} \mid z = \xi(k), k \in K\}$ and is a convex body in \mathbb{C} .

Proof. As K is a convex body in W and ξ is linear, we have that $\xi(K)$ is also a convex body. Moreover, the function $\xi \mapsto \mu(\xi(K))$ is 1-homogeneous in ξ . It is also subadditive, and hence the support function of some convex body $ZK \subset W$. Indeed, we have $(\xi_1 + \xi_2)(K) \subset \xi_1(K) + \xi_2(K)$, hence the monotonicity of μ gives

$$\mu((\xi_1 + \xi_2)(K)) \leq \mu(\xi_1(K) + \xi_2(K)) = \mu(\xi_1(K)) + \mu(\xi_2(K)),$$

where the last equality holds since μ has degree of homogeneity 1, and, thus it is Minkowski additive (cf. [11, Theorem 3.2]).

To prove the valuation property, let $K, L, K \cup L \in \mathcal{K}(W)$. Then, since μ is a valuation,

$$\mu(\xi(K \cup L)) + \mu(\xi(K \cap L)) = \mu(\xi(K) \cup \xi(L)) + \mu(\xi(K) \cap \xi(L)) = \mu(\xi(K)) + \mu(\xi(L)).$$

The translation invariance and the continuity of Z follow from the linearity of ξ and the corresponding properties of μ . Moreover, Z is $\mathrm{SL}(W, \mathbb{C})$ -covariant, since for every $g \in \mathrm{SL}(W, \mathbb{C})$ and $\xi \in W^*$ it holds, from the properties of the support function (see [31]),

$$h(ZgK, w) = \mu(\xi(gK)) = \mu((g^*\xi)(K)) = h(ZK, g^*\xi) = h(gZK, \xi).$$

□

Remark 2.2. *The examples provided by the previous proposition are (monotone) Minkowski valuations satisfying the properties of Theorem A. Thus, given μ there exists a convex body C with $h(D_C K, \xi) = \mu(\xi(K))$.*

The monotone valuation μ appearing above can be given more explicitly using mixed volumes of planar convex bodies. This gives a more geometric expression for the support function of the complex difference body, as the following proposition shows.

Proposition 2.3. For $K \in \mathcal{K}(W)$ and $C \in \mathcal{K}(\mathbb{C})$, the following equality holds

$$(6) \quad h(D_C K, \xi) = V_2(\xi(K), \overline{C}).$$

If W is endowed with a hermitian scalar product, then

$$h(D_C K, \xi) = V_2(K|_{\xi_C}, \overline{C}),$$

where $K|_{\xi_C}$ denotes the projection of K onto the 2-dimensional real space in W generated by $\{\xi, -J\xi\}$, identified with \mathbb{C} .

If $m = 1$, then for $\xi \in S^1$,

$$(7) \quad h(D_C K, \xi) = V_2(R_{\overline{\xi}}K, \overline{C}),$$

where R_{ξ} denotes the rotation of K defined by ξ .

Proof. By definition, we have

$$\begin{aligned} h(D_C K, \xi) &= \int_{S^1} h(\alpha K, \xi) d\mathcal{S}(C, \alpha) = \int_{S^1} h(K, \overline{\alpha\xi}) d\mathcal{S}(C, \alpha) \\ &= \int_{S^1} h(\xi(K), \alpha) d\mathcal{S}(\overline{C}, \alpha) = V_2(\xi(K), \overline{C}), \end{aligned}$$

where we have used

$$\begin{aligned} h(K, \overline{\alpha\xi}) &= \sup_{k \in K} \{\operatorname{Re}(\overline{\alpha\xi}(k))\} = \sup_{k \in K} \{\operatorname{Re}(\overline{\alpha}(\xi(k)))\} = \sup_{l \in \xi(K)} \{\operatorname{Re}(\overline{\alpha}(l))\} \\ &= h(\xi(K), \overline{\alpha}), \end{aligned}$$

with $\xi(K) = \{z \in \mathbb{C} \mid z = \xi(k), k \in K\}$, as in Proposition 2.1.

Assume that W is endowed with a hermitian scalar product (\cdot, \cdot) . With the identification of \mathbb{C}^m with \mathbb{R}^{2m} given in (5), there exists a scalar product $\langle \cdot, \cdot \rangle$ in \mathbb{R}^{2m} for which

$$(x, y) = \langle x, y \rangle + i\langle -Jx, y \rangle.$$

Therefore,

$$\xi(k) = (\xi, k) = \langle \xi, k \rangle + i\langle -J\xi, k \rangle \in \mathbb{C},$$

and $\xi(k)\xi \in \mathbb{C}^m$ is the orthogonal projection of k onto the subspace generated by ξ and $-J\xi$.

For the case $m = 1$ it remains to prove that $\xi(K) = R_{\overline{\xi}}(K)$. This follows from

$$\xi(K) = \{\xi(k) : k \in K \subset \mathcal{K}(\mathbb{C})\} = \{(\xi_1 k_1 - \xi_2 k_2) + i(\xi_1 k_2 + \xi_2 k_1)\} = R_{\overline{\xi}}K.$$

□

Using the above description of the support function of $D_C K$ we obtain the following result.

Corollary 2.4. The operator $D: \mathcal{K}(\mathbb{C}) \times \mathcal{K}(W) \rightarrow \mathcal{K}(W)$ is monotone in its two variables.

Now we prove Theorem 1.2.

Proof of Theorem 1.2. Let $g \in \operatorname{SL}(W, \mathbb{C})$, K an l -dimensional convex body in W and $E = \operatorname{aff}(K)$ the real affine hull of K . As $\dim D_C gK = \dim g D_C K = \dim D_C K$, and $\operatorname{SL}(W, \mathbb{C})$ acts transitively on the subspaces of dimension l having $\dim(E \cap JE)$ fixed, we can suppose that

$$E = \operatorname{span}_{\mathbb{R}}\{e_1, Je_1, \dots, e_a, Je_a, e_{a+1}, e_{a+2}, \dots, e_{l-a}\}$$

for some vectors e_1, \dots, e_{l-a} linearly independent over \mathbb{C} .

We claim that for any direction $e_i \in E$, the direction Je_i belongs to $\operatorname{aff}(D_C K)$, the minimal subspace in W containing $D_C K$. Indeed, without loss of generality, we can assume that $0 \in \operatorname{relint}(K)$, where $\operatorname{relint}(K)$ denotes the relative interior of K . Thus, there exists $\epsilon > 0$ such that $L_i := [-\epsilon e_i, \epsilon e_i] \subset K$ for every $i = 1, \dots, l - a$. Using the monotonicity of D_C , we have that $D_C L_i \subset D_C K$. Hence, it suffices to prove that there exists $\lambda > 0$ for which $\lambda Je_i \in D_C L_i$ to have that $Je_i \in \operatorname{aff}(D_C K)$. From the monotonicity of the support

function, this is equivalent to show that $0 < \lambda \leq h(D_C L_i, J e_i)$ (since 0 is in the relative interior of $D_C L_i$).

The nonnegativity of the area measure of C yields

$$h(D_C L_i, J e_i) = \epsilon \int_{S^1} |\langle \alpha e_i, J e_i \rangle| d\mathcal{S}(C, \alpha) = \epsilon \int_{S^1} |\operatorname{Im}(\alpha)| d\mathcal{S}(C, \alpha) \geq 0,$$

with equality if and only if $\mathcal{S}(C, \cdot)$ is concentrated on $1, -1$, that is, if and only if, C is an interval on the imaginary axis of $\mathbb{C} \equiv \mathbb{R}^2$.

Thus, the claim follows for every convex body $C \in \mathcal{K}(\mathbb{C})$ with interior points. An analogous argument shows that for every $e_i \in E = \operatorname{aff}(K)$, it also holds $e_i \in \operatorname{aff}(D_C K)$, C with interior points. Therefore, we get

$$\dim D_C K \geq 2a + 2(l - 2a) = 2(l - a),$$

with $2a$ the number of directions $\{e_1, J e_1, \dots, e_a, J e_a\}$ and $l - 2a$ the number of directions $\{e_{a+1}, \dots, e_{l-a}\}$.

Let $\tilde{E} := \{e_1, J e_1, \dots, e_{l-a}, J e_{l-a}\}$ and $F := \{e_{l-a+1}, J e_{l-a+1}, \dots, e_m, J e_m\}$ be so that $\{e_1, \dots, e_{l-a}, e_{l-a+1}, \dots, e_m\}$ constitutes an orthonormal basis of the complex vector space W . It remains to prove that $D_C K \subset \tilde{E}$. Since $W = \tilde{E} \oplus F$, it suffices to show that $h(D_C K, v) = 0$ for every $v \in F$ (see [31, Section 1.7]).

Let $v \in F$. Using that $K \subset E$ and $\alpha K \subset \alpha E \subset \tilde{E}$, we have

$$h(D_C K, v) = \int_{S^1} h(\alpha K, v) d\mathcal{S}(C, \alpha) = 0,$$

and $\dim D_C K = 2(l - a)$. □

For $\rho \in \mathbb{C}$, using the definition of *scaling* a convex body by a complex number mentioned in the introduction, we can ask for the behavior of the complex difference body operator when either, $C \in \mathcal{K}(\mathbb{C})$ or $K \in \mathcal{K}(W)$ are *scaled* by ρ . This is the content of the next result, which shall be used without further mention along of the paper.

Lemma 2.5.

i) Let $K \in \mathcal{K}(W)$, $C \in \mathcal{K}(\mathbb{C})$ and $\rho \in \mathbb{C}$ (not necessarily in S^1). Then,

$$D_{\rho C} K = D_C(\rho K) = \rho D_C K.$$

ii) If $\rho \in S^1$ and $C = B_2$ is the unit ball in \mathbb{C} , then

$$\rho D_{B_2} K = D_{B_2} K,$$

that is, $D_{B_2} K$ is S^1 -invariant for every $K \in \mathcal{K}(W)$.

iii) If $\rho_0 C = C$ for some $\rho_0 \in \mathbb{C}$, then, for every $K \in \mathcal{K}(W)$,

$$\rho_0 D_C K = D_C K.$$

iv) Let $\varphi = T + p$ with $T \in \operatorname{GL}(W, \mathbb{C})$ and $p \in W$. Then,

$$D_C(\varphi K) = D_C(TK) = T D_C K.$$

v) If $K = B_{2m}$, where $m = \dim_{\mathbb{C}} W$, then $D_C K = \operatorname{length}(C) B_{2m}$.

Proof. To prove i), let $\rho = |\rho| \tilde{\rho}$, with $\tilde{\rho} \in S^1$. Then, we have

$$h(D_{\rho C} K, \xi) = \int_{S^1} h(\alpha K, \xi) d\mathcal{S}(\rho C, \alpha) = \int_{S^1} |\rho| h(\tilde{\rho} \beta K, \xi) d\mathcal{S}(C, \beta) = h(D_C(\rho K), \xi).$$

Statements ii) and iii) follow directly from i); iv) from the translation invariance and the $\operatorname{SL}(W, \mathbb{C})$ -covariance of D_C ; and v) from the rotation invariance of B_{2m} and the fact that $\int_{S^1} d\mathcal{S}(C, \alpha) = \operatorname{length}(C)$. □

Next, we observe the following fact which easily arises from the previous results.

Corollary 2.6. $D : \mathcal{K}(\mathbb{C}) \times \mathcal{K}(W) \longrightarrow \mathcal{K}(W)$ is a continuous, translation invariant, $\operatorname{SL}(W, \mathbb{C})$ -covariant Minkowski valuation in each component.

In the following, we would like to focus on a special type of convex bodies, namely polytopes. We shall need the following lemma which holds in \mathbb{C} for any convex body.

Lemma 2.7. *Let $C, K \in \mathcal{K}(\mathbb{C})$. Then,*

$$\mathcal{S}(D_C K, \cdot) = \mathcal{S}(C, \cdot) * \mathcal{S}(K, \cdot),$$

where $*$ denotes the convolution between the (nonnegative) surface area measures $\mathcal{S}(C, \cdot)$ and $\mathcal{S}(K, \cdot)$.

Proof. In the sense of distributions, if necessary, we can write (see, for instance [5] or [31, p. 110])

$$\mathcal{S}(D_C K, \cdot) = h''_{D_C K} + h_{D_C K},$$

where we are interpreting $h_{D_C K}$ as a 2π -periodic function on \mathbb{R} . Let $\varphi \in \mathcal{C}^\infty(\mathbb{R})$ be a 2π -periodic test function. Using the definition of the derivative of a distribution, the definition of $D_C K$ and the Fubini theorem, we get

$$\begin{aligned} (h''_{D_C K} + h_{D_C K})(\varphi) &= \int_0^{2\pi} (h''_{D_C K} + h_{D_C K})(t)\varphi(t)dt = \int_0^{2\pi} h_{D_C K}(t)(\varphi'' + \varphi)(t)dt \\ &= \int_0^{2\pi} \int_0^{2\pi} h_K(t-s)(\varphi'' + \varphi)(t)dt d\mathcal{S}(C, s) \\ &= \int_0^{2\pi} \int_0^{2\pi} \varphi(\tilde{t}-s)(h''_K + h_K)(\tilde{t})d\tilde{t} d\mathcal{S}(C, s) \\ &= \int_0^{2\pi} \int_0^{2\pi} \varphi(\tilde{t}-s)d\mathcal{S}(K, \tilde{t})d\mathcal{S}(C, s) \\ &= (\mathcal{S}(C, \cdot) * \mathcal{S}(K, \cdot))(\varphi). \end{aligned}$$

□

Next we prove the stated property that $D_C K$ is a polytope if and only if both K and C are polytopes.

Proof of Theorem 1.3. The only if part is easy. Indeed, let $C \in \mathcal{K}(\mathbb{C})$ be the polytope with outer normal vector α_i to the facet (edge) f_i having length s_i , $i = 1, \dots, N$. Then

$$(8) \quad h(D_C K, \cdot) = \int_{S^1} h(\alpha K, \cdot) d\mathcal{S}(C, \alpha) = \sum_{i=1}^N s_i h(\alpha_i K, \cdot) = h\left(\sum_{i=1}^N s_i \alpha_i K, \cdot\right).$$

Since K is a polytope, $\alpha_i K$ is a polytope for every $i = 1, \dots, N$ and $\sum_{i=1}^N s_i \alpha_i K$ is a polytope too.

For the if part, assume first that $m = 1$. It is known (see [31, Theorem 4.6.4]) that a convex body $L \in \mathcal{K}(\mathbb{C})$ with surface area measure $\mathcal{S}(L, \cdot)$ is a polygon if and only if

$$\mathcal{S}(L, \cdot) = \sum_{i=1}^N r_i \delta_{u_i},$$

where $r_i > 0$, $u_i \in S^1$ such that $\sum_{i=1}^N r_i u_i = 0$ and the linear subspace generated by $\{u_i\}_{i=1}^N$ satisfies $\text{lin}\{u_i, i = 1, \dots, N\} = \mathbb{R}^2$. Thus, using Lemma 2.7, $D_C K$ is a polygon if and only if $\mathcal{S}(K, \cdot) * \mathcal{S}(C, \cdot)$ is a finite positive linear combination of Dirac measures, that is, the measure $\mathcal{S}(D_C K, \cdot)$ has discrete support. It is known (see [9, p. 194]) that for positive Borel measures S, T , we have

$$\text{supp}(S * T) = \text{cl}\{\text{supp}(S) + \text{supp}(T)\},$$

where $\text{cl}\{A\}$ denotes the closure of the set A . Then, we have that $\mathcal{S}(D_C K, \cdot)$ is a finite linear combination of Dirac measures if and only if $\mathcal{S}(C, \cdot)$ and $\mathcal{S}(K, \cdot)$ are also of this form, which concludes the proof for $m = 1$.

For general $m \geq 2$, let $\xi \in S^{2m-1}$. By assumption, the 2-dimensional convex body $(D_C K)|_{\xi_C}$ is a polygon and

$$h((D_C K)|_{\xi_C}, \alpha\xi) = h(D_C K, \alpha\xi) = V_2(K|_{(\alpha\xi)_C}, \overline{C}) = V_2(R_{\overline{\alpha}}(K|_{\xi_C}), \overline{C}) = h(D_C(K|_{\xi_C}), \alpha).$$

Thus, using the case $m = 1$, we can assure that \overline{C} is a polytope. Hence, C is a polytope too and the support function of $D_C K$ is given by (8). Therefore, $D_C K$ is a finite linear combination of rotations and dilations of K . Since the only convex bodies which are summands of polytopes, are polytopes, K is itself a polytope (see [14, Chapter 15.1]). \square

Restricted to zonotopes, the following statement holds.

Proposition 2.8.

- i) *Let K be a zonotope. Then, $D_C K$ is a zonotope if and only if C is a polygon.*
- ii) *Given C a zonotope, there exists $K \in \mathcal{K}(W)$ such that $D_C K$ is a zonotope, but K is not.*

Proof. i) Without loss of generality let $K = \sum_{i=1}^N r_i[-u_i, u_i]$ with $r_i > 0$ and $u_i \in S^{2m-1}$.

It holds,

$$h(D_C K, \cdot) = \sum_{i=1}^N r_i \int_{S^1} |\langle \alpha u_i, \cdot \rangle| d\mathcal{S}(C, \alpha),$$

which is the support function of a zonotope if and only if $\mathcal{S}(C, \cdot)$ is a discrete measure (see for example [31, Section 3.5]), that is, if and only if C is a polytope.

- ii) By linearity of $C \mapsto D_C K$, it is enough to consider in the plane the case $C = [-i/2, i/2]$, a segment and $K = \text{conv} \left\{ \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\}$ a triangle, which, of course is not a zonotope. Then,

$$D_C K = K - K = \left[\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right] + \left[\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \end{pmatrix} \right] + \left[\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ -1 \end{pmatrix} \right].$$

\square

Although it is not connected with polytopes, we include next result, which ensures that if K is a convex body of class \mathcal{C}_+^2 , then $D_C K$ is of class \mathcal{C}_+^2 . A convex body K is said to be of class \mathcal{C}_+^2 if its boundary is a submanifold of class \mathcal{C}^2 and all principal curvatures are strictly positive at every point of the boundary.

Proposition 2.9.

- i) *Let $K \in \mathcal{K}(W)$ be of class \mathcal{C}_+^2 . Then, $D_C K$ is of class \mathcal{C}_+^2 for every $C \in \mathcal{K}(\mathbb{C})$.*
- ii) *Let $C \in \mathcal{K}(\mathbb{C})$ be of class \mathcal{C}_+^2 . Then, $D_C K$ is of class \mathcal{C}_+^2 for every $K \in \mathcal{K}(W)$ if and only if $\dim_{\mathbb{C}} W = 1$.*
- iii) *Let $C \in \mathcal{K}(\mathbb{C})$ and $K \in \mathcal{K}(W)$ be so, that $D_C K = \text{length}(C)K$ (for instance, if K is S^1 -invariant, cf. Theorem 6.1). Then $D_C K$ is of class \mathcal{C}_+^2 for every $C \in \mathcal{K}(\mathbb{C})$ if and only if K is of class \mathcal{C}_+^2 .*

Proof. i) Since K is of class \mathcal{C}_+^2 , we have that the Hessian matrix $d^2(h_K)_u$ is positive definite for every $u \in S^{2m-1}$ (see [31, p. 107]). From the definition of the support function of $D_C K$, we have, by linearity,

$$d^2(h_{D_C K})_u = \int_{S^1} d^2(h_{\alpha K})_u d\mathcal{S}(C, \alpha),$$

which is again a positive definite matrix, since $d^2(h_{\alpha K})_u$ is positive definite for every $\alpha \in S^1$ and $\mathcal{S}(C, \cdot)$ a non-negative measure.

ii) If $\dim_{\mathbb{C}} W = 1$, then we have, by Lemma 2.7 and Corollary 7.25 in [31] that $D_C K = D_K C$ (cf. Lemma 3.1). Using i), we obtain the result.

For $\dim_{\mathbb{C}} W > 1$, it suffices to take $K = B_2 \times B_2 \subset \mathbb{C}^2$, the cartesian product of two balls contained in orthogonal subspaces E_1 and E_2 with $E_1, E_2 \cong \mathbb{C}$. Since K is a S^1 -invariant convex body, we have $D_C K = \text{length}(C)K$ for every C . Thus, the statement follows since K is not of class \mathcal{C}_+^2 .

iii) It follows directly from the fact $D_C K = \text{length}(C)K$ and i). □

In the same line, it follows directly from Lemma 2.5, that if $K = L + \epsilon B_{2m}$, $\epsilon > 0$, then $D_C K = D_C L + \epsilon \text{length}(C)B_{2m}$.

The analogue study for symmetric or S^1 -invariant convex bodies needs the tools contained in Section 4 and will be treated after that.

3. GEOMETRIC INEQUALITIES FOR $D_C K$

In this section we intend to study some of the most classical magnitudes of convex bodies for the complex difference body $D_C K$ in dependence of the corresponding magnitudes for K and C .

Lemma 3.1. *Let W be a complex vector space with $\dim_{\mathbb{C}} W = m$. Let $u \in S^{2m-1}$ and $C \in \mathcal{K}(\mathbb{C})$.*

i) *If $m = 1$, then*

$$D_C K = D_K C \quad \forall K \in \mathcal{K}(\mathbb{C}).$$

ii) *For $m \geq 1$, if K is a line segment, i.e., $K = [au, bu]$, $a < b$, $u \in S^{2m-1}$, we have*

$$D_C K = \frac{b-a}{2} D_{[-1,1]}(C \cdot u) = (b-a)D(iC \cdot u),$$

where $C \cdot u := \{cu \in W : c \in C \subset \mathbb{C}\}$.

Proof. i) The statement follows directly from Lemma 2.7 and Corollary 7.2.5 in [31]. We include a more geometric proof of this fact which uses Proposition 2.3. Let $K \in \mathcal{K}(\mathbb{C})$ be a convex body. Then, using (7) we can write

$$h(D_C K, \xi) = V_2(R_{\xi} K, \overline{C})$$

and

$$h(D_K C, \xi) = V_2(R_{\xi} C, \overline{K}).$$

Using known properties of mixed volumes (see for example [31, Chapter 5]), it follows, on the one hand, that $h(D_C K, \xi) = V_2(R_{\xi} K, \overline{C}) = V_2(K, R_{\xi} \overline{C})$. On the other hand, $h(D_K C, \xi) = V_2(R_{\xi} C, \overline{K}) = V_2(\overline{R_{\xi} C}, K)$. Thus, it is enough to prove that $\overline{R_{\xi} C} = R_{\xi} \overline{C}$ which follows directly taking $R_{\xi} C = e^{i\theta} C$ for some $\theta \in [0, 2\pi)$.

ii) Theorem 1.2 yields that $D_C K \subset \text{lin}_{\mathbb{C}}\{u\}$. Thus, it suffices to compute the support function of $D_C K$ in the directions βu , $\beta \in S^1$ and to compare it with the support function of $D_{[-1,1]}(C \cdot u)$. Using the translation invariance of $D_C K$ and the first statement of this lemma, we get

$$\begin{aligned} h(D_C K, \beta u) &= \frac{b-a}{2} \int_{S^1} h([- \alpha u, \alpha u], \beta u) d\mathcal{S}(C, \alpha) = \frac{b-a}{2} \int_{S^1} h([- \alpha, \alpha], \beta) d\mathcal{S}(C, \alpha) \\ &= \frac{b-a}{2} h(D_C[-1, 1], \beta) = \frac{b-a}{2} h(D_{[-1,1]} C, \beta), \end{aligned}$$

and

$$h(D_{[-1,1]}(C \cdot u), \beta u) = 2(h(i(C \cdot u), \beta u) + h(-i(C \cdot u), \beta u)) = h(D_{[-1,1]} C, \beta).$$

□

For the usual difference body of K , basic properties of the Minkowski sum ensure that, up to a translation, K is contained in $K - K$. The question thus arises, whether there is a positive constant λ , maybe depending on K and C , such that $\lambda K \subset D_C K$.

Proposition 3.2. *Let $K \in \mathcal{K}(W)$ be a convex body of dimension l contained in an l -dimensional real subspace $E \subset W$, and $C \in \mathcal{K}(\mathbb{C})$ with interior points. Then, there exists $\lambda(K, C) > 0$ so that, up to translation, $\lambda(K, C)K \subset D_C K$.*

Proof. We distinguish two cases. First, if $l = 2m$ and $r(K)$ denotes the inradius of K , then $r(K)B_{2m} \subseteq K$ (up to a translation of K) and using Corollary 2.4 we have that

$$\text{length}(C)r(K)B_{2m} = r(K)D_C B_{2m} \subseteq D_C K.$$

In order to get $\lambda(K, C)$, we use the notion of relative inradius. The *relative inradius* $r(K; L)$ of K with respect to L , $K, L \in \mathcal{K}(\mathbb{R}^n)$, is defined by

$$r(K; L) = \sup\{r : \exists x \in \mathbb{R}^n \text{ with } x + rL \subset K\}.$$

Thus, it is enough to consider $\lambda(K, C) = \text{length}(C)r(K)r(B_{2m}; K) > 0$, being $r(B_{2m}; K)$ the relative inradius of B_{2m} with respect to K . We notice that in this case, $\lambda(K, C)$ depends only on K and the length of C but not its shape.

For the case $l < 2m$, let $E = \text{aff}(K) \subset W$. We denote by $r(K; B_l)$ the relative inradius of K with respect to B_l (in the ambient space E). The monotonicity of $D_C K$ yields

$$r(K; B_l)D_C B_l \subseteq D_C K.$$

To prove the existence of $\lambda(K, C)$, arguing as in the previous case, it will be enough to prove that $D_C B_l$ has interior points in E .

As B_l is a zonoid, we can approximate it by finite sums of segments, namely $B_l = \lim_N \sum_{j=1}^N [-v_j, v_j]$, where $v_j \in E$ are so that they span E . Using the continuity and the Minkowski additivity of the complex difference body operator and Lemma 3.1, we have

$$D_C B_l = D_C \lim_N \sum_{j=1}^N [-v_j, v_j] = \lim_N \sum_{j=1}^N D_{[-1,1]}(C \cdot v_j).$$

Since $D_{[-1,1]}(C \cdot v_j)$ is $(iC - iC) \cdot v_j \subset \text{lin}_{\mathbb{C}}\{v_j\}$, C has interior points, and the set of vectors $v_j \in E$ must span E , there will be interior points (relative to E) in $\sum_{j=1}^N D_{[-1,1]} C \cdot v_j$ and hence in $D_C B_l$ too. Now, it is enough to consider $r(K; B_l)r(D_C B_l; K)$. Notice, that the dimension of $D_C B_l$ may be strictly larger than the dimension of K . \square

Let $K \in \mathcal{K}(W)$ and $u \in S^{2m-1}$. The *width of K in the direction u* is $w(K, u) = h(K, u) + h(K, -u)$. The minimum and maximum of the widths of K are, respectively, the *minimal width* $w(K)$ of K and the *diameter* $\text{diam}(K)$ of it. In the next lemma we give bounds for the width of $D_C K$.

Lemma 3.3. *Let $K \in \mathcal{K}(W)$ and $C \in \mathcal{K}(\mathbb{C})$. Then*

i) *for every $u \in S^{2m-1}$,*

$$w(K)\text{length}(C) \leq w(D_C K, u) \leq \text{diam}(K)\text{length}(C).$$

ii) $w(K)\text{length}(C) \leq w(D_C K) \leq \text{diam}(D_C K) \leq \text{diam}(K)\text{length}(C)$.

If C is a segment, we have equality everywhere.

Proof. It follows from the fact that

$$w(D_C K, u) = h(D_C K, u) + h(D_C K, -u) = \int_{S^1} (h(\alpha K, u) + h(\alpha K, -u)) d\mathcal{S}(C, \alpha).$$

\square

Remark 3.4. We observe that when constructing the complex difference body of a convex body K with respect to $C \in \mathcal{K}(\mathbb{C})$ with $\text{length}(C) = 1$, the diameter is not increasing and the width is not decreasing. Furthermore, the mean width remains unchanged. When C is one dimensional, i.e., when $D_C K$ is the usual difference body (up to rigid motions), the ratio between diameter and width remains unchanged. In some sense these facts support the idea stated in the introduction that the complex difference body should be rounder than the original K .

As a corollary of Lemma 3.3, we have in particular the following result for convex bodies of constant width.

Corollary 3.5. Let $K \in \mathcal{K}(W)$ be a convex body of constant width $w(K)$ and $C \in \mathcal{K}(\mathbb{C})$. Then, $D_C K$ has constant width equal to $\text{length}(C)w(K)$.

If, in addition, C is symmetric, then $D_C K$ is a ball of radius $\text{length}(C)w(K)/2$.

Proof. The first assertion follows straightforward using Lemma 3.3.

For the second one, if C is symmetric, then $D_C K$ is also symmetric (cf. Lemma 2.5). In addition, it is well known that the only symmetric convex bodies with constant width are the balls. \square

If instead of the maximum and minimum of the widths of a convex body $K \in \mathcal{K}(\mathbb{R}^n)$, its average is considered, we are dealing (up to constants) with the so called *mean width of K* , $b(K)$. Indeed, again up to a constant, $b(K)$ coincides with the $(n - 1)$ -st quermassintegral of $K \in \mathcal{K}(\mathbb{R}^n)$. Quermassintegrals are particular cases of the more general mixed volumes (for which we refer to [31, Chapter 5]) obtained when only one convex body and a ball are involved.

Next we provide bounds for the quermassintegrals, $W_i(K)$, $i = 0, 1, \dots, 2m$ of $D_C K$, in terms of the quermassintegrals of K and the length of C . Notice that $i = 0$ is just the volume and $i = 2m$ is the volume of the unit ball, denoted by κ_{2m} .

Proposition 3.6. Let $K \in \mathcal{K}(W)$ and $C \in \mathcal{K}(\mathbb{C})$. For any $i = 0, \dots, 2m - 1$,

$$(9) \quad W_i(D_C K) \geq W_i(K) \text{length}(C)^{2m-i}.$$

Proof. Using the multilinearity of the mixed volumes and applying the Brunn-Minkowski inequality and the Minkowski first inequality we obtain

$$\begin{aligned} W_i(D_C K) &= \int_{S^1} \dots \int_{S^1} V(\alpha_1 K, \dots, \alpha_{2m-i} K, B_{2m}[i]) d\mathcal{S}(C, \alpha_1) \dots d\mathcal{S}(C, \alpha_{2m-i}) \\ &\geq \int_{S^1} \dots \int_{S^1} \prod_{j=1}^{2m-i} V(\alpha_j K[2m-i], B_{2m}[i])^{\frac{1}{2m-i}} d\mathcal{S}(C, \alpha_1) \dots d\mathcal{S}(C, \alpha_{2m-i}) \\ &= \int_{S^1} \dots \int_{S^1} W_i(K) d\mathcal{S}(C, \alpha_1) \dots d\mathcal{S}(C, \alpha_{2m-i}) \\ &= W_i(K) \text{length}(C)^{2m-i}, \end{aligned}$$

where we have also used that mixed volumes are invariant under (the same) rigid motions of all their components. \square

When $i = 2m - 1$ we have equality in the above inequality for any $K \in \mathcal{K}(W)$ and $C \in \mathcal{K}(\mathbb{C})$, since W_{2m-1} is linear. Furthermore, if $\alpha K = K$ for all $\alpha \in S^1$, we will have equality for any $i = 0, 1, \dots, 2m - 1$. For a fixed convex body C , the equality holds for every convex body K satisfying $h(\alpha_i K, u) = h(\alpha_j K, u)$ for every $u \in W$ and α_i, α_j normal vectors to C .

As a consequence of (9) and Theorem 1.2, the following lower bound for the volume of $D_C K$ holds.

Proposition 3.7. *Let $K \in \mathcal{K}(W)$ be a convex body of dimension l contained in E . Let $a := \dim_{\mathbb{C}}(E \cap JE)$. Then, for any $C \in \mathcal{K}(\mathbb{C})$ we have*

$$\text{length}(C)^d \text{Vol}_d(K) \leq \text{Vol}_d(D_C K),$$

where $d = 2(l - a)$. S^1 -invariant convex bodies provide equality.

Despite the mentioned equality cases, the bound for the volume contained in the result is certainly bad for convex bodies K which are not $2m$ -dimensional and $l \neq 2a$.

Unfortunately using the above idea we have been able only to bring the dimension of K into play when bounding a precise quermassintegral of $D_C K$. For, we use the normalization of quermassintegrals known as intrinsic volumes. If, furthermore, the classical inradius $r(K)$, and circumradius, $R(K)$, come into play, we get further lower bounds, as well as upper bounds involving both, the volume of $D_C K$ and the volume (in the corresponding dimension) of K . For completeness we include the mentioned results in the following lemma.

Lemma 3.8. *Let $K \in \mathcal{K}(W)$ with $\dim K = l \leq 2m$ and $C \in \mathcal{K}(\mathbb{C})$. Then*

- i) $W_{2m-l}(D_C K) \geq \frac{\kappa_{2m-l}}{\binom{2m}{l}} \text{length}(C)^l \text{Vol}_l(K)$,
- ii) $\text{Vol}(D_C K) \geq \frac{\kappa_{2m-l}}{\binom{n}{l}} r(D_C K)^{2m-l} \text{length}(C)^l \text{Vol}_l(K)$,
- iii) $\text{Vol}(D_C K) \geq \frac{\kappa_{2m-l}}{\binom{n}{l}} r(K)^{2m-l} \text{length}(C)^{2m} \text{Vol}_l(K)$,
- iv) $\text{Vol}(D_C K) \leq \kappa_{2m} R(K)^{2m} \text{length}(C)^{2m}$.

Equality holds in i) and ii) if $l = 2m$ and $K = \alpha K$ for all $\alpha \in S^1$, i.e., K is a S^1 -invariant convex body. If $K = B_{2m}$ there is equality in iii) and iv).

Proof. Notice that from the definition of inradius and circumradius we have that, up to translation,

$$r(D_C K)B_{2m} \subseteq D_C K \subseteq R(D_C K)B_{2m}.$$

- i) Using Proposition 3.6 for the case $i = 2m - l$, we have that $W_{2m-l}(D_C K) \geq W_{2m-l}(K) \text{length}(C)^l$. Since K has dimension l , using the relation between quermassintegrals and intrinsic volumes ([31, (4.2.26)]) we have that

$$W_{2m-l}(K) = \frac{\kappa_{2m-l}}{\binom{2m}{l}} \text{Vol}_l(K),$$

which gives i).

- ii) Since mixed volumes are monotone, we have that

$$\text{Vol}(D_C K) \geq V(r(D_C K)B_{2m}[2m - l], D_C K[l]) = r(D_C K)^{2m-l} W_{2m-l}(D_C K).$$

Then the inequality follows from i).

- iii) It is enough to prove it for $\dim K = 2m$. In this case, $r(K)B_{2m} \subseteq K$. Now, i), the monotonicity of the complex difference body operator and the inradius ensure that $r(D_C r(K)B_{2m}) \subseteq r(D_C K)$, from which the second inequality follows by observing that $r(D_C r(K)B_{2m}) = r(K) \text{length}(C)$.
- iv) From the monotonicity of D_C we have $D_C K \subseteq D_C(R(K)B_{2m}) = \text{length}(C)R(K)B_{2m}$ and the inequality follows. □

An immediate consequence coming from the classical Brunn-Minkowski inequality, since D_C is Minkowski additive, i.e., $D_C(K + L) = D_C K + D_C L$, is the following

Corollary 3.9. *Let $K, L \in \mathcal{K}(W)$. Then*

$$\text{Vol}(D_C(K + L))^{1/2m} \geq \text{Vol}(D_C K)^{1/2m} + \text{Vol}(D_C L)^{1/2m}.$$

Next we deal with some very classical magnitudes, such as diameter diam and circumradius R of the complex difference body $D_C K$.

Proposition 3.10. *Let $K \in \mathcal{K}(W)$ and $C \in \mathcal{K}(\mathbb{C})$ be a polygon. Then, there is a constant $\mu_C > 0$ independent of K so that*

$$\mu_C \text{diam}(K) \text{length}(C) \leq \text{diam}(D_C K) \leq \text{diam}(K) \text{length}(C),$$

and

$$\mu_C \sqrt{\frac{2m+1}{m}} R(K) \text{length}(C) \leq R(D_C K) \leq R(K) \text{length}(C).$$

The upper bound holds, in both cases, for every $C \in \mathcal{K}(\mathbb{C})$.

Proof. The upper bound for $\text{diam}(D_C K)$ was proved in Lemma 3.3. For the lower bound, we use that for $K, K' \in \mathcal{K}(W)$ (see [10, Theorem 1.2])

$$\sqrt{2} \text{diam}(K + K') \geq \text{diam}(K) + \text{diam}(K').$$

Applying this inequality to $D_C K = \sum_{i=1}^N r_i \alpha_i K$, where $\alpha_i \in S^1$ are the normal vectors of C to the edges s_i with length r_i , $i = 1, \dots, N$ we have

$$\text{diam}(D_C K) \geq \sum_{i=1}^{N-1} \frac{1}{2^{i/2}} \text{diam}(\alpha_i r_i K) + \frac{1}{2^{(N-1)/2}} \text{diam}(\alpha_N r_N K) \geq \sum_{i=1}^N \frac{1}{2^{i/2}} \text{diam}(\alpha_i r_i K).$$

Then, taking $s = \sum_{i=1}^N 2^{i/2}$, since the diameter is 1-homogeneous, we get

$$\text{diam}(D_C K) \geq \frac{\text{length}(C)}{s} \text{diam}(K).$$

The lower bound of the circumradius, follows from previous inequality for the diameter and Jung's inequality between circumradius and diameter (see [17]). We get

$$R(D_C K) \geq \frac{\text{diam}(D_C K)}{2} \geq \frac{\text{length}(C)}{s} \text{diam}(K) \geq \frac{\text{length}(C)}{s} \sqrt{\frac{2m+1}{m}} R(K).$$

We prove the upper bound for $R(D_C K)$, for C a polygon. The general result for the upper bound follows by approximation. It holds

$$R(D_C K) = R\left(\sum_{i=1}^N r_i \alpha_i K\right) \leq \sum_{i=1}^N R(r_i \alpha_i K) = \sum_{i=1}^N r_i R(K) = \text{length}(C) R(K).$$

□

Of course, if C is not a polygon, approximating it by polytopes, $\mu_C = 0$ and the trivially zero as lower bound is obtained.

To finish this section we include an inequality for the mixed volume of j times K and $2m - j$ times $D_C K$. For a conjecture on an upper bound in the classical case we refer to [31, Notes Section 7.3].

Lemma 3.11. *Let $K \in \mathcal{K}(W)$ and $C \in \mathcal{K}(\mathbb{C})$. Then*

$$V(K[j], D_C K[2m - j]) \geq \text{length}(C)^{2m-j} \text{Vol}(K).$$

Proof. The proof follows as a consequence of a special case of the Aleksandrov-Fenchel inequality. Indeed, using the multilinearity of the mixed volumes, we have

$$V(K[j], D_C K[2m - j]) = \int_{S^1} \cdots \int_{S^1} V(K[j], \alpha_1 K, \dots, \alpha_{2m-j} K) d\mathcal{S}(C, \alpha_1) \cdots d\mathcal{S}(C, \alpha_{2m-j}).$$

From the Aleksandrov-Fenchel inequality, it follows (see [8, (B.18)])

$$V(K[j], \alpha_1 K, \dots, \alpha_{2m-j} K)^{2m} \geq \text{Vol}(K)^j \text{Vol}(\alpha_1 K) \cdots \text{Vol}(\alpha_{2m-j} K) = \text{Vol}(K)^{2m},$$

which yields straightforward the result. □

4. SPHERICAL HARMONICS IN S^{2m-1}

In this section we introduce the notions and results we shall use from the theory of spherical harmonics in the unit sphere of \mathbb{C}^m endowed with the standard hermitian product. We first review the basic notions in a real vector space. For a description of the theory of spherical harmonics with applications to convex geometry we refer to the book [13]. The results for a complex vector space can be found in [20, 25, 26, 29] and references therein.

Let \mathbb{R}^n be endowed with the usual scalar product. A *spherical harmonic on S^{n-1}* is the restriction to S^{n-1} of a harmonic polynomial p on \mathbb{R}^n , i.e., $\Delta p = 0$ where Δ denotes the Laplace operator on \mathbb{R}^n . It is said *of degree k* if it is the restriction of a harmonic polynomial of degree of homogeneity k with $k \in \mathbb{N}$. In the following we assume that $0 \in \mathbb{N}$. We denote by \mathcal{H}_k^n the vector space of spherical harmonics in S^{n-1} of degree k and by \mathcal{H}^n the vector space of the spherical harmonics in S^{n-1} . It is known that \mathcal{H}^n can be decomposed as

$$\mathcal{H}^n = \bigoplus_{k=0}^{\infty} \mathcal{H}_k^n,$$

and $\dim \mathcal{H}_k^n =: N(n, k) = \frac{n+2k-2}{n+k-2} \binom{n+k-2}{n-2}$. Here and in the following, the orthogonality is understood with respect to the standard product on $L^2(S^{n-1})$:

$$\langle f, g \rangle_{L^2} := \int_{S^{n-1}} f(u) \overline{g(u)} d\sigma_{n-1}(u), \quad f, g \in L^2(S^{n-1}),$$

where $d\sigma_{n-1}$ denotes the standard measure on the sphere S^{n-1} .

If $f \in L^2(S^{n-1})$ and $\{Y_{kj} \in \mathcal{H}_k^n : k \in \mathbb{N}, j = 1, \dots, N(n, k)\}$ is an orthonormal system, then the *harmonic expansion of f* with respect to this orthonormal system is given by

$$f \sim \sum_{k=0}^{\infty} \sum_{j=1}^{N(n, k)} \langle f, Y_{kj} \rangle Y_{kj} = \sum_{k=0}^{\infty} \pi_k f,$$

where $\pi_k f := \sum_{j=1}^{N(n, k)} \langle f, Y_{kj} \rangle Y_{kj}$ is the orthogonal projection of f onto \mathcal{H}_k^n .

We observe that the spherical harmonics of even (respectively odd) degree are even (respectively odd) functions on the sphere. Moreover, the space \mathcal{H}_k^n is rotation-invariant (i.e., if $Y \in \mathcal{H}_k^n$, then $\varphi Y \in \mathcal{H}_k^n$ for every $\varphi \in \text{SO}(n)$) and irreducible, i.e., the only subspaces of \mathcal{H}_k^n which are $\text{SO}(n)$ -invariant are the trivial and the total space \mathcal{H}_k^n . Fixing a point $v_0 \in S^{n-1}$, we can describe S^{n-1} as the homogeneous space

$$\text{SO}(n)/\text{SO}(n-1)$$

where $\text{SO}(n-1)$ is the isotropy group of v_0 . Using this description of S^{n-1} and the irreducibility of \mathcal{H}_k^n , $k \in \mathbb{N}$, we get from the representation theory that there exists a unique spherical harmonic Q_k of degree k , $\text{SO}(n-1)$ -invariant and such that $Q_k(v_0) = 1$. The polynomials $\{Q_k\}_k$ define the *Legendre polynomials*, which are essential in the theory of spherical harmonics.

In our situation, we have $\mathbb{C}^m \cong \mathbb{R}^{2m}$ endowed with the standard hermitian product (\cdot, \cdot) as an ambient space. Considering the sphere S^{2m-1} in \mathbb{R}^{2m} , all the previous facts remain valid. However, as a vector space over \mathbb{C} , the multiplication by complex numbers gives rise to a richer structure on the space \mathcal{H}^{2m} .

Definition 4.1. A spherical harmonic $Y \in \mathcal{H}^{2m}$ is homogeneous of bi-degree (k, l) if $Y \in \mathcal{H}_{k+l}^{2m}$ and

$$Y(\alpha u) = \alpha^k \bar{\alpha}^l Y(u) \quad \forall \alpha \in S^1, u \in S^{2m-1}.$$

We denote by $\mathcal{H}_{k,l}^{2m}$ the space of spherical harmonics on S^{2m-1} of bi-degree (k, l) .

It holds

$$(10) \quad \mathcal{H}_k^{2m} = \bigoplus_{l=0}^k \mathcal{H}_{l, k-l}^{2m} \quad \forall k \in \mathbb{N}.$$

Each space $\mathcal{H}_{l,k-l}^{2m}$ is invariant under the unitary group $U(m)$ and also irreducible (see [21]). Thus, a function $f \in \mathcal{C}(S^{2m-1})$ can be expressed as

$$f \sim \sum_{k=0}^{\infty} \sum_{l=0}^k \pi_{l,k-l}(f),$$

where $\pi_{k,l}(f)$ is the orthogonal projection of f onto $\mathcal{H}_{k,l}^{2m}$, and $f \in \mathcal{C}(S^{2m-1})$ is univocally determined by $\pi_{k,l}(f)$, $(k, l) \in \mathbb{N}^2$. Fixed a point $e \in S^{2m-1}$ with isotropy group $U(m-1)$, we have $S^{2m-1} \cong U(m)/U(m-1)$. Similar to the real case, we get that for every $(k, l) \in \mathbb{N}^2$ there exists a unique spherical harmonic $P_{k,l}$ of degree $k+l$, $U(m-1)$ -invariant and such that $P_{k,l}(e) = 1$. These polynomials, analogues to the Legendre polynomials, are the so-called *Jacobi polynomials*.

Definition 4.2. A function $\tilde{P} \in \mathcal{C}(S^{2m-1})$ is called spherical function of order (k, l) if $\tilde{P} \in \mathcal{H}_{k,l}^{2m}$, \tilde{P} is $U(m-1)$ -invariant and $\tilde{P}(e) = 1$. Its associated Jacobi polynomial $P : B_2 \rightarrow \mathbb{C}$ of order (k, l) is defined by $P(z) = \tilde{P}(w)$, where $w \in S^{2m-1}$ is such that $(e, w) = z$ and B_2 denotes the unit disc in \mathbb{C} .

Jacobi polynomials satisfy the following properties.

Proposition 4.3 ([16, 26]). For every $k, l \in \mathbb{N}$ the vector space $\mathcal{H}_{k,l}^{2m}$ contains a unique spherical function given by $P_{k,l}((e, \cdot))$ and satisfying

$$P_{k,l}(z) = \overline{P_{k,l}(\bar{z})}, \quad k, l \in \mathbb{N},$$

$$P_{k,l}(z) = r^{|k-l|} e^{i\theta(k-l)} Q_{\min\{k,l\}}(|k-l|, m-2, |z|^2), \quad k, l \in \mathbb{N}, z = re^{i\theta}, r \geq 0, \theta \in [0, 2\pi),$$

where $\{Q_l(a, b, t) : l \in \mathbb{N}\}$ constitutes a complete set of polynomials in $t \in \mathbb{R}$ orthogonal on $[0, 1]$ in weight $t^a(1-t)^b dt$ and satisfying $Q_l(a, b, 1) = 1$, $a, b > -1$.

The set $\{P_{k,l}\}_{k,l}$ constitutes a complete orthogonal set of functions in $L^2(B_2)$.

One of the main results in the theory of spherical harmonics is the Funk-Hecke theorem. A complex version of it was proved in [26]. We shall denote by $L^2(B_2, ((1-|z|^2)^{m-2})dz)$ the weighted L^2 space on B_2 with respect to the density measure $((1-|z|^2)^{m-2})dz$.

Theorem 4.4 (Complex Funk-Hecke Theorem, [26]). Let $K : B_2 \rightarrow \mathbb{R}$ be in $L^2(B_2, ((1-|z|^2)^{m-2})dz)$ and $Y_{k,l} \in \mathcal{H}_{k,l}^{2m}$. Then,

$$\int_{S^{2m-1}} K((u, v)) Y_{k,l}(v) d\sigma_{2m-1}(v) = \lambda_{k,l} Y_{k,l}(u),$$

with

$$\lambda_{k,l} = \sigma_{2m-3} \int_{B_2} K(z) \overline{P_{k,l}(z)} (1-|z|^2)^{m-2} dz,$$

where σ_{2m-3} denotes the surface of S^{2m-3} .

Definition 4.5. A linear map $A : \mathcal{C}(S^{2m-1}) \rightarrow \mathcal{C}(S^{2m-1})$ is a multiplier transformation, if for every $k, l \in \mathbb{N}$ there exists a complex number $\lambda_{k,l}$ such that

$$\pi_{k,l}(Af) = \lambda_{k,l} \pi_{k,l}(f) \quad \forall f \in \mathcal{C}(S^{2m-1}).$$

The complex numbers $\lambda_{k,l}$ are called multipliers of the map A .

Similar to the real case, a multiplier transformation is injective if and only if $\lambda_{k,l} \neq 0$ for every $k, l \in \mathbb{N}$ (cf. [24]).

Note that if the linear map $A : \mathcal{C}(S^{2m-1}) \rightarrow \mathcal{C}(S^{2m-1})$ is given by a kernel $K : B_2 \rightarrow \mathbb{R}$ in $L^2(B_2, ((1-|z|^2)^{m-2})dz)$, then by the complex Funk-Hecke theorem we have that A is a multiplier transformation. In the following, we shall use the harmonic expansion of the support function of a convex body, seen as a function on the sphere S^{2m-1} .

In the previous lines, we have introduced the theory of spherical harmonics for the space of continuous functions on the sphere. However, this theory can be established for its

dual space, the space of Borel measures on the sphere. In this paper, we shall only treat Borel measures on S^1 , for which we can consider the classical Fourier series expansion without using spherical harmonics. We identify S^1 with $[0, 2\pi]$ and the functions on S^1 with 2π -periodic functions on \mathbb{R} . In the identification of S^1 with $[0, 2\pi]$, we denote the elements of both spaces by α , but it should be clear from the context where α is meant. Using the standard notation, we define the j -th Fourier coefficient $c_j(f)$ of $f \in \mathcal{C}(S^1)$ and, analogously, of a Borel measure μ on S^1 by

$$\begin{aligned} c_0(f) &:= \frac{1}{2\pi} \int_0^{2\pi} f(\alpha) d\alpha, & c_0(\mu) &:= \frac{1}{2\pi} \int_0^{2\pi} d\mu(\alpha), \\ c_j(f) &:= \frac{1}{\pi} \int_0^{2\pi} e^{ij\alpha} f(\alpha) d\alpha, & c_j(\mu) &:= \frac{1}{\pi} \int_0^{2\pi} e^{ij\alpha} d\mu(\alpha), \quad j \in \mathbb{Z}. \end{aligned}$$

We shall use the following fact for the surface area measure of a planar convex body.

Lemma 4.6. *Let $C \in \mathcal{K}(\mathbb{R}^2)$. The Fourier coefficients of the surface area measure of C , \mathcal{S}_C , are related with the Fourier coefficients of the support function of C by*

$$c_j(\mathcal{S}_C) = (1 - j^2)c_j(h_C), \quad j \in \mathbb{Z}.$$

Proof. By definition,

$$c_j(\mathcal{S}_C) = \frac{1}{\pi} \int_0^{2\pi} e^{ij\alpha} d\mathcal{S}(C, \alpha) = \frac{1}{\pi} \int_0^{2\pi} e^{ij\alpha} (h_C(\alpha) + h_C''(\alpha)) d\alpha.$$

Interpreting $h_C + h_C''$ as a distribution and $\alpha \mapsto e^{ij\alpha}$ as a 2π -periodic test function on \mathbb{R} , we can use twice integration by parts and we obtain the result. \square

Remark 4.7. *Note that $c_{-j}(h_C) = \overline{c_j(h_C)}$ since h_C takes only real values.*

Next, we collect the main geometric cases we will treat. For, we need the following lemma.

Lemma 4.8. *A function $f \in L^2(S^{2m-1})$ is S^1 -invariant, i.e., $f(\alpha u) = f(u)$ for every $\alpha \in S^1$, $u \in S^{2m-1}$, if and only if its harmonic expansion is given by*

$$f \sim \sum_{j=0}^{\infty} \pi_{j,j}(f).$$

Proof. From the definition of the spherical harmonics of bi-degree (k, l) , it follows directly that $Y \in \mathcal{H}_{k,l}^{2m}$ is a S^1 -invariant function on S^{2m-1} if and only if $k = l$. \square

Examples 4.9.

- i) Balls. *It is well known that $\pi_j(h_{B_{2m}}) = 0$ for every $j \in \mathbb{N}$, $j > 0$. From the orthogonal decomposition in (10) it also holds $\pi_{k,l}(B_{2m}) = 0$ for every $(k, l) \in \mathbb{N}^2 \setminus \{(0, 0)\}$.*
- ii) S^1 -invariant convex bodies. *Let $K \in \mathcal{K}(\mathbb{C}^m)$ be a S^1 -invariant convex body. Then, its support function is invariant under the action of S^1 , i.e.,*

$$h(K, \alpha u) = h(K, u) \quad \forall \alpha \in S^1, u \in S^{2m-1}.$$

Thus, using Lemma 4.8, we have that the harmonic expansion of the support function of a S^1 -invariant convex body only contains terms of the form $\pi_{j,j}(h_K)$, $j \in \mathbb{N}$.

- iii) Symmetric convex bodies. *The Fourier coefficients $c_j(h_C)$ of a symmetric convex body $C \in \mathcal{K}(\mathbb{R}^2)$ satisfy $c_{2j+1}(h_C) = 0$, for every $j \in \mathbb{Z}$ (since $h_C(\alpha + \pi) = h_C(\alpha)$ for every $\alpha \in [0, 2\pi]$).*

In higher dimensions, for a symmetric convex body K , it holds $\pi_{2j+1}(h_K) = 0$ since $h_K(-u) = h_K(u)$ for every $u \in S^{2m-1}$. Hence, we also have $\pi_{k,l}(h_K) = 0$ for every $(k, l) \in \mathbb{N}^2$ such that $k + l$ is odd.

- iv) Convex bodies of constant width. For a convex body K of constant width, it holds $\pi_{2j}(h_K) = 0$, $j > 0$ (see [13, Proposition 5.7.1]). If $K \in \mathcal{K}(\mathbb{C}^m)$ has constant width, then $\pi_{k,l}(h_K) = 0$ for every $(k,l) \in \mathbb{N}^2 \setminus \{(0,0)\}$ such that $k+l$ is even.
- v) Interval. Assume that $I \subset \mathbb{C}$ is the interval $[-i/2, i/2]$. One can easily compute the Fourier coefficients of \mathcal{S}_I obtaining $c_0(\mathcal{S}_I) = 1/\pi$ and $c_j(\mathcal{S}_I) = (1 + (-1)^j)/\pi$. Notice that the perimeter of a segment in \mathbb{R}^2 is twice its length.
- vi) Difference body. The coefficients of the harmonic expansion of the difference body DK are given by

$$\pi_j(h_{DK}) = \pi_j(h_K) + \pi_j(h_{-K}) = \begin{cases} 2\pi_j(h_K), & \text{if } j \text{ is even,} \\ 0, & \text{if } j \text{ is odd.} \end{cases}$$

Hence, the difference body operator (extended from the space of support functions to the space of continuous functions on the sphere) is a multiplier transformation with multipliers $\lambda_{2j} = 2$ and $\lambda_{2j+1} = 0$, $j \in \mathbb{N}$, which coincides with the Fourier coefficients of \mathcal{S}_I up to its normalization factor $1/2\pi$ or $1/\pi$.

5. THE OPERATOR D_C AS A MULTIPLIER TRANSFORMATION

In view of the last example above, we would like to interpret the operator D_C , for a general $C \in \mathcal{K}(\mathbb{C})$, as a multiplier transformation and express its multipliers in terms of the Fourier coefficients of h_C . This is the content of the next result.

Theorem 5.1. *Let $C \in \mathcal{K}(\mathbb{C})$ be a convex body. The extension of the operator D_C to the space of continuous functions on S^{2m-1} given by*

$$A_C : \begin{array}{ccc} \mathcal{C}(S^{2m-1}) & \longrightarrow & \mathcal{C}(S^{2m-1}) \\ f & \longmapsto & \int_{S^1} f(\bar{\alpha} \cdot) d\mathcal{S}(C, \alpha) \end{array}$$

is a multiplier transformation and the multipliers are

$$(11) \quad \lambda_{k,l}(A_C) = \int_0^{2\pi} e^{i(k-l)\alpha} d\mathcal{S}(C, \alpha).$$

Proof. Assume first that $C \in \mathcal{K}(\mathbb{C})$ is of class \mathcal{C}^2 , that is, its support function is of class \mathcal{C}^2 . In this case, we have that $g := h_{\bar{C}} + h_C''$ is a continuous function on S^1 . We denote by \check{g} its homogeneous extension of degree 0 on \mathbb{R}^2 . Let $\epsilon > 0$ and $v \in S^{2m-1}$. We define the map $K_{\epsilon,g}((v, \cdot))$ on S^{2m-1} by

$$K_{\epsilon,g}((v, \cdot)) := \frac{m-1}{2^{m-2} \sigma_{2m-3} \epsilon^{m-1}} \chi_{[1-\epsilon, 1]}(|(v, \cdot)|) \check{g}((v, \cdot)).$$

We claim that the function

$$v \mapsto \lim_{\epsilon \rightarrow 0} \int_{S^{2m-1}} K_{\epsilon,g}((v, w)) f(w) d\sigma_{2m-1}(w)$$

converges uniformly to $A_C f$.

For $v \in S^{2m-1}$ fixed, any vector $w \in S^{2m-1}$ can be uniquely decomposed as

$$w = zv + \sqrt{1 - |z|^2} \xi, \quad z \in B_2, \quad \xi \in S^{2m-1}, \quad (\xi, v) = 0.$$

The parametrization of the sphere corresponding to this decomposition, is given by

$$d\sigma_{2m-1}(w) = (1 - |z|^2)^{m-2} dz d\sigma_{2m-3}(\xi),$$

where $d\sigma_j$ is the standard measure on the sphere S^j . Then, we have

$$\int_{S^{2m-1}} K_{\epsilon,g}((w, v)) f(w) d\sigma_{2m-1}(w) = \int_{S^{2m-3}} \int_{B_2} K_{\epsilon,g}(z) f(zv + \sqrt{1 - |z|^2} \xi) (1 - |z|^2)^{m-2} dz d\sigma_{2m-3}(\xi).$$

Using the definition of $K_{\epsilon,g}$, polar coordinates on B_2 , the 0-homogeneity of \check{g} and the change of variable $t = (1 - r)/\epsilon$, we get

$$\begin{aligned} & \int_{S^{2m-1}} K_{\epsilon,g}((w, v))f(w)d\sigma_{2m-1}(w) \\ &= \frac{m-1}{2^{m-2}\sigma_{2m-3}\epsilon^{m-1}} \int_{S^{2m-3}} \int_{B_2} \chi_{[1-\epsilon,1]}(|z|)\check{g}(z)(1-|z|^2)^{m-2}f(zv + \sqrt{1-|z|^2}\xi)dzd\sigma_{2m-3}(\xi) \\ &= \frac{m-1}{2^{m-2}\sigma_{2m-3}\epsilon^{m-1}} \int_{S^{2m-3}} \int_0^{2\pi} \int_{1-\epsilon}^1 (1-r^2)^{m-2}rf(re^{i\theta}v + \sqrt{1-r^2}\xi)g(e^{i\theta})drd\theta d\sigma_{2m-3}(\xi) \\ &= \frac{m-1}{2^{m-2}\sigma_{2m-3}} \int_{S^{2m-3}} \int_0^{2\pi} \int_0^1 t^{m-2}f_\epsilon(t, \theta, \xi)g(e^{i\theta})dtd\theta d\sigma_{2m-3}(\xi), \end{aligned}$$

where

$$f_\epsilon(t, \theta, \xi) := (2 - t\epsilon)^{m-2}(1 - t\epsilon)f((1 - t\epsilon)e^{i\theta}v + \sqrt{t\epsilon(2 - t\epsilon)}\xi).$$

For $0 < \epsilon < 1$ we have $|f_\epsilon(t, \theta, \xi)| \leq 2^{m-2}\|f\|_\infty < +\infty$ since $0 \leq t \leq 1$ and f is bounded on S^{2m-1} . Moreover,

$$\lim_{\epsilon \rightarrow 0} f_\epsilon(t, \theta, \xi) = 2^{m-2}f(e^{i\theta}v).$$

Applying the theorem of the dominated convergence, we get

$$\begin{aligned} \lim_{\epsilon \rightarrow 0} \int_{S^{2m-1}} K_{\epsilon,g}((w, v))f(w)d\sigma(w) &= \frac{m-1}{\sigma_{2m-3}} \int_{S^{2m-3}} \int_0^{2\pi} \int_0^1 t^{m-2}f(e^{i\theta}v)g(e^{i\theta})dtd\theta d\sigma_{2m-3}(\xi) \\ &= \int_0^{2\pi} f(e^{i\theta}v)d\mathcal{S}(\overline{C}, \theta) = \int_0^{2\pi} f(e^{-i\theta}v)d\mathcal{S}(C, \theta), \end{aligned}$$

and the claim follows.

If $C \in \mathcal{K}(\mathbb{C})$ is not of class \mathcal{C}^2 , it can be approximated by convex bodies of class \mathcal{C}^2 and, by continuity, we get the result.

From the complex Funk-Hecke theorem, we have that A_C is a multiplier transformation, and the multipliers are given by

$$\lambda_{k,l}(A_C) = \sigma_{2m-3} \lim_{\epsilon \rightarrow 0} \int_{B_2} K_{\epsilon,g}(z)\overline{P_{k,l}}(z)(1-|z|^2)^{m-2}dz.$$

Using polar coordinates on B_2 , Proposition 4.3, the 0-homogeneity of \check{g} , again the change of variable $t = (1 - r)/\epsilon$, the dominated convergence theorem and $Q_l(a, b, 1) = 1$, $a, b > -1$, we get

$$\begin{aligned} \lambda_{k,l}(A_C) &= \lim_{\epsilon \rightarrow 0} \frac{c}{\epsilon^{m-1}} \int_{1-\epsilon}^1 r^{|k-l|+1} Q_{\min\{k,l\}}(|k-l|, m-2, r^2)(1-r^2)^{m-2}dr \int_0^{2\pi} e^{i\theta(k-l)}d\mathcal{S}(C, \theta) \\ &= c \lim_{\epsilon \rightarrow 0} \int_0^1 Q_{\min\{k,l\}}(|k-l|, m-2, (1-t\epsilon)^2)t^{m-2}(1-t\epsilon)^{|k-l|+2}(2-t\epsilon)^{m-2}dt \int_0^{2\pi} e^{i\theta(k-l)}d\mathcal{S}(C, \theta) \\ &= \int_0^{2\pi} e^{i\theta(k-l)}d\mathcal{S}(C, \theta), \end{aligned}$$

where $c = (m-1)2^{2-m}$. Hence, the result follows. \square

Remark 5.2. The multipliers $\lambda_{k,l}(A_C)$ can be easily related with the Fourier coefficients of the area measure of C as

$$\lambda_{k,k}(A_C) = 2\pi c_0(\mathcal{S}_C), \quad \lambda_{k,l}(A_C) = \pi c_{k-l}(\mathcal{S}_C), \quad k \neq l.$$

In order to simplify the notation we will redefine

$$c_{k-l}(\mathcal{S}_C) := \lambda_{k,l}(A_C), \quad k, l \in \mathbb{N}.$$

From now on $c_j(\mathcal{S}_C)$ has to be interpreted as the j -th Fourier coefficient of \mathcal{S}_C with the normalization described in this remark. The same renormalization will be considered for the Fourier coefficients of the support function h_C .

Now, using Lemma 4.6 we can explicitly give the coefficients of the harmonic expansion of the support function of the complex difference body, as contained in the next result.

Corollary 5.3. *The coefficients of the harmonic expansion of the support function of $D_C K$, $C \in \mathcal{K}(\mathbb{C})$, $K \in \mathcal{K}(\mathbb{C}^m)$ are given by*

$$\pi_{k,l}(h_{D_C K}) = c_{k-l}(\mathcal{S}_C)\pi_{k,l}(h_K) = (1 - (k-l)^2)c_{k-l}(h_C)\pi_{k,l}(h_K).$$

Remark 5.4. *For every $C \in \mathcal{K}(\mathbb{C})$ and $K \in \mathcal{K}(\mathbb{C}^m)$, the convex body $D_C K$ has Steiner point at the origin. Indeed, the Steiner point $s(L)$ of a convex body $L \in \mathcal{K}(\mathbb{C}^m)$ is given by the first coefficient of the harmonic expansion of its support function, more precisely, by (see [31, p. 430])*

$$\pi_1(h_L) = \langle s(L), \cdot \rangle_{\mathbb{R}^{2m}},$$

since $s(L) = \frac{1}{\omega_{2m}} \int_{S^{2m-1}} h_K(u)u d\sigma(u)$. From Corollary 5.3 we have the claim.

6. FIXED POINTS OF $D_C K$. CONNECTIONS OF $D_C K$ TO CLASSES OF CONVEX BODIES

It is well-known that $DK = 2K$ if and only if K is a symmetric convex body. Next, we study the set of fixed points of the operator D_C , $C \in \mathcal{K}(\mathbb{C})$.

Theorem 6.1. *Let $C \in \mathcal{K}(\mathbb{C})$ and $K \in \mathcal{K}(\mathbb{C}^m)$. Then, $D_C K = \lambda K$ if and only if $\lambda = \text{length}(C)$ and either $\pi_{k,l}(h_K) = 0$ or $c_{k-l}(\mathcal{S}_C) = \lambda$, $\forall (k,l) \in \mathbb{N}^2$.*

Proof. We have

$$c_0(\mathcal{S}_C) = \int_{S^1} d\mathcal{S}(C, \alpha) = \text{length}(C).$$

Using Corollary 5.3 we have $\pi_{0,0}(h_{D_C K}) = c_0(\mathcal{S}_C)\pi_{0,0}(h_K)$. Hence, $\pi_{0,0}(h_{D_C K}) = \lambda\pi_{0,0}(h_K)$ if and only if $\lambda = \text{length}(C)$.

The second condition also follows directly from Corollary 5.3. \square

Notice that if $\pi_{0,0}(h_K) = 0$, then K is a point and $D_C K = 0$, for every $C \in \mathcal{K}(\mathbb{C})$.

From the previous result, we recover the classical case (cf. 4.9.v) and we have these two particular facts.

Corollary 6.2. *Let K be a S^1 -invariant convex body in \mathbb{C}^m . Then, $D_C K = \text{length}(C)K$ for every $C \in \mathcal{K}(\mathbb{C})$.*

Let C be the unit disc in \mathbb{C} . Then, $D_C K = 2\pi K$ if and only if K is S^1 -invariant.

A question related to the study of the fixed points of an operator concerns the study of the iteration of the operator, i.e., in our case, the study of D_C^N for $N \geq 1$. For the classical case, we clearly have $D \circ \dots \circ D = D^N = 2^{N-1}D$, that is, the iteration of the operator D does not change the “shape” of the image of $K \in \mathcal{K}(\mathbb{R}^n)$. From Theorem 6.1 we have that this fact generalizes only for few $C \in \mathcal{K}(\mathbb{C})$; for instance, if C is a ball.

Corollary 6.3. *Let $C \in \mathcal{K}(\mathbb{C})$. Then, $D_C^N = \text{length}(C)^{N-1}D_C$ if and only if for every $j \in \mathbb{N}$, $c_j(\mathcal{S}_C)^{N-1} = \text{length}(C)^{N-1}$ or $c_j(\mathcal{S}_C) = 0$.*

Proof. Let $K \in \mathcal{K}(\mathbb{C}^m)$. From Theorem 6.1, we have that $D_C^N K = \text{length}(C)^N K$ if and only if $\pi_{k,l}(h_K) = 0$ or $c_{k-l}(\mathcal{S}_C) = \text{length}(C)$ for every $(k,l) \in \mathbb{N}^2$. Indeed, using recursively that $D_C^N K = D_C(D_C^{N-1} K)$, we have

$$\pi_{k,l}(h_{D_C^N K}) = c_{k-l}(\mathcal{S}_C)\pi_{k,l}(h_{D_C^{N-1} K}) = c_{k-l}^N(\mathcal{S}_C)\pi_{k,l}(h_K),$$

and the claim follows. \square

We study next the set of fixed points of D_{C_N} being C_N a regular polygon with N sides. This case enlightens how the “symmetry” of C_N is inherited by $D_{C_N} K$.

Proposition 6.4. *Let $C_N \subset \mathbb{C}$ be a regular polygon with N sides each of length $1/N$ and normal vector α_i , $i = \{1, \dots, N\}$. Then,*

$$D_{C_N} K = K \Leftrightarrow \alpha_i K = K \quad \forall i \in \{1, \dots, N\}.$$

Proof. Using that $D_C K$ is invariant under translations of C , we can assume that C_N is a regular polygon centered at the origin. Hence, it has rotational symmetry of order N , and we have that the normal vectors, as elements in S^1 , constitute a finite group G_N , i.e., for every $i, j \in \{1, \dots, N\}$ we have $\alpha_i \alpha_j = \alpha_k$ for some $k \in \{1, \dots, N\}$. Here the multiplication on the group G_N is given by the usual product of complex numbers.

We claim that $\alpha_i D_{C_N} K = D_{C_N} K$ for every $K \in \mathcal{K}(\mathbb{C}^m)$ and every $i \in \{1, \dots, N\}$. Indeed,

$$\begin{aligned} h(\alpha_i D_{C_N} K, u) &= \int_{S^1} h(\alpha K, \overline{\alpha_i} u) d\mathcal{S}(C_N, \alpha) = \int_{S^1} h(\alpha_i \alpha K, u) d\mathcal{S}(C_N, \alpha) \\ &= \frac{1}{N} \sum_{j=1}^N h(\alpha_i \alpha_j K, u) = \frac{1}{N} \sum_{k=1}^N h(\alpha_k K, u) = h(D_{C_N} K, u). \end{aligned}$$

Thus, $D_{C_N} K = K$ implies that $\alpha_i K = K$ for every $i \in \{1, \dots, N\}$. If K satisfies the previous condition, then

$$h(D_{C_N} K, u) = \frac{1}{N} \sum_{j=1}^N h(\alpha_j K, u) = h(K, u)$$

and the result follows. \square

Next we study different *families* of convex bodies, such as Minkowski classes, rotation means and universal convex bodies, in relation to the complex difference body.

Definition 6.5. A Minkowski class of convex bodies in \mathbb{R}^n is a subset $\mathcal{M} \subset \mathcal{K}(\mathbb{R}^n)$ closed in the Hausdorff metric and closed under Minkowski addition and nonnegative dilations.

Let G be a group acting on the set of convex bodies. A Minkowski class \mathcal{M} is G -invariant if for every $K \in \mathcal{M}$ and $g \in G$ we have $gK \in \mathcal{M}$.

Examples of Minkowski classes are the symmetric convex bodies and the zonoids (see [32] and [31, p.164]). Since the complex difference body operator is linear (for every $C \in \mathcal{K}(\mathbb{C})$), we get the following result.

Proposition 6.6. For every $C \in \mathcal{K}(\mathbb{C})$, the set $\mathcal{M}_C := \text{Im } D_C \subset \mathcal{K}(\mathbb{C}^m)$ is a Minkowski class. Moreover, if G is a group acting on \mathbb{C} under which C is invariant (i.e., $gC = C$ for every $g \in G$), then \mathcal{M}_C is G -invariant, where the action of G on W is the diagonal action described in the introduction.

For $C = I$ an interval, the class \mathcal{M}_I coincides with the class of symmetric convex bodies. If $C = B_2$ is a centered ball, then \mathcal{M}_{B_2} coincides with the class of S^1 -invariant convex bodies (cf. Section 7).

Definition 6.7 ([30]). A convex body $K \in \mathcal{K}(\mathbb{R}^n)$ is said to be universal if $\pi_j(h_K) \neq 0$ for every $j \in \mathbb{N}$, $j \neq 1$.

We define a convex body $K \in \mathcal{K}(\mathbb{C}^m)$ to be complex universal if $\pi_{k,l}(h_K) \neq 0$ for every $(k, l) \in \mathbb{N}^2$, $|k - l| \neq 1$. From Corollary 5.3 we get the following fact.

Corollary 6.8. $D_C K$ is complex universal if and only if K and C are both complex universal convex bodies.

Definition 6.9 ([31]). For a convex body $K \in \mathcal{K}(\mathbb{R}^n)$, we say that K' is a rotation mean of K if there are $N \geq 1$, $N \in \mathbb{N}$, and $\rho_i \in \text{SO}(n)$, $i = 1, \dots, N$ rotations so that

$$K' = \frac{1}{N} (\rho_1 K + \dots + \rho_N K).$$

Hadwiger proved that for any convex body K with positive dimension there is a sequence of rotation means of K converging to a ball (see [15]).

If $C \in \mathcal{K}(\mathbb{C})$ is a polygon whose edges have length one, then, up to a constant, the complex difference body $D_C K$ is a rotation mean of K . As Corollary 7.4 shall show, rotations in

$\mathrm{SL}(W, \mathbb{C})$ are not enough in order $D_C K$ to be a ball. Indeed, this result characterizes the pairs $K \in \mathcal{K}(\mathbb{C}^m)$ and $C \in \mathcal{K}(\mathbb{C})$ for which this can be done.

7. SURJECTIVITY AND INJECTIVITY OF D_C AND RELATED QUESTIONS

In this section, we use the interpretation of the operator D_C as a multiplier transformation in order to obtain statements about its range of injectivity and image for some geometrically interesting cases of C or restricted to some classes of convex bodies.

Definition 7.1 ([31]). *A non-empty convex body $K \subset \mathbb{R}^n$ is said to be indecomposable if $K = K_1 + K_2$ with $K_1, K_2 \in \mathcal{K}(\mathbb{R}^n)$ implies that K_1 and K_2 are homothetic to K .*

Now we can prove the following

Corollary 7.2. *The map $D: \mathcal{K}(\mathbb{C}) \times \mathcal{K}(\mathbb{C}^m) \rightarrow \mathcal{K}(\mathbb{C}^m)$ is neither injective nor surjective.*

Proof. The map D is not injective since for the real case, i.e., C a line segment, it is not.

For the surjectivity, let us consider, in the plane, a triangle T , all whose edges have different lengths. It is known that such triangles are indecomposable (cf. [31, Theorem 3.2.11]), i.e., there exist no convex bodies K, L so that $K + L = T$ and K, L are no dilates of T . Since we know, from Theorem 1.3, that if $D_C K = T$, then both C, K are polytopes, we have that

$$T = \sum_{i=1}^N \alpha_i r_i K$$

where α_i are the unit normal vectors to the edges of C , with corresponding lengths r_i , $i = 1, \dots, N$. Since T is indecomposable $\alpha_i r_i K = a_i T$ for $a_i \geq 0$, $i = 1, \dots, N$. Then, K is, up to a dilation, a rotation of T , namely $K = \overline{\alpha_i} r_i^{-1} a_i T$, what yields

$$T = \sum_{i=1}^N \alpha_i r_i \overline{\alpha_i} r_i^{-1} a_i T = \sum_{i=1}^N a_i T.$$

But this is not $D_C T$ for any $C \in \mathcal{K}(\mathbb{C})$ since all the edges of T have different length. \square

Next we deal with **convex bodies of constant width**. We denote by $\mathcal{W}^{2m} \subset \mathcal{K}(\mathbb{C}^m)$ the class of convex bodies of constant width.

Corollary 7.3. *Let $C \in \mathcal{K}(\mathbb{C})$ and $K \in \mathcal{K}(\mathbb{C}^m)$. Then, $D_C K \in \mathcal{W}^{2m}$ if and only if for every $(k, l) \in \mathbb{N}^2$ with $k + l \geq 2$ even, either $c_{k-l}(h_C) = 0$ or $\pi_{k,l}(h_K) = 0$.*

In particular, $\mathrm{Im} D_C \subseteq \mathcal{W}^{2m}$ if and only if C has constant width. In this case, D_C is injective among \mathcal{W}^{2m} if and only if $c_{2j+1}(h_C) \neq 0$ for every $j \in \mathbb{N} \setminus \{0\}$.

Proof. The convex body $D_C K$ has constant width if and only if $\pi_{2j}(h_{D_C K}) = 0$ for every $j > 0$ (see Example 4.9.iv). Using the orthogonal decomposition (10) of \mathcal{H}_{2j}^{2m} , Funk-Hecke theorem and Equation (11), we get

$$\pi_{2j}(h_{D_C K}) = \sum_{l=0}^{2j} \pi_{l, 2j-l}(h_{D_C K}) = \sum_{l=0}^{2j} \lambda_{l, 2j-l}(A_C) \pi_{l, 2j-l}(h_K) = \sum_{l=0}^{2j} c_{2(l-j)}(S_C) \pi_{l, 2j-l}(h_K),$$

and the results follow. \square

For the classical case, we have that the image of the class of bodies with constant width is the class of balls. This fact also follows directly from 4.9.vi, and it remains true for any symmetric convex body C with $c_{2j}(h_C) \neq 0$, $j \in \mathbb{N}$. For a general C , $\mathrm{Im} D_C(\mathcal{W}^{2m}) \subseteq \mathcal{W}^{2m}$.

Next we devote ourselves to the more restrictive case of **balls**. First we study for which $C \in \mathcal{K}(\mathbb{C})$ and $K \in \mathcal{K}(\mathbb{C}^m)$, the complex difference body is a ball, obtaining the following result, which is a consequence of Example 4.9.i) and Corollary 5.3, and gives the reciprocal of Corollary 3.5.

Corollary 7.4. *Let $C \in \mathcal{K}(\mathbb{C})$ and $K \in \mathcal{K}(\mathbb{C}^m)$. Then, $D_C K$ is a ball if and only if for every $(k, l) \in \mathbb{N}^2 \setminus \{(0, 0)\}$, either $c_{k-l}(h_C) = 0$ or $\pi_{k,l}(h_K) = 0$.*

In particular, D_{B_2} is injective among the class of S^1 -invariant convex bodies.

Remark 7.5. *If we consider $m = 1$ in the previous corollary, i.e., $D_C : \mathcal{K}(\mathbb{C}) \rightarrow \mathcal{K}(\mathbb{C})$, it can be restated as follows:*

$D_C K$ is a ball if and only if for every $j \in \mathbb{N} \setminus \{0\}$, either $c_j(h_C) = 0$ or $c_j(h_K) = 0$.

The previous condition coincides with the one found by Görtler in [12] studying the relation between two (planar) convex bodies K_1 and K_2 in order

$$(12) \quad V_2(K_1, K_2) = V_2(K_1, \rho K_2) \quad \forall \rho \in \text{SO}(2),$$

to hold, i.e., the mixed volume to be invariant with respect to rotations on just one of the variables of the mixed volume (see also [13, Theorem 4.6.2]).

Görtler showed that (12) holds if and only if for every $j \geq 2$, either $c_j(h_{K_1}) = 0$ or $c_j(h_{K_2}) = 0$. Hence, we have that $D_C K$ is a ball if and only if $V_2(C, K) = V_2(C, \rho K)$ for every $\rho \in \text{SO}(2)$. Using the expression of the support function $h_{D_C K}$ given in Proposition 2.3, in particular (7), we clearly have the equivalence between both conditions.

Concerning the class of **S^1 -invariant convex bodies**, we get from Lemma 4.8 the following result, which, in particular, studies the reciprocal of the trivial fact that if K is S^1 -invariant, then $D_C K$ is so too.

Corollary 7.6. *Let $C \in \mathcal{K}(\mathbb{C})$ and $K \in \mathcal{K}(\mathbb{C}^m)$. Then, $D_C K$ is a S^1 -invariant convex body if and only if for every $(k, l) \in \mathbb{N}^2$ with $k \neq l$, either $c_{k-l}(h_C) = 0$ or $\pi_{k,l}(h_K) = 0$.*

In particular, $\text{Im } D_C \subseteq \{\text{class of } S^1\text{-invariant convex bodies}\}$ if and only if C is a ball. In this case, we have indeed equality.

Moreover, the operator D_C is injective among the class of S^1 -invariant convex bodies for every C .

Analog to the study of convex bodies of constant width, we have, for the class of **symmetric convex bodies**:

Corollary 7.7. *Let $C \in \mathcal{K}(\mathbb{C})$ and $K \in \mathcal{K}(\mathbb{C}^m)$. Then, $D_C K$ is symmetric if and only if for every $(k, l) \in \mathbb{N}^2$ with $k + l$ odd, either $c_{k-l}(h_C) = 0$ or $\pi_{k,l}(h_K) = 0$.*

In particular, $\text{Im } D_C \subseteq \{\text{symmetric convex bodies}\}$ if and only if C is symmetric. In this case, the operator D_C is injective among all the symmetric convex bodies $K \in \mathcal{K}(\mathbb{C}^m)$ if and only if $c_{2j}(h_C) \neq 0$, $j \in \mathbb{N}$.

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