THE PRECISE REPRESENTATIVE FOR THE GRADIENT OF THE RIESZ POTENTIAL OF A FINITE MEASURE

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ABSTRACT. Given a finite nonnegative Borel measure m in $\mathbb{R}^d$, we identify the Lebesgue set $\mathcal{L}(V_s) \subset \mathbb{R}^d$ of the vector-valued function

$$V_s(x) = \int_{\mathbb{R}^d} \frac{x-y}{|x-y|^{s+1}} \mathrm{d}m(y),$$

for any order $0 < s < d$. We prove that $a \in \mathcal{L}(V_s)$ if and only if the integral above has a principal value at a and

$$\lim_{r \to 0} \frac{m(B_r(a))}{r^s} = 0.$$

In that case, the precise representative of V_s at a coincides with the principal value of the integral.

1. INTRODUCTION

Given $0 < s < d$ and a finite nonnegative Borel measure m in $\mathbb{R}^d$, we consider in $\mathbb{R}^d$ the vector-valued function

$$V_s(x) := \int_{\mathbb{R}^d} \frac{x-y}{|x-y|^{s+1}} \, \mathrm{d}m(y) \tag{1.1}$$

that is well-defined for every point x in the set

$$\operatorname{dom} V_s := \left\{ x \in \mathbb{R}^d : \int_{\mathbb{R}^d} \frac{\mathrm{d}m(y)}{|x-y|^s} < \infty \right\}.$$

The domain $\operatorname{dom} V_s$ is rather large as its complement $\mathbb{R}^d \setminus \operatorname{dom} V_s$ is negligible with respect to the Lebesgue measure. Indeed, as a consequence of Fubini's theorem one has

$$\int_{\mathbb{R}^d} \left(\int_{\mathbb{R}^d} \frac{\mathrm{d}m(y)}{|x-y|^s} \right) \mathrm{e}^{-|x|^2} \, \mathrm{d}x < \infty.$$

A finer analysis shows that the Hausdorff dimension of $\mathbb{R}^d \setminus \operatorname{dom} V_s$ is actually not greater that s ; see Theorem 1 in Chapter 4 of [1].

Up to multiplicative constants, V_s is the gradient of the Riesz potential of order $s-1$ and, in particular, V_{d-1} is the gradient of the Newtonian potential

$$\int_{\mathbb{R}^d} \frac{\mathrm{d}m(y)}{|x-y|^{d-2}} \quad \text{for } x \in \mathbb{R}^d$$

1991 *Mathematics Subject Classification.* 31B15 (primary); 26B05 (secondary).

generated by a distribution of mass described by m. Hence, for every function $u \in L^1_{\mathrm{loc}}(\mathbb{R}^d)$ such that

$$-\Delta u = m \quad \text{in the sense of distributions in } \mathbb{R}^d, \tag{1.2}$$

by Weyl's lemma on weakly harmonic functions it follows that

$$\nabla u = cV_{d-1} + \Phi \quad \text{almost everywhere in } \mathbb{R}^d,$$

where $\Phi : \mathbb{R}^d \to \mathbb{R}^d$ is the gradient of a harmonic function and $c \in \mathbb{R}$ is a nonzero constant that depends on d. As a result, a reasonable pointwise definition of ∇u for any solution of (1.2) can be obtained from a good understanding of V_{d-1}.

More generally, we are interested in the existence of the *precise representative* of V_s at a given point $a \in \mathbb{R}^d$. We recall that the precise representative in this case is a vector $\alpha \in \mathbb{R}^d$ that satisfies

$$\lim_{r \to 0} \fint_{B_r(a)} |V_s - \alpha| = 0, \tag{1.3}$$

where the symbol $\fint$ stands for the average integral with respect to the d-dimensional Lebesgue measure and $B_r(a)$ for the ball of radius r centered at a. When such an α exists, we say that a is a *Lebesgue point* of V_s and we denote

$$\widehat{V}_s(a) := \alpha.$$

It follows from the fundamental property (1.3) that

$$\widehat{V}_s(a) = \lim_{r \to 0} \fint_{B_r(a)} V_s.$$

As a consequence of the classical Lebesgue Differentiation Theorem, $\widehat{V}_s(a)$ exists for almost every $a \in \mathbb{R}^d$ with respect to the Lebesgue measure and

$$\widehat{V}_s = V_s \quad \text{almost everywhere in } \mathbb{R}^d.$$

When the function one is dealing with has better properties, like being in some Sobolev space, the exceptional set $\mathcal{E}$ (that is, the complement of the set $\mathcal{L}$ of Lebesgue points) is typically smaller than merely having zero Lebesgue measure; see e.g. Section 4.8 in [4] and Chapter 8 in [8]. It is then natural to expect that the same property holds for the exceptional set $\mathcal{E}(V_s)$ of the potential V_s.

This note stems from the recent work [2] by the first and third authors concerning capacitary differentiability of the Newtonian potential; see also [13]. In analogy with [2], we show that existence of a principal value for the integral, combined with a density property of m at a, allows one to decide whether a is a Lebesgue point of V_s.

Before stating our result, we recall that the principal value of V_s at a is

$$\text{p.v.}\, V_s(a) := \lim_{\epsilon \to 0} \int_{\mathbb{R}^d \setminus B_\epsilon(a)} \frac{a - y}{|a - y|^{s+1}} \, \mathrm{d}m(y)$$

whenever this limit exists. We relate the question of existence of a principal value for V_s with that of a precise representative in the following

Theorem 1.1. *A point $a \in \mathbb{R}^d$ is a Lebesgue point of V_s if and only if the principal value of V_s exists at a and*

$$\lim_{r \to 0} \frac{m(B_r(a))}{r^s} = 0. \tag{1.4}$$

One then has

$$\widehat{V}_s(a) = \mathrm{p.v.}\, V_s(a).$$

We prove Theorem 1.1 in Sections 2 and 3. That every $a \in \operatorname{dom} V_s$ is a Lebesgue point of V_s and $\widehat{V}_s(a) = V_s(a)$ can be seen independently from Theorem 1.1 by observing that, for every $y \in \mathbb{R}^d$ and $r > 0$,

$$\fint_{B_r(a)} \frac{\mathrm{d}x}{|x-y|^s} \le \frac{C_{d,s}}{|a-y|^s}$$

and then

$$\fint_{B_r(a)} \left| \frac{x-y}{|x-y|^{s+1}} - \frac{a-y}{|a-y|^{s+1}} \right| \mathrm{d}x \le \frac{C_{d,s}+1}{|a-y|^s} \quad \text{for } y \neq a.$$

An application of Fubini's Theorem and the Dominated Convergence Theorem now gives

$$\fint_{B_r(a)} |V_s - V_s(a)| \le \int_{\mathbb{R}^d} \left(\fint_{B_r(a)} \left| \frac{x-y}{|x-y|^{s+1}} - \frac{a-y}{|a-y|^{s+1}} \right| \mathrm{d}x \right) \mathrm{d}m(y) \to 0 \quad \text{as } r \to 0.$$

As $\operatorname{dom} V_s$ is contained in the set of Lebesgue points of V_s, we get

$$\dim_{\mathcal{H}} (\mathcal{E}(V_s)) \le \dim_{\mathcal{H}} (\mathbb{R}^d \setminus \operatorname{dom} V_s) \le s. \tag{1.5}$$

To obtain a more precise quantification of the size of $\mathcal{E}(V_s)$, we observe that, by Theorem 1.1, failure of having Lebesgue points may occur by either that the principal value of V_s does not exist at a or

$$\limsup_{r \to 0} \frac{m(B_r(a))}{r^s} > 0.$$

The latter condition holds on a set of points a with at most σ-finite Hausdorff measure $\mathcal{H}^s$.

The question of existence of principal values is more subtle and has been investigated by Mattila and the third author [6]. The answer involves a capacity κ_s that is defined on every compact subset $E \subset \mathbb{R}^d$ as

$$\kappa_s(E) = \sup \mu(\mathbb{R}^d),$$

where the supremum is computed over all finite nonnegative Borel measures μ supported in E such that

(a) $\mu(A) = 0$ for every Borel set $A \subset \mathbb{R}^d$ with σ-finite $\mathcal{H}^s$ measure,

(b) the maximal Riesz transform R_s^* is a bounded operator from $L^2(\mu)$ into itself with

$$\|R_s^*(f\mu)\|_{L^2(\mu)} \le \|f\|_{L^2(\mu)} \quad \text{for every } f \in (L^1 \cap L^2)(\mu),$$

where

$$R_s^*(f\mu)(x) := \sup_{\epsilon>0} \left| \int_{\mathbb{R}^d \setminus B_\epsilon(x)} \frac{x-y}{|x-y|^{s+1}} f(y)\, \mathrm{d}\mu(y) \right| \quad \text{for every } x \in \mathbb{R}^d.$$

As a consequence of [6] and using a straightforward adaptation of the argument in [2], we show that

Theorem 1.2. *For every compact subset $E \subset \mathcal{E}(V_s)$, one has $\kappa_s(E) = 0$.*

Other properties of κ_s for any $0 < s < d$ are presented in Section 4, where we also prove Theorem 1.2 and explain why the latter implies (1.5). A variant of κ_s was introduced by Prat in [9], where the author obtains analogous estimates for the Hausdorff dimension of sets of zero capacity via comparison between capacity and Hausdorff content.

A deep result by Ruiz de Villa and Tolsa [11] ensures that κ_{d-1} is equivalent to the C^1-harmonic capacity. Using this interpretation, the fact that $\kappa_{d-1}(E) = 0$ means that every C^1 function in $\mathbb{R}^d$ that is harmonic in $\mathbb{R}^d \setminus E$ must be harmonic in the entire space $\mathbb{R}^d$. In dimension $d = 2$, it follows from work by Tolsa [12] that the C^1-harmonic capacity is equivalent to the classical continuous analytic capacity. We then deduce from Theorem 1.2 for $s = 1$ that

Corollary 1.3. *For every finite complex Borel measure ν in $\mathbb{C}$, the exceptional set for Lebesgue points of the Cauchy integral of ν defined by*

$$\int_{\mathbb{C}} \frac{\mathrm{d}\nu(w)}{z-w} \quad \textit{for almost every } z \in \mathbb{C}$$

has zero continuous analytic capacity.

The preceding corollary is sharp in the sense that there exists a compact set K of zero continuous analytic capacity that is exactly the exceptional set of the Cauchy integral of some finite measure. For instance, take as K the corner quarters planar Cantor set, which has positive and finite one dimensional Hausdorff measure m. Then, the Cauchy integral f of m is continuous in the complement of K and each point of K is an exceptional point of f because (1.4) is not satisfied. It would be interesting to know whether each compact set with zero continuous analytic capacity can be expressed as the exceptional set of the Cauchy integral of some finite Borel measure.

2. PROOF OF THE REVERSE IMPLICATION OF THEOREM 1.1

We first need a couple of estimates related to V_s.

Lemma 2.1. *For every $a \in \mathbb{R}^d$ and every $r > 0$,*

$$\fint_{B_r(a)} \left(\int_{B_{2r}(a)} \frac{\mathrm{d}m(y)}{|z-y|^s} \right) \mathrm{d}z \le C \, \frac{m(B_{2r}(a))}{r^s},$$

for some constant $C > 0$ depending on d.

Proof. By Fubini's theorem,

$$\fint_{B_r(a)} \left(\int_{B_{2r}(a)} \frac{\mathrm{d}m(y)}{|z-y|^s} \right) \mathrm{d}z = \int_{B_{2r}(a)} \left(\fint_{B_r(a)} \frac{\mathrm{d}z}{|z-y|^s} \right) \mathrm{d}m(y).$$

For every $y \in \mathbb{R}^d$,

$$\fint_{B_r(a)} \frac{\mathrm{d}z}{|z-y|^s} \le \fint_{B_r(a)} \frac{\mathrm{d}z}{|z-a|^s} = \frac{C}{r^s},$$

which gives the conclusion. □

Lemma 2.2. *For every $a \in \mathbb{R}^d$ and every $r > 0$,*

$$\fint_{B_r(a)} \left(\int_{\mathbb{R}^d \setminus B_{2r}(a)} \left| \frac{z-y}{|z-y|^{s+1}} - \frac{a-y}{|a-y|^{s+1}} \right| \mathrm{d}m(y) \right) \mathrm{d}z \le C' r \int_{2r}^{\infty} \frac{m(B_t(a))}{t^{s+2}} \, \mathrm{d}t,$$

for some constant $C' > 0$ depending on d.

Proof. To simplify the notation, we may assume that $a = 0$. For $y \in \mathbb{R}^d \setminus B_{2r}(0)$ and $z \in B_r(0)$, by the Mean Value Theorem there exists $0 \le \theta \le 1$ such that

$$\left| \frac{z-y}{|z-y|^{s+1}} + \frac{y}{|y|^{s+1}} \right| \le \frac{C_1 |z|}{|\theta z - y|^{s+1}} \le \frac{C_2 r}{|y|^{s+1}}.$$

We thus have

$$\fint_{B_r(0)} \left(\int_{\mathbb{R}^d \setminus B_{2r}(0)} \left| \frac{z-y}{|z-y|^{s+1}} + \frac{y}{|y|^{s+1}} \right| \mathrm{d}m(y) \right) \mathrm{d}z$$

$$\le C_2 r \fint_{B_r(0)} \left(\int_{\mathbb{R}^d \setminus B_{2r}(0)} \frac{\mathrm{d}m(y)}{|y|^{s+1}} \right) \mathrm{d}z = C_2 r \int_{\mathbb{R}^d \setminus B_{2r}(0)} \frac{\mathrm{d}m(y)}{|y|^{s+1}}.$$

Using Cavalieri's principle, see e.g. Corollary 2.2.34 in [14], one gets

$$\int_{\mathbb{R}^d \setminus B_{2r}(0)} \frac{\mathrm{d}m(y)}{|y|^{s+1}} = (s+1) \int_{2r}^{\infty} \frac{m(B_t(0))}{t^{s+2}} \, \mathrm{d}t$$

and the lemma follows. □

To handle the limit as $r \to 0$ of the right-hand side in the estimate in Lemma 2.2, we need an elementary fact from Real Analysis:

Lemma 2.3. *Let $h : [0, \infty) \to \mathbb{R}$ be a bounded continuous function. If $h(0) = 0$, then, for every $\beta > 0$,*

$$\lim_{r \to 0} r^{\beta} \int_r^{\infty} \frac{h(t)}{t^{1+\beta}} \, \mathrm{d}t = 0.$$

Proof. Given $\epsilon > 0$, take $\delta > 0$ such that $|h(t)| \leq \epsilon$ for every $0 < t \leq \delta$. For $0 < r < \delta$, we then have

$$\left|\int_r^\infty \frac{h(t)}{t^{1+\beta}}\,\mathrm{d}t\right| \leq \left|\int_r^\delta\right| + \left|\int_\delta^\infty\right| \leq \frac{\epsilon}{\beta}\Big(\frac{1}{r^\beta} + \frac{\|h\|_{L^\infty}}{\delta^\beta}\Big).$$

Therefore,

$$\limsup_{r\to 0} r^\beta \left|\int_r^\infty \frac{h(t)}{t^{1+\beta}}\,\mathrm{d}t\right| \leq \frac{\epsilon}{\beta}.$$

Since this inequality holds for every $\epsilon > 0$, the limit equals zero. □

Proof of Theorem 1.1. "$\Longleftarrow$". We assume that the principal value $\mathrm{p.v.}\, V_s(a)$ exists and the limit (1.4) holds. For almost every $z \in \mathbb{R}^d$,

$$\begin{aligned}
&\left|V_s(z) - \int_{\mathbb{R}^d\setminus B_{2r}(a)} \frac{a-y}{|a-y|^{s+1}}\,\mathrm{d}m(y)\right| \\
&\quad\leq \left|\int_{\mathbb{R}^d\setminus B_{2r}(a)} \frac{z-y}{|z-y|^{s+1}}\,\mathrm{d}m(y) - \int_{\mathbb{R}^d\setminus B_{2r}(a)} \frac{a-y}{|a-y|^{s+1}}\,\mathrm{d}m(y)\right| \\
&\qquad\qquad + \left|\int_{B_{2r}(a)} \frac{z-y}{|z-y|^{s+1}}\,\mathrm{d}m(y)\right|.
\end{aligned}$$

Hence,

$$\begin{aligned}
&\fint_{B_r(a)} \left|V_s(z) - \int_{\mathbb{R}^d\setminus B_{2r}(a)} \frac{a-y}{|a-y|^{s+1}}\,\mathrm{d}m(y)\right| \mathrm{d}z \\
&\quad\leq \fint_{B_r(a)} \left(\int_{\mathbb{R}^d\setminus B_{2r}(a)} \left|\frac{z-y}{|z-y|^{s+1}}\,\mathrm{d}m(y) - \frac{a-y}{|a-y|^{s+1}}\right| \mathrm{d}m(y)\right) \mathrm{d}z \\
&\qquad\qquad + \fint_{B_r(a)} \left(\int_{B_{2r}(a)} \frac{\mathrm{d}m(y)}{|z-y|^s}\right) \mathrm{d}z.
\end{aligned}$$

We then have by Lemmas 2.1 and 2.2,

$$\begin{aligned}
&\fint_{B_r(a)} \left|V_s(z) - \int_{\mathbb{R}^d\setminus B_{2r}(a)} \frac{a-y}{|a-y|^{s+1}}\,\mathrm{d}m(y)\right| \mathrm{d}z \\
&\qquad\qquad \leq C_1\left(r\int_{2r}^\infty \frac{m(B_t(a))}{t^{s+2}}\,\mathrm{d}t + \frac{m(B_{2r}(a))}{r^s}\right).
\end{aligned} \tag{2.1}$$

Denoting by α the principal value of V_s at a, we get

$$\begin{aligned}
&\fint_{B_r(a)} |V_s(z) - \alpha|\,\mathrm{d}z \\
&\leq C_1\left(r\int_{2r}^\infty \frac{m(B_t(a))}{t^{s+2}}\,\mathrm{d}t + \frac{m(B_{2r}(a))}{r^s}\right) + \left|\int_{\mathbb{R}^d\setminus B_{2r}(a)} \frac{a-y}{|a-y|^{s+1}}\,\mathrm{d}m(y) - \alpha\right|.
\end{aligned}$$

As $r \to 0$, the last term converges to zero by definition of principal value. The quantity in parentheses also converges to zero from the assumption on the measure m, as we can apply Lemma 2.3 with $\beta = 1$ and $h(t) = m(B_t(a))/t^s$ for $t > 0$. □

3. Proof of the direct implication of Theorem 1.1

We begin with the following estimate:

Proposition 3.1. *Let $0 < s \leq d-1$. For every $\alpha \in \mathbb{R}^d$ and every $r > 0$,*
$$\frac{m(B_r(a))}{r^s} \leq C \fint_{B_{2r}(a)} |V_s - \alpha|,$$
for some constant $C > 0$ depending on d.

Proof. We may assume that $a = 0$. We take the inner product of $V_s(x)$ with $x/|x|$ and integrate over the ball $B_{2r}(0)$ for any $r > 0$. By integrability of the function $(x, y) \mapsto 1/|x-y|^s$ with respect to the product measure $\mathcal{H}^d \otimes m$ in $B_{2r}(0) \times \mathbb{R}^d$ we can interchange the order of integration:
$$\int_{B_{2r}(0)} V_s(x) \cdot \frac{x}{|x|} \, \mathrm{d}x = \int_{\mathbb{R}^d} \biggl(\int_{B_{2r}(0)} \frac{x-y}{|x-y|^{s+1}} \cdot \frac{x}{|x|} \, \mathrm{d}x \biggr) \, \mathrm{d}m(y). \tag{3.1}$$
By the integration formula in polar coordinates,
$$\int_{B_{2r}(0)} \frac{x-y}{|x-y|^{s+1}} \cdot \frac{x}{|x|} \, \mathrm{d}x = \int_0^{2r} \biggl(\int_{\partial B_\rho(0)} \frac{x-y}{|x-y|^{s+1}} \cdot \frac{x}{|x|} \, \mathrm{d}\sigma(x) \biggr) \, \mathrm{d}\rho, \tag{3.2}$$
where σ denotes the surface measure on the sphere $\partial B_\rho(0)$.

We claim that
$$\int_{\partial B_\rho(0)} \frac{x-y}{|x-y|^{s+1}} \cdot \frac{x}{|x|} \, \mathrm{d}\sigma(x) \geq \epsilon \, \rho^{d-s-1} \chi_{B_\rho(0)}(y), \tag{3.3}$$
for some $\epsilon > 0$ independent of ρ and y. To this end, note that
$$\frac{x-y}{|x-y|^{s+1}} = \nabla g(x) \quad \text{where } g(x) := -\frac{1}{(s-1)} \frac{1}{|x-y|^{s-1}}.$$
When $s < d-1$, it follows from the Divergence Theorem in $B_\rho(0)$ that
$$\int_{\partial B_\rho(0)} \frac{x-y}{|x-y|^{s+1}} \cdot \frac{x}{|x|} \, \mathrm{d}\sigma(x) = \int_{B_\rho(0)} \Delta g = (d-s-1) \int_{B_\rho(0)} \frac{\mathrm{d}x}{|x-y|^{s+1}},$$
which implies (3.3). When $s = d-1$, the function g is harmonic in $\mathbb{R}^d \setminus \{y\}$. An application of the Divergence Theorem on $B_\rho(0) \setminus B_\eta(y)$ with $\eta \to 0$ yields
$$\int_{\partial B_\rho(0)} \frac{x-y}{|x-y|^{s+1}} \cdot \frac{x}{|x|} \, \mathrm{d}\sigma(x) = \begin{cases} \mathcal{H}^{d-1}(\partial B_1(0)) & \text{if } y \in B_\rho(0), \\ 0 & \text{if } y \notin \overline{B_\rho(0)}, \end{cases}$$
which also gives (3.3) and completes the proof of the claim.

Combining (3.2) and (3.3), we get
$$\int_{B_{2r}(0)} \frac{x-y}{|x-y|^{s+1}} \cdot \frac{x}{|x|} \, \mathrm{d}x \geq c \, r^{d-s} \chi_{B_r(0)}(y),$$
which by (3.1) then gives
$$\int_{B_{2r}(0)} V_s(x) \cdot \frac{x}{|x|} \, \mathrm{d}x \geq c \, r^{d-s} m(B_r(0)).$$

Since $x/|x|$ has zero average we can subtract any constant vector α in the integrand in the left-hand side without changing the value of the integral. The conclusion readily follows since $x/|x|$ is a unit vector. □

When $s > d - 1$, one has

$$\int_{\partial B_\rho(0)} \frac{x-y}{|x-y|^{s+1}} \cdot \frac{x}{|x|} \, \mathrm{d}\sigma(x) = \begin{cases} (s+1-d) \displaystyle\int_{\mathbb{R}^d \setminus B_\rho(0)} \frac{\mathrm{d}x}{|x-y|^{s+1}} & \text{if } y \notin \overline{B_\rho(0)}, \\ (d-s-1) \displaystyle\int_{B_\rho(0)} \frac{\mathrm{d}x}{|x-y|^{s+1}} & \text{if } y \in B_\rho(0). \end{cases}$$

In particular, these values have opposite signs, in contrast with the previous proof. We prove in that case the following counterpart of Proposition 3.1 that is enough for our purposes:

Proposition 3.2. *Let $d - 1 < s < d$. For every $\alpha \in \mathbb{R}^d$ and every $r > 0$,*

$$\frac{m(B_r(a))}{r^s} \leq C' \left(\fint_{B_r(a)} |V_s - \alpha| + r^{d-s} \int_{\mathbb{R}^d \setminus B_r(a)} \frac{|V_s(x) - \alpha|}{|x-a|^{2d-s}} \, \mathrm{d}x \right),$$

for some constant $C' > 0$ depending on s and d.

As the vector-field $(x-a)/|x-a|$ does not seem to be a convenient choice to test V_s in this range of s, we rely on a different one that is provided by our next

Lemma 3.3. *Let $d - 1 < s < d$. For every $\varphi \in C_c^\infty(\mathbb{R}^d)$, there exists a summable smooth vector-field $\Psi : \mathbb{R}^d \to \mathbb{R}^d$ such that $\int_{\mathbb{R}^d} \Psi = 0$,*

$$\varphi(x) = \int_{\mathbb{R}^d} \frac{x-y}{|x-y|^{s+1}} \cdot \Psi(y) \, \mathrm{d}y \quad \textit{for every } x \in \mathbb{R}^d,$$

and

$$|\Psi(x)| \leq \frac{C''}{(1+|x|)^{2d-s}} \quad \textit{for every } x \in \mathbb{R}^d,$$

for some constant $C'' > 0$ depending on s, d and φ.

In terms of the Fourier transform, the vector-field Ψ satisfies

$$\widehat{\varphi} = \widehat{\frac{z}{|z|^{s+1}}} \cdot \widehat{\Psi} = c \, \frac{\xi}{|\xi|^{d-s+1}} \cdot \widehat{\Psi}$$

for a constant $c \in \mathbb{C} \setminus \{0\}$. This identity is satisfied for example with

$$\widehat{\Psi} = \frac{1}{c} |\xi|^{d-s-1} \xi \, \widehat{\varphi}, \tag{3.4}$$

which indicates a convenient choice of Ψ.

Proof of Lemma 3.3. Take $\Psi : \mathbb{R}^d \to \mathbb{R}^d$ defined for $x \in \mathbb{R}^d$ by

$$\Psi(x) = c' \int_{\mathbb{R}^d} \frac{\nabla\varphi(y)}{|x-y|^{2d-s-1}} \, \mathrm{d}y,$$

with $c' \in \mathbb{R}$. Since $s > d - 1$, the integral above is well-defined and one shows that Ψ satisfies identity (3.4) for a suitable choice of c'. That Ψ satisfies the desired relation with φ can also be checked without the Fourier transform, by an explicit computation of the convolution between the homogeneous functions $z/|z|^\ell$ and $1/|z|^j$; see Lemma 15.10 and Exercise 15.4 in [8] for dimensions $d \geq 2$ and $d = 1$, respectively.

We observe that Ψ is bounded and summable. Moreover, smoothness of Ψ follows from a standard change of variable,

$$\Psi(x) = c' \int_{\mathbb{R}^d} \frac{\nabla\varphi(x - z)}{|z|^{2d-s-1}} \, \mathrm{d}z. \tag{3.5}$$

Since $\int_{\mathbb{R}^d} \nabla\varphi = 0$, we also get by integration on both sides of (3.5) and Fubini's theorem that

$$\int_{\mathbb{R}^d} \Psi = 0.$$

We now take $R > 0$ such that $\operatorname{supp}\varphi \subset B_R(0)$. To conclude the proof, it suffices to show that

$$|\Psi(x)| \leq \frac{C_1}{|x|^{2d-s}} \quad \text{for every } |x| \geq 2R. \tag{3.6}$$

To this end, we use again that $\int_{\mathbb{R}^d} \nabla\varphi = 0$ to write

$$\Psi(x) = c' \int_{\mathbb{R}^d} \nabla\varphi(y) \Big(\frac{1}{|x - y|^{2d-s-1}} - \frac{1}{|x|^{2d-s-1}} \Big) \, \mathrm{d}y.$$

For $y \in B_R(0)$ and $x \in \mathbb{R}^d \setminus B_{2R}(0)$, an application of the Mean Value Theorem gives $0 \leq \theta \leq 1$ with

$$\left| \frac{1}{|x - y|^{2d-s-1}} - \frac{1}{|x|^{2d-s-1}} \right| \leq \frac{C_2|y|}{|x - \theta y|^{2d-s}} \leq \frac{C_3}{|x|^{2d-s}},$$

where $C_3 > 0$ depends on s, d and R. Hence, when $|x| \geq 2R$ we get

$$|\Psi(x)| \leq \int_{B_R(0)} |\nabla\varphi(y)| \left| \frac{1}{|x - y|^{2d-s-1}} - \frac{1}{|x|^{2d-s-1}} \right| \mathrm{d}y \leq \Big(\int_{\mathbb{R}^d} |\nabla\varphi| \Big) \frac{C_3}{|x|^{2d-s}},$$

which implies (3.6). The estimate of $|\Psi(x)|$ in $\mathbb{R}^d$ then follows from the boundedness of Ψ. □

Proof of Proposition 3.2. We may assume that $a = 0$. Fix a nonnegative function $\varphi \in C_c^\infty(\mathbb{R}^d)$ such that $\varphi \geq 1$ in $B_1(0)$ and let $\Psi : \mathbb{R}^d \to \mathbb{R}^d$ be the vector-field given by Lemma 3.3. For $r > 0$, one has by scaling that

$$\varphi\Big(\frac{x}{r}\Big) = \int_{\mathbb{R}^d} \frac{x - y}{|x - y|^{s+1}} \cdot \Psi_r(y) \, \mathrm{d}y \quad \text{for every } x \in \mathbb{R}^d,$$

where $\Psi_r(x) := \frac{1}{r^{d-s}} \Psi(\frac{x}{r})$. Since $\varphi \geq 1$ on $B_1(0)$ and m is a nonnegative measure, we have

$$m(B_r(0)) \leq \int_{\mathbb{R}^d} \varphi\Big(\frac{x}{r}\Big) \, \mathrm{d}m(x) = \int_{\mathbb{R}^d} \Big(\int_{\mathbb{R}^d} \frac{x - y}{|x - y|^{s+1}} \cdot \Psi_r(y) \, \mathrm{d}y \Big) \, \mathrm{d}m(x).$$

Then, by Fubini's theorem,

$$m(B_r(0)) \leq - \int_{\mathbb{R}^d} V_s \cdot \Psi_r \, .$$

Since $\int_{\mathbb{R}^d} \Psi_r = 0$, for every $\alpha \in \mathbb{R}^d$ we get

$$m(B_r(0)) \le -\int_{\mathbb{R}^d} (V_s - \alpha) \cdot \Psi_r \le \int_{\mathbb{R}^d} |V_s - \alpha| |\Psi_r|. \tag{3.7}$$

The pointwise estimate satisfied by Ψ gives

$$|\Psi_r(x)| = \frac{1}{r^{d-s}} \Big|\Psi\Big(\frac{x}{r}\Big)\Big| \le C_1 \Big(\frac{1}{r^{d-s}} \chi_{B_r(0)}(x) + \frac{r^d}{|x|^{2d-s}} \chi_{\mathbb{R}^d \setminus B_r(0)}(x) \Big). \tag{3.8}$$

The conclusion follows by inserting (3.8) in (3.7). □

To handle the additional term that appears in Proposition 3.2, compared to Proposition 3.1, we need the following

Lemma 3.4. *If $f \in L^1(\mathbb{R}^d)$ is such that*

$$\lim_{r \to 0} \fint_{B_r(a)} f = 0,$$

then, for every $\beta > 0$,

$$\lim_{r \to 0} r^\beta \int_{\mathbb{R}^d \setminus B_r(a)} \frac{f(x)}{|x-a|^{d+\beta}} \, \mathrm{d}x = 0.$$

Proof. We first prove that, for every $r > 0$,

$$\int_{\mathbb{R}^d \setminus B_r(a)} \frac{f(x)}{|x-a|^{d+\beta}} \, \mathrm{d}x = C_1 \int_r^\infty \frac{1}{\rho^{1+\beta}} \Big(\fint_{B_\rho(a)} f \Big) \, \mathrm{d}\rho - \frac{C_2}{r^\beta} \fint_{B_r(a)} f, \tag{3.9}$$

for some constants $C_1, C_2 > 0$. By the integration formula in polar coordinates,

$$\begin{aligned} \int_{\mathbb{R}^d \setminus B_r(a)} \frac{f(x)}{|x-a|^{d+\beta}} \, \mathrm{d}x &= \int_r^\infty \Big(\int_{\partial B_\rho(a)} \frac{f(x)}{|x-a|^{d+\beta}} \, \mathrm{d}\sigma(x) \Big) \, \mathrm{d}\rho \\ &= \int_r^\infty \frac{1}{\rho^{d+\beta}} \Big(\frac{\mathrm{d}}{\mathrm{d}\rho} \int_{B_\rho(a)} f \Big) \, \mathrm{d}\rho. \end{aligned}$$

One then gets (3.9) by integration by parts. To conclude, it suffices to apply Lemma 2.3 with $h(t) = \fint_{B_t(a)} f$ for $t > 0$. □

Proof of Theorem 1.1. "$\Longrightarrow$". Let a be a Lebesgue point of V_s. We first show that the limit (1.4) holds. To this end, denote by α the precise representative of V_s at a. When $0 < s \le d-1$, we deduce from Proposition 3.1 that

$$\limsup_{r \to 0} \frac{m(B_r(a))}{r^s} \le C \lim_{r \to 0} \fint_{B_{2r}(a)} |V_s - \alpha| = 0,$$

which implies (1.4). When $d - 1 < s < d$, we apply Proposition 3.2. In this case, Lemma 3.4 with $f = |V_s - \alpha|$ and $\beta = d - s$ handles the additional term in the estimate as $r \to 0$ and we get (1.4) as before.

We next recall that

$$\alpha = \lim_{r \to 0} \fint_{B_r(a)} V_s.$$

Moreover, by estimate (2.1) in the proof of Theorem 1.1, we have

$$\left| \fint_{B_r(a)} V_s - \int_{\mathbb{R}^d \setminus B_{2r}(a)} \frac{a-y}{|a-y|^{s+1}} \, \mathrm{d}m(y) \right| \\ \leq C_1 \left(r \int_{2r}^{\infty} \frac{m(B_t(a))}{t^{s+2}} \, \mathrm{d}t + \frac{m(B_{2r}(a))}{r^s} \right).$$

As $r \to 0$, using (1.4) we deduce from Lemma 2.3 that the right-hand side converges to zero and then

$$\alpha = \lim_{r \to 0} \fint_{B_r(a)} V_s = \lim_{r \to 0} \int_{\mathbb{R}^d \setminus B_{2r}(a)} \frac{a-y}{|a-y|^{s+1}} \, \mathrm{d}m(y).$$

Thus, by definition, α is the principal value of V_s at a. □

4. THE CAPACITY κ_s AND PROOF OF THEOREM 1.2

We begin by showing that κ_s is subadditive:

Proposition 4.1. *For every compact subsets $E_1, E_2 \subset \mathbb{R}^d$,*

$$\kappa_s(E_1 \cup E_2) \leq \kappa_s(E_1) + \kappa_s(E_2).$$

The capacity κ_s is equivalent to another one that was known to be semi-additive (that is, the estimate above is verified with a constant $C \geq 1$); see p. 3643 in [11] and Section 2 in [10].

Before proving the proposition, we start with the following observation: If μ is a Borel measure in $\mathbb{R}^d$ that is admissible in the definition of $\kappa_s(E)$ and if $F \subset E$ is a compact subset, then, for any Borel subset $A \subset F$, the Borel measure $\mu\lfloor_A$ defined by

$$\mu\lfloor_A(B) = \mu(A \cap B)$$

is admissible for $\kappa_s(F)$. Indeed, it suffices to verify the L^2 boundedness of the maximal Riesz transform. Since R_s^* is bounded in $L^2(\mu)$, we can estimate

$$\int_{\mathbb{R}^d} |R_s^*(f\mu\lfloor_A)|^2 \, \mathrm{d}\mu\lfloor_A \leq \int_{\mathbb{R}^d} |R_s^*(f\chi_A\mu)|^2 \, \mathrm{d}\mu \\ \leq \int_{\mathbb{R}^d} |f\chi_A|^2 \, \mathrm{d}\mu = \int_{\mathbb{R}^d} |f|^2 \, \mathrm{d}\mu\lfloor_A,$$

which justifies our assertion.

Proof of Proposition 4.1. Let μ be a nonnegative Borel measure supported in $E_1 \cup E_2$ that satisfies (a) and (b) in the definition of κ_s for this set. By the computation above, the measure $\mu\lfloor_{E_1}$ is admissible for $\kappa_s(E_1)$ and $\mu\lfloor_{E_2 \setminus E_1}$ is admissible for $\kappa_s(E_2)$. Thus,

$$\mu(\mathbb{R}^d) = \mu(E_1 \cup E_2) = \mu(E_1) + \mu(E_2 \setminus E_1) \leq \kappa_s(E_1) + \kappa_s(E_2)$$

and it suffices to take the supremum with respect to μ. □

It follows readily from (a) in the definition of κ_s that $\kappa_s(E) = 0$ for every compact set E with $\mathcal{H}^s(E) < \infty$. Dimension s is critical for κ_s :

Proposition 4.2. *If $E \subset \mathbb{R}^d$ is compact and $\kappa_s(E) = 0$, then $\dim_{\mathcal{H}}(E) \leq s$.*

Proof. It suffices to prove that if $E \subset \mathbb{R}^d$ is a compact set with $\mathcal{H}^t(E) > 0$ for some $t > s$, then $\kappa_s(E) > 0$. By Frostman's lemma, there exists a nontrivial finite nonnegative Borel measure μ supported by E such that

$$\mu(B_r(x)) \leq r^t \quad \text{for every } x \in \mathbb{R}^d \text{ and } r > 0. \tag{4.1}$$

Hence, $\mu \leq C_1 \mathcal{H}^t_\infty$, where $\mathcal{H}^t_\infty$ is the Hausdorff content of dimension t. For every set $A \subset \mathbb{R}^d$ with σ-finite $\mathcal{H}^s$ measure, we have $\mathcal{H}^t_\infty(A) = 0$. Then, $\mu(A) = 0$ and the first requirement in the definition of $\kappa_s(E)$ is satisfied.

Next, for every $f \in (L^1 \cap L^2)(\mu)$ and every $x \in \mathbb{R}^d$,

$$|R_s^*(f\mu)(x)| \leq \int_{\mathbb{R}^d} \frac{|f(y)|}{|x-y|^s} \,\mathrm{d}\mu(y).$$

Then, by Young's inequality,

$$\|R_s^*(f\mu)\|_{L^2(\mu)} \leq \left(\int_{\mathbb{R}^d} \frac{\mathrm{d}\mu(z)}{|z|^s}\right) \|f\|_{L^2(\mu)}.$$

By Cavalieri's principle,

$$\int_{\mathbb{R}^d} \frac{\mathrm{d}\mu(z)}{|z|^s} = s \int_0^\infty \frac{\mu(B_r(0))}{r^{s+1}} \,\mathrm{d}r.$$

Since μ is finite and satisfies (4.1) with exponent $t > s$,

$$\int_0^\infty \frac{\mu(B_r(0))}{r^{s+1}} \,\mathrm{d}r \leq \int_0^1 \frac{r^t}{r^{s+1}} \,\mathrm{d}r + \int_1^\infty \frac{\mu(\mathbb{R}^d)}{r^{s+1}} \,\mathrm{d}r < \infty.$$

Therefore,

$$\|R_s^*(f\mu)\|_{L^2(\mu)} \leq C_2 \|f\|_{L^2(\mu)},$$

which implies that μ/C_2 is a nontrivial admissible measure in the definition of $\kappa_s(E)$. In particular, $\kappa_s(E) > 0$. □

The proof of Theorem 1.2 relies on Theorem 1.6 of [6] concerning the existence of principal values for the Riesz transform. We recall that from [6] one knows that, for every finite nonnegative Borel measure μ in $\mathbb{R}^d$ such that

$$\lim_{r \to 0} \frac{\mu(B_r(x))}{r^s} = 0 \quad \text{for } \mu\text{-almost every } x \in \mathbb{R}^d \tag{4.2}$$

and

$$\|R_s^*(f\mu)\|_{L^2(\mu)} \leq C \|f\|_{L^2(\mu)} \quad \text{for every } f \in (L^1 \cap L^2)(\mu), \tag{4.3}$$

the principal value of

$$x \longmapsto \int_{\mathbb{R}^d} \frac{x-y}{|x-y|^{s+1}} \,\mathrm{d}\mu(y) \tag{4.4}$$

exists μ-almost everywhere.

We observe that in Theorem 1.6 of [6] one also assumes

$$\mu(B_r(x)) \le C' r^s \quad \text{for every } x \in \mathbb{R}^d \text{ and } r > 0,$$

but such a property is a consequence of the uniform boundedness of the Riesz transform given by (4.3) and the fact that, by (4.2), the measure μ cannot charge points; see Proposition 1.4 in Part III of [3].

We also need a standard property from Measure Theory:

Proposition 4.3. *If μ is a finite nonnegative Borel measure in $\mathbb{R}^d$ such that $\mu(A) = 0$ for every Borel set $A \subset \mathbb{R}^d$ with σ-finite $\mathcal{H}^s$ measure, then* (4.2) *holds.*

Proof. Since μ does not charge sets with finite $\mathcal{H}^s$ measure, by Proposition 3.2 in [7] for every $\epsilon > 0$ one finds a compact set $K \subset \mathbb{R}^d$ with $\mu(\mathbb{R}^d \setminus K) \le \epsilon$ such that, for every $c > 0$, there exists $\delta > 0$ with

$$\mu\lfloor_K(B_r(x)) \le c r^s \quad \text{for every } x \in \mathbb{R}^d \text{ and } 0 < r \le \delta.$$

Thus,

$$\lim_{r \to 0} \frac{\mu\lfloor_K(B_r(x))}{r^s} = 0 \quad \text{for every } x \in \mathbb{R}^d. \tag{4.5}$$

On the other hand, by the Besicovitch Differentiation Theorem,

$$\lim_{r \to 0} \frac{\mu\lfloor_K(B_r(x))}{\mu(B_r(x))} = 1 \quad \text{for } \mu\text{-almost every } x \in K. \tag{4.6}$$

Combining (4.5) and (4.6), one gets the limit in (4.2) for μ-almost every $x \in K$. Since $\mu(\mathbb{R}^d \setminus K) \le \epsilon$ and $\epsilon > 0$ is arbitrary, the conclusion follows. □

Proof of Theorem 1.2. Let μ be a finite nonnegative Borel measure that is admissible for $\kappa_s(E)$, where E is any compact subset of $\mathcal{E}(V_s)$. By Theorem 1.1, we can write this set as $E = E_1 \cup E_2$, where

$$E_1 := \left\{ x \in E : \limsup_{r \to 0} \frac{\mu(B_r(x))}{r^s} > 0 \right\}$$

and

$$E_2 := \left\{ x \in E : \mathrm{p.v.}\, V_s(x) \text{ does not exist} \right\}.$$

On one hand, since E_1 is σ-finite for $\mathcal{H}^s$ and μ does not charge those sets, $\mu(E_1) = 0$. On the other hand, from Proposition 4.3 above and Theorem 1.6 of [6] we know that the principal value of V_s exists μ-almost everywhere in $\mathbb{R}^d$. Hence, $\mu(E_2) = 0$. We thus have

$$\mu(\mathbb{R}^d) = \mu(E) \le \mu(E_1) + \mu(E_2) = 0,$$

for every measure μ that is admissible for $\kappa_s(E)$. Therefore, $\kappa_s(E) = 0$. □

As a final observation, (1.5) can also be deduced as a consequence of Theorem 1.2. Indeed, if we had $\dim \mathcal{E}(V_s) > s$, then for any $s < t < \dim \mathcal{E}(V_s)$ one could find a compact subset $F \subset \mathcal{E}(V_s)$ such that $\mathcal{H}^t(F) > 0$; see Theorem 8.13 in [5]. Such a property would then contradict Proposition 4.2 above.

ACKNOWLEDGEMENTS

The authors would like to thank L. Grafakos for an interesting correspondence. J. Cufí acknowledges partial support from grant 2017SGR358 (Generalitat de Catalunya) and J. Verdera from grants 2017SGR395 (Generalitat de Catalunya) and MTM2016–75390 (Ministerio de Educación y Ciencia). A. C. Ponce is grateful for the invitation, hospitality and support of the Departament de Matemàtiques of the Universitat Autònoma de Barcelona. He also acknowledges support of the Fonds de la Recherche scientifique (F.R.S.–FNRS) from grant J.0020.18.

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