Schubert calculus and cohomology of flag manifolds

Haibao Duan
Institute of Mathematics, Chinese Academy of Sciences

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Abstract

In the context of Schubert calculus, we present an approach to the cohomology rings $H^*(G/P)$ of all flag manifold G/P that is free of the types of the group G and the parabolic subgroup P.

1 Introduction to Enumerative Geometry

Let G be a compact connected Lie group and let $\alpha : \mathbb{R} \to G$ be a group homomorphism. The centralizer P_α of the one parameter subgroup $\alpha(\mathbb{R})$ of G is called a *parabolic subgroup* of G. The corresponding homogeneous space G/P_α is canonically a projective variety, called a *flag manifold* of G.

In his fundamental treaty [17] A.Weil attributed the classical Schubert calculus to the "*determination of cohomology ring $H^*(G/P)$ of flag manifolds G/P* ". The aim of the present lectures is to present a unified approach to the cohomology rings $H^*(G/P)$ of all flag manifolds G/P.

In order to show how the geometry and topology properties of certain flag varieties are involved in the original work [16] of Schubert in 1873–1879, we start with a review on some problems of the classical enumerative geometry.

1.1 Enumerative problem of a polynomial system

A basic enumerative problem of algebra is:

Problem 1.1 (Apollonius, 200. BC). Given a system of polynomials over the field $\mathbb{C}$ of complexes

$$\begin{cases} f_1(x_1, \cdots, x_n) = 0 \\ \qquad \vdots \\ f_n(x_1, \cdots, x_n) = 0 \end{cases}$$

find the number of solutions to the system.

In the context of intersection theory Problem 1 has the next appearance:

Problem 1.2. Given a set $N_i \subset M$, $i = 1, \cdots, k$ of subvarieties in a (smooth) variety M that satisfies the dimension constraint

$$\sum \dim N_i = (k-1) \dim M,$$

find the number $|\cap N_i|$ of intersection points

$$\cap N_i = \{x \in M \mid x \in N_i \text{ for all } i = 1, \cdots, k\}.$$

In cohomology theory Problem 1.2 takes the following form

Problem 1.3. Given a set $\{\alpha_i \in H^*(M) \mid i = 1, \cdots, k\}$ of cohomology classes of an oriented closed manifold M that satisfies the degree constraint $\sum \deg \alpha_i = \dim M$, compute the Kronnecker pairing

$$\langle \alpha_1 \cup \cdots \cup \alpha_k, [M] \rangle = ?$$

The analogue of Problem 1.3 in De Rham theory is

Problem 1.4. Given a set $\{\alpha_i \in \Omega^*(M) \mid i = 1, \cdots, k\}$ of differential forms on of an oriented smooth manifold M satisfying the degree constraint $\sum \deg \alpha_i = \dim M$, compute the integration along M

$$\int_M \alpha_1 \wedge \cdots \wedge \alpha_k = ?.$$

We may regard the above problems as mutually equivalent ones. This brings us the next question:

Among the four problems stated above, which one is more easier to solve?

1.2 Examples from enumerative geometry

Let $\mathbb{C}P^n$ be the n–dimensional complex projective space. A *conic* is a curve on $\mathbb{C}P^2$ defined by a quratic polynomial $\mathbb{C}P^2 \to \mathbb{C}$. A *quadric* is a surface on $\mathbb{C}P^3$ defined by a quratic polynomial $\mathbb{C}P^3 \to \mathbb{C}$. A *twisted cubic space* curve is the image of an algebraic map $\mathbb{C}P^1 \to \mathbb{C}P^3$ of degree 3.

The following problems, together, with their solutions, can be found in Schubert's book [16, 1879].

The 8-quadric problem: Given 8 quadrics in space ($\mathbb{C}P^3$) in general position, how many conics tangent to all of them?

Solution: 4,407,296

The 9-quadric problem: Given 9 quadrics in space how many quadrics tangent to all of them?

Solution: 666,841,088

The 12-quadric problem: Given 12 quadrics in space how many twisted cubic space curves tangent to all of them?

Solution: 5,819,539,783,680.

The above cited works of Schubert are controversial at his time [12, 1976]. In particular, Hilbert asked in his problem 15 for a rigorous foundation of this calculation, and for an actual verification of those geometric numbers that constitute solutions to such problems of enumerative geometry.

1.3 Rigorous treatment

Detailed discussion of content in this section can be found in [9]

What is the variety of all conics on $\mathbb{C}P^2$?

The 3×3 matrix space has a ready made decomposition:

$$M(3, \mathbb{C}) = Sym(3) \oplus Skew(3)$$

or in a more useful form

$$\mathbb{C}^3 \otimes \mathbb{C}^3 = Sym(\mathbb{C}^3) \oplus Skew(\mathbb{C}^3).$$

Each non–zero vector $u = (a_{ij})_{3\times 3} \in Sym(\mathbb{C}^3)$ gives rise to a conic C_u on $\mathbb{C}P^2$ defined by

$$f_u : \mathbb{C}P^2 \to \mathbb{C},\ f_u[x_1, x_2, x_3] = \sum_{1\leq i,j\leq 3} a_{ij} x_i x_j$$

that satisfies $C_u = C_{\lambda u}$ for all $\lambda \in \mathbb{C}\backslash\{0\}$. Therefore, the space $\mathbb{C}P^5 = \mathbb{P}(Sym(\mathbb{C}^3))$ is the parameter space of all conics on $\mathbb{C}P^2$, called *the variety of conics on* $\mathbb{C}P^2$.

It should be aware that the map

$$s : \mathbb{C}^3 \to Sym(\mathbb{C}^3) \text{ by } s(u) = u \otimes u$$

induces an embedding $\mathbb{C}P^2 \to \mathbb{C}P^5$ whose image is the degenerate locus of all double lines. So the blow–up of $\mathbb{C}P^5$ along the center $\mathbb{C}P^2$ is called *the variety of complete conics* on $\mathbb{C}P^2$.

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rational functions of x which can be expressed in the form

$$\frac{G(X_1,\cdots,X_m)}{p^h},$$

where G is a rational integral function of the arguments $X_1,\cdots,X_m$ and p^h is any power of the prime number p. Earlier investigations of mine* show immediately that all such expressions for a fixed exponent h form a finite domain of integrality. But the question here is whether the same is true for all exponents h, *i. e.*, whether a finite number of such expressions can be chosen by means of which for every exponent h every other expression of that form is integrally and rationally expressible.

From the boundary region between algebra and geometry, I will mention two problems. The one concerns enumerative geometry and the other the topology of algebraic curves and surfaces.

15. Rigorous Foundation of Schubert's Enumerative Calculus.

The problem consists in this: *To establish rigorously and with an exact determination of the limits of their validity those geometrical numbers which Schubert† especially has determined on the basis of the so-called principle of special position, or conservation of number, by means of the enumerative calculus developed by him.*

Although the algebra of to-day guarantees, in principle, the possibility of carrying out the processes of elimination, yet for the proof of the theorems of enumerative geometry decidedly more is requisite, namely, the actual carrying out of the process of elimination in the case of equations of special form in such a way that the degree of the final equations and the multiplicity of their solutions may be foreseen.

16. Problem of the Topology of Algebraic Curves and Surfaces.

The maximum number of closed and separate branches which a plane algebraic curve of the nth order can have has been determined by Harnack.‡ There arises the further

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* *Math. Annalen*, vol. 36 (1890), p. 485.
† Kalkül der abzählenden Geometrie, Leipzig, 1879.
‡ *Math. Annalen*, vol. 10.

What is the variety of conics on $\mathbb{C}P^3$?

Let γ be the Hopf complex line bundle over $\mathbb{C}P^3$

$$\gamma = \{(l, v) \in \mathbb{C}P^3 \times \mathbb{C}^4 \mid v \in l\}$$

and let α be the orthogonal complement of the subbundle $\gamma \subset \mathbb{C}P^3 \times \mathbb{C}^4$. One has the decomposition of vector bundles

$$\alpha \otimes \alpha = Sym(\alpha) \oplus Skew(\alpha).$$

The projective bundle $\mathbb{P}(Sym(\alpha))$ associated to the subbundle $Sym(\alpha)$ is a $\mathbb{C}P^5$–bundle on $\mathbb{C}P^3$, called *the variety of conics on* $\mathbb{C}P^3$.

The bundle map $s : \alpha \to Sym(\alpha)$ by $v \to v \otimes v$ over the identity of $\mathbb{C}P^3$ satisfies $s(\lambda v) = \lambda^2 s(v)$ for $\lambda \in \mathbb{C}$, hence induces a smooth embedding of the associated projective bundles

$$i : \mathbb{P}(\alpha) \to \mathbb{P}(Sym(\alpha))$$

whose image is the degenerate locus of all double lines.

Definition 1.5: The blow-up M of $\mathbb{P}(Sym(\alpha))$ along the subvariety $\mathbb{P}(\alpha)$ is called *the variety of complete conics on* $\mathbb{P}^3$.

With these preparation let us show

Theorem 1.6. *Given 8 quadrics in space in general position, there are 4,407,296 conics tangent to all of them.*

Proof. A quadric S on the space $\mathbb{C}P^3$ defines a hyperplane on M :

$$V(S) = \{x \in M \mid x \text{ is tangent to } S\}$$

with the property that

if S and S' are two quadrics then $V(S)$ and $V(S')$ are homotopic in M.

Given 8 quadrics S_i, $1 \le i \le 8$, in general position we need to find the number of interesection points

$$|(V(S_1) \cap \cdots \cap V(S_8))| =?$$

or equivalently, let $\alpha_i \in H^2(M)$ be the Poincare dual of the cycle class $[V(S_1)] \in H_{14}(M)$

$$\langle \alpha_1 \cup \cdots \cup \alpha_8, [M] \rangle = \langle \alpha_1^8, [M] \rangle =?$$

From the theory of Chern characteristic classes the cohomologies of $\mathbb{P}(\alpha) \subset \mathbb{P}(Sym(\alpha))$ can be easily calculated as follows

$$H^*(\mathbb{P}(\alpha)) = \mathbb{Z}[x, \rho] / \left\langle x^4, \rho^3 + \rho^2 x + \rho x^2 + x^3 \right\rangle;$$
$$H^*(Sym(\alpha)) = \mathbb{Z}[x, y] / \left\langle x^4, y^6 + 4xy^5 + 10x^2y^4 + 20x^3y^3 \right\rangle.$$

By a general formula computing the cohomology of the blow–up $\widetilde{N}$ of variety N along a subvariety X one obtains that

$$H^*(M) = \frac{\mathbb{Z}[x,y]}{\langle x^4; y^6+4xy^5+10x^2y^4+20x^3y^3 \rangle} \oplus \frac{\mathbb{Z}[x,\rho]}{\langle x^4, \rho^3+\rho^2x+\rho x^2+x^3 \rangle} \{z, z^2\}$$

with the relation:

i) $4y^3 + 8xy^2 + 8x^2y = (30\rho^2 + 20\rho x + 6x^2)z - (3x + 9\rho)z^2 + z^3$.

ii) $yz = 2\rho z$.

Moreover, with respect to this presentation one can show that

$$\alpha_1 = 8x + 6y - 2z.$$

Consequently,

$$\left\langle (8x + 6y - 2z)^8, [M] \right\rangle = 4,407,296. \square$$

What is the space of all quadrics on $\mathbb{C}P^3$?

Consider the decomposition

$$\mathbb{C}^4 \otimes \mathbb{C}^4 = Sym(\mathbb{C}^4) \oplus Skew(\mathbb{C}^4).$$

Each non–zero vector $u = (a_{ij})_{4\times 4} \in Sym(\mathbb{C}^4)$ gives rise to a quadric S_u on $\mathbb{C}P^3$ defined by

$$f_u : \mathbb{C}P^3 \to \mathbb{C},\ f_u[x_1, x_2, x_3, x_4] = \sum_{1\le i,j\le 4} a_{ij}x_ix_j$$

that satisfies $S_u = S_{\lambda u}$ for all $\lambda \in \mathbb{C}\backslash\{0\}$. Therefore, the space

$$\mathbb{C}P^9 = P(Sym(\mathbb{C}^4))$$

is the *parameter space of all quadric on* $\mathbb{C}P^3$.

Consider the map $s : \mathbb{C}^4 \times \mathbb{C}^4 \to Sym(\mathbb{C}^4) \subset \mathbb{C}^4 \otimes \mathbb{C}^4$ defined by $s(u, v) = u \otimes v$. Since $s(\lambda u, v) = s(u, \lambda v) = \lambda s(u, v)$ it induces a smooth embedding on the quotients

$$\varphi : \mathbb{CP}^3 \times \mathbb{CP}^3 \to \mathbb{CP}^9.$$

Geometrically,

$$\varphi(l_1, l_2) = L_1 \cup L_2 \text{ (the degenerate quadrics of two planes)}$$

with L_i the hyperplane perpendicular to l_i. Let M be the variety obtained from $\mathbb{P}^9$ by first blow up along $\Delta \subset \mathbb{P}^3 \times \mathbb{P}^3$, then along $\mathbb{P}^3 \times \mathbb{P}^3 \subset \mathbb{P}^9$.

Definition 1.7. Let M be the variety obtained from $\mathbb{P}^9$ by first blow up along the diagonal $\Delta \subset \mathbb{P}^3 \times \mathbb{P}^3$, then along $\mathbb{P}^3 \times \mathbb{P}^3 \subset \mathbb{P}^9$. The space M is called the *variety of complete quadrics on* $\mathbb{P}^3$.□

The integral cohomology ring of the variety M has the presentation

$$H^*(M) = \mathbb{Z}[u]/\left\langle u^{10}\right\rangle \oplus \mathbb{Z}[y]/\left\langle y^4\right\rangle \{v, v^2, \cdots, v^5\}$$
$$\oplus \frac{\mathbb{Z}[c_1,c_2,t]}{\left\langle 2c_1c_2-c_1^3; c_2^2-c_1^2c_2; t^3+3t^2c_1+t(2c_1^2+4c_2)+2c_1^3\right\rangle}\{w, w^2\}$$

that is subject to the relations

i) $v^6 + 16v^5y + 110v^4y^2 + 420v^3y^3 + 8u^6 = 0$;

ii) $uv = 2yv$;

iii) $uw = tw$, $vw = -2(c_1 + t)w$;

iv) $10u^3 + 22u^2v + 16uv^2 + 4v^3 = (30c_1^2 + 18c_1t + 3t^2 - 4c_2)w$
$+(9c_1 + 3t)w^2 + w^3$.

Let us show

Theorem 1.8. *Given 9 quadrics in space in general position, there are 666,841,088 quadrics tangent to all of them.*

Proof. A quadric S on the space $\mathbb{C}P^3$ defines a hyperplane on M :

$$V(S) = \{x \in M \mid x \text{ is tangent to } S\}$$

with the property that

if S and S' are two quadrics then $V(S)$ and $V(S')$ are homotopic in M.

Given 9 quadrics S_i, $1 \le i \le 9$, in general position we need to find the number

$$|(V(S_1) \cap \cdots \cap V(S_9))| = ?$$

or equivalently, let $\alpha_i \in H^2(M)$ be the Poincare dual of the cycle class $[V(S_i)] \in H_{16}(M)$

$$\langle \alpha_1 \cup \cdots \cup \alpha_9, [M] \rangle = \langle \alpha_1^9, [M] \rangle = ?.$$

Moreover, with respect to this presentation one can show that

$$\alpha_1 = 12u + 6v - 2w.$$

Consequently,

$$\langle (12u + 6v - 2w)^9, [M] \rangle = 666,841,088.\square$$

Summarizing calculation above we may conclude that, in order to solve an enumerative problem we need

i) to describe the parameter space M of the geometric figures concerned in term of flag manifolds;

ii) compute the cohomology ring of the parameter space M;

iii) solve the problem by computation in the ring $H^*(M)$.

1.4 Appendix: Topology of blow–ups

Let $X \subset M$ be a submanifold whose normal bundle γ_X has a complex structure, and let $\pi : E = \mathbb{P}(\gamma_X) \to X$ be the complex projective bundle associated with γ_X. The tautological line bundle on E is denoted by λ_E, viewed as an 1–dimensional complex subbundle of the pull–back $\pi^*\gamma_X$.

Fix a metric on M and consider the associated spherical bundles

$$S(\lambda_E) = \{(l, v) \in E \times \pi^*\gamma_X \mid v \in l, \|v\|^2 = 1\};$$
$$S(\gamma_X) = \{(x, v) \in X \times \gamma_X \mid v \in \gamma_X \mid x, \|v\|^2 = 1\}.$$

The map $g : S(\gamma_X) \to S(\lambda_E)$ defined by $(x, v) \to (\langle v \rangle, v)$ is clearly a diffeomorphism, where $\langle v \rangle$ is the complex line spanned by the non–zero normal vector v. The adjoint manifold

$$\widetilde{M} = (M \setminus \overset{\circ}{D}(\gamma_X)) \cup_g D(\lambda_E)$$

obtained by gluing $D(\lambda_E)$ to $(M \setminus \overset{\circ}{D}(\gamma_X))$ along the boundary $S(\lambda_E)$ using the diffeomorphism g is called the *blow–up* of M along the submanifold X with *exceptional divisor* E [15].

The next result tells how the cohomology of $\widetilde{M}$ can be formulated from that of X and M together with the total Chern class $C(\gamma_X) = 1 + c_1 + \cdots + c_m$ of the normal bundle γ_X.

Theorem 1.9 [9]. *The integral cohomology of the blow–up M has the additive decomposition*

$$H^*(\widetilde{M}) = H^*(M) \oplus H^*(X)\{\omega_E, \cdots, \omega_E^{k-1}\},\ 2k = \dim_{\mathbb{R}} \gamma_X$$

that is subject to the relations:

i) $\omega_X = \sum_{1 \leq r \leq k} (-1)^{r-1} c_{k-r} \cdot \omega_E^r$,

ii) for any $y \in H^r(M)$, $y \cdot \omega_E = i^*(y) \cdot \omega_E$,

where $\omega_X, \omega_E \in H^*(\widetilde{M})$ *are the Poincare duals of the fundamental classes* $[X] \in H_*(M)$, $[E] \in H_*(\widetilde{M})$, *respectively, and where* $i : X \to M$ *is the inclusion.*

2 Topology of Bott manifolds

The cohomology of Bott manifolds will provide us with a simple module in which Schubert calculus can be simplified.

In this section we work in the category of pointed spaces and continuous maps preserving the base point.

2.1 Bott manifolds

We single out the class spaces which we will concern in this section.

Definition 2.1. A smooth manifold M is called a *Bott manifold of rank* n if there is a tower of smooth maps

$$M \overset{p_{n-1}}{\to} M_{n-1} \overset{p_{k-2}}{\to} \cdots \overset{p_2}{\to} M_2 \overset{p_1}{\to} M_1$$

in which

i) M_1 is diffeomorphic to S^2 with an orientation;

ii) each p_i is a projection of an oriented smooth S^2–bundle over M_i with a fixed section σ_i, $1 \leq i \leq n-1$.□

Let $X \vee Y$ be the one point union of two pointed spaces X and Y. For a given Bott manifold M set $S_1^2 = M_1$, and let S_i^2 be the fiber of p_{i-1} over the base point. The natural inclusion $\iota_i : S_i^2 \to M$ given by the fiber inclusion $S_i^2 \to M_i$ followed by the composition $\sigma_{n-1} \circ \cdots \circ \sigma_{i+1} \circ \sigma_i$ yields an embedding

$$\iota : S_1^2 \vee \cdots \vee S_k^2 \to M \text{ with } \iota \mid S_i^2 = \iota_i.$$

Put $y_i = \iota_{i*}[S_i^2] \in H_2(M)$.

Lemma 2.2. The 2–dimensional homology classes $y_1, \cdots, y_n \in H_2(M)$ form an additive basis for $H_2(M)$.

Consequently, let $x_i \in H^2(M) = Hom(H_2(M);\mathbb{Z})$ be the class dual to y_i via the Kronecker paring. Then the cohomology $H^2(M)$ has basis $\{x_1, \cdots, x_n\}$.□

Generally, for each subsequence $I = \{i_1, \cdots, i_r\} \subset \{1, \cdots, n\}$ with length r we set $x_I = \prod_{i \in I} x_i \in H^{2r}(M)$. Then we have

Theorem 2.3. The cohomology ring $H^*(M)$ has additive basis $\{x_I \mid I \subseteq \{1, \cdots, n\}\}$. In particular,

$\chi(M) = 2^n$, $\beta_r(M) = C_n^r$.

Proof. For the direct product of n copies of 2–dimensional spheres

$$N = S^2 \times \cdots \times S^2 \text{ (n–copies)}$$

one has the ready made cell decomposition:

$$N = \bigcup_{I \subseteq \{1, \cdots, n\}} S(I), \dim S(I) = 2\,|I|,$$

with

$$S(I) = \{(x_1, x_2, \cdots, x_n) \in S^2 \times \cdots \times S^2 \mid x_i = * \text{ for } i \notin I\}.$$

Similarly, a Bott manifold M of rank n has the cell decomposition

$$(2.1)\quad M = \bigcup_{I \subseteq \{1, \cdots, n\}} S(I), \dim S(I) = 2\,|I|$$

with each $S(I)$ defined inductively on the value of $|I|$ as follows:

1) $S(1) = M_1 = S^2$;

2) if $i > 1$, $S(i) \subset M_i$ is the fiber sphere of p_{i-1} over the base point.

Assume that $S(L^{'}) \subset M_{i_{r-1}}$ with $L' = [i_1, \cdots, i_{r-1}]$ has been defined and consider the case $L = [i_1, \cdots, i_{r-1}, j]$. Then

3) $S(L) \subset M_j$ is the total space of the restricted bundle of $p_{j-1} : M_j \to M_{j-1}$ to the subspace $S(L^{'}) \subset M_{j-1}$.

The natural bundle map over the inclusion $S(L^{'}) \subset M_{i_{r-1}} \to M_{j-1}$ gives rise to the desired embedding $S(L) \subseteq M_j \subseteq M$.

The proof of the theorem is done by noticing that the cohomology class

$$x_I \in H^{2|I|}(M) = Hom(H_{2|I|}(M), \mathbb{Z})$$

is the Kronnecker dual of the fundamental classes $[S(I)]$ in the sense that

$$x_I([S(J)]) = \delta_{I,J}.\square$$

We remark that in the decomposition (2.1) of a Bott manifold M each cell $S(I)$ is again a Bott manifold but with rank $|I|$.

2.2 The cohomology ring of a Bott manifold

In a Bott manifold each section $\sigma_i : M_i \to M_{i+1}$ is a co–dimension 2 embedding. Its normal bundle, denoted by γ_i, is an oriented 2–dimensional real bundle over M_i. Hence its Euler class

$$e_{i+1} = e(\gamma_i) \in H^2(M_i)$$

is well defined. Since the group $H^2(M_i)$ is generated by $\{x_1, \cdots, x_i\}$ by Lemma 2.2 there is a set of integers $a_{1,i+1}, \cdots, a_{i,i+1}$ so that one has the unique presentation

$$e_{i+1} = a_{1,i+1}x_1 + \cdots + a_{i,i+1}x_i.$$

Definition 2.4. The $n \times n$ integral strictly upper triangular matrix $A = (a_{i,j})_{n\times n}$ (i.e. $a_{i,j} = 0$ for all $i \geq j$) is called *the structure matrix* of the Bott manifold M.$\square$

Theorem 2.5. Let M be a Bott manifold of rank n with structure matrix $A = (a_{i,j})_{n\times n}$. Then, with respect to the basis $\{x_1, \cdots, x_n\}$ of $H^2(M)$, one has the presentation

$$H^*(M) = \mathbb{Z}[x_1, \cdots, x_n]/\left\langle x_r^2 - e_r x_r; 1 \leq r \leq n\right\rangle,$$

where $e_r = a_{1,r+1}x_1 + \cdots + a_{r,r+1}x_r$.

Proof. In general, for an oriented S^2–bundle $p : M \to X$ over a manifold X with a section $\sigma : X \to M$, let γ be the (oriented) normal bundle of the embedding σ with Euler class $e = e(\gamma) \in H^2(X)$. Then there exists a unique class $x \in H^2(M)$ satisfying

i) $i^*(x) \in H^2(S^2)$ is the orientation class;

ii) $\sigma^*(x) = 0 \in H^2(X)$.

Moreover

$$H^*(M) = H^*(X)[x]/\left\langle x^2 - ex\right\rangle.\square$$

2.3 Construction of Bott manifolds

The proof of the next result tells the way by which all Bott manifolds can be constructed.

Theorem 2.6. Any strictly upper triangular matrix $A = (a_{i,j})_{n\times n}$ with integer entries can be realized as the structure matrix of a Bott manifold.

Proof. Let $\gamma \oplus \varepsilon \to \mathbb{C}P^\infty$ be the 3–dimensional real vector bundle over $\mathbb{C}P^\infty$ with γ the Hopf complex line bundle over $\mathbb{C}P^\infty$ and with $\varepsilon = \mathbb{C}P^\infty \times \mathbb{R}$ the 1–dimensional trivial bundle. Consider the associated spherical fibration of $\gamma \oplus \varepsilon$

$$\gamma^s : S^2 \subset S(\gamma \oplus \varepsilon) \to \mathbb{C}P^\infty.$$

It has a canonical section $\sigma : \mathbb{C}P^\infty \to S(\gamma \oplus \varepsilon)$ given by $\sigma(l) = (0, 1)$. Recall that for any topological space X one has

$$H^2(X) = [X, \mathbb{C}P^\infty].$$

For a given $A = (a_{i,j})_{n\times n}$ ($a_{i,j} = 0$ for $i \geq j$), we set $M_1 = S^2$ and let $f_1 : M_1 \to \mathbb{C}P^\infty$ be the classifying map of the class $a_{1,2}x_1 \in H^2(M_1) = [M_1, \mathbb{C}P^\infty]$. The pull–back of γ^s via f_1 gives a S^2–bundle over M_1:

$$f_1^*\gamma^s : M_2 \to M_1$$

with a section $\sigma_1 : M_1 \to M_2$ given by σ. That is M_2 is a Bott manifold of rank 2 with structure matrix $A_2 = (a_{i,j})_{2\times 2}$.

Similarly, letting $f_2 : M_2 \to \mathbb{C}P^\infty$ be the classifying map for the class $-a_{1,3}x_1 - a_{2,3}x_2 \in H^2(M_2) = [M_2, \mathbb{C}P^\infty]$, we get the S^2–bundle over M_2

$$f_2^*\gamma^s : M_3 \to M_2$$

whose total space M_3 is a Bott manifold with structure matrix $A_3 = (a_{i,j})_{3\times 3}$. Repeating this procedure until all columns of A have been used, one obtains a Bott manifold $M = M_n$ whose structure matrix is A.□

Recently, the next problem appears to be popular in toric topology, which has been solved for Bott manifolds up to rank 4, see [6]

Rigidity problem of Bott manifolds (conjecture): Given two Bott manifolds M and N with isomorphic cohomology rings, does $M \cong N$?

For instance let M and N be two Bott manifold of rank 2 with structure matrix

$$\begin{pmatrix} 0 & a \\ 0 & 0 \end{pmatrix} \text{ and } \begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix}$$

respectively, then it can be shown that

$$H^*(M) \cong H^*(N) \Leftrightarrow a \equiv b \bmod 2 \Leftrightarrow M \cong N.$$

More precisely

$$M = \begin{cases} S^2 \times S^2 \text{ if } a \equiv 0 \bmod 2 \\ \mathbb{C}P^2 \# \overline{\mathbb{C}P}^2 \text{ if } a \equiv 1 \bmod 2. \end{cases}$$

2.4 Integration along Bott manifolds

Let $\mathbb{Z}[x_1, \cdots, x_n] = \bigoplus_{r \geq 0} \mathbb{Z}[x_1, \cdots, x_n]^{(r)}$ be the ring of integral polynomials in $x_1, \cdots, x_k$, graded by $\mid x_i \mid= 1$.

Definition 2.7. Given an $n \times n$ strictly upper triangular integer matrix $A = (a_{i,j})_{n\times n}$ the *triangular operator* associated to A is the composed homomorphism

$$\begin{aligned} T_A : \mathbb{Z}[x_1, \cdots, x_n]^{(n)} &\overset{T_n}{\to} \mathbb{Z}[x_1, \cdots, x_{n-1}]^{(n-1)} \overset{T_{n-1}}{\to} \cdots \to \mathbb{Z}[x_1, \cdots, x_r]^{(r)} \\ &\overset{T_r}{\to} \mathbb{Z}[x_1, \cdots, x_{r-1}]^{(r-1)} \overset{T_{r-1}}{\to} \cdots \to \mathbb{Z}[x_1]^{(1)} \overset{T_1}{\to} \mathbb{Z} \end{aligned}$$

defined recurrently by the following rule:

$$T_1(cx_1) = c;$$
$$T_r(cx_1^{s_1} \cdots x_{r-1}^{s_{r-1}} x_r^{s_r}) = \begin{cases} 0 \text{ if } s_r = 0; \\ cx_1^{s_1} \cdots x_{r-1}^{s_{r-1}} (a_{1,r}x_1 + \cdots + a_{r-1,r}x_{r-1})^{s_r - 1} \text{ if } s_r > 0, \end{cases}$$

where $c \in \mathbb{Z}$.

Example 2.8. Definition 2.7 gives an effective algorithm to evaluate T_A.

For $n = 2$ and $A = \begin{pmatrix} 0 & a \\ 0 & 0 \end{pmatrix}$, the homomorphism $T_A : \mathbb{Z}[x_1, x_2]^{(2)} \to \mathbb{Z}$ is given by

$$T_A(x_1^2) = 0,$$
$$T_A(x_1x_2) = T_1(x_1) = 1 \text{ and}$$
$$T_A(x_2^2) = T_1(ax_1) = a.$$

For $n = 3$ and $A = \begin{pmatrix} 0 & a & b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{pmatrix}$, set $A_1 = \begin{pmatrix} 0 & a \\ 0 & 0 \end{pmatrix}$. The homomorphism $T_A : \mathbb{Z}[x_1, x_2, x_3]^{(3)} \to \mathbb{Z}$ is given by

$$T_A(x_1^{r_1}x_2^{r_2}x_3^{r_3}) = \{ \begin{array}{l} 0, \text{ if } r_3 = 0 \text{ and} \\ T_{A_1}(x_1^{r_1}x_2^{r_2}(bx_1+cx_2)^{r_3-1}), \text{ if } r_3 \geq 1, \end{array}$$

where $r_1 + r_2 + r_3 = 3$, and where T_{A_1} is calculated before.□

The cohomology of a Bott manifold M is a simple ring

$$H^*(M) = \mathbb{Z}[x_1, \cdots, x_n]/\left\langle x_r^2 + e_r x_r,\ 1 \leq i \leq n \right\rangle,$$

in the following sense:

i) it is generated by elements with homogeneous degree 2;

ii) subject to relations with homogeneous degree 4.

Moreover, in our latter course to reduce Schubert calculus in the cohomology of a general flag manifold G/P to computation in this simple ring the following problem appears to be crucial.

Write $\mathbb{Z}[x_1, \cdots, x_n]^{(r)} \subset \mathbb{Z}[x_1, \cdots, x_n]$ for the subset of all homogeneous polynomials of degree r and let

$$p_M : \mathbb{Z}[x_1, \cdots, x_n]^{(r)} \to H^{2r}(M)$$

be the obvious quotient ring map. Consider the additive correspondence

$$\textstyle\int_M : \mathbb{Z}[x_1, \cdots, x_n]^{(n)} \to H^{2n}(M) = \mathbb{Z}$$

defined by $\int_M h = \langle p_M(h), [M] \rangle$, where $[M] \in H_2(M) = \mathbb{Z}$ is the orientation class. As indicated by the notation, the operator $\int_M$ can be interpreted as "*integration along* M" in De Rham theory, see also Problem 1.3 in Section 1.

Theorem 2.9. Let M be a Bott manifold with structure matrix $A = (a_{ij})_{n\times n}$. Then

$$\textstyle\int_M = T_A : \mathbb{Z}[x_1, \cdots, x_n]^{(2n)} \to \mathbb{Z}.$$

Proof. Consider the fibration $S^2 \subset M_k \overset{p_{k-1}}{\to} M_{k-1}$ in the definition of Bott manifold. Clearly M_{k-1} is a Bott manifold of rank $k-1$, whose structure matrix A' can be obtained from A by deleting the last $n-k$ columns and rows. The natural inclusion

$$\mathbb{Z}[x_1, \cdots, x_{k-1}] \to \mathbb{Z}[x_1, \cdots, x_k]$$

by $x_i \longmapsto x_i$, $i \leq k-1$, preserves both the grade and ideal, hence yields, when passing to the quotients, the induced map

$$p^*_{k-1} : H^*(M_{k-1}) = \mathbb{Z}[x_1, \cdots, x_{k-1}]/r_{M_{k-1}} \to H^*(M_k) = \mathbb{Z}[x_1, \cdots, x_k]/r_{M_k}.$$

Concerning this ring map we have

i) *Integration by part:* $\int_{M_k} p^*_{k-1}(a) \cup x_k = \int_{M_{k-1}} a.$

Next, in the ring $H^*(M_k)$ we have

ii) $x_k^r = (a_{1,k}x_1 + \cdots + a_{k-1,k}x_{k-1})^{r-1}x_k$

because of the relation $x_k^2 = (a_{1,k}x_1 + \cdots + a_{k-1,k}x_{k-1})x_k$.

Repeatedly applying the relations i) and ii) reduces the computation of $\int_M$ to $\int_{S^2}$, which is given as T_1 in Definition 2.7.□

3 Geometry of Lie groups

We introdce the Stiefel diagram for semi–simple Lie groups G, and recall the story of E. Cartan to classify Lie groups by their Cartan matrixes. We bring also a passage from the geometry of the Cartan subalgebra $L(T)$ to certain topological properties of the flag manifold G/T.

3.1 Lie groups and examples

Definition 3.1. A *Lie group* is a smooth manifold G which is furnished with a group structure

i) a product μ: $G \times G \to G$

ii) an inverse γ: $G \to G$

iii) a group unit: $e \in G$

in which the group operations μ and γ are smooth as maps between smooth manifolds.

Immediately from the definition, one has following familiar examples of Lie groups.

Example 3.2. The n–dimensional Euclidean space $\mathbb{R}^n$ is is a non–compact Lie group $(\mathbb{R}^n, +, 0)$ with dimension n.□

The n–dimensional torus $T^n = S^1 \times \cdots \times S^1$ is a compact Lie group with dimension n, where S^1 is the circle group

$$S^1 = \{e^{i\theta} \in \mathbb{C} \mid \theta \in \mathbb{R}\}$$

with product given by multiplying complex numbers.□

Let $M(n;\mathbb{F})$ be the $n \times n$ matrix space $\{A = (a_{ij})_{n\times n} \mid a_{ij} \in \mathbb{F}\}$ with entries in

$$\mathbb{F} = \begin{cases} \mathbb{R} \text{ (the field of reals)} \\ \mathbb{C} \text{ (the field of complexes)} \\ \mathbb{H} \text{ (the algebra of quaternions).} \end{cases}$$

As an Euclidean space we have

$$\dim_{\mathbb{R}} M(n;\mathbb{F}) = \begin{cases} n^2 \text{ if } \mathbb{F} = \mathbb{R} \\ (2n)^2 \text{ if } \mathbb{F} = \mathbb{C} \\ (4n)^2 \text{ if } \mathbb{F} = \mathbb{H}. \end{cases}$$

Consider the subspace of $M(n;\mathbb{F})$

$$O(n;\mathbb{F}) = \{A \in M(n;\mathbb{F}) \mid A\overline{A}^{\tau} = I_n\}.$$

The usual matrix operations

$$O(n;\mathbb{F}) \times O(n;\mathbb{F}) \to O(n;\mathbb{F}) \ (A,B) \to A \cdot B$$
$$O(n;\mathbb{F}) \to O(n;\mathbb{F}),\ A \to \overline{A}^{\tau}$$

furnishes $O(n;\mathbb{F})$ with the structure of a Lie group with group unit the identity matrix I_n, called "*the classical Lie groups*." Precisely we have

$$O(n;\mathbb{F}) = \begin{cases} O(n) \text{ the orthogonal group of order } n \text{ if } \mathbb{F} = \mathbb{R}, \\ U(n) \text{ the unitary group of order } n \text{ if } \mathbb{F} = \mathbb{C} \\ Sp(n) \text{ the symplectic group of order } n \text{ if } \mathbb{F} = \mathbb{H}.\square \end{cases}$$

If G_1, G_2 are two Lie groups, their product gives the third one

$$G = G_1 \times G_2$$

in which G_i is called a factor of G.

Definition 3.3. A Lie group G is called

i) *compact* if it has no factor $\mathbb{R}^n$;

ii) *semi–simple* if it has no factor T^n;

iii) *simple* if G is compact, semi–simple and $G = G_1 \times G_2$ implies that one of G_1, G_2 is a trivial group.□

3.2 Stiefel diagram of a semi–simple Lie group

Let G be a simple Lie group. Up to conjugate G contains a unique maximal connected abelian subgroup T, called a *maximal torus* of G. The dimension of T is called the *rank* of the Lie group G.

Fix a maximal torus T in G, consider the commutative diagram induced by the exponential map of G

$$\begin{array}{ccc} L(T) & \to & L(G) \\ \exp\downarrow & & \exp\downarrow \\ T & \to & G \end{array}$$

where

$L(G)$:=the tangent space to G at the group unit e (the Lie algebra of G)

$L(T)$:=the tangent space to T at the group unit e (the Cartan subalgebra of T).

For a non–zero vector $u \in L(T)$ the map exp carries the straight line $l_u = \{tu \mid t \in \mathbb{R}\}$ on the space $L(T)$ to a 1–parameter subgroup (or a geodesic) on G

$$\{\exp(tu) \in G \mid t \in \mathbb{R}\}.$$

Let C_u be the centralizer of this subgroup of G. Clearly one has $T \subseteq C_u$.

Definition 3.4. A point $u \in L(T)$ is called *singular* (resp. *regular*) if

$$\dim T < \dim C_u \text{ (resp. } \dim T = \dim C_u).$$

Let $\mathcal{S}(G) \subset L(T)$ be the set of all singular points. The pair $(L(T), \mathcal{S}(G))$ is called the *Stiefel diagram* of G.□

Theorem 3.5 (Geometry of Stiefel diagram). *Let G be a semi–simple Lie group with rank n, and set $m = \frac{1}{2}(\dim G - n)$. Then*

i) there are precisely m hyperplanes $L_1, \cdots, L_m$ in $L(T)$ through the origin $0 \in L(T)$ so that $\mathcal{S}(G) = L_1 \cup \cdots \cup L_m$;

ii) let l_k be the line normal to L_k and through the origin 0, then the exponential map $\exp : L(T) \to G$ carries l_k to a circle subgroup of G;

iii) let $K_i \subset G$ be the centralizer of the subset $\exp(L_i) \subset G$, then $T \subset K_i$ and $K_i/T = S^2$, $1 \leq i \leq m$.□

Instead of giving a proof of this general result I would like to point out that, if G is one of the classical groups $SU(n), SO(n)$ or $Sp(n)$, the theorem can be directly verified using linear algebra.

Example 3.6. For $G = U(n)$ we have

$$T = \{diag\{e^{i\theta_1}, \cdots, e^{i\theta_n}\} \in U(n) \mid \theta_r \in \mathbb{R}\};$$
$$L(U(n)) = \{B \in M(n; C) \mid \overline{B}^{\tau} = -B\};$$
$$L(T) = \{diag\{i\theta_1, \cdots, i\theta_n\} \mid \theta_r \in \mathbb{R}\}$$

and

$$\exp(B) = I + B + \tfrac{1}{2!}B^2 + \cdots + \tfrac{1}{n!}B^n + \cdots.$$

Moreover, if we set

$$L_{s,t} = \{diag\{i\theta_1, \cdots, i\theta_n\} \in L(T) \mid \theta_s = \theta_t \in \mathbb{R}\},\ s < t.$$

Then Theorem 3.5 is verified by

i) $\mathcal{S}(U(n)) = \bigcup_{1 \le s < t \le n} L_{s,t}$;

ii) the normal line $l_{s,t}$ to the hyperplane $L_{s,t}$ is

$$l_{s,t} = \{t\alpha_{s,t} \mid \alpha_{s,t} = diag\{0, \cdots, 0, i, 0, \cdots, 0, -i, 0, \cdots, 0\}\};$$

iii) the centralizer of $\exp(L_{s,t}) \subset U(n)$ is isomorphic to $T^{n-1} \times S^3$.□

3.3 The Cartan matrix and Weyl group of a Lie group

Based on the geometry of the Stiefel diagram we introduce basic notation about Lie groups theory.

Definition. 3.7. Let $\sigma_i \in Aut(L(T)$ be the reflection in the hyperplane $L_i \in \mathcal{S}(G)$. The subgroup $W(G) \subset Aut(L(T))$ generated by σ_i, $1 \le i \le m$, is called *Weyl group* of G.

By definition each element $w \in W(G)$ admits a factorization

$$w = \sigma_{i_1} \circ \cdots \circ \sigma_{i_r},\ 1 \le i_1, \cdots, i_r \le m. \qquad (3.1)$$

The *length* $l(w)$ of an element $w \in W(G)$ is the least number of factors in all decompositions of w in form (3.1). It gives rise to a function

$$l : W(G) \to \mathbb{Z}$$

called the *length function* on $W(G)$. The decomposition (3.1) is said *reduced* if $r = l(w)$.□

Definition 3.8. Let $l_k \subset L(T)$ be the line normal to the singular plane L_k and $\pm\alpha_k \in l_k$ be the non–zero vectors with minimal length so that $\exp(\pm\alpha_k) = e$, $1 \leq k \leq m$. The subset

$$\Phi_G = \{\pm\alpha_k \in L(T) \mid 1 \leq k \leq m\}$$

of $L(T)$ is called the *root system of* G.

For a pair $a_i, \alpha_j \in \Phi_G$ of roots the number $2(a_i, \alpha_j)/(\alpha_j, \alpha_j)$ is called the *Cartan number* of G relative to a_i, α_j (only $0, \pm 1, \pm 2, \pm 3$ can occur.)□

The planes in $\mathcal{S}(G)$ divide $L(T)$ into finitely many convex regions, each one is called a *Weyl chamber* of G.

Fix a regular point $x_0 \in L(T)$, and let $\mathcal{F}(x_0)$ be the closure of the Weyl chamber containing x_0. Assume that $L(x_0) = \{L_1, \cdots, L_n\}$ is the subset of $\mathcal{S}(G)$ consisting of the walls of $\mathcal{F}(x_0)$, where $n = \dim T$ because of G is semi–simple.

Definition 3.9. Let $\alpha_i \in \Phi_G$ be the root normal to the wall $L_i \in L(x_0)$ and pointing toward x_0. Then the subset $S(x_0) = \{\alpha_1, \cdots, \alpha_n\}$ of the root system Φ_G is called *the system of simple roots* of G relative to x_0.

The *Cartan matrix* of G (relative to x_0) is the $n \times n$ matrix defined by

$$A = (b_{ij})_{n\times n},\ b_{ij} = 2(a_i, \alpha_j)/(\alpha_j, \alpha_j).$$

The reflection $\sigma_i \in Aut(L(T)$ in the hyperplane $L_i \in L(x_0)$ is called a simple reflection.□

Just from the geometric fact that $W(G)$ acts transitively on the set of all Weyl chambers one can get

Corollary 3.10. *A system of simple roots of G is a basis of the vector space $L(T)$.*

The Weyl group $W(G)$ is generated by a set of simple reflections.□

Moreover, the next result due to E. Cartan tells that the local types of simple Lie groups are classified by their Cartan matrix:

Theorem 3.11 (Cartan). *The isomorphism types of all 1–connected simple Lie groups are in on to on correspondence with their Cartan matrixes listed below*

the classical types: A_n, B_n, C_n, D_n;

the exceptional types G_2, F_4, E_6, E_7, E_8.

Moreover, any compact connected Lie group G has the canonical presentation

$$G \cong (G_1 \times \cdots \times G_k \times T^r)/K$$

in which

i) each G_t is one of the 1–connected simple Lie groups enumerated above;

ii) the denominator K is a finite subgroup of the center of the numerator group.$\square$

As a supplyment to Theorem 3.11 we list the types and centers of all 1–connected simple Lie groups in the table below

G	$SU(n)$	$Sp(n)$	$Spin(2n+1)$	$Spin(2n)$	G_2	F_4	E_6	E_7	E_8
Φ_G	A_{n-1}	B_n	C_n	D_n	G_2	F_4	E_6	E_7	E_8
$\mathcal{Z}(G)$	$\mathbb{Z}_n$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_4$, $n = 2k+1$ $\mathbb{Z}_2 \oplus \mathbb{Z}_2$, $n = 2k$	$\{e\}$	$\{e\}$	$\mathbb{Z}_3$	$\mathbb{Z}_2$	$\{e\}$

Table 1. The types and centers of 1–connected simple Lie groups

3.4 Bott–Samelson K–cycles on G/T

For a singular plane $L_i \in \mathcal{S}(G)$ let $K_i \subset G$ be the centralizer of the subset $\exp(L_i) \subset G$. By property iii) of Theorem 3.7 we have $T \subset K_i$ with $K_i/T = S^2$. This indicates that we have a family of embeddings of 2–dimensional sphere

$$K_i/T = S^2 \to G/T,\ 1 \leq i \leq m.$$

Generalizing these maps gives rise to so called *Bott–Samelson K–cycles* on the flag manifold G/T.

Give a sequence $(i_1, \cdots, i_r)$ of integers with $1 \leq i_1, \cdots, i_r \leq m$ consider the map

$$K(i_1, \cdots, i_r) = K_{i_1} \times \cdots \times K_{i_r} \to G$$

defined by $(g_1, \cdots, g_r) \to g_1 \cdots g_r$. The product group $(T)^{r-1}$ of $r-1$ copies of the maximal torus T acts on the group $K(i_1, \cdots, i_r)$ from left by the rule

$$(g_1, \cdots, g_r)(t_1, \cdots, t_{r-1}) = (g_1 t_1, t_1^{-1} g_2 t_2, \cdots, t_{r-1}^{-1} g_r).$$

The map above induces the quotient maps

$$\begin{array}{ccc} K_{i_1} \times_T \cdots \times_T K_{i_r} & \to & G \\ p \downarrow & & \pi \downarrow \\ K_{i_1} \times_T \cdots \times_T K_{i_r}/T & \to & G/T \end{array}$$

Denote the map on the bottom as

$$g_{i_1,\cdots,i_r} : \Gamma(i_1,\cdots,i_r) = K_{i_1} \times_T \cdots \times_T K_{i_r}/T \to G/T.$$

It will be called *the Bott–Samelson K–cycle* on G/T associated to the sequence $(i_1, \cdots, i_r)$.

Theorem 3.15. *The quotient space $\Gamma(i_1, \cdots, i_r)$ is a Bott manifold of rank r whose structure matrix is $A = (a_{s,t})_{r\times r}$, where*

$$a_{s,t} = \begin{cases} 0 \text{ if } s \geq t; \\ -2(a_{i_s}, \alpha_{i_t})/(\alpha_{i_t}, \alpha_{i_t}) \text{ if } s < t \end{cases}.$$

Proof. For a point $(g_1, \cdots, g_r) \in K_{i_1} \times \cdots \times K_{i_r}$ write $[g_1, \cdots, g_r]$ for the equivalent class in the quotient space $\Gamma(i_1, \cdots, i_r)$. Then the map

$$\Gamma(i_1, \cdots, i_r) \to \Gamma(i_1, \cdots, i_{r-1}) \text{ by } [g_1, \cdots, g_r] \to [g_1, \cdots, g_{r-1}]$$

is a smooth fibration with fiber K_{i_r}/T a 2–dimensional sphere. It has a canonical section

$$\Gamma(i_1, \cdots, i_{r-1}) \to \Gamma(i_1, \cdots, i_r) \text{ by } [g_1, \cdots, g_{r-1}] \to [g_1, \cdots, g_{r-1}, e]$$

with $e \in K_{i_r} \subset G$ the group unit. These show that $\Gamma(i_1, \cdots, i_r)$ is a Bott manifold with rank r.

To compute the structure matrix A of $\Gamma(i_1, \cdots, i_r)$ consider the Cartan decomposition of the Lie algebra

$$L(G) = L(T) \bigoplus_{\alpha\in\Phi^+(G)} \gamma_\alpha,$$

where γ_α is the root space (an oriented 2–dimensional real vector space) associated to the positive root $\alpha \in \Phi^+(G)$. It gives rise to the decomposition of the tangent bundle of G/T

$$T(G/T) = \bigoplus_{\alpha\in\Phi^+(G)} \gamma_\alpha,$$

here γ_α is an oriented 2–dimensional real vector bundle of G/T corresponds to the 2–plane γ_α in the Cartan decomposition of the algebra $L(G)$. Then we have

$$\langle e_\alpha, [K_i/T]\rangle = -2(\alpha, \alpha_i)/(\alpha_i, \alpha_i) \text{ (the Cartan number)},$$

where $e_\alpha \in H^2(G/T)$ is the Euler class of the bundle γ_α, and where $[K_i/T] \in H_2(G/T)$ is the fundamental class of the embedding $K_i/T = S^2 \to G/T$.□

3.5 Relationship between $L(T)$ and $H^2(G/T)$

Let G be a semisimple Lie group with a system $S(x_0) = \{\alpha_1, , \cdots, \alpha_n\}$ of simple roots relative to a regular point $x_0 \in L(T)$.

Definition 3.16. The subset of the Cartan subalgebra $L(T)$

$$\Omega_G = \{\omega_i \in L(T) \mid 2(\omega_i, \alpha_j)/(\alpha_j, \alpha_j) = \delta_{i,j},\ \alpha_j \in S(x_0)\}$$

is called the set of *fundamental dominant weights* of G relative to x_0, where $\delta_{i,j}$ is the Kronecker symbol.□

Lemma 3.17. *Let G be a semisimple Lie group with Cartan matrix A, and let $\Omega_G = \{\omega_1, \cdots, \omega_n\}$ be the set of fundamental dominant weights relative to the regular point x_0. Then*

i) for each $1 \le i \le n$ the half line $\{t\omega_i \in L(T) \mid t \in \mathbb{R}^+\}$ is the edge of the Weyl chamber $\mathcal{F}(x_0)$ opposite to the wall L_i;

ii) the system of simple roots $\{\alpha_1, \cdots, \alpha_n\}$ can be expressed in term of the fundamental dominant weights $\omega_1, \cdots, \omega_n$ as

$$(3.2)\quad \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_n \end{pmatrix} = A \begin{pmatrix} \omega_1 \\ \omega_2 \\ \vdots \\ \omega_n \end{pmatrix}.$$

Proof. By Definition 3.16 each weight $\omega_i \in \Omega_G$ is perpendicular to all the roots α_j (i.e. $\omega_i \in L_j$) with $j \ne i$. This shows i). ii) comes directly from the definition.□

The next idea is due to Borel and Hirzebruch [3, 1958]. Consider the fibration $G/T \overset{\psi}{\hookrightarrow} BT \overset{\pi}{\to} BG$ induced by the inclusion $T \subset G$ and examine the induced map (also known as the *Borel's characteristic map)*

$$\psi^* : H^2(BT) \to H^2(G/T).$$

On the otherhand let $\Lambda = \exp^{-1}(e)$ be the unit lattice in $L(T)$. Then each root $\alpha \in S(x_0)$ induces the commutative diagram

$$\begin{array}{ccc} L(T) & \overset{\alpha^*}{\to} & \mathbb{R} \\ \exp \downarrow & & \downarrow \\ T = L(T)/\Lambda_G & \overset{\alpha^*}{\to} & S^1 = \mathbb{R}/\mathbb{Z} \end{array}$$

where $\alpha^*(u) = 2(u, \alpha)/(\alpha, \alpha)$, since $\alpha^*(\Lambda) \subset \mathbb{Z}$. The homomorphism α^* at the bottom determines a map between classifying space

$$B\alpha^* : BT \to BS^1 = K(2, \mathbb{Z})$$

and consequently $B\alpha^* \in H^2(BT)$. Let us set

$$\alpha =: \psi^* B\alpha^* \in H^2(G/T).$$

In this way we can regard the set Φ_G of roots of G as a set of cohomology classes in $H^2(G/T)$.

Theorem 3.18 [3, 1958]. *Let γ_α be the oriented 2–dimensional real bundle on G/T with Euler class $\alpha \in H^2(G/T)$. Then*

i) $T(G/T) = \bigoplus_{\alpha \in \Phi^+(G)} \gamma_\alpha$, where $\Phi^+(G)$ is the set of positive roots relative to the reqular point $x_0 \in L(T)$;

ii) the set $\Omega_G = \{\omega_1, \cdots, \omega_n\}$ of fundamental dominant weights is a basis for the group $H^2(G/T)$;

iii) the action of a simple reflection σ_i on $H^2(G/T)$ is given by

$$\sigma_i(\omega_k) = \begin{cases} \omega_i \text{ if } k \neq i; \\ \omega_i - \sum_{1 \le j \le n} a_{ij}\omega_j \text{ if } k = i, \end{cases}$$

where $A = (a_{ij})_{n \times n}$ is the Cartan matrix of G.□

4 Schubert calculus

In this section we bring together the classical works of Bott–Samelson [4, 1955], Chevalley [5, 1958] and Hansen [13, 1973] concerning the decomposition of flag manifolds into Schubert cells (varieties), introduce the fundamental problem of Schubert calculus, and present a solution to it.

4.1 Bott–Samelson cycles on G/T

For a compact Lie group G with a maximal torus T consider the fibration

(4.1) $G/T \overset{\psi}{\hookrightarrow} BT \overset{\pi}{\to} BG$

induced by the inclusion $T \subset G$, where BT (resp. BG) is the classifying space of T (resp. G). The ring map

$$\psi^* : H^*(BT) \to H^*(G/T)$$

induced by the fiber inclusion ψ is known as the *Borel's characteristic map*. Earlier in 1952, Borel [2] proved that

Theorem 4.1. *Over the field $\mathbb{R}$ of reals the map ψ^* is surjective and induces an isomorphism of algebras*

$$H^*(G/T;\mathbb{R}) = H^*(BT;\mathbb{R})/\left\langle H^+(BT;\mathbb{R})^W \right\rangle$$

where $\left\langle H^+(BT;\mathbb{R})^W \right\rangle$ *is the ideal in* $H^*(BT;\mathbb{R})$ *generated by Weyl invariants in positive degrees.*

Subsequently, Bott and Samelson [4, 1955] studied the following question:

> *What happens to the structure of the integral cohomology* $H^*(G/T)$ *?*

Consider the commutative diagram induced by the exponential map of the group G

$$\begin{array}{ccc} L(T) & \rightarrow & L(G) \\ \exp\downarrow & & \downarrow\exp \\ T & \rightarrow & G \end{array}$$

where the horizontal maps are the obvious inclusions. Equip $L(G)$ (hence also $L(T)$) an inner product invariant under the adjoint action of G on $L(G)$.

Inside the Euclidean space $L(G)$ where are two geometric objects which we will be interested in:

i) the linear subspace $L(T) \subset L(G)$ which is furnished with the Stiefel diagram $\mathcal{S}(G)$ of G;

ii) taking a regular point $\alpha \in L(T)$ the adjoint representation of G gives rise to a map

$G \to L(G)$ by $g \to Ad_g(\alpha)$

which identifies G/T as a submanifold of the Euclidean space $L(G)$

$G/T = \{Ad_g(\alpha) \in L(G) \mid g \in G\}$.

From the fixed regular point $\alpha \in L(T)$ given in ii) we get also

iii) the W–orbit through the point $\alpha \in L(T)$

$W(\alpha) = \{w(\alpha) \in L(T) \mid w \in W\}$,

iv) the Euclidean distance function: $f_a : G/T \to \mathbb{R}$ from the point α

$f_\alpha(x) = \parallel x - \alpha \parallel^2$

The following result of Bott and Samelson [4, 1955] tells how to read the critical points of the function f_a from the linear geometry of the vector space $L(T)$:

Theorem 4.2. *The function* f_a *is a Morse function on* G/T *with* $W(\alpha)$ *as the set of critical points.*

The index function Ind: $W(\alpha) \to \mathbb{Z}$ *is given by*

$$Ind(w(a)) = 2\#\{L_i \mid L_i \cap [a, w(a)] \neq \emptyset],$$

where $[a, w(a)]$ *is the line segment in* $L(T)$ *joining* a *and* $w(a)$.

Proof. The linear subspace $L(T) \subset L(G)$ meets the submanifold $G/T \subset L(G)$ perpendicularly at the W–orbit $W(\alpha)$ of α:

$$L(T) \cap G/T = W(\alpha).$$

This shows that the set of critical points of the function f_a is $W(\alpha)$.

To compute the index of f_a at a critical point $w(\alpha) \in W(\alpha)$ we need to decide the centers of curvatures along the segment $[a, w(a)]$ normal to G/T at the point $w(a)$. They are in one to one correspondent to the intersection point of L_i and $[a, w(a)]$, each counted with multiplicity 2.□

Consider the partition on the Weyl group of G defined by the length function

$$W = \coprod_{0 \leq r \leq m} W^r \text{ with } l(W^r) = r.$$

Let us define $\beta_{2r} = |W^r|$. Theorem 4.2 implies that

Corollary 4.3. *The cohomology of* G/T *is torsion free, vanishes in odd degrees, and has Poincare polynomial*

$$P_t(G/T) = 1 + \beta_2 t^2 + \cdots + \beta_{2m} t^{2m},$$

where $m = \frac{\dim G - n}{2}$.□

Moreover, Bott and Samelson constructed a set of geometric cycles in G/T that realizes an additive basis of $H_*(G/T; \mathbb{Z})$ as follows.

For a $w \in W$ assume that the singular planes that meet the directed segment $[a, w(a)]$ are in the order $L_1, \cdots, L_r$. Let $K_i \subset G$ be the centralizer of the subset $\exp(L_i)$ of G and put $\Gamma_w = K_1 \times_T \cdots \times_T K_r/T$. Let

$$g_w : \Gamma_w \to G/T$$

be the Bott–Samelson K–cycles associated to the sequence $(1, \cdots, r)$.

Theorem 4.4. *The homology* $H_*(G/T; \mathbb{Z})$ *is torsion free with the additive basis*

$$\{g_{w*}[\Gamma_w] \in H_*(G/T; \mathbb{Z}) \mid w \in W\}.$$

Proof. Let $e \in K_i (\subset G)$ be the group unit and put $\overline{e} = [e, \cdots, e] \in \Gamma_w$. It were actually shown by Bott and Samelson that

(1) $g_w^{-1}(w(a))$ consists of the single point $\overline{e}$;

(2) the composed function $f_a \circ g_w : \Gamma_w \to \mathbb{R}$ attains its maximum only at $\overline{e}$;

(3) the tangent map of g_w at $\overline{e}$ maps the tangent space of Γ_w at $\overline{e}$ isomorphically onto the negative part of $H_{w(a)}(f_a)$, the Hessian form of the function f_a at the point $w(a) \in G/T$.

These completes the proof.□

4.2 Basis Theorem of Chevalley

Let K be a linear algebraic group over the field $\mathbb{C}$ of complex numbers, and let $B \subset K$ be a Borel subgroup. The homogeneous space K/B is a projective variety on which the group K acts by left multiplication. Historically, Schubert varieties were introduced in terms of the orbits of B action on K/B.

Let T be a maximal torus containing in B and let $N(T)$ be the normalizer of T in K. The Weyl group of K (relative to T) is $W = N(T)/T$. For a $w \in W$ take an $n(w) \in N(T)$ such that its residue class mod T is w.

The following result was first discovered by Bruhat for classical Lie groups K in 1954, and proved to be the case for all reductive algebraic linear groups by Chevalley [5, 1958].

Theorem 4.5. *One has the disjoint union decomposition*
$$K/B = \underset{w \in W}{\cup} Bn(w) \cdot B$$
in which each orbit $Bn(w) \cdot B$ is isomorphic to an affine space of complex dimension $l(w)$.

The Zariski closure of the open cell $Bn(w) \cdot B$ in K/B with the canonical reduced structure, denoted by X_w, is called the *Schubert variety associated to w.*

Corollary 4.6 (Basis Theorem of Schubert calculus). *Let $[X_w] \in H_{2l(w)}(K/B)$ be the fundamental class of the Schubert variety associated to w. Then the homology $H_*(K/B)$ has additive basis $\{[X_w] \in H_*(K/B) \mid w \in W\}$.*

Consequently, let $s_w \in H^*(K/B)$ be the Kronnecker dual of the class $[X_w]$ in cohomology. Then the cohomology $H^*(K/B)$ has additive basis $\{s_w \in H^*(K/B) \mid w \in W\}$.□

In view of second part of Corollary 4.6, the product $s_u \cdot s_v$ of two arbitrary Schubert classes can be expanded in terms of the Schubert basis

$$s_u \cdot s_v = \sum_{l(w)=l(u)+l(v), w \in \overline{W}} a_{u,v}^w \cdot s_w,\ a_{u,v}^w \in \mathbb{Z},$$

where the coefficients $a^w_{u,v}$ are called *the structure constants* on G/P.

Fundamental problem[1] of Schubert calculus: *Given a flag manifold G/T determine the structure constants $a^w_{u,v}$ on G/T for all $w, u, v \in W$ with $l(w) = l(u) + l(v)$.*$\square$

For a compact connected Lie group G with a maximal torus T let K be the complexification of G, and let B be a Borel subgroup in K containing T. It is well known that the natural inclusion $G \to K$ induces an isomorphism

$$G/T = K/B.$$

Conversely, the reductive algebraic linear groups are exactly the complexifications of the compact real Lie groups [14].

Up to now the homology $H_*(G/T)$ has two *canonical additive bases*: one is given by the K–cycles constructed by Bott-Samelson in order to describe the stable manifolds of a perfect Morse function on G/T; and the other consists of Schubert varieties, and both of them are indexed by the Weyl group of G. The following result was obtained by Hansen [**?**, 1973].

Theorem 4.7. *Under the natural isomorphism $G/T = K/B$, the K–cycle $g_w : \Gamma_w \to G/T$ of Bott-Samelson in Theorem 4.4 is a degree 1 map onto the Schubert variety X_w.*

Because of this result the map $g_w : \Gamma_w \to G/T$ is also known as *the Bott–Samelson resolution* of the Schubert variety X_w.

4.3 Multiplicative rule of Schubert classes

Let G be a simple Lie group of rank n with a system of simple roots $\{\alpha_1, \cdots, \alpha_n\}$, a set $\{\sigma_1, \cdots, \sigma_n\}$ of simple reflections. For a sequence $(i_1, \cdots, i_k)$ of k integers, $1 \leq i_1, \cdots, i_r \leq n$, consider the corresponding K–cycle on G/T

$$g_{i_1,\cdots,i_k} : \Gamma(i_1, \cdots, i_r) = K_{i_1} \times_T \cdots \times_T K_{i_r}/T \to G/T,$$

and its induced the cohomology map:

$$g^*_{i_1,\cdots,i_r} : H^{2s}(G/T) \to H^{2s}(\Gamma(i_1, \cdots, i_r)).$$

Recall that

[1] For the historical backgroud in this problem, we quote from J. L. Coolidge [7, (1940)] "the fundamental problem which occupies Schubert is to express the product of two of these symbols in terms of others linearly. He succeeds in part"; and from A. Weil [17, p.331]: "The classical Schubert calculus amounts to the determination of cohomology rings of flag manifolds."

i) the group $H^{2s}(G/T)$ has additive basis $\{s_w \mid w \in W, l(w) = s\}$ by the basis theorem of Chevalley;

ii) the group $H^{2s}(\Gamma(i_1, \cdots, i_k))$ has additive basis $\{x_I \mid I \subseteq [1, \cdots, r], |I| = s\}$.

Theorem 4.8. *The induced map* $g^*_{i_1,\cdots,i_k} : H^{2s}(G/T) \to H^{2s}(\Gamma(i_1, \cdots, i_k))$ *is given by*

$$g^*_{i_1,\cdots,i_k}(s_w) = \sum_{I\subseteq\ [1,\cdots,k],|I|=s,\sigma_I=w} x_I$$

where $\sigma_{(j_1,\cdots,j_r)} = \sigma_{i_{j_1}} \circ \cdots \circ \sigma_{i_{j_r}}$.

Proof. With respect to the cell decomposition of the manifolds $\Gamma(i_1, \cdots, i_k)$ and G/T the map $g_{i_1,\cdots,i_k}$ has nice behavior

$$g_{i_1,\cdots,i_k} : \Gamma(i_1, \cdots, i_k) = \bigcup_{I\subset(i_1,\cdots,i_k)} \Gamma(I) \to G/T = \bigcup_{w\in W} X_w$$

in the sense that

$$g_{i_1,\cdots,i_k}(\Gamma(I)) = X_{\sigma_I} \text{ with } \sigma_{(j_1,\cdots,j_r)} = \sigma_{j_1} \circ \cdots \circ \sigma_{j_r}.$$

This completes the proof.□

Granted with Theorem 4.8 we present a solution to the

Fundamental problem of Schubert calculus: *Given a flag manifold* G/T *determine the structure constants* $a^w_{u,v}$ *of* G/T *in the product*

$$s_u \cdot s_v = \sum_{l(w)=l(u)+l(v),w\in\overline{W}} a^w_{u,v} \cdot s_w,\ a^w_{u,v} \in \mathbb{Z},$$

where $w, u, v \in W$ *with* $l(w) = l(u) + l(v)$.□

Take a reduced decomposition for $w \in W$

$$w = \sigma_{i_1} \circ \cdots \circ \sigma_{i_k}$$

and let A_w be the Cartan matrix of the Bott–Samelson resolution

$$g_w = g_{i_1,\cdots,i_k} : \Gamma_w = \Gamma(i_1, \cdots, i_k) \to X_w \subset G/T$$

of the Schubert variety X_w.

Theorem 4.9 [8].

$$a^w_{u,v} = T_{A_w}[(\sum_{L\subset\{1,\cdots,k\},|L|=l(u),\sigma_L=u} x_L)(\sum_{K\subset\{1,\cdots,k\},|K|=l(v),\sigma_K=v} x_K)],$$

where $\sigma_L \in W$ (resp. $\sigma_K \in W$) is the *subword* of w corresponding to the sequence L (K).

Proof. Assume in the ring $H^*(G/T)$ we have that

$$s_u \cdot s_v = \sum_{l(w)=l(u)+l(v), w\in\overline{W}} a_{u,v}^w \cdot s_w,\ a_{u,v}^w \in \mathbb{Z}.$$

Then

$$\begin{aligned} a_{u,v}^w &= \langle s_u \cdot s_v, [X_w] \rangle = \langle s_u \cdot s_v, g_w^* [\Gamma_w] \rangle \\ &= \langle (g_w^* s_u \cdot g_w^* s_v), [\Gamma_w] \rangle \\ &= \left\langle (\sum_{|L|=l(u),\sigma_L=u} x_L) \cdot (\sum_{|K|=l(v),\sigma_K=v} x_K), [\Gamma_w] \right\rangle \text{ (by Theorem 4.8)} \\ &= \int_{\Gamma_w} (\sum_{|L|=l(u),\sigma_L=u} x_L) \cdot (\sum_{|K|=l(v),\sigma_K=v} x_K) \\ &= T_{A_w} (\sum_{|L|=l(u),\sigma_L=u} x_L) \cdot (\sum_{|K|=l(v),\sigma_K=v} x_K) \text{ (by Theorem 2.9).}\square \end{aligned}$$

Example 4.10. We emphasize that the above formula express the number $a_{u,v}^w$ as a polynomial in the Cartan numbers of G.

For instance the Cartan matrix of the second exceptional group F_4 is:

$$\begin{pmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -2 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{pmatrix}.$$

Consider the following elements of $W(F_4)$ given by reduced decompositions:

$$w_1 = \sigma_1\sigma_2\sigma_3;\ w_2 = \sigma_2\sigma_3\sigma_4.$$

We have

$$A_{w_1} = \begin{pmatrix} 0 & -1 & 0 \\ 0 & 0 & -2 \\ 0 & 0 & 0 \end{pmatrix};\ A_{w_2} = \begin{pmatrix} 0 & -2 & 0 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{pmatrix}.$$

That is, one can read off the matrix A_w directly from a reduced decomposition of w and the Cartan matrix of G.$\square$

4.4 General cases

Generally if G is a compact connected Lie group and if $\alpha : \mathbb{R} \to G$ be a group homomorphism, then centralizer P_α of the one parameter subgroup $\alpha(\mathbb{R})$ of G is a *parabolic subgroup* of G, and the corresponding homogeneous space G/P_α is canonically a projective variety, called a *flag manifold* of G.

Theorem 4.12 below indicates that the basis theorem of Chevalley (i.e. Corollary 4.6) and the multiplcative formula (i.e. Theorem 4.9) applies equally well to determine the integral cohomology ring $H^*(G/P_\alpha)$.

Assume that the group G is semi–simple with rank n. For a subset $I \subseteq \{1, \cdots, n\}$ let P_I be the centralizer of the 1–parameter subgroup

$$\alpha : \mathbb{R} \to G,\ \alpha(t) = \exp(t \sum_{i \in I} \omega_i)$$

on G, where $\{\omega_1, \cdots, \omega_n\} \subset L(T)$ is a set of fundamental dominant weights of G (i.e. lying on the edges of a fixed Weyl chamber of G). Note that if $I = \{1, \cdots, n\}$, then $P_I = T$.

Lemma 4.11 [1, 5.1]. *The centralizer of any* 1*–parameter subgroup on* G *is isomorphic to a subgroup* P_I *for some* $I \subseteq \{1, \cdots, n\}$. *Moreover,*

i) P_I *is a parabolic subgroup of* G *whose Dynkin diagram can obtained from that of* G *by deleting the vertices* β_i *with* $i \in I$*, as well as the edges adjoining to it;*

ii) the Weyl group W_I *of* P_I *is the subgroup of* W *generated by the set* $\{\sigma_j \mid j \notin I\}$ *of simple reflections on* $L(T)$*;*

iii) identifying the set W/W_I *of left cosets of* $W_I \subset W$ *with the subset of* W *[1, 5.1]*

$$W/W_I = \{w \in W \mid l(w_1) \geq l(w),\ w_1 \in wW\},$$

then the flag manifold G/P_I *has the cell decomposition into Schubert varieties*

$$G/P_I = \underset{w \in W/W_I}{\cup} \pi(X_w),$$

where $\pi : G/T \to G/P_I$ *is the fibration induced by the inclusion of subgroups* $T \subset P_I \subset G$.□

For a proper subset $I \subset \{1, \cdots, n\}$ consider the fibration in flag manifolds induced by the inclusion $T \subset P_I \subset G$ of subgroups

(4.2) $P_I/T \overset{i}{\hookrightarrow} G/T \overset{\pi}{\to} G/P_I$.

We observe that, with respect to the Schubert bases on the three flag manifolds P_I/T, G/T and G/P_I, the induced maps π^* and i^* have nice behavior.

Theorem 4.12. *With respect to the inclusion* $W_I \subset W$ *the induced map*

$$i^* : H^*(G/T) \to H^*(P_I/T)$$

identifies the subset $\{s_w\}_{w\in W_I\subset W}$ *of the Schubert basis of* $H^*(G/T)$ *with the Schubert basis* $\{s_w\}_{w\in W_I}$ *of* $H^*(P_I/T)$.

With respect to the inclusion $W/W_I \subset W$ *the induced map*

$$\pi^* : H^*(G/P_I) \to H^*(G/T)$$

identifies the Schubert basis $\{s_w\}_{w\in W/W_I}$ *of* $H^*(G/P_I)$ *with the subset* $\{s_w\}_{w\in W/W_I}$ *of the Schubert basis* $\{s_w\}_{w\in W}$ *of* $H^*(G/T)$.

Proof. This lemma comes from the next two geometric properties that follow directly from the definition of Schubert varieties. With respect to the cell decompositions (4.1) on the three flag varieties P_I/T, G/T and G/P_I one has:

i) for each $w \in W_I \subset W$ the fiber inclusion $i : P_I/T \to G/T$ carries the Schubert variety X_w on P_I/T identically onto the Schubert variety X_w on G/T;

ii) for each $w \in W/W_I \subset W$ the projection $\pi : G/T \to G/P_I$ restricts to a degree 1 map from the Schubert variety X_w on G/T to the corresponding Schubert variety on G/P_I.□

Based on the formula in Theorem 4.4 for multipliying rule of Schubert classes, a program entitled "*Littlewood-Richardson Coefficients*" has been compiled in [11], whose function is briefed below.

> **Algorithm:** *L-R coefficients.*
>
> **In:** *A Cartan matrix* $A = (a_{ij})_{n\times n}$ *(to specify a Lie group* G*) and a subset* $I \subseteq \{1, \cdots, n\}$ *(to specify a parabolic subgroup* $P \subset G$*)*
>
> **Out:** *The structure constants* $a^w_{u,v} \in \mathbb{Z}$ for all $w, u, v \in W/W_I$ *with* $l(w) = l(u) + l(v)$.

5 Applications

The computational examples in this section are taken from [10].

In principle, the basis theorem of Chevalley (i.e. Corollary 4.6) and the formula for multiplying Schubert classes (i.e. Theorem 4.9) consist of a complete characterization of the ring $H^*(G/P)$. However, concerning the needs of many relevant studies such a characterization is hardly a practical one, since the number of Schubert classes on G/P is usually very large, not to mention the number of the corresponding structure constants $a^w_{u,v}$ involved. It is therefore natural to ask for such a compact presentation of the ring $H^*(G/P)$ as that demonstrated in the next example.

Given a set $\{x_1, \cdots, x_k\}$ of k elements let $\mathbb{Z}[x_1, \cdots, x_k]$ be the ring of polynomials in $x_1, \cdots, x_k$ over the ring $\mathbb{Z}$ of integers. For a set $\{f_1, \cdots, f_m\} \subset \mathbb{Z}[x_1, \cdots, x_k]$ of polynomials write $\langle f_1, \cdots, f_m \rangle$ for the ideal generated by $f_1, \cdots, f_m$.

Example 5.1. If $G = U(n)$ is the unitary group of rank n and if $P = U(k) \times U(n-k)$, the flag manifold G/P is the *Grassmannians* $G_{n,k}$ of k–planes through the origin in the n–dimensional complex vector space $\mathbb{C}^n$. In addition to the characterization of the ring $H^*(G_{n,k})$ by $\binom{n}{k}$ Schubert classes, one has the compact presentation due to Borel [2]

(5.1) $H^*(G_{n,k}) = \mathbb{Z}[c_1, \cdots, c_k]/\langle \overline{c}_{n-k+1}, \cdots, \overline{c}_n \rangle$,

where $c_r \in H^{2r}(G_{n,k})$, $1 \leq r \leq k$, are the *special Schubert classes* on $G_{n,k}$, and where $\overline{c}_r$ is the component of the formal inverse of $1 + c_1 + \cdots + c_k$ in degree r.□

Motivated by the result in Exmple 1.4 we introduce the following notation.

Definition 5.2. A *Schubert presentation* of the cohomology ring of a flag manifold G/P is an isomorphism

(5.2) $H^*(G/P) = \mathbb{Z}[x_1, \cdots, x_k]/\langle f_1, \cdots, f_m \rangle$,

where

i) $\{x_1, \cdots, x_k\}$ is a minimal set of Schubert classes on G/P that generates the ring $H^*(G/P)$ multiplicatively;

ii) the number m of the generating polynomials $f_1, \cdots, f_m$ of the ideal $\langle f_1, \cdots, f_m \rangle$ is minimum subject to the isomorphism (1.3).□

This section is devoted to study the next problem for the exceptional Lie groups.

Problem 5.3. Given a flag manifold G/P find a Schubert presentation of its cohomology ring $H^*(G/P)$.

If G is exceptional with rank n, we assume that the set $\Omega = \{\omega_1, \ldots, \omega_n\}$ is so ordered as the root–vertices in the Dynkin diagram of G pictured in [**?**, p.58]. With this convention we single out, for given G and $\omega \in \Omega$, seven parabolic H, as well as their semi–simple part H_s, in the table below:

G	F_4	F_4	E_6	E_6	E_7	E_7	E_8
ω	ω_1	ω_4	ω_2	ω_6	ω_1	ω_7	ω_8
P	$C_3 \cdot S^1$	$B_3 \cdot S^1$	$A_6 \cdot S^1$	$D_5 \cdot S^1$	$D_6 \cdot S^1$	$E_6 \cdot S^1$	$E_7 \cdot S^1$
P_s	C_3	B_3	A_6	D_5	D_6	E_6	E_7

5.1 Cohomology of the homogeneous spaces G/P_s

We calculate the rings $H^*(G/H_s)$ for the seven homogeneous spaces

(5.3) F_4/C_3, F_4/B_3, E_6/A_6, E_6/D_5, E_7/D_6, E_7/E_6, E_8/E_7.

The results are stated in Theorems 5.4–5.10 below.

Given a set $\{d_1, \dots, d_t\}$ of elements graded by $|d_i| > 0$, let $\Gamma(1, d_1, \dots, d_t)$ be the graded free abelian group spanned by $1, d_1, \dots, d_t$, and considered as a graded ring with the trivial products $1 \cdot d_i = d_i$; $d_i \cdot d_j = 0$.

For a graded commutative ring A, let $A \widehat{\otimes} \Gamma(1, d_1, \dots, d_t)$ be the quotient of the tensor product $A \otimes \Gamma(1, d_1, \dots, d_t)$ by the relations $\mathrm{Tor}(A) \cdot d_i = 0$, $1 \le i \le t$.

Let $s_{r,i}$ for the i^{th} Schubert class on G/P in degree r. If $y \in H^*(G/P)$ we write $\overline{y} := p^*(y) \in H^*(G/P_s)$.

Theorem 5.4. *Let y_3, y_4, y_6 be the Schubert classes on $F_4/C_3 \cdot S^1$ with Weyl coordinates $\sigma[3,2,1]$, $\sigma[4,3,2,1]$, $\sigma[3,2,4,3,2,1]$ respectively, and let $d_{23} \in H^{23}(F_4/C_3)$ be with $\beta(d_{23}) = 2s_{11,1} - s_{11,2}$. Then*

$$H^*(F_4/C_3) = \mathbb{Z}[\overline{y}_3, \overline{y}_4, \overline{y}_6] / \langle h_3, h_6, h_8, h_{12} \rangle \widehat{\otimes} \Gamma(1, d_{23}),$$

where $h_3 = 2\overline{y}_3$, $h_6 = 2\overline{y}_6 + \overline{y}_3^2$, $h_8 = 3\overline{y}_4^2$, $h_{12} = \overline{y}_6^2 - \overline{y}_4^3$.
Proof.

nontrivial $H^k(F_4/C_3)$	basis elements	relations
$H^6 \cong \mathbb{Z}_2$	$\bar{s}_{3,1}$	
$H^8 \cong \mathbb{Z}$	$\bar{s}_{4,2}$	
$H^{12} \cong \mathbb{Z}_4$	$\bar{s}_{6,2}$	$-2\bar{s}_{6,2} = \bar{s}_{3,1}^2$
$H^{14} \cong \mathbb{Z}_2$	$\bar{s}_{7,1}$	$= \bar{s}_{3,1}\bar{s}_{4,2}$
$H^{16} \cong \mathbb{Z}_3$	$\bar{s}_{8,1}$	$= -\bar{s}_{4,2}^2$
$H^{18} \cong \mathbb{Z}_2$	$\bar{s}_{9,2}$	$= \bar{s}_{3,1}\bar{s}_{6,2}$
$H^{20} \cong \mathbb{Z}_4$	$\bar{s}_{10,2}$	$= \bar{s}_{4,2}\bar{s}_{6,2}$
$H^{26} \cong \mathbb{Z}_2$	$\bar{s}_{13,1}$	$= \bar{s}_{3,1}\bar{s}_{4,2}\bar{s}_{6,2}$
$H^{23} \cong \mathbb{Z}$	$d_{23} = \beta^{-1}(2\, s_{11,1} - s_{11,2})$	
$H^{31} \cong \mathbb{Z}$	$d_{31} = \beta^{-1}(s_{15,1})$	$= \pm \bar{s}_{4,2} d_{23}$

Theorem 5.5. *Let y_4 be the Schubert class on $F_4/B_3 \cdot S^1$ with Weyl coordinate $\sigma[3,2,3,4]$; and let $d_{23} \in H^{23}(F_4/B_3)$ be with $\beta(d_{23}) = -s_{11,1} + s_{11,2}$. Then*

$$H^*(F_4/B_3) = \mathbb{Z}[\overline{y}_4] / \langle h_8, h_{12} \rangle \widehat{\otimes} \Gamma(1, d_{23}),$$

where $h_8 = 3\overline{y}_4^2$, $h_{12} = \overline{y}_4^3$.
Proof.

nontrivial $H^k(F_4/B_3)$	basis elements	relations
$H^8 \cong \mathbb{Z}$	$\bar{s}_{4,2}$	
$H^{16} \cong \mathbb{Z}_3$	$\bar{s}_{8,1}$	$= \bar{s}_{4,2}^2$
$H^{23} \cong \mathbb{Z}$	$d_{23} = \beta^{-1}(-s_{11,1} + s_{11,2})$	
$H^{31} \cong \mathbb{Z}$	$d_{31} = \beta^{-1} s_{15,1}$	$= \pm \bar{s}_{4,2} d_{23}$

.□

Theorem 5.6. *Let y_3, y_4, y_6 be the Schubert classes on $E_6/A_6 \cdot S^1$ with Weyl coordinates $\sigma[5,4,2]$, $\sigma[6,5,4,2]$, $\sigma[1,3,6,5,4,2]$ respectively, and let $d_{23}, d_{29} \in H^{\text{odd}}(E_6/A_6)$ be with*

$$\beta(d_{23}) = 2s_{11,1} - s_{11,2},\ \beta(d_{29}) = s_{14,1} + s_{14,2} + s_{14,4} - \ s_{14,5}.$$

Then

$$H^*(E_6/A_6) = \\ \{\mathbb{Z}[\overline{y}_3, \overline{y}_4, \overline{y}_6] / \langle h_6, h_8, h_9, h_{12} \rangle \widehat{\otimes} \Gamma(1, d_{23}, d_{29})\} / \langle 2d_{29} = \overline{y}_3 d_{23} \rangle,$$

where $h_6 = 2\overline{y}_6 + \overline{y}_3^2$, $h_8 = 3\overline{y}_4^2$, $h_9 = 2\overline{y}_3\overline{y}_6$, $h_{12} = \overline{y}_6^2 - \overline{y}_4^3$.

Proof.

nontrivial H^k	basis elements	relations
$H^6 \cong \mathbb{Z}$	$\bar{s}_{3,2}$	
$H^8 \cong \mathbb{Z}$	$\bar{s}_{4,3}$	
$H^{12} \cong \mathbb{Z}$	$\bar{s}_{6,1}$	$-2\bar{s}_{6,1} = \bar{s}_{3,2}^2$
$H^{14} \cong \mathbb{Z}$	$\bar{s}_{7,1}$	$\bar{s}_{3,2}\,\bar{s}_{4,3}$
$H^{16} \cong \mathbb{Z}_3$	$\bar{s}_{8,1}$	$\bar{s}_{4,3}^2$
$H^{18} \cong \mathbb{Z}_2$	$\bar{s}_{9,1}$	$\bar{s}_{3,2}\,\bar{s}_{6,1}$
$H^{20} \cong \mathbb{Z}$	$\bar{s}_{10,1}$	$-\bar{s}_{4,3}\,\bar{s}_{6,1}$
$H^{22} \cong \mathbb{Z}_3$	$\bar{s}_{11,1}$	$\bar{s}_{4,3}^2\bar{s}_{3,2}$
$H^{26} \cong \mathbb{Z}_2$	$\bar{s}_{13,2}$	$\bar{s}_{3,2}\,\bar{s}_{4,3}\bar{s}_{6,1}$
$H^{28} \cong \mathbb{Z}_3$	$\bar{s}_{14,1}$	$-\,\bar{s}_{4,3}^2\bar{s}_{6,1}$
$H^{23} \cong \mathbb{Z}$	$d_{23} = \beta^{-1}(s_{11,1} - s_{11,2} - s_{11,3} + s_{11,4}$ $-s_{11,5} + s_{11,6})$	
$H^{29} \cong \mathbb{Z}$	$d_{29} = \beta^{-1}(-s_{14,1} + s_{14,2} + s_{14,4} - \ s_{14,5})$	$2d_{29} = \pm\bar{s}_{3,2}d_{23}$
$H^{31} \cong \mathbb{Z}$	$d_{31} = \beta^{-1}(s_{15,1} - 2\,s_{15,2} + s_{15,3} - \ s_{15,4})$	$\pm\bar{s}_{4,3}d_{23}$
$H^{35} \cong \mathbb{Z}$	$d_{35} = \beta^{-1}(-s_{17,1} + s_{17,2} + s_{17,3})$	$\pm\bar{s}_{6,1}d_{23}$
$H^{37} \cong \mathbb{Z}$	$d_{37} = \beta^{-1}(-s_{18,1} + s_{18,2})$	$\pm\bar{s}_{4,3}d_{29}$
$H^{43} \cong \mathbb{Z}$	$d_{43} = \beta^{-1}(s_{22,1})$	$\pm\bar{s}_{4,3}\bar{s}_{6,1}d_{23}$

.□

Theorem 5.7. *Let y_4 be the Schubert class on $E_6/D_5 \cdot S^1$ with Weyl coordinate $\sigma[2,4,5,6]$, and let $d_{17} \in H^{\text{odd}}(E_6/D_5)$ be with $\beta(d_{17}) = s_{8,1} - s_{8,2} - s_{8,3}$. Then*

$$H^*(E_6/D_5) = \mathbb{Z}[\overline{y}_4]/\langle h_{12}\rangle \widehat{\otimes}\Gamma(1, d_{17}),$$

where $h_{12} = \overline{y}_4^3$.

Proof.

nontrivial H^k	basis elements	relations
$H^8 \cong \mathbb{Z}$	$\overline{s}_{4,1}$	
$H^{16} \cong \mathbb{Z}$	$\overline{s}_{8,1}$	$\overline{s}_{4,1}^2$
$H^{17} \cong \mathbb{Z}$	$d_{17} = \beta^{-1}(s_{8,1} - s_{8,2} - s_{8,3})$	
$H^{25} \cong \mathbb{Z}$	$d_{25} = \beta^{-1}(s_{12,1} - s_{12,2})$	$\pm\overline{s}_{4,1}d_{17}$
$H^{33} \cong \mathbb{Z}$	$d_{33} = \beta^{-1}(s_{16,1})$	$\pm\overline{s}_{4,1}^2 d_{17}$

.□

Theorem 5.8. *Let y_5, y_9 be the Schubert classes on $E_7/E_6 \cdot S^1$ with Weyl coordinates $\sigma[2,4,5,6,7]$, $\sigma[1,5,4,2,3,4,5,6,7]$ respectively, and let d_{37}, $d_{45} \in H^{\text{odd}}(E_7/E_6)$ be with*

$$\beta(d_{37}) = s_{18,1} - s_{18,2} + s_{18,3},\ \beta(d_{45}) = s_{22,1} - s_{22,2}.$$

Then

$$H^*(E_7/E_6) = \{\mathbb{Z}[\overline{y}_5, \overline{y}_9]/\langle h_{10}, h_{14}, h_{18}\rangle \widehat{\otimes}\Gamma(1, d_{37}, d_{45})\}/\langle \overline{y}_9 d_{37} = \overline{y}_5 d_{45}\rangle,$$

where $h_{10} = \overline{y}_5^2$; $h_{14} = 2\overline{y}_5\overline{y}_9$; $h_{18} = \overline{y}_9^2$.

Proof.

nontrivial H^k	basis elements	relations
$H^{10} \cong \mathbb{Z}$	$\overline{s}_{5,1}$	$\overline{s}_{5,1}$
$H^{18} \cong \mathbb{Z}$	$\overline{s}_{9,1}$	$\overline{s}_{9,1}$
$H^{28} \cong \mathbb{Z}_2$	$\overline{s}_{14,1}$	$\overline{s}_{5,1}\overline{s}_{9,1}$
$H^{37} \cong \mathbb{Z}$	$d_{37} = \beta^{-1}(s_{18,1} - s_{18,2} + s_{18,3})$	
$H^{45} \cong \mathbb{Z}$	$d_{45} = \beta^{-1}(s_{22,1} - s_{22,2})$	
$H^{55} \cong \mathbb{Z}$	$d_{55} = \beta^{-1}(s_{27,1})$	$\overline{s}_{9,1}d_{37} = \pm\overline{s}_{5,1}d_{45}$

.□

Theorem 5.9. *Let y_4, y_6, y_9 be the Schubert classes on $E_7/D_6 \cdot S^1$ with Weyl coordinates $\sigma[2,4,3,1]$, $\sigma[2,6,5,4,3,1]$, $\sigma[3,4,2,7,6,5,4,3,1]$ respectively, and let $d_{35}, d_{51} \in H^{\text{odd}}(E_7/D_6)$ be with*

$$\beta(d_{35}) = s_{17,1} - s_{17,2} - s_{17,3} + s_{17,4} - s_{17,5} + s_{17,6} - s_{17,7};$$

$$\beta(d_{51}) = s_{25,1} - s_{25,2} - s_{25,4}.$$

Then

$$H^*(E_7/D_6) = \{\mathbb{Z}[\overline{y}_4,\overline{y}_6,\overline{y}_9]/\langle h_9,h_{12},h_{14},h_{18}\rangle \widehat{\otimes}\Gamma(1,d_{35},d_{51})\}/\left\langle 3d_{51}=\overline{y}_4^2 d_{35}\right\rangle,$$

where $h_9 = 2\overline{y}_9$, $h_{12} = 3\overline{y}_6^2 - \overline{y}_4^3$, $h_{14} = 3\overline{y}_4^2\overline{y}_6$, $h_{18} = \overline{y}_9^2 - \overline{y}_6^3$.

Proof.

nontrivial H^k	basis elements	relations
$H^8 \cong \mathbb{Z}$	$\overline{s}_{4,1}$	
$H^{12} \cong \mathbb{Z}$	$\overline{s}_{6,1}$	
$H^{16} \cong \mathbb{Z}$	$\overline{s}_{8,1}$	$\overline{s}_{4,1}^2$
$H^{18} \cong \mathbb{Z}_2$	$\overline{s}_{9,2}$	
$H^{20} \cong \mathbb{Z}$	$\overline{s}_{10,1}$	$\overline{s}_{4,1}\overline{s}_{6,1}$
$H^{24} \cong \mathbb{Z}$	$\overline{s}_{12,2}$	$\overline{s}_{12,2} = \overline{s}_{6,1}^2; 3\overline{s}_{12,2} = \overline{s}_{4,1}^3$
$H^{26} \cong \mathbb{Z}_2$	$\overline{s}_{13,1}$	$\overline{s}_{4,1}\overline{s}_{9,2}$
$H^{28} \cong \mathbb{Z}_3$	$\overline{s}_{14,1}$	$-\overline{s}_{4,1}^2\overline{s}_{6,1}$
$H^{30} \cong \mathbb{Z}_2$	$\overline{s}_{15,1}$	$\overline{s}_{6,1}\overline{s}_{9,2}$
$H^{32} \cong \mathbb{Z}$	$\overline{s}_{16,1}$	$\overline{s}_{4,1}\overline{s}_{6,1}^2$
$H^{34} \cong \mathbb{Z}_2$	$\overline{s}_{17,2}$	$\overline{s}_{4,1}^2\overline{s}_{9,2}$
$H^{38} \cong \mathbb{Z}_2$	$\overline{s}_{19,2}$	$\overline{s}_{4,1}\overline{s}_{6,1}\overline{s}_{9,2}$
$H^{40} \cong \mathbb{Z}_3$	$\overline{s}_{20,1}$	$\overline{s}_{4,1}^2\overline{s}_{6,1}^2$
$H^{42} \cong \mathbb{Z}_2$	$\overline{s}_{21,3}$	$\overline{s}_{4,1}^3\overline{s}_{9,2}$
$H^{50} \cong \mathbb{Z}_2$	$\overline{s}_{25,1}$	$\overline{s}_{4,1}^4\overline{s}_{9,2}$
$H^{35} \cong \mathbb{Z}$	$d_{35} = \beta^{-1}(s_{17,1} - s_{17,2} - s_{17,3} + s_{17,4} - s_{17,5} + s_{17,6} - s_{17,7})$	
$H^{43} \cong \mathbb{Z}$	$\beta^{-1}(s_{21,1} - 2\,s_{21,2} + s_{21,3} - 3\,s_{21,4} + 2\,s_{21,5} - s_{21,6})$	$\pm\overline{s}_{4,1}d_{35}$
$H^{47} \cong \mathbb{Z}$	$\beta^{-1}(2\,s_{23,1} - s_{23,2} + s_{23,3} - s_{23,4} + s_{23,5})$	$\pm\overline{s}_{6,1}d_{35}$
$H^{51} \cong \mathbb{Z}$	$d_{51} = \beta^{-1}(s_{25,1} - s_{25,2} - s_{25,4})$	$3d_{51} = \pm\overline{s}_{4,1}^2 d_{35}$
$H^{55} \cong \mathbb{Z}$	$\beta^{-1}(s_{27,1} + s_{27,2} - s_{27,3})$	$\pm\overline{s}_{4,1}\overline{s}_{6,1}d_{35}$
$H^{59} \cong \mathbb{Z}$	$\beta^{-1}(s_{29,1} - s_{29,2})$	$\pm\overline{s}_{6,1}^2 d_{35}, \pm\overline{s}_{4,1}d_{51}$
$H^{67} \cong \mathbb{Z}$	$\beta^{-1}(s_{33,1})$	$\overline{s}_{4,1}\overline{s}_{6,1}^2 d_{35} = \pm\overline{s}_{4,1}^2 d_{51}$

.□

Theorem 5.10. *Let* y_6,y_{10}, y_{15} *be the Schubert classes on* $E_8/E_7\cdot S^1$ *with Weyl coordinates* $\sigma[3,4,5,6,7,8]$, $\sigma[1,5,4,2,3,4,5,6,7,8]$, $\sigma[5,4,3,1,7,6,5,4,2,3,4,5,6,7,8]$ *respectively, and let* $d_{59} \in H^{\mathrm{odd}}(E_8/E_7)$ *be with*

$$\beta(d_{59}) = s_{29,1} - s_{29,2} - s_{29,3} + s_{29,4} - s_{29,5} + s_{29,6} - s_{29,7} + s_{29,8}.$$

Then

$$H^*(E_8/E_7) = \mathbb{Z}[\overline{y}_6, \overline{y}_{10}, \overline{y}_{15}]/\langle h_{15}, h_{20}, h_{24}, h_{30}\rangle \widehat{\otimes}\Gamma(1, d_{59}),$$

where $h_{15} = 2\overline{y}_{15}$, $h_{20} = 3\overline{y}_{10}^2$, $h_{24} = 5\overline{y}_6^4$, $h_{30} = \overline{y}_6^5 + \overline{y}_{10}^3 + \overline{y}_{15}^2 = 0$.

Proof.

nontrivial H^k	basis elements	relations
$H^{12} \cong \mathbb{Z}$	$\bar{s}_{6,2}$	
$H^{20} \cong \mathbb{Z}$	$\bar{s}_{10,1}$	
$H^{24} \cong \mathbb{Z}$	$\bar{s}_{12,1}$	$\pm\bar{s}_{6,2}^2$
$H^{30} \cong \mathbb{Z}_2$	$\bar{s}_{15,4}$	
$H^{32} \cong \mathbb{Z}$	$\bar{s}_{16,1}$	$\pm\bar{s}_{6,2}\bar{s}_{10,1}$
$H^{36} \cong \mathbb{Z}$	$\bar{s}_{18,2}$	$\pm\bar{s}_{6,2}^3$
$H^{40} \cong \mathbb{Z}_3$	$\bar{s}_{20,1}$	$\pm\bar{s}_{10,1}^2$
$H^{42} \cong \mathbb{Z}_2$	$\bar{s}_{21,3}$	$\pm\bar{s}_{6,2}\bar{s}_{15,4}$
$H^{44} \cong \mathbb{Z}$	$\bar{s}_{22,1}$	$\pm\bar{s}_{6,2}^2\bar{s}^{10,1}$
$H^{48} \cong \mathbb{Z}_5$	$\bar{s}_{24,1}$	$\pm\bar{s}_{6,2}^4$
$H^{50} \cong \mathbb{Z}_2$	$\bar{s}_{25,1}$	$\pm\bar{s}_{10,1}\bar{s}_{15,4}$
$H^{52} \cong \mathbb{Z}_3$	$\bar{s}_{26,1}$	$\pm\bar{s}_{6,2}\bar{s}_{10,1}^2$
$H^{54} \cong \mathbb{Z}_2$	$\bar{s}_{27,1}$	$\pm\bar{s}_{6,2}^2\bar{s}_{15,4}$
$H^{56} \cong \mathbb{Z}$	$\bar{s}_{28,1}$	$\pm\bar{s}_{6,2}^3\bar{s}_{10,1}$
$H^{62} \cong \mathbb{Z}_2$	$\bar{s}_{31,2}$	$\pm\bar{s}_{6,2}\bar{s}_{10,1}\bar{s}_{15,4}$
$H^{64} \cong \mathbb{Z}_3$	$\bar{s}_{32,1}$	$\pm\bar{s}_{6,2}^2\bar{s}_{10,1}^2$
$H^{66} \cong \mathbb{Z}_2$	$\bar{s}_{33,3}$	$\pm\bar{s}_{6,2}^3\bar{s}_{15,4}$
$H^{68} \cong \mathbb{Z}_5$	$\bar{s}_{34,1}$	$\pm\bar{s}_{6,2}^4\bar{s}_{10,1}$
$H^{74} \cong \mathbb{Z}_2$	$\bar{s}_{37,2}$	$\pm\bar{s}_{6,2}^2\bar{s}_{10,1}\bar{s}_{15,4}$
$H^{76} \cong \mathbb{Z}_3$	$\bar{s}_{38,1}$	$\pm\bar{s}_{6,2}^3\bar{s}_{10,1}^2$
$H^{86} \cong \mathbb{Z}_2$	$\bar{s}_{43,1}$	$\pm\bar{s}_{6,2}^3\bar{s}_{10,1}^2\bar{s}_{15,4}$
$H^{59} \cong \mathbb{Z}$	$d_{59} = \beta^{-1}(s_{29,1} - s_{29,2} - s_{29,3} + s_{29,4} - s_{29,5} + s_{29,6} - s_{29,7} + s_{29,8})$	
$H^{71} \cong \mathbb{Z}$	$\beta^{-1}(2s_{35,1} - 3s_{35,2} - s_{35,3} + s_{35,4} + s_{35,5} - s_{35,6} + s_{35,7})$	$\pm\bar{s}_{6,2}d_{59}$
$H^{79} \cong \mathbb{Z}$	$\beta^{-1}(2s_{39,1} - s_{39,2} - s_{39,3} - s_{39,4} + s_{39,5} - 2s_{39,6})$	$\pm\bar{s}_{10,1}d_{59}$
$H^{83} \cong \mathbb{Z}$	$\beta^{-1}(2\,s_{41,1} - s_{41,2} + s_{41,3} - s_{41,4} + s_{41,5})$	$\pm\bar{s}_{6,2}d_{59}$
$H^{91} \cong \mathbb{Z}$	$\beta^{-1}(s_{45,1} - s_{45,2} - s_{45,3} + s_{45,4})$	$\pm\bar{s}_{6,2}\bar{s}_{10,1}d_{59}$
$H^{95} \cong \mathbb{Z}$	$\beta^{-1}(s_{47,1} - s_{47,2} + s_{47,3})$	$\pm\bar{s}_{6,2}^3d_{59}$
$H^{103} \cong \mathbb{Z}$	$\beta^{-1}(-s_{51,1} + s_{51,2})$	$\pm\bar{s}_{6,2}^2\bar{s}_{10,1}d_{59}$

. □

5.2 Cohomology ring of generalized Grassmannians G/P

Theorem 1. Let y_1, y_3, y_4, y_6 be the Schubert classes on $F_4/C_3 \cdot S^1$ with Weyl coordinates $\sigma[1], \sigma[3,2,1], \sigma[4,3,2,1], \sigma[3,2,4,3,2,1]$ respectively. Then

$$H^*(F_4/C_3 \cdot S^1) = \mathbb{Z}[y_1, y_3, y_4, y_6]/\langle r_3, r_6, r_8, r_{12}\rangle,$$

where

$$r_3 = 2y_3 - y_1^3;$$
$$r_6 = 2y_6 + y_3^2 - 3y_1^2 y_4;$$
$$r_8 = 3y_4^2 - y_1^2 y_6;$$
$$r_{12} = y_6^2 - y_4^3.$$

Theorem 2. Let y_1, y_4 be the Schubert classes on $F_4/B_3 \cdot S^1$ with Weyl coordinates $\sigma[4], \sigma[3,2,3,4]$ respectively, Then

$$H^*(F_4/B_3 \cdot S^1) = \mathbb{Z}[y_1, y_4]/\langle r_8, r_{12}\rangle,$$

where

$$r_8 = 3y_4^2 - y_1^8;$$
$$r_{12} = 26y_4^3 - 5y_1^{12}.$$

Theorem 3. Let y_1, y_3, y_4, y_6 be the Schubert classes on $E_6/A_6 \cdot S^1$ with Weyl coordinates $\sigma[2], \sigma[5,4,2], \sigma[6,5,4,2], \sigma[1,3,6,5,4,2]$ respectively. Then

$$H^*(E_6/A_6 \cdot S^1) = \mathbb{Z}[y_1, y_3, y_4, y_6]/\langle r_6, r_8, r_9, r_{12}\rangle,$$

where

$$r_6 = 2y_6 + y_3^2 - 3y_1^2 y_4 + 2y_1^3 y_3 - y_1^6;$$
$$r_8 = 3y_4^2 - 6y_1 y_3 y_4 + y_1^2 y_6 + 5y_1^2 y_3^2 - 2y_1^5 y_3;$$
$$r_9 = 2y_3 y_6 - y_1^3 y_6;$$
$$r_{12} = y_4^3 - y_6^2.$$

Theorem 4. Let y_1, y_4 be the Schubert classes on $E_6/D_5 \cdot S^1$ with Weyl coordinates $\sigma[6], \sigma[2,4,5,6]$ respectively. Then

$$H^*(E_6/D_5 \cdot S^1) = \mathbb{Z}[y_1, y_4]/\langle r_9, r_{12}\rangle,$$

where

$$r_9 = 2y_1^9 + 3y_1 y_4^2 - 6y_1^5 y_4;$$
$$r_{12} = y_4^3 - 6y_1^4 y_4^2 + y_1^{12}.$$

Theorem 5. Let y_1, y_5, y_9 be the Schubert classes on $E_7/E_6 \cdot S^1$ with Weyl coordinates $\sigma[7], \sigma[2,4,5,6,7], \sigma[1,5,4,2,3,4,5,6,7]$ respectively. Then

$$H^*(E_7/E_6\cdot S^1)=\mathbb{Z}[y_1,y_5,y_9]/\langle r_{10},r_{14},r_{18}\rangle,$$

where

$$\begin{aligned} r_{10} &= y_5^2-2y_1y_9;\\ r_{14} &= 2y_5y_9-9y_1^4y_5^2+6y_1^9y_5-y_1^{14};\\ r_{18} &= y_9^2+10y_1^3y_5^3-9y_1^8y_5^2+2y_1^{13}y_5. \end{aligned}$$

Theorem 6. Let y_1,y_4,y_6,y_9 be the Schubert classes on $E_7/D_6\cdot S^1$ with Weyl coordinates $\sigma[1],\sigma[2,4,3,1],\sigma[2,6,5,4,3,1],\sigma[3,4,2,7,6,5,4,3,1]$ respectively. Then

$$H^*(E_7/D_6\cdot S^1)=\mathbb{Z}[y_1,y_4,y_6,y_9]/\langle r_9,r_{12},r_{14},r_{18}\rangle,$$

where

$$\begin{aligned} r_9 &= 2y_9+3y_1y_4^2+4y_1^3y_6+2y_1^5y_4-2y_1^9;\\ r_{12} &= 3y_6^2-y_4^3-3y_1^4y_4^2-2y_1^6y_6+2y_1^8y_4;\\ r_{14} &= 3y_4^2y_6+3y_1^2y_6^2+6y_1^2y_4^3+6y_1^4y_4y_6+2y_1^5y_9-y_1^{14};\\ r_{18} &= 5y_9^2+29y_6^3-24y_1^6y_6^2+45y_1^2y_4y_6^2+2y_1^9y_9. \end{aligned}$$

Theorem 7. Let y_1,y_6,y_{10},y_{15} be the Schubert classes on $E_8/E_7\cdot S^1$ with Weyl coordinates $\sigma[8]$, $\sigma[3,4,5,6,7,8]$, $\sigma[1,5,4,2,3,4,5,6,7,8]$, $\sigma[5,4,3,1,$ $7,6,5,4,2,\ 3,4,5,6,7,8]$ respectively. Then

$$H^*(E_8/E_7\cdot S^1)=\mathbb{Z}[y_1,y_6,y_{10},y_{15}]/\langle r_{15},r_{20},r_{24},r_{30}\rangle,$$

where

$$\begin{aligned} r_{15} &= 2y_{15}-16y_1^5y_{10}-10y_1^3y_6^2+10y_1^9y_6-y_1^{15};\\ r_{20} &= 3y_{10}^2+10y_1^2y_6^3+18y_1^4y_6y_{10}-2y_1^5y_{15}-8y_1^8y_6^2+4y_1^{10}y_{10}-y_1^{14}y_6;\\ r_{24} &= 5y_6^4+30y_1^2y_6^2y_{10}+15y_1^4y_{10}^2-2y_1^9y_{15}-5y_1^{12}y_6^2+y_1^{14}y_{10};\\ r_{30} &= y_{15}^2-8y_{10}^3+y_6^5-2y_1^3y_6^2y_{15}+3y_1^4y_6y_{10}^2-8y_1^5y_{10}y_{15}+6y_1^9y_6y_{15}\\ &\quad -9y_1^{10}y_{10}^2-y_1^{12}y_6^3-2y_1^{14}y_6y_{10}-3y_1^{15}y_{15}+8y_1^{20}y_{10}+y_1^{24}y_6-y_1^{30}. \end{aligned}$$

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