COVERING DIMENSION OF CUNTZ SEMIGROUPS

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Abstract. We introduce a notion of covering dimension for Cuntz semigroups of C^*-algebras. This dimension is always bounded by the nuclear dimension of the C^*-algebra, and for subhomogeneous C^*-algebras both dimensions agree.

Cuntz semigroups of $\mathcal{Z}$-stable C^*-algebras have dimension at most one. Further, the Cuntz semigroup of a simple, $\mathcal{Z}$-stable C^*-algebra is zero-dimensional if and only if the C^*-algebra has real rank zero or is stably projectionless.

1. Introduction

The Cuntz semigroup of a C^*-algebra is a powerful invariant in the structure and classification theory of C^*-algebras. We define a notion of covering dimension for Cuntz semigroups, thus introducing a second-level invariant for C^*-algebras; see Definition 3.1. More generally, we define covering dimension for abstract Cuntz semigroups, usually called Cu-semigroups, as introduced in [CEI08] and extensively studied in [APT18, APT19, APT20, APRT18, APRT19].

Our definition really captures a notion of covering dimension: for every compact, metrizable space X, the Cu-semigroup $\mathrm{Lsc}(X,\overline{\mathbb{N}})$ of lower-semicontinuous functions $X \to \overline{\mathbb{N}} = \{0,1,2,\ldots,\infty\}$ has dimension agreeing with the covering dimension of X; see Example 3.4. More interestingly, we show that a similar result holds for Cuntz semigroups of commutative C^*-algebras:

Proposition A (4.3)**.** *Let X be a compact, Hausdorff space. Then*

$$\dim(\mathrm{Cu}(C(X))) = \dim(X).$$

We prove the expected permanence properties: The covering dimension does not increase when passing to ideals or quotients of a Cu-semigroup (Proposition 3.5); the covering dimension of a direct sum of Cu-semigroups is the maximum of the covering dimensions of the summands (Proposition 3.5); and if $S = \varinjlim_\lambda S_\lambda$ is an inductive limit of Cu-semigroups, then $\dim(\varinjlim_\lambda S_\lambda) \leq \liminf_\lambda \dim(S_\lambda)$ (Proposition 3.9).

In Section 4, we study the connection between the dimension of the Cuntz semigroup of a C^*-algebra and the nuclear dimension [WZ10] of the C^*-algebra.

Theorem B (4.1, 4.10)**.** *Every C^*-algebra A satisfies* $\dim(\mathrm{Cu}(A)) \leq \dim_{\mathrm{nuc}}(A)$. *If A is subhomogeneous, then* $\dim(\mathrm{Cu}(A)) = \dim_{\mathrm{nuc}}(A)$.

We note that $\dim(\mathrm{Cu}(A))$ can be strictly smaller than $\dim_{\mathrm{nuc}}(A)$. For example, the irrational rotation algebra A_θ satisfies $\dim(\mathrm{Cu}(A_\theta)) = 0$ while $\dim_{\mathrm{nuc}}(A_\theta) = 1$; see Example 4.11.

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The dimension of the Cuntz semigroup of a C^*-algebra A can also be computed in many situations of interest beyond the subhomogeneous case:

(1) If A has real rank zero, then $\dim(\mathrm{Cu}(A)) = 0$; see Proposition 5.4.
(2) If A is unital and of stable rank one, then $\dim(\mathrm{Cu}(A)) = 0$ if and only if A has real rank zero; see Corollary 5.8.
(3) If A is $\mathcal{W}$-stable[1], then $\dim(\mathrm{Cu}(A)) = 0$. If A is $\mathcal{Z}$-stable[2], then we have $\dim(\mathrm{Cu}(A)) \leq 1$; see Proposition 3.22.
(4) If A is purely infinite (not necessarily simple), then $\dim(\mathrm{Cu}(A)) = 0$; see Proposition 3.21.

Our results allow us to compute the dimension of the Cuntz semigroup of many simple C^*-algebras. In particular, by Corollary 5.9, if A is a separable, simple, $\mathcal{Z}$-stable C^*-algebra, then

$$\dim(\mathrm{Cu}(A)) = \begin{cases} 0, & \text{if } A \text{ has real rank zero or if } A \text{ is stably projectionless} \\ 1, & \text{otherwise.} \end{cases}$$

This should be compared to the computation of the nuclear dimension of a separable, simple C^*-algebra A as accomplished in [CET$^+$19, CE19]:

$$\dim_{\mathrm{nuc}}(A) = \begin{cases} 0, & \text{if } A \text{ is an AF-algebra} \\ 1, & \text{if } A \text{ is nuclear, } \mathcal{Z}\text{-stable, but not an AF-algebra} \\ \infty, & \text{of } A \text{ is nuclear and not } \mathcal{Z}\text{-stable, or non-nuclear} \end{cases}$$

It will be interesting to tackle the following problem:

Problem C. Compute the dimension of the Cuntz semigroups of simple C^*-algebras. In particular, what dimensions can occur (beyond zero and one)?

In the last two sections, we thoroughly investigate the class of countably based, simple, weakly cancellative Cu-semigroups S satisfying (O5) and (O6) (which includes the Cuntz semigroups of separable, simple C^*-algebras of stable rank one). We show that S is zero-dimensional if and only if S is algebraic (that is, the compact elements are sup-dense) or soft (that is, S contains no nonzero compact elements); see Lemma 7.1.

To describe the structure of S in the second case, we introduce the class of elements with *thin boundary* (see Definition 6.3), which turn out to play a similar role to that of the compact elements in the algebraic case. We show that an element x has thin boundary if and only if it is *complementable* in the sense that for every y satisfying $x \ll y$ there exists z such that $x + z = y$; see Theorem 6.12. Further, the elements with thin boundary form a cancellative monoid; see Theorem 6.13.

Theorem D (7.8)**.** *Let S be a countably based, simple, soft, weakly cancellative* Cu*-semigroup satisfying (O5) and (O6). Then S is zero-dimensional if and only if the elements with thin boundary are sup-dense.*

We finish Section 7 by briefly studying the relation between zero-dimensionality, almost divisibility and the Riesz interpolation property; see Proposition 7.13.

2. Preliminaries

Let a, b be two positive elements in a C^*-algebra A. Recall that a is said to be *Cuntz subequivalent* to b, in symbols $a \precsim b$, if there exists a sequence $(r_n)_n$ in A such that $a = \lim_n r_n b r_n^*$. One writes $a \sim b$ if $a \precsim b$ and $b \precsim a$, and denotes the class of $a \in A$ by $[a]$.

[1] A is $\mathcal{W}$-stable if $A \cong A \otimes \mathcal{W}$, for the Jacelon-Razac algebra $\mathcal{W}$

[2] A is $\mathcal{Z}$-stable if $A \cong A \otimes \mathcal{Z}$, for the Jiang-Su algebra $\mathcal{Z}$

The Cuntz semigroup of A, denoted by $\mathrm{Cu}(A)$, is defined as the quotient of $(A\otimes\mathcal{K})_+$ by the equivalence relation $\sim$. Endowed with the addition induced by $[a]+[b]=[\left(\begin{smallmatrix} a & 0\\ 0 & b\end{smallmatrix}\right)]$ and the order induced by $\precsim$, the Cuntz semigroup $\mathrm{Cu}(A)$ becomes a positively ordered monoid.

2.1. Given a pair of elements x,y in a partially ordered set, we say that x is *way-below* y, in symbols $x\ll y$, if for any increasing sequence $(y_n)_n$ for which the supremum exists and is greater than y one can find n such that $x\leq y_n$.

It was shown in [CEI08] that the Cuntz semigroup of any C^*-algebra satisfies the following properties:

- (O1) Every increasing sequence has a supremum.
- (O2) Every element can be written as the supremum of an $\ll$-increasing sequence.
- (O3) Given $x'\ll x$ and $y'\ll y$, we have $x'+y'\ll x+y$.
- (O4) Given increasing sequences $(x_n)_n$ and $(y_n)_n$, we have $\sup_n x_n+\sup_n y_n=\sup_n(x_n+y_n)$.

In a more abstract setting, any positively ordered monoid satisfying (O1)-(O4) is called a Cu-*semigroup*.

A map between two Cu-semigroups is called a *generalized* Cu-*morphism* if it is a positively ordered monoid homomorphism that preserves suprema of increasing sequences. We say that a generalized Cu-morphism is a Cu-*morphism* if it also preserves the way-below relation. Every *-homomorphism $A\to B$ between C^*-algebras induces a Cu-morphism $\mathrm{Cu}(A)\to\mathrm{Cu}(B)$; see [CEI08, Theorem 1].

We denote by Cu the category whose objects are Cu-semigroups and whose morphisms are Cu-morphisms.

The reader is referred to [CEI08] and [APT18] for a further detailed exposition.

2.2. In addition to (O1)-(O4), it was proved in [APT18, Proposition 4.6] and [Rob13] that the Cuntz semigroup of a C^*-algebra always satisfies the following additional properties:

- (O5) Given $x+y\leq z$, $x'\ll x$ and $y'\ll y$, there exists c such that $x'+c\leq z\leq x+c$ and $y'\ll c$.
- (O6) Given $x'\ll x\leq y+z$ there exist $v\leq x,y$ and $w\leq x,z$ such that $x'\leq v+w$.

Axiom (O5) is often used with $y=0$. In this case, it states that, given $x'\ll x\leq z$, there exists c such that $x'+c\leq z\leq x+c$.

Recall that a Cu-semigroup is said to be *weakly cancellative* if $x\ll y$ whenever $x+z\ll y+z$ for some element z. Stable rank one C^*-algebras have weakly cancellative Cuntz semigroups by [RW10, Theorem 4.3].

2.3. A subset $D\subseteq S$ in a Cu-semigroup S is said to be *sup-dense* if whenever $x',x\in S$ satisfy $x'\ll x$, there exists $y\in D$ with $x'\leq y\ll x$. Equivalently, every element in S is the supremum of an increasing sequence of elements in D.

We say that a Cu-semigroup is *countably based* if it contains a countable sup-dense subset. Cuntz semigroups of separable C*-algebras are countably based (see, for example, [APS11]).

3. Dimension of Cuntz semigroups

In this section we introduce a notion of covering dimension for Cu-semigroups and study some of its main permanence properties while providing a variety of examples; see Proposition 3.5, Proposition 3.9 and Proposition 3.15.

In Proposition 3.17 we investigate the relation between the dimension of a simple Cu-semigroup and its soft part, while in Proposition 3.20 we study how the dimension behaves in the presence of certain R-multiplications. This result is then applied to the Cuntz semigroups of purely infinite, $\mathcal{W}$-stable and $\mathcal{Z}$-stable C^*-algebras; see Proposition 3.21 and Proposition 3.22.

Definition 3.1. Let S be a Cu-semigroup. Given $n \in \mathbb{N}$, we write $\dim(S) \leq n$ if, whenever $x' \ll x \ll y_1 + \ldots + y_r$ in S, then there exist $z_{j,k} \in S$ for $j = 1, \ldots, r$ and $k = 0, \ldots, n$ such that:

(i) $z_{j,k} \ll y_j$ for each j and k;
(ii) $x' \ll \sum_{j,k} z_{j,k}$;
(iii) $\sum_{j=1}^{r} z_{j,k} \ll x$ for each $k = 0, \ldots, n$.

We set $\dim(S) = \infty$ if there exists no $n \in \mathbb{N}$ with $\dim(S) \leq n$. Otherwise, we let $\dim(S)$ be the smallest $n \in \mathbb{N}$ such that $\dim(S) \leq n$. We call $\dim(S)$ the *(covering) dimension* of S.

Remark 3.2. Recall that the *(covering) dimension* $\dim(X)$ of a topological space X is defined as the smallest $n \in \mathbb{N}$ such that every finite open cover of X admits a finite open refinement $\mathcal{V}$ such that at most $n + 1$ distinct elements in $\mathcal{V}$ have nonempty intersection; see for example [Pea75, Definition 3.1.1, p.111].

By [KW04, Proposition 1.5], a normal space X satisfies $\dim(X) \leq n$ if and only if every finite open cover of X admits a finite open refinement $\mathcal{V}$ that is $(n + 1)$-*colorable*, that is, there is a decomposition $\mathcal{V} = \mathcal{V}_0 \sqcup \ldots \sqcup \mathcal{V}_n$ such that the sets in $\mathcal{V}_j$ are pairwise disjoint for $j = 0, \ldots, n$. (The sets in $\mathcal{V}_j$ have color j, and sets of the same color are disjoint.)

Definition 3.1 is modeled after the above characterization of covering dimension in terms of colorable refinements. We interpret the expression '$x \ll y_1 + \ldots + y_r$' as saying that x is 'covered' by $\{y_1, \ldots, y_r\}$. Then, condition (i) from Definition 3.1 means that $\{z_{j,k}\}$ is a 'refinement' of $\{y_1, \ldots, y_r\}$; condition (ii) means that $\{z_{j,k}\}$ is a cover of x' (which is an approximation of x); and condition (iii) means that $\{z_{j,k}\}$ is $(n + 1)$-colorable.

In Definition 3.1, some of the $\ll$-relations may be changed for $\leq$.

Lemma 3.3. *Let S be a* Cu*-semigroup and $n \in \mathbb{N}$. Then we have $\dim(S) \leq n$ if and only if, whenever $x' \ll x \ll y_1 + \ldots + y_r$ in S, there exist $z_{j,k} \in S$ for $j = 1, \ldots, r$ and $k = 0, \ldots, n$ such that:*

(1) $z_{j,k} \leq y_j$ for each j and k;
(2) $x' \leq \sum_{j,k} z_{j,k}$;
(3) $\sum_{j=1}^{r} z_{j,k} \leq x$ for each $k = 0, \ldots, n$.

Proof. The forward implication is clear. To show the converse, let $x' \ll x \ll y_1 + \ldots + y_r$ in S. Choose $s', s, y_1', \ldots, y_r' \in S$ such that

$$x' \ll s' \ll s \ll x \ll y_1' + \ldots + y_r', \quad y_1' \ll y_1, \quad \ldots, \quad \text{and} \quad y_r' \ll y_r.$$

Applying the assumption, we obtain elements $z_{j,k}$ for $j = 1, \ldots, r$ and $k = 0, \ldots, n$ satisfying properties (1)-(3) for $s' \ll s \ll y_1' + \ldots + y_r'$. Then the same elements satisfy (i)-(iii) in Definition 3.1 for $x' \ll x \ll y_1 + \ldots + y_r$, thus verifying $\dim(S) \leq n$. □

Example 3.4. Let X be a compact, metrizable space. We use $\mathrm{Lsc}(X, \overline{\mathbb{N}})$ to denote the set of functions $f \colon X \to \overline{\mathbb{N}}$ that are lower-semicontinuous, that is, for each $n \in \mathbb{N}$ the set $f^{-1}(\{n, n + 1, \ldots, \infty\}) \subseteq X$ is open. We equip $\mathrm{Lsc}(X, \overline{\mathbb{N}})$ with pointwise addition and order. Then $\mathrm{Lsc}(X, \overline{\mathbb{N}})$ is Cu-semigroup. We have

$$\dim(\mathrm{Lsc}(X, \overline{\mathbb{N}})) = \dim(X).$$

We will omit the elaborate verification of the inequality '$\leq$' since it follows from the computation of $\dim(C(X))$; see Corollary 4.4.

Let us prove the inequality '$\geq$'. Set $n := \dim(\mathrm{Lsc}(X,\overline{\mathbb{N}}))$, which we may assume to be finite. To verify that $\dim(X) \leq n$, let $\mathcal{U} = \{U_1, \ldots, U_r\}$ be a finite open cover of X. We need to find a $(n+1)$-colourable, finite, open refinement of $\mathcal{U}$.

We use χ_U to denote the characteristic function of a subset $U \subseteq X$. Then

$$\chi_X \ll \chi_X \ll \chi_{U_1} + \ldots + \chi_{U_r}.$$

Applying that $\dim(\mathrm{Lsc}(X,\overline{\mathbb{N}})) \leq n$, we obtain elements $z_{j,k} \in \mathrm{Lsc}(X,\overline{\mathbb{N}})$ for $j = 1, \ldots, r$ and $k = 0, \ldots, n$ such that

(i) $z_{j,k} \ll \chi_{U_j}$ for every j, k;
(ii) $\chi_X \ll \sum_{j,k} z_{j,k}$;
(iii) $\sum_j z_{j,k} \ll \chi_X$ for every k.

For each j and k, condition (i) implies that $z_{j,k} = \chi_{V_{j,k}}$ for some open subset $V_{j,k} \subseteq U_j$. Condition (ii) implies that X is covered by the sets $V_{j,k}$. Thus, the family $\mathcal{V} := \{V_{j,k}\}$ is a finite, open refinement of $\mathcal{U}$. For each k, condition (iii) implies that the sets $V_{1,k}, \ldots, V_{r,k}$ are pairwise disjoint. Thus, $\mathcal{V}$ is $(n+1)$-colourable, as desired.

Recall that an *ideal* I of a Cu-semigroup S is a downward-hereditary submonoid closed under suprema of increasing sequences; see [APT18, Section 5].

Given $x, y \in S$, we write $x \leq_I y$ if there exists $z \in I$ such that $x \leq y + z$. We set $x \sim_I y$ if $x \leq_I y$ and $y \leq_I x$. The quotient $S/\sim_I$ endowed with the induced sum and order $\leq_I$ is denoted by S/I.

As shown in [APT18, Lemma 5.1.2], S/I is a Cu-semigroup and the quotient map $S \to S/I$ is a Cu-morphism.

Proposition 3.5. *Let S be a* Cu*-semigroup, and let $I \subseteq S$ be an ideal. Then:*

$$\dim(I) \leq \dim(S), \quad \textit{and} \quad \dim(S/I) \leq \dim(S).$$

Proof. Set $n := \dim(S)$, which we may assume to be finite, since otherwise there is nothing to prove. It is straightforward to show that $\dim(I) \leq n$ using that I is downward-hereditary. Given $x \in S$, we use $[x]$ to denote its equivalence class in S/I.

To verify $\dim(S/I) \leq n$, let $[u] \ll [x] \ll [y_1] + \ldots + [y_r]$ in S/I. Then there exists $y_{r+1} \in I$ such that $x \leq y_1 + \ldots + y_r + y_{r+1}$ in S. Using that the quotient map $S \to S/I$ preserves suprema of increasing sequences, we can choose $x'', x' \in S$ such that

$$x'' \ll x' \ll x, \quad \text{and} \quad [u] \leq [x''].$$

Applying the definition of $\dim(S) \leq n$ to $x'' \ll x' \ll y_1 + \ldots + y_r + y_{r+1}$, we obtain elements $z_{j,k} \in S$ for $j = 1, \ldots, r+1$ and $k = 0, \ldots, n$ such that $z_{j,k} \ll y_j$ for every j, k, such that $x'' \ll \sum_{j,k} z_{j,k}$, and such that $\sum_j z_{j,k} \ll x'$ for every k.

Since $y_{r+1} \in I$, we have $z_{r+1,k} \in I$ and thus $[z_{r+1,k}] = 0$ in S/I for $k = 0, \ldots, n$. Using also that the quotient map $S \to S/I$ is $\ll$-preserving, we see that the elements $[z_{j,k}]$ for $j = 1, \ldots, r$ and $k = 0, \ldots, n$ have the desired properties. □

Problem 3.6. Let S be a Cu-semigroup, and let $I \subseteq S$ be an ideal. Can we bound $\dim(S)$ in terms of $\dim(I)$ and $\dim(S/I)$? In particular, do we always have $\dim(S) \leq \dim(I) + \dim(S/I) + 1$?

Given Cu-semigroups S and T, we use $S \oplus T$ to denote the Cartesian product $S \times T$ equipped with elementwise addition and order. It is straightforward to verify that $S \oplus T$ is a Cu-semigroup and that $S \oplus T$ is both the product and coproduct of S and T in the category Cu; see also [APT19, Proposition 3.10]. We omit the straightforward proof of the next result.

Proposition 3.7. *Let S and T be* Cu*-semigroups. Then*

$$\dim(S \oplus T) = \max\{\dim(S), \dim(T)\}.$$

3.8 (Inductive limits)**.** By [APT18, Corollary 3.1.11], the category Cu has inductive limits. (The sequential case was previously shown in [CEI08, Theorem 2].)

To recall the construction, let $((S_\lambda)_{\lambda\in\Lambda}, (\varphi_{\mu,\lambda})_{\lambda\leq\mu \text{ in } \Lambda})$ be a directed system in Cu, that is, Λ is a directed set, each S_λ is a Cu-semigroup, and for $\lambda \leq \mu$ in Λ we have a connecting Cu-morphism $\varphi_{\mu,\lambda}\colon S_\lambda \to S_\mu$ such that $\varphi_{\lambda,\lambda} = \mathrm{id}_{S_\lambda}$ for each $\lambda \in \Lambda$ and $\varphi_{\nu,\mu} \circ \varphi_{\mu,\lambda} = \varphi_{\nu,\lambda}$ for all $\lambda \leq \mu \leq \nu$ in Λ.

It is shown in [APT19, Theorem 2.9] that Cu is a full, reflective subcategory of a more algebraic category W defined in [APT19, Definition 2.5]. The inductive limit in Cu can therefore be constructed by applying the reflection functor W $\to$ Cu to the inductive limit in W.

Consider the equivalence relation $\sim$ on the disjoint union $\bigsqcup_\lambda S_\lambda$ given by $x_\lambda \sim x_\mu$ (for $x_\lambda \in S_\lambda$ and $x_\mu \in S_\mu$) if there exists $\nu \geq \lambda, \mu$ such that $\varphi_{\nu,\lambda}(x_\lambda) = \varphi_{\nu,\mu}(x_\mu)$. The set of equivalence classes is the set-theoretic inductive limit, which we denote by S_{alg}. We write $[x_\lambda]$ for the equivalence class of $x_\lambda \in S_\lambda$.

We define an addition $+$ and a binary relation $\prec$ on S_{alg} as follows: Given $x_\lambda \in S_\lambda$ and $x_\mu \in S_\mu$, set

$$[x_\lambda] + [x_\mu] := [\varphi_{\nu,\lambda}(x_\lambda) + \varphi_{\nu,\mu}(x_\mu)]$$

for any $\nu \geq \lambda, \mu$. Further, set $[x_\lambda] \prec [x_\mu]$ if there exists $\nu \geq \lambda, \mu$ such that $\varphi_{\nu,\lambda}(x_\lambda) \ll \varphi_{\nu,\mu}(x_\mu)$ in S_ν. This gives S_{alg} the structure of a W-semigroup, which together with the natural maps $S_\lambda \to S_{\mathrm{alg}}, x_\lambda \mapsto [x_\lambda]$, is the inductive limit in W.

The reflection of S_{alg} in Cu is a Cu-semigroup S together with a (universal) W-morphism $\alpha\colon S_{\mathrm{alg}} \to S$. Using [APT18, Theorem 3.1.8], S and α are characterized by the following conditions:

- (R1) α is an embedding in the sense that $[x_\lambda] \prec [x_\mu]$ if (and only if) $\alpha([x_\lambda]) \ll \alpha([x_\mu])$, for any $x_\lambda \in S_\lambda$ and $x_\mu \in S_\mu$;
- (R2) α has dense image in the sense that for all $x', x \in S$ satisfying $x' \ll x$ there exists $x_\lambda \in S_\lambda$ such that $x' \ll \alpha([x_\lambda]) \ll x$.

It follows that a Cu-semigroup S together with Cu-morphisms $\varphi_\lambda\colon S_\lambda \to S$ for $\lambda \in \Lambda$ is the inductive limit in Cu of the system $((S_\lambda)_{\lambda\in\Lambda}, (\varphi_{\mu,\lambda})_{\lambda\leq\mu \text{ in } \Lambda})$ if and only if the following conditions are satisfied:

- (L0) we have $\varphi_\mu \circ \varphi_{\mu,\lambda} = \varphi_\lambda$ for all $\lambda \leq \mu$ in Λ;
- (L1) if $x_\lambda \in S_\lambda$ and $x_\mu \in S_\mu$ satisfy $\varphi_\lambda(x_\lambda) \ll \varphi_\mu(x_\mu)$, then there exists $\nu \geq \lambda, \mu$ such that $\varphi_{\nu,\lambda}(x_\lambda) \ll \varphi_{\nu,\mu}(x_\mu)$;
- (L2) for all $x', x \in S$ satisfying $x' \ll x$ there exists $x_\lambda \in S_\lambda$ such that $x' \ll \varphi_\lambda(x_\lambda) \ll x$.

Proposition 3.9. *Let $S = \varinjlim_{\lambda\in\Lambda} S_\lambda$ be an inductive limit of* Cu*-semigroups. Then $\dim(S) \leq \liminf_\lambda \dim(S_\lambda)$.*

Proof. Let $\varphi_\lambda\colon S_\lambda \to S$ be the Cu-morphisms into the inductive limit. We use that S and the φ_λ's satisfy (L0)-(L2) from Paragraph 3.8. Set $n := \liminf_\lambda \dim(S_\lambda)$, which we may assume to be finite. To verify $\dim(S) \leq n$, let $x' \ll x \ll y_1 + \ldots + y_r$ in S. Choose $y_1', \ldots, y_r' \in S$ such that

$$x \ll y_1' + \ldots + y_r', \quad y_1' \ll y_1, \quad \ldots, \quad \text{and} \quad y_r' \ll y_r.$$

Using (L2), we obtain $a_\lambda \in S_\lambda$ such that $x' \ll \varphi_\lambda(a_\lambda) \ll x$. Analogously, we obtain $b_{\lambda_k} \in S_{\lambda_k}$ such that $y_k' \ll \varphi_{\lambda_k}(b_{\lambda_k}) \ll y_k$ for $k = 1, \ldots, r$.

Using that φ_λ is a Cu-morphism, we obtain $a_\lambda' \in S_\lambda$ such that

$$x' \ll \varphi_\lambda(a_\lambda') \ll \varphi_\lambda(a_\lambda) \ll x, \quad \text{and} \quad a_\lambda' \ll a_\lambda.$$

Choose $\mu \in \Lambda$ such that $\mu \geq \lambda, \lambda_1, \ldots, \lambda_r$, and set

$$a' := \varphi_{\mu,\lambda}(a'_\lambda), \quad a := \varphi_{\mu,\lambda}(a_\lambda), \quad b_1 := \varphi_{\mu,\lambda_1}(b_{\lambda_1}), \quad \ldots, \quad \text{and} \quad b_r := \varphi_{\mu,\lambda_r}(b_{\lambda_r}).$$

Hence,

$$\varphi_\mu(a) = \varphi_\mu(\varphi_{\mu,\lambda}(a_\lambda)) = \varphi_\lambda(a_\lambda) \ll x \ll y'_1 + \ldots + y'_r \ll \varphi_\mu(b_1 + \ldots + b_r).$$

Applying (L1), we obtain $\nu \geq \mu$ such that $\varphi_{\nu,\mu}(a) \ll \varphi_{\nu,\mu}(b_1 + \ldots + b_r)$.

Using that $\liminf_\lambda \dim(S_\lambda) \leq n$, we may also assume that $\dim(S_\nu) \leq n$. Applying $\dim(S_\nu) \leq n$ to

$$\varphi_{\nu,\mu}(a') \ll \varphi_{\nu,\mu}(a') \ll \varphi_{\nu,\mu}(b_1) + \ldots + \varphi_{\nu,\mu}(b_r),$$

we obtain elements $z_{j,k} \in S_\nu$ for $j = 1, \ldots, r$ and $k = 0, \cdots, n$ satisfying properties (i)-(iii) from Definition 3.1. It is now easy to check that the elements $\varphi_\nu(z_{j,k}) \in S$ have the desired properties to verify $\dim(S) \leq n$. □

Proposition 3.10. *Given a C^*-algebra A and a (closed, two-sided) ideal $I \subseteq A$, we have*

$$\dim(\mathrm{Cu}(I)) \leq \dim(\mathrm{Cu}(A)), \quad \textit{and} \quad \dim(\mathrm{Cu}(A/I)) \leq \dim(\mathrm{Cu}(A)).$$

Given C^*-algebras A and B, we have

$$\dim(\mathrm{Cu}(A \oplus B)) = \max\{\dim(\mathrm{Cu}(A)), \dim(\mathrm{Cu}(B))\}.$$

Given an inductive limit of C^*-algebras $A = \varinjlim_\lambda A_\lambda$, we have

$$\dim(\mathrm{Cu}(A)) \leq \liminf_\lambda \dim(\mathrm{Cu}(A_\lambda)).$$

Proof. The first statement follows from Proposition 3.5 using that $\mathrm{Cu}(I)$ is naturally isomorphic to an ideal of $\mathrm{Cu}(A)$, and that $\mathrm{Cu}(A/I)$ is naturally isomorphic to $\mathrm{Cu}(A)/\mathrm{Cu}(I)$; see [APT18, Section 5.1]. The second statement follows from Proposition 3.7 using that $\mathrm{Cu}(A \oplus B)$ is isomorphic to $\mathrm{Cu}(A) \oplus \mathrm{Cu}(B)$. Finally, the third statements follows from Proposition 3.9 and the fact that the Cuntz semigroup of an inductive limit of C^*-algebras is naturally isomorphic to the inductive limit of the C^*-algebras; see [APT18, Corollary 3.2.9]. □

Example 3.11. Recall that $\mathrm{Cu}(\mathbb{C})$ is naturally isomorphic to $\overline{\mathbb{N}} := \{0, 1, 2, \ldots, \infty\}$. We say that a Cu-semigroup S is *simplicial* if $S \cong \overline{\mathbb{N}}^k = \overline{\mathbb{N}} \oplus \ldots_k \oplus \overline{\mathbb{N}}$ for some $k \geq 1$. If A is a finite-dimensional C^*-algebra, then $\mathrm{Cu}(A)$ is simplicial.

It is easy to verify that $\dim(\overline{\mathbb{N}}) = 0$. By Proposition 3.7, we get $\dim(\overline{\mathbb{N}}^k) = 0$ for every $k \geq 1$. Thus, if S is an inductive limit of simplicial Cu-semigroups, then $\dim(S) = 0$ by Proposition 3.9. Further, it follows from Proposition 3.10 that $\dim(\mathrm{Cu}(A)) = 0$ for every AF-algebra A. In Proposition 5.4, we will generalize this to C^*-algebras of real rank zero (which include all AF-algebras).

By applying the Cu-semigroup version of the Effros-Handelman-Shen theorem, [APT18, Corollary 5.5.13], it also follows that every countably-based, weakly cancellative, unperforated, algebraic Cu-semigroup satisfying (O5) and (O6) is zero-dimensional. In Corollary 5.3, we will generalize this to weakly cancellative, algebraic Cu-semigroups satisfying (O5) and (O6).

Example 3.12. Recall that a Cu-semigroup is said to be *elementary* if it is isomorphic to $\{0\}$, or if it is simple and contains a minimal nonzero element; see [APT18, Paragraph 5.1.16]. Typical examples of elementary Cu-semigroups are $\overline{\mathbb{N}}$ and $E_k = \{0, 1, 2, \ldots, k, \infty\}$ for $k \in \mathbb{N}$, where the sum of two elements in E_k is defined as ∞ if their usual sum would exceed k; see [APT18, Paragraph 5.1.16]. By [APT18, Proposition 5.1.19], these are the only elementary Cu-semigroups that satisfy (O5) and (O6).

It is easy to see that every elementary Cu-semigroup satisfying (O5) and (O6) is zero-dimensional. In Example 3.13 below, we show that this is no longer the case without (O5). To see that (O6) is also necessary, consider $S := \overline{\mathbb{N}} \cup \{1'\}$, with $1'$ a compact element not comparable with 1 and such that $1'+1' = 2$ and $1+k = 1'+k$ for every $k \in \overline{\mathbb{N}} \setminus \{0\}$. We claim that $\dim(S) = \infty$.

Assume, for the sake of contradiction, that $\dim(S) \leq n$ for some $n \in \mathbb{N}$. Then, since $1' \ll 1' \ll 2 = 1+1$, there exist elements $z_{1,k}, z_{2,k} \in S$ for $k = 0, \ldots, n$ satisfying conditions (i)-(iii) from Definition 3.1. By condition (i), we have $z_{j,k} \ll 1$ and therefore $z_{j,k} = 0$ or $z_{j,k} = 1$ for every j,k. By condition (ii), we have $1' \ll \sum_{j,k} z_{j,k}$, and so there exist $j' \in \{1,2\}$ and $k' \in \{0, \ldots, n\}$ such that $z_{j',k'} = 1$. However, by condition (iii), we have $z_{j',k'} \ll 1'$, which is a contradiction because the elements 1 and $1'$ are not comparable.

Example 3.13. Let $k, l \in \mathbb{N}$, and let E_k and E_l be the elementary Cu-semigroups as in Example 3.12. Then the abstract bivariant Cu-semigroup $[\![E_k, E_l]\!]$, as defined in [APT20], has dimension one whenever $l > k$ and dimension zero otherwise.

Indeed, by [APT20, Proposition 5.18], we know that $[\![E_k, E_l]\!] = \{0, r, \ldots, l, \infty\}$ with $r = \lceil (l+1)/(k+1) \rceil$. Thus, if $l \leq k$, then $[\![E_k, E_l]\!] = E_l$, which is zero-dimensional by Example 3.12. Note that $[\![E_k, E_l]\!]$ is an elementary Cu-semigroup satisfying (O6). Further, $[\![E_k, E_l]\!]$ satisfies (O5) if and only if $l \leq k$.

Let us now assume that $l > k$, that is $r > 1$. Then, even though $r+1 \ll r+1 \ll r+r$, one cannot find $z_1, z_2 \ll r$ such that $r+1 = z_1 + z_2$. This shows that $\dim([\![E_k, E_l]\!]) \neq 0$.

To verify $\dim([\![E_k, E_l]\!]) \leq 1$, let $x \ll x \ll y_1 + \ldots + y_r$ in $[\![E_k, E_l]\!]$. We may assume that y_j is nonzero for every j. If there exists $i \in \{1, \ldots, r\}$ with $x \leq y_i$, then $z_{i,0} := x$ and $z_{j,k} := 0$ for $j \neq i$ or $k = 1$ have the desired properties.

So we may assume that $y_j < x$ for every j. Let k be the least integer such that $x \leq y_1 + \ldots + y_k$. Define $z_{j,0} := y_j$ for every $j < k$ and $z_{j,0} := 0$ for $j \geq k$. Further, define $z_{k,1} := y_k$ and $z_{j,1} := 0$ for $j \neq k$. By choice of k, we have $\sum_j z_{j,0} \ll x$. We also have $\sum_j z_{j,1} = y_k \ll x$. Finally, $x \ll \sum_j z_{j,0} + \sum_j z_{j,1}$, as desired.

Definition 3.14. Let S and T be Cu-semigroups. We say that S is a *retract* of T if there exist a Cu-morphism $\iota\colon S \to T$ and a generalized Cu-morphism $\sigma\colon T \to S$ such that $\sigma \circ \iota = \mathrm{id}_S$.

Many properties of Cu-semigroups pass to retracts. In Lemma 7.12 we show this for the Riesz interpolation property and for almost divisibility. The next result shows that the dimension does not increase when passing to a retract.

Proposition 3.15. *Let S and T be* Cu*-semigroups and assume that S is a retract of T. Then $\dim(S) \leq \dim(T)$.*

Proof. Let $\iota\colon S \to T$ be a Cu-morphism, and let $\sigma\colon T \to S$ be a generalized Cu-morphism such that $\sigma \circ \iota = \mathrm{id}_S$. Set $n := \dim(T)$, which we may assume to be finite. To verify the assumptions of Lemma 3.3, let $x' \ll x \ll y_1 + \ldots + y_r$ in S. Then

$$\iota(x') \ll \iota(x) \ll \iota(y_1) + \ldots + \iota(y_r)$$

in T. Using that $\dim(T) \leq n$, we obtain elements $z_{j,k}$ in T satisfying conditions (i)-(iii) of Definition 3.1. Applying σ, we see that the elements $\sigma(z_{j,k})$ satisfy conditions (1)-(3) in Lemma 3.3, from which the result follows. ☐

Given a simple Cu-semigroup S, let us now show that its sub-Cu-semigroup of soft elements S_{soft}, as defined in Paragraph 6.1, is a retract of S. As we will see in Proposition 3.17 below, such elements play an important role in the study of the dimension of S.

Proposition 3.16. *Let S be a countably based, simple, weakly cancellative* Cu-*semigroup satisfying (O5) and (O6). Then S_{soft} is a retract of S.*

Proof. By [APT18, Proposition 5.3.18], S_{soft} is a Cu-semigroup. By [Thi20b, Proposition 2.9], for each $x \in S$ there exists a (unique) maximal soft element dominated by x and the map $\sigma\colon S \to S_{\mathrm{soft}}$ given by

$$\sigma(x) := \max\left\{x' \in S_{\mathrm{soft}} : x' \leq x\right\}, \quad \text{for } x \in S,$$

is a generalized Cu-morphism. Further, the inclusion $\iota\colon S_{\mathrm{soft}} \to S$ is a Cu-morphism and the composition $\sigma \circ \iota$ is the identity on S_{soft}, as desired. □

Proposition 3.17. *Let S be a countably based, simple, weakly cancellative* Cu-*semigroup satisfying (O5) and (O6). Then*

$$\dim(S_{\mathrm{soft}}) \leq \dim(S) \leq \dim(S_{\mathrm{soft}}) + 1.$$

Proof. The first inequality follows from Propositions 3.15 and 3.16. To show the second inequality, set $n := \dim(S_{\mathrm{soft}})$, which we may assume to be finite. If S is elementary, then $\dim(S) = 0$ as noted in Example 3.12. Thus, we may assume that S is nonelementary. By [APT18, Proposition 5.3.16], every nonzero element of S is either soft or compact. To verify $\dim(S) \leq n+1$, let $x' \ll x \ll y_1 + \ldots + y_r$ in S. We may assume that x and y_1 are nonzero. If x is soft, then we let $s', s \in S$ be any pair of soft elements satisfying $x' \ll s' \ll s \ll x$. If x is compact, then we apply Lemma 6.4 to obtain a nonzero element $w \in S$ satisfying $w \leq x, y_1$. Then $x \ll \sigma(x) + w$, which allows us to choose soft elements $s' \ll s$ such that $s \ll \sigma(x)$ and $x \ll s' + w$. In both cases, we have

$$s' \ll s \ll \sigma(x) \leq \sigma(y_1) + \ldots + \sigma(y_r)$$

in S_{soft}. Using that $\dim(S_{\mathrm{soft}}) \leq n$, we obtain (soft) elements $z_{j,k} \in S$ for $j = 1, \ldots, r$ and $k = 0, \ldots, n$ such that

(i) $z_{j,k} \ll \sigma(y_j)$ (and thus, $z_{j,k} \ll y_j$) for each j and k;
(ii) $s' \ll \sum_{j,k} z_{j,k}$;
(iii) $\sum_{j=1}^r z_{j,k} \ll s \ll \sigma(x)$ (and thus, $\sum_{j=1}^r z_{j,k} \ll x$) for each $k = 0, \ldots, n$.

If x is soft, then $x' \ll s' \ll \sum_{j,k} z_{j,k}$, which shows that the elements $z_{j,k}$ have the desired properties. If x is compact, then set $z_{1,n+1} := w$ and $z_{j,n+1} := 0$ for $j = 2, \ldots, r$. Then $z_{j,n+1} \ll y_j$ for each j. Further,

$$x' \ll x \ll s' + w \leq \left(\sum_{k=0}^{n}\sum_{j=1}^{r} z_{j,k}\right) + \sum_{j=1}^{r} z_{j,n+1} = \sum_{k=0}^{n+1}\sum_{j=1}^{r} z_{j,k}.$$

Lastly, $\sum_{j=1}^r z_{j,n+1} = w \ll x$, which shows that the elements $z_{j,k}$ have the desired properties. □

Remark 3.18. Proposition 3.17 applies in particular to the Cuntz semigroups of separable, simple C^*-algebras of stable rank one (see [Rob13, Proposition 5.1.1]). More generally, Engbers showed in [Eng14] that for every separable, simple, stably finite C^*-algebra A, every compact element in $\mathrm{Cu}(A)$ has a predecessor. The proof of Proposition 3.17 can be generalized to this situation and we obtain

$$\dim(\mathrm{Cu}(A)_{\mathrm{soft}}) \leq \dim(\mathrm{Cu}(A)) \leq \dim(\mathrm{Cu}(A)_{\mathrm{soft}}) + 1.$$

Example 3.19. Let $Z = \mathrm{Cu}(\mathcal{Z})$, the Cuntz semigroup of the Jiang-Su algebra $\mathcal{Z}$. Then, $\dim(Z) = 1$. Indeed, since Z is a simple, weakly cancellative Cu-semigroup satisfying (O5) that is neither algebraic nor soft, it follows from Lemma 7.1 that $\dim(Z) > 0$. On the other hand, we have $Z_{\mathrm{soft}} \cong [0,\infty]$, and it is easy to verify that $\dim([0,\infty]) = 0$. Therefore, we have $\dim(Z) \leq \dim([0,\infty]) + 1 = 1$ by Proposition 3.17.

A similar argument shows that $\dim(Z') = 1$, where Z' is the Cu-semigroup considered in [APT18, Question 9(8)], that is, $Z' := Z \cup \{1''\}$ with $1''$ a compact element not comparable with 1 and such that $1'' + 1'' = 2$ and $1 + x = 1'' + x$ for every $x \in Z \setminus \{0\}$.

The notion of R-multiplication on a Cu-semigroup for a Cu-semiring R was introduced in [APT18, Definition 7.1.3]. Given a solid Cu-semiring R (such as $\{0, \infty\}$, $[0, \infty]$ or Z), any two R-multiplications on a Cu-semigroup are equal, and therefore having an R-multiplication is a property; see [APT18, Remark 7.1.9].

It was shown in [APT18, Theorem 7.2.2] that a Cu-semigroup has $\{0, \infty\}$-multiplication if and only if every element in the semigroup is idempotent. By [APT18, Theorem 7.3.8], a Cu-semigroup has Z-multiplication if and only if it is almost unperforated and almost divisible. By [APT18, Theorem 7.5.4], a Cu-semigroup has $[0, \infty]$-multiplication if and only if it has Z-multiplication and every element in S is soft.

Proposition 3.20. *Let S be a* Cu*-semigroup satisfying (O5) and (O6). Then:*

(1) If S has $\{0, \infty\}$-multiplication, then $\dim(S) = 0$.
(2) If S has $[0, \infty]$-multiplication, then $\dim(S) = 0$.
(3) If S has Z-multiplication, then $\dim(S) \leq 1$.

Proof. (1) Given elements $x' \ll x \ll y_1 + \ldots + y_r$ in a Cu-semigroup with $\{0, \infty\}$-multiplication, apply (O6) to obtain elements $z_j \leq x, y_j$ such that

$$x' \leq z_1 + \ldots + z_r.$$

Using that every element in S is idempotent, one also has

$$z_1 + \ldots + z_r \leq x + \ldots_r + x = rx = x.$$

This shows that the elements z_j satisfy the conditions in Lemma 3.3, as required.

(2) Note that S is isomorphic to its realification S_R by Theorem 7.5.4 and Proposition 7.5.9 in [APT18]. We can now use the decomposition property of S_R proven in [Rob13, Theorem 4.1.1] to deduce that S is zero-dimensional.

(3) Assume that S has Z-multiplication. By [APT18, Proposition 7.3.13], an element $x \in S$ is soft if and only if $x = 1'x$ (where $1'$ denotes the soft one in Z). Further, the Cu-semigroup $S_{\mathrm{soft}} := 1'S$ of soft elements in S is isomorphic to the realification of S; see [APT18, Corollary 7.5.10]. Since the realification of S has $[0, \infty]$-multiplication, we get $\dim(S_{\mathrm{soft}}) = 0$ by (2).

To verify $\dim(S) \leq 1$, let $x' \ll x \ll y_1 + \ldots + y_r$ in S. Using that S has Z-multiplication, one gets

$$\tfrac{5}{8}x' \ll \tfrac{6}{8}x \ll \tfrac{7}{8}y_1 + \ldots + \tfrac{7}{8}y_r.$$

Note that all elements in the previous expression belong to S_{soft}. Since $\dim(S_{\mathrm{soft}}) = 0$, we obtain (soft) elements $z_1, \ldots, z_r \in S$ such that $z_j \ll \tfrac{7}{8}y_j$ for each j, and such that

$$\tfrac{5}{8}x' \ll z_1 + \ldots + z_r \ll \tfrac{6}{8}x.$$

Define $z_{j,0} := z_j$ and $z_{j,1} := z_j$ for $j = 1, \ldots, r$. We trivially have $z_{j,k} \ll y_j$ for each j and k. Further,

$$x' \leq \tfrac{10}{8}x' \ll 2(z_1 + \cdots + z_r) = \sum_{j,k} z_{j,k},$$

and

$$\sum_j z_{j,k} \ll \tfrac{6}{8}x \leq x.$$

for each $k = 0, 1$, as desired. ∎

Let A be a C^*-algebra. Then, we know from [APT18, Proposition 7.2.8] that A is purely infinite if and only if $\mathrm{Cu}(A)$ has $\{0,\infty\}$-multiplication.

Proposition 3.21. *Let A be a purely infinite C^*-algebra. Then* $\dim(A)=0$.

Let $\mathcal{W}$ denote the Jacelon-Racak algebra. Given a C^*-algebra A, it follows from [APT18, Proposition 7.6.3] that $\mathrm{Cu}(A\otimes\mathcal{W})$ has $[0,\infty]$-multiplication, and that $\mathrm{Cu}(A\otimes\mathcal{Z})$ has Z-multiplication.

Proposition 3.22. *Let A be a C^*-algebra. Then*

$$\dim(\mathrm{Cu}(A\otimes\mathcal{W}))=0, \quad \textit{and} \quad \dim(\mathrm{Cu}(A\otimes\mathcal{Z}))\leq 1.$$

In particular, Cuntz semigroups of $\mathcal{W}$-stable C^*-algebras are zero-dimensional, and Cuntz semigroups of $\mathcal{Z}$-stable C^*-algebras have dimension at most one.

Example 3.23. Let X be a compact, metrizable space containing at least two points, and let $S:=\mathrm{Lsc}(X,\overline{\mathbb{N}})_{++}\cup\{0\}$ be the sub-Cu-semigroup of $\mathrm{Lsc}(X,\overline{\mathbb{N}})$ consisting of strictly positive functions and 0. Then $\dim(S)=\infty$.

Indeed, assume for the sake of contradiction that $\dim(S)\leq n$ for some $n\in\mathbb{N}$, and take $r>n$. Since X contains at least two points, we can choose open subsets $U',U\subset X$ such that

$$\emptyset\neq U', \quad \overline{U'}\subseteq U, \quad \text{and} \quad U\neq X.$$

Let $\chi_{U'}$ and χ_U denote the corresponding characteristic functions. Consider the elements $x':=1+(n+1)\chi_{U'}$ and $x:=1+(n+1)\chi_U$ in S. Then, we have $x'\ll x\ll r+1=1+\ldots_{r+1}+1$ in S.

Using that $\dim(S)\leq n$, we obtain elements $z_{j,k}\in S$ for $j=1,\ldots,r+1$ and $k=0,\ldots,n$ satisfying (i)-(iii) from Definition 3.1. By condition (i), we have $z_{j,k}\ll 1$ and therefore $z_{j,k}=0$ or $z_{j,k}=1$ for each j,k.

Given $k\in\{0,\ldots,n\}$, we have $\sum_j z_{j,k}\ll x$ by condition (iii), and thus all but possibly one of the elements $z_{1,k},\ldots,z_{r,l}$ are zero. Thus, $\sum_j z_{j,k}\leq 1$. Using this at the last step, and using condition (ii) at the first step, we get

$$x'\ll\sum_{j,k}z_{j,k}=\sum_{k=0}^{n}\big(\sum_{j=1}^{r}z_{j,k}\big)\leq n+1,$$

a contradiction.

4. Commutative and subhomogeneous C^*-algebras

In this section, we first prove that the dimension of the Cuntz semigroup of a C^*-algebra A is bounded by the nuclear dimension of A; see Theorem 4.1. For every compact, Hausdorff space X, we show that the dimension of the Cuntz semigroup of $C(X)$ agrees with the dimension of X; see Proposition 4.3. More generally, on the class of subhomogeneous C^*-algebras, the dimension of the Cuntz semigroup agrees with the topological dimension, which in turn is equal to the nuclear dimension; see Theorem 4.10.

Theorem 4.1. *Let A be a C^*-algebra. Then* $\dim(\mathrm{Cu}(A))\leq\dim_{\mathrm{nuc}}(A)$.

Proof. Set $n:=\dim_{\mathrm{nuc}}(A)$, which we may assume to be finite. By [Rob11, Proposition 2.2], there exists an ultrafilter $\mathcal{U}$ on an index set Λ, and finite-dimensional C*-algebras $F_{\lambda,k}$ for $\lambda\in\Lambda$ and $k=0,\ldots,n$, and completely positive, contractive (cpc.) order-zero maps $\psi_k\colon A\to\prod_{\mathcal{U}}F_{\lambda,k}$ and $\varphi_k\colon\prod_{\mathcal{U}}F_{\lambda,k}\to A_{\mathcal{U}}$ such that

$$\iota=\sum_{k=0}^{n}\varphi_k\circ\psi_k,$$

where $\iota\colon A\to A_{\mathcal{U}}$ denotes the natural inclusion map.

To verify $\dim(\mathrm{Cu}(A)) \leq n$, let $x', x, y_1, \ldots, y_r \in \mathrm{Cu}(A)$ satisfy

$$x' \ll x \ll y_1 + \ldots + y_r.$$

A cpc. order-zero map $\alpha\colon C \to D$ induces a generalized Cu-morphism $\bar{\alpha}\colon \mathrm{Cu}(C) \to \mathrm{Cu}(D)$; see, for example, [APT18, Paragraph 3.2.5].

For each $k \in \{0, \ldots, n\}$, set $x_k := \bar{\psi}_k(x) \in \mathrm{Cu}(\prod_{\mathcal{U}} F_{\lambda,k})$. We have

$$\bar{\iota}(x') \ll \bar{\iota}(x) = \sum_{k=0}^{n} \bar{\varphi}_k(\bar{\psi}_k(x)) = \sum_{k=0}^{n} \bar{\varphi}_k(x_k).$$

Using that $\bar{\varphi}_k$ preserves suprema of increasing sequences, we can choose an element $x_k' \in \mathrm{Cu}(\prod_{\mathcal{U}} F_{\lambda,k})$ such that $x_k' \ll x_k$ and

$$\bar{\iota}(x') \ll \sum_{k=0}^{n} \bar{\varphi}_k(x_k').$$

Given $k \in \{0, \ldots, n\}$, we have

$$x_k' \ll x_k = \bar{\psi}_k(x) \leq \bar{\psi}_k(\sum_{j=1}^{r} y_j) = \sum_{j=1}^{r} \bar{\psi}_k(y_j).$$

Since $\prod_{\mathcal{U}} F_{\lambda,k}$ has real rank zero, we obtain $z_{1,k}, \ldots, z_{r,k} \in \mathrm{Cu}(\prod_{\mathcal{U}} F_{\lambda,k})$ such that $z_{j,k} \leq \bar{\psi}_k(y_j)$ for $j = 1, \ldots, r$ and

$$x_k' \leq \sum_{j=1}^{r} z_{j,k} \leq x_k.$$

We now consider the elements $\bar{\varphi}_k(z_{j,k}) \in \mathrm{Cu}(A_{\mathcal{U}})$. For each j and k, we have

$$\bar{\varphi}_k(z_{j,k}) \leq \bar{\varphi}_k(\bar{\psi}_k(y_j)) \leq \sum_{k'=0}^{n} \bar{\varphi}_{k'}(\bar{\psi}_{k'}(y_j)) = \bar{\iota}(y_j).$$

Further, we have

$$\bar{\iota}(x') \ll \sum_{k=0}^{n} \bar{\varphi}_k(x_k') \leq \sum_{k=0}^{n} \bar{\varphi}_k(\sum_{j=1}^{r} z_{j,k}) = \sum_{k=0}^{n}\sum_{j=1}^{r} \bar{\varphi}_k(z_{j,k}).$$

For each $k \in \{0, \ldots, n\}$, we also have

$$\sum_{j=1}^{r} \bar{\varphi}_k(z_{j,k}) = \bar{\varphi}_k(\sum_{j=1}^{r} z_{j,k}) \leq \bar{\varphi}_k(x_k) = \bar{\varphi}_k(\bar{\psi}_k(x)) \leq \bar{\iota}(x).$$

Since the classes of elements in $\bigcup_{N\in\mathbb{N}} (A_{\mathcal{U}} \otimes M_N)_+$ are sup-dense in $\mathrm{Cu}(A_{\mathcal{U}})$, there exist $N \in \mathbb{N}$ and positive elements $c_{j,k} \in A_{\mathcal{U}} \otimes M_N$ such that $[c_{j,k}] \ll \bar{\varphi}_k(z_{j,k})$ and $\bar{\iota}(x') \ll \sum_{j,k} [c_{j,k}]$.

We have $A_{\mathcal{U}} = \prod_\lambda A / c_{\mathcal{U}}$, where

$$c_{\mathcal{U}} = \{(a_\lambda)_\lambda \in \prod_\lambda A : \lim_{\lambda\to\mathcal{U}} \|a_\lambda\| = 0\}.$$

We let $\pi\colon \prod_\lambda A \to A_{\mathcal{U}}$ denote the quotient map.

We have $A_{\mathcal{U}} \otimes M_N \cong (A \otimes M_N)_{\mathcal{U}}$. We also use π to denote its amplification to matrix algebras. Choose positive elements $c_{j,k,\lambda} \in A \otimes M_N$ such that $\pi((c_{j,k,\lambda})_\lambda) = c_{j,k}$. Then, for a sufficiently large λ, the elements $[c_{j,k,\lambda}] \in \mathrm{Cu}(A)$ satisfy the properties of Lemma 3.3 for $x' \ll x \ll y_1 + \cdots + y_r$, as desired. $\square$

Lemma 4.2. *Let X be a compact, Hausdorff space. Then*

$$\dim(X) \leq \dim(\mathrm{Cu}(C(X))).$$

Proof. Set $n := \dim(\mathrm{Cu}(C(X)))$, which we may assume to be finite. To verify that $\dim(X) \leq n$, let $\mathcal{U} = \{U_1, \dots, U_r\}$ be a finite open cover of X. We need to find a $(n+1)$-colourable, finite, open refinement of $\mathcal{U}$; see Remark 3.2.

Since X is a normal space, we can find an open cover $\mathcal{V} = \{V_1, \dots, V_r\}$ of X such that $\overline{V_j} \subseteq U_j$ for each j; see for example [Pea75, Proposition 1.3.9, p.20]. For each j, by Urysohn's lemma we obtain a continuous function $f_j \colon X \to [0,1]$ that takes the value 1 on $\overline{V_j}$ and that vanishes on $X \setminus U_j$.

We have $1 \leq f_1 + \ldots + f_r$, and therefore

$$[1] \ll [1] \leq [f_1 + \ldots + f_r] \leq [f_1] + \ldots + [f_r]$$

in $\mathrm{Cu}(C(X))$. Using that $\dim(\mathrm{Cu}(C(X))) \leq n$, we obtain elements $z_{j,k} \in \mathrm{Cu}(C(X))$ for $j = 1, \dots, r$ and $k = 0, \dots, n$ satisfying (i)-(iii) in Definition 3.1.

For each j and k, choose $g_{j,k} \in (C(X) \otimes \mathcal{K})_+$ such that $z_{j,k} = [g_{j,k}]$. Viewing $g_{j,k}$ as a positive, continuous function $g_{j,k} \colon X \to \mathcal{K}$, we set

$$W_{j,k} := \{x \in X : g_{j,k}(x) \neq 0\}.$$

Then $W_{j,k}$ is an open set. Condition (i) implies that $g_{j,k} = \lim_n h_n f_j h_n^*$ for some sequence $(h_n)_n$ in $C(X) \otimes \mathcal{K}$. Thus, $g_{j,k}(x) = 0$ whenever $f_j(x) = 0$, which shows that $W_{j,k} \subseteq U_j$. Condition (ii) implies that X is covered by the sets $W_{j,k}$. Thus, the family $\mathcal{W} := \{W_{j,k}\}$ is a finite, open refinement of $\mathcal{U}$.

Let $k \in \{0, \dots, n\}$. Given $x \in X$, it follows from condition (iii) that the rank of $g_{1,k}(x) \oplus \ldots \oplus g_{r,k}(x)$ is at most one. This implies that at most one of $g_{1,k}(x), \dots, g_{r,k}(x)$ is nonzero. Thus, the sets $W_{1,k}, \dots, W_{r,k}$ are pairwise disjoint.

Hence, $\mathcal{W}$ is $(n+1)$-colourable, as desired. □

Proposition 4.3. *Let X be a compact, Hausdorff space. Then*

$$\dim(\mathrm{Cu}(C(X))) = \dim(X).$$

Proof. The inequality '$\geq$' is shown in Lemma 4.2. By [WZ10, Proposition 2.4], we have $\dim(X) = \dim_{\mathrm{nuc}}(C(X))$ if X is second-countable. By Theorem 4.8, this also holds for arbitrary compact, Hausdorff spaces. Thus, the inequality '$\leq$' follows from Theorem 4.1. □

Corollary 4.4. *Let X be a compact, metrizable space. Then*

$$\dim(\mathrm{Lsc}(X, \overline{\mathbb{N}})) = \dim(X)$$

Proof. It is enough to see that $\mathrm{Lsc}(X, \overline{\mathbb{N}})$ is a retract of $\mathrm{Cu}(C(X))$, since the inequality '$\geq$' has already been proven in Example 3.4 and the inequality '$\leq$' will follow from Lemma 3.15 and Proposition 4.3.

Thus, set $S = \mathrm{Lsc}(X, \overline{\mathbb{N}})$ and $T = \mathrm{Cu}(C(X))$. Define $\iota \colon \mathrm{Lsc}(X, \overline{\mathbb{N}}) \to \mathrm{Cu}(C(X))$ as the unique Cu-morphism mapping the characteristic function χ_U to the class of a positive function in $C(X)$ with support U for every open subset $U \subset X$.

Also, let $\sigma \colon T \to S$ be the generalized Cu-morphism mapping the class of an element $a \in C(X) \otimes \mathcal{K}$ to its rank function $\sigma(a) \colon X \to \overline{\mathbb{N}}$, $\sigma(a)(x) = \mathrm{rank}(a(x))$.

It is easy to check that $\sigma \circ \iota = \mathrm{id}_S$, as desired. □

Recall that the *local dimension* $\mathrm{locdim}(X)$ of a topological space X is defined as the smallest $n \in \mathbb{N}$ such that every point in X has a closed neighborhood of covering dimension at most n; see [Pea75, Definition 5.1.1, p.188]. If X is a locally compact, Hausdorff space, then

$$\mathrm{locdim}(X) = \sup\big\{ \dim(K) : K \subseteq X \text{ compact } \big\}.$$

If X is σ-compact, locally compact and Hausdorff, then $\mathrm{locdim}(X) = \dim(X)$, but in general $\mathrm{locdim}(X)$ can be strictly smaller than $\dim(X)$. If X is locally compact, Hausdorff but not compact, then it follows from [Pea75, Proposition 3.5.6] that

$\operatorname{locdim}(X)$ agrees with the the dimension of αX, the one-point compactification of X.

Theorem 4.5. *Let X be a locally compact, Hausdorff space. Then*

$$\dim(\mathrm{Cu}(C_0(X))) = \operatorname{locdim}(X).$$

Proof. Let $K \subseteq X$ be a compact subset. Then $C(K)$ is a quotient of $C_0(X)$. Using Proposition 4.3 at the first step and Proposition 3.10 at the second step, we get

$$\dim(K) = \dim(\mathrm{Cu}(C(K))) \leq \dim(\mathrm{Cu}(C_0(X))).$$

It follows that $\operatorname{locdim}(X) \leq \dim(\mathrm{Cu}(C_0(X)))$.

Conversely, we use that $C_0(X)$ is an ideal in $C(\alpha X)$. Applying Proposition 3.10 at the first step, and using Proposition 4.3 and $\dim(\alpha X) = \operatorname{locdim}(X)$ at the second step, we get

$$\dim(\mathrm{Cu}(C_0(X))) \leq \dim(\mathrm{Cu}(C(\alpha X))) = \operatorname{locdim}(X).$$

This show the converse inequality and finishes the proof. ☐

4.6. Let $d \in \mathbb{N}$ with $d \geq 1$. Recall that a C^*-algebra A is said to be *d-(sub)homogeneous* if every irreducible representation of A has dimension (at most) d. Further, A is *(sub)homogeneous* if it is d-(sub)homogeneous for some d. If A is d-subhomogeneous, then so is every sub-C^*-algebra of A.

Let us briefly recall the main structure theorems for (sub)homogeneous C^*-algebras. For details, we refer to [Bla06, Sections IV.1.4, IV.1.7]. Given a locally trivial $M_d(\mathbb{C})$-bundle over a locally compact, Hausdorff space X, the algebra of sections vanishing at infinity is a d-homogeneous C^*-algebra with primitive ideal space homeomorphic to X. Moreover, every homogeneous C^*-algebra arises this way.

Let A be a d-subhomogeneous C^*-algebra. For each $k \geq 2$, let $I_{\geq k} \subseteq A$ be the set of elements $a \in A$ such that $\pi(a) = 0$ for every irreducible representation π of A of dimension at most $k-1$. Set $I_{\geq 1} = A$. Then

$$\{0\} = I_{\geq d+1} \subseteq I_{\geq d} \subseteq \ldots \subseteq I_{\geq 2} \subseteq I_{\geq 1} = A$$

is an increasing chain of (closed, two-sided) ideals of A. For each $k \geq 1$, the canonical k-homogeneous ideal-quotient (that is, an ideal of a quotient) of A is

$$A_k := I_{\geq k}/I_{\geq k+1}.$$

Note that $A_k = \{0\}$ for $k \geq d+1$.

For each $k \geq 1$, we have a short exact sequence

$$0 \to A_{k+1} \to A/I_{\geq k+1} \to A/I_{\geq k} \to 0.$$

In particular, $A/I_{\geq 3}$ is an extension of $A/I_{\geq 2} = A_1$ by A_2. Then $A/I_{\geq 4}$ is an extension of $A/I_{\geq 3}$ by A_3, and so on. Finally, A is an extension of $A/I_{\geq d-1}$ by A_d. Thus, every subhomogeneous C^*-algebra is obtained as a finite successive extension of homogeneous C^*-algebras.

In [BP09], Brown and Pedersen introduced the *topological dimension* for certain C^*-algebras, including all type I C^*-algebras. We only recall the definition for subhomogeneous C^*-algebras. First, if A is homogeneous, then its primitive ideal space $\mathrm{Prim}(A)$ is locally compact and Hausdorff, and then the topological dimension of A is defined as $\operatorname{topdim}(A) := \operatorname{locdim}(\mathrm{Prim}(A))$.

If A is subhomogeneous, then the topological dimension of A is defined as the maximum of the topological dimensions of the canonical homogeneous ideal-quotients:

$$\operatorname{topdim}(A) := \max_{k=1,\ldots,d} \operatorname{topdim}(A_k) = \max_{k=1,\ldots,d} \operatorname{locdim}(\mathrm{Prim}(A_k)).$$

Given a C^*-algebra A, we use $\mathrm{Sub}_{\mathrm{sep}}(A)$ to denote the collection of separable sub-C^*-algebras of A. A family $\mathcal{S} \subseteq \mathrm{Sub}_{\mathrm{sep}}(A)$ is said to be *σ-complete* if for every countable, directed subfamily $\mathcal{T} \subseteq \mathcal{S}$ we have $\overline{\bigcup\{B : B \in \mathcal{T}\}} \in \mathcal{S}$. Further, a family $\mathcal{S} \subseteq \mathrm{Sub}_{\mathrm{sep}}(A)$ is said to be *cofinal* if for every $B_0 \in \mathrm{Sub}_{\mathrm{sep}}(A)$ there exists $B \in \mathcal{S}$ with $B_0 \subseteq B$.

Proposition 4.7. *Let $n \in \mathbb{N}$. Then for every subhomogeneous C^*-algebra A satisfying $\mathrm{topdim}(A) \leq n$, the set*

$$\big\{B \in \mathrm{Sub}_{\mathrm{sep}}(A) : \mathrm{topdim}(B) \leq n\big\}$$

is σ-complete and cofinal.

Proof. We will use the following facts. The first is a consequences of [Thi20a, Proposition 3.5], the second follows from [BP09, Proposition 2.2].

Fact 1: *Given a homogeneous C^*-algebra B with $\mathrm{locdim}(B) \leq n$, the collection*

$$\big\{C \in \mathrm{Sub}_{\mathrm{sep}}(B) : \mathrm{topdim}(C) \leq n\big\}$$

is σ-complete and cofinal.

Fact 2: *If B is subhomogeneous and $I \subseteq B$ is an ideal, then*

$$\mathrm{topdim}(B) = \max\{\mathrm{topdim}(I), \mathrm{topdim}(B/I)\}.$$

We prove the result for d-subhomogeneous C^*-algebras by induction over d. First, note that a C^*-algebra is 1-subhomogeneous if and only if it is 1-homogeneous (if and only if it is commutative). In this case, the result follows directly from Fact 1.

Next, let $d \geq 1$ and assume that the result holds for every d-subhomogeneous C^*-algebra. Let A be $(d+1)$-subhomogeneous. We need to show that the set $\mathcal{S} := \{B \in \mathrm{Sub}_{\mathrm{sep}}(A) : \mathrm{topdim}(B) \leq n\}$ is σ-complete and cofinal.

To verify that $\mathcal{S}$ is σ-complete, let $\mathcal{T} \subseteq \mathcal{S}$ be a countable, directed family. Set $C := \overline{\bigcup\{B : B \in \mathcal{T}\}}$. Then C is a separable C^*-algebra that is approximated by the sub-C^*-algebras $B \subseteq C$ for $B \in \mathcal{T}$ that each satisfy $\mathrm{topdim}(B) \leq n$. By [Thi13, Proposition 8], we have $\mathrm{topdim}(C) \leq n$. Thus, $C \in \mathcal{S}$, as desired.

Next, we verify that $\mathcal{S}$ is cofinal. Set $I := I_{\geq d+1} \subseteq A$, the ideal of all elements in A that vanish under all irreducible representations of dimension at most d. Then I is $(d+1)$-homogeneous and A/I is d-subhomogeneous. By Fact 2, we have $\mathrm{topdim}(I) \leq n$ and $\mathrm{topdim}(A/I) \leq n$. By Fact 1 and by the assumption of the induction, the collections

$$\mathcal{T}_1 := \big\{C \in \mathrm{Sub}_{\mathrm{sep}}(I) : \mathrm{topdim}(C) \leq n\big\},$$
$$\mathcal{T}_2 := \big\{D \in \mathrm{Sub}_{\mathrm{sep}}(A/I) : \mathrm{topdim}(D) \leq n\big\},$$

are σ-complete and cofinal. By [Thi20a, Lemma 3.2], it follows that the families

$$\mathcal{S}_1 := \big\{B \in \mathrm{Sub}_{\mathrm{sep}}(A) : \mathrm{topdim}(B \cap I) \leq n\big\},$$
$$\mathcal{S}_2 := \big\{B \in \mathrm{Sub}_{\mathrm{sep}}(A) : \mathrm{topdim}(B/(B \cap I)) \leq n\big\},$$

are σ-complete and cofinal. Then $\mathcal{S}_1 \cap \mathcal{S}_2$ is σ-complete and cofinal as well. Given $B \in \mathcal{S}_1 \cap \mathcal{S}_2$, it follows from Fact 2 that

$$\mathrm{topdim}(B) = \max\{\mathrm{topdim}(B \cap I), \mathrm{topdim}(B/(B \cap I))\} \leq n.$$

Thus, $\mathcal{S}_1 \cap \mathcal{S}_2 \subseteq \mathcal{S}$. Since $\mathcal{S}_1 \cap \mathcal{S}_2$ is cofinal, so is $\mathcal{S}$. □

We deduce a result that is probably known to the experts, but which does not appear in the literature so far. The equality of the topological dimension and the decomposition rank $\mathrm{dr}(A)$ of a *separable* subhomogeneous C^*-algebra A was shown in [Win04].

Theorem 4.8. *Let A be a subhomogeneous C^*-algebra. Then*

$$\dim_{\mathrm{nuc}}(A) = \mathrm{dr}(A) = \mathrm{topdim}(A).$$

Proof. As noted in [WZ10, Remarks 2.2(ii)], the inequality $\dim_{\mathrm{nuc}}(B) \leq \mathrm{dr}(B)$ holds for every C^*-algebra B. To verify $\mathrm{dr}(A) \leq \mathrm{topdim}(A)$, set $n := \mathrm{topdim}(A)$. We may assume that n is finite. By Proposition 4.7, the family

$$\mathcal{S} := \big\{ B \in \mathrm{Sub}_{\mathrm{sep}}(A) : \mathrm{topdim}(B) \leq n \big\}$$

is cofinal. Each $B \in \mathcal{S}$ is a separable, subhomogeneous C^*-algebra, whence we can apply [Win04, Theorem 1.6] to deduce that $\mathrm{dr}(B) = \mathrm{topdim}(B) \leq n$. Thus, A is approximated by the collection $\mathcal{S}$ consisting of C^*-algebras with decomposition rank at most n. It is straightforward to verify that this implies $\mathrm{dr}(A) \leq n$.

To verify $\mathrm{topdim}(A) \leq \dim_{\mathrm{nuc}}(A)$, set $m := \dim_{\mathrm{nuc}}(A)$, which we may assume to be finite. It follows from [WZ10, Proposition 2.6] that the family

$$\mathcal{T} := \big\{ B \in \mathrm{Sub}_{\mathrm{sep}}(A) : \dim_{\mathrm{nuc}}(B) \leq m \big\}$$

is cofinal. Let $B \in \mathcal{T}$. Then B is a separable, subhomogeneous C^*-algebra. For each $k \geq 1$, let B_k be the canonical k-homogeneous ideal-quotient of B; see Paragraph 4.6. Using [WZ10, Corollary 2.10] at the first step, and using that the nuclear dimension does not increase when passing to ideals ([WZ10, Proposition 2.5]) or quotients ([WZ10, Proposition 2.3(iv)]) at the second step, we get

$$\mathrm{topdim}(B_k) = \dim_{\mathrm{nuc}}(B_k) \leq \dim_{\mathrm{nuc}}(B) \leq m.$$

Using that B is obtained as a successive extension of B_1 by B_2, and then by B_3, and so on, it follows from [BP09, Proposition 2.2] (see Fact 2 in the proof of Proposition 4.7) that $\mathrm{topdim}(B) \leq m$. Thus, A is approximated by the collection $\mathcal{T}$ consisting of C^*-algebras with topological dimension at most m. By [Thi13, Proposition 8], we get $\mathrm{topdim}(A) \leq m$, as desired. □

Lemma 4.9. *Let A be a homogeneous C^*-algebra. Then*

$$\dim_{\mathrm{nuc}}(A) \leq \dim(\mathrm{Cu}(A)).$$

Proof. Let $d \geq 1$ such that A is d-homogeneous. Set $X := \mathrm{Prim}(A)$, which is locally compact and Hausdorff. Then $\mathrm{topdim}(A) = \mathrm{locdim}(X)$, and we need to show that $\mathrm{locdim}(X) \leq \dim(\mathrm{Cu}(A))$.

Let $x \in X$. Since A is the algebra of sections vanishing at infinity of a locally trivial $M_d(\mathbb{C})$-bundle over X, there exists a compact neighborhood Y of x over which the bundle is trivial. Let $I \subseteq A$ be the ideal of all sections in A that vanish on $X \setminus Y$. Then A/I is the algebra of sections of the trivial $M_d(\mathbb{C})$-bundle over Y, and so $A/I \cong C(Y) \otimes M_d$. Using Lemma 4.2 at the first step, using that $C(Y)$ and $C(Y) \otimes M_d$ have isomorphic Cuntz semigroup at the second step, and using Proposition 3.10 at the last step, we get

$$\dim(Y) \leq \dim(\mathrm{Cu}(C(Y))) = \dim(\mathrm{Cu}(C(Y) \otimes M_d)) \leq \dim(\mathrm{Cu}(A)).$$

Thus, every point in X has a closed neighborhood of dimension at most $\dim(\mathrm{Cu}(A))$, whence $\mathrm{locdim}(X) \leq \dim(\mathrm{Cu}(A))$, as desired. □

Theorem 4.10. *Let A be a subhomogeneous C^*-algebra. Then*

$$\dim(\mathrm{Cu}(A)) = \dim_{\mathrm{nuc}}(A) = \mathrm{dr}(A) = \mathrm{topdim}(A).$$

Proof. The second and third equalities are shown in Theorem 4.8. By Theorem 4.1, the inequality $\dim(\mathrm{Cu}(A)) \leq \dim_{\mathrm{nuc}}(A)$ holds in general. It remains to verify that $\mathrm{topdim}(A) \leq \dim(\mathrm{Cu}(A))$.

For each $k \geq 1$, let A_k be the canonical k-homogeneous ideal-quotient of A as in Paragraph 4.6. Using Lemma 4.9 at the first step, and using Proposition 3.10 at the second step, we get

$$\mathrm{topdim}(A_k) \leq \dim(\mathrm{Cu}(A_k)) \leq \dim(\mathrm{Cu}(A)).$$

Consequently,

$$\mathrm{topdim}(A) = \max_{k\geq 1} \mathrm{topdim}(A_k) \leq \dim(\mathrm{Cu}(A)),$$

as desired. □

Example 4.11. There are many examples showing that Theorem 4.10 does not hold for all C^*-algebras. In Proposition 5.4, we will show that every C^*-algebra A of real rank zero satisfies $\dim(\mathrm{Cu}(A)) = 0$. On the other hand, a separable C^*-algebra A satisfies $\dim_{\mathrm{nuc}}(A) = 0$ if and only if A is an AF-algebra; see [WZ10, Remarks 2.2(iii)]. Thus, every separable C^*-algebra A of real rank zero that is not an AF-algebra is an example where $\dim(\mathrm{Cu}(A))$ is strictly smaller than $\dim_{\mathrm{nuc}}(A)$. More extremely, every non-nuclear C^*-algebra A of real rank zero, such as $\mathcal{B}(\ell^2(\mathbb{N}))$, satisfies $\dim(\mathrm{Cu}(A)) = 0$ while $\dim_{\mathrm{nuc}}(A) = \infty$. Another example is the irrational rotation algebra A_θ, which satisfies $\dim(\mathrm{Cu}(A_\theta)) = 0$ while $\dim_{\mathrm{nuc}}(A_\theta) = 1$.

5. Algebraic, zero-dimensional Cuntz semigroups

In this section we begin our systematic study of zero-dimensional Cu-semigroups. After giving a useful characterization of zero-dimensionality (Lemma 5.1), we provide a sufficient criterion: A Cu-semigroup is zero-dimensional whenever it contains a sup-dense subsemigroup that satisfies the Riesz decomposition property with respect to the pre-order induced by the way-below relation; see Proposition 5.2. We deduce that the Cuntz semigroup of every C^*-algebra of real rank zero is zero-dimensional; see Proposition 5.4. Conversely, we show that every unital C^*-algebra of stable rank one and with zero-dimensional Cuntz semigroup has real rank zero; Corollary 5.8.

We also show that every weakly cancellative, zero-dimensional Cu-semigroup satisfying (O5) contains a largest algebraic ideal, which contains all compact elements; see Corollary 5.6. In Section 7, we study certain zero-dimensional Cu-semigroups that contain no compact elements.

Lemma 5.1. *Let S be a* Cu*-semigroup. Then* $\dim(S) = 0$ *if and only if, whenever $x' \ll x \ll y_1 + y_2$ in S, there exist $z_1, z_2 \in S$ such that*

$$z_1 \ll y_1, \quad z_2 \ll y_2, \quad \textit{and} \quad x' \ll z_1 + z_2 \ll x.$$

Proof. The forward implication is clear, so we are left to prove the converse. Given $r \geq 1$ and $x' \ll x \ll y_1 + \ldots + y_r$ in S, we need to find $z_1, \ldots, z_r \in S$ such that

$$z_1 \ll y_1, \quad \ldots, \quad z_r \ll y_r, \quad \text{and} \quad x' \ll z_1 + \ldots + z_r \ll x.$$

We prove this by induction on r. The case $r = 1$ is clear and the case $r = 2$ holds by assumption.

Thus, let $r > 2$ and assume that the result holds for $r - 1$. Given $x' \ll x \ll y_1 + \ldots + y_r$, apply the assumption to

$$x' \ll x \ll (y_1 + \ldots + y_{r-1}) + y_r$$

to obtain $u_1, u_2 \in S$ such that

$$u_1 \ll y_1 + \ldots + y_{r-1}, \quad u_2 \ll y_r, \quad \text{and} \quad x' \ll u_1 + u_2 \ll x.$$

Choose u_1' such that $u_1' \ll u_1$ and $x' \ll u_1' + u_2$. Applying the induction hypothesis to

$$u_1' \ll u_1 \ll y_1 + \ldots + y_{r-1},$$

we obtain $z_1, \ldots, z_{r-1} \in S$ such that

$$z_1 \ll y_1, \quad \ldots, \quad z_{r-1} \ll y_{r-1}, \quad \text{and} \quad u_1' \ll z_1 + \ldots + z_{r-1} \ll u_1.$$

Set $z_r := u_2$. Then $z_1, \ldots, z_r$ have the desired properties. □

Recall that a semigroup S with a pre-order $\prec$ is said to satisfy the *Riesz decomposition property* if whenever $x, y, z \in S$ satisfy $x \prec y + z$, then there exist $e, f \in S$ such that $x = e + f$, $e \prec y$ and $f \prec z$.

Proposition 5.2. *Let S be a* Cu*-semigroup, and let $D \subseteq S$ be a sup-dense subsemigroup such that D satisfies the Riesz decomposition property for the pre-order induced by $\ll$. Then* $\dim(S) = 0$.

Proof. To verify the condition in Lemma 5.1, let $x' \ll x \ll y_1 + y_2$ in S. Using that D is sup-dense, we find $\tilde{x}, \tilde{y}_1 \tilde{y}_2 \in D$ such that

$$x' \ll \tilde{x} \ll x \leq \tilde{y}_1 + \tilde{y}_2, \quad \tilde{y}_1 \ll y_1, \quad \text{and} \quad \tilde{y}_2 \ll y_2.$$

Then $\tilde{x} \ll \tilde{y}_1 + \tilde{y}_2$. Using that D satisfies the Riesz decomposition property, we obtain $x_1, x_2 \in D$ such that

$$\tilde{x} = x_1 + x_2, \quad x_1 \ll \tilde{y}_1, \quad \text{and} \quad x_2 \ll \tilde{y}_2.$$

Then x_1 and x_2 have the desired properties to verify the condition of Lemma 5.1. □

Recall that a Cu-semigroup is said to be *algebraic* if its compact elements are sup-dense; see [APT18, Section 5.5].

Corollary 5.3. *Let S be a weakly cancellative, algebraic* Cu*-semigroup satisfying (O5) and (O6). Then* $\dim(S) = 0$.

Proof. Set $D := \{x \in S : x \ll x\}$, the semigroup of compact elements. By assumption, D is sup-dense. By [APT18, Corollary 5.5.10], D satisfies the Riesz decomposition property. Hence, $\dim(S) = 0$ by Proposition 5.2. □

Proposition 5.4. *Let A be a C^*-algebra of real rank zero. Then* $\dim(\mathrm{Cu}(A)) = 0$.

Proof. Let $C \subseteq \mathrm{Cu}(A)$ be the set of Cuntz equivalence classes of projections in $A \otimes \mathcal{K}$. Then C is a submonoid consisting of compact elements, and using that A has real rank zero it follows that $C \subseteq \mathrm{Cu}(A)$ is sup-dense.

To verify that C satisfies the Riesz decomposition property, let $x, y, z \in C$ satisfy $x \ll y + z$. Choose projections $p, q, r \in A \otimes \mathcal{K}$ such that $x = [p]$, $y = [q]$ and $z = [r]$. Then p is Cuntz subequivalent to $q \oplus r$. Since among projections Cuntz subequivalence agrees with Murray-von Neumann subequivalence, we obtain that p is Murray-von Neumann subequivalent to $q \oplus r$.

By [BP91, Corollary 3.3], $A \otimes \mathcal{K}$ has real rank zero. Hence, it follows from [Zha90, Theorem 1.1] that the Murray-von Neumann semigroup of projections satisfies the Riesz decomposition property. Thus, there exist projections $q' \leq q$ and $r' \leq r$ such that p is Murray-von Neumann equivalent to $q' \oplus r'$. Using at the first step that Murray-von Neumann equivalence is stronger than Cuntz equivalence, we get

$$x = [p] = [q'] + [r'], \quad [q'] \ll [q] = y, \quad \text{and} \quad [r'] \ll [r] = z.$$

Now it follows from Proposition 5.2 that $\dim(\mathrm{Cu}(A)) = 0$. □

Lemma 5.5. *Let S be a weakly cancellative* Cu*-semigroup satisfying (O5) and* $\dim(S) = 0$. *Let $c \in S$ be compact. Then the ideal generated by c is algebraic.*

Proof. Let I be the ideal generated by c. Note that $x \in S$ belongs to I if and only if $x \leq \infty c$. To verify that I is algebraic, let $x', x \in I$ satisfy $x' \ll x$. We need to find a compact element z such that $x' \ll z \ll x$.

Choose $x'' \in S$ such that $x' \ll x'' \ll x$. Then $x'' \ll x \leq \infty c$, which allows us to choose $n \in \mathbb{N}$ such that $x'' \leq nc$. Applying (O5) to $x' \ll x'' \leq nc$, we obtain $y \in S$ such that

$$x' + y \leq nc \leq x'' + y.$$

Using that $\dim(S) = 0$ for $nc \ll nc \ll x'' + y$, we obtain $z_1, z_2 \in S$ such that

$$nc = z_1 + z_2, \quad z_1 \ll x'', \quad \text{and} \quad z_2 \ll y.$$

By weak cancellation, z_1 and z_2 are compact. We now have

$$x' + y \ll nc = z_1 + z_2 \leq z_1 + y.$$

Using weak cancellation, we get $x' \ll z_1$. Thus, z_1 has the desired properties. □

Corollary 5.6. *Let S be a weakly cancellative* Cu*-semigroup satisfying (O5) and* $\dim(S) = 0$*. Then S contains a largest algebraic ideal, which agrees with the ideal generated by all compact elements of S.*

Proposition 5.7. *Let S be a weakly cancellative* Cu*-semigroup satisfying (O5). Then the following are equivalent:*

(1) We have $\dim(S) = 0$*, and the compact elements of S are full (that is, there is no proper ideal of S containing all compact elements);*
(2) S is algebraic and satisfies (O6).

Proof. Assuming (1), it follows from Lemma 5.5 that S is algebraic. Further, it is clear that $\dim(S) = 0$ implies that S satisfies (O6). Conversely, assuming (2), it follows from Corollary 5.3 that $\dim(S) = 0$, and since S is algebraic it is clear that compact elements of S are full. □

Corollary 5.8. *Let A be a unital C^*-algebra of stable rank one. Then we have* $\dim(\mathrm{Cu}(A)) = 0$ *if and only if A has real rank zero.*

Proof. If A has real rank zero, then $\dim(\mathrm{Cu}(A)) = 0$ by Proposition 5.4. (This implication does not require stable rank one.) Conversely, assume that A has stable rank one and $\dim(\mathrm{Cu}(A)) = 0$. Since A is unital, the compact elements in $\mathrm{Cu}(A)$ are full. Thus, by Proposition 5.7, $\mathrm{Cu}(A)$ is algebraic. Now it follow from [CEI08, Corollary 5] that A has real rank zero. □

Corollary 5.9. *Let A be a separable, simple, $\mathcal{Z}$-stable C^*-algebra. Then we have* $\dim(\mathrm{Cu}(A)) \leq 1$*. Moreover,* $\dim(\mathrm{Cu}(A)) = 0$ *if and only if A has real rank zero or if A is stably projectionless.*

Proof. It follows from [Rør02, Theorem 4.1.10] that A is either purely infinite or stably finite. Thus, we can distinguish three cases: A is either purely infinite or stably projectionless, or stably finite and not stably projectionless.

The first statement follows from Proposition 3.22. To show the forward implication of the second statement, assume that $\dim(\mathrm{Cu}(A)) = 0$. We need to show that A has real rank zero or is stable projectionless. First, if A is purely infinite, then A has real rank zero; see [Bla06, Proposition V.3.2.12]. Second, if A is stably projectionless, then there is nothing to show. Third, we consider the case that A is stably finite and not stably projectionless. Let $p \in A \otimes \mathcal{K}$ be a nonzero projection. Then $p(A \otimes \mathcal{K})p$ is a separable, unital, simple, stably finite, $\mathcal{Z}$-stable C^*-algebra and therefore has stable rank one by [Rør04, Theorem 6.7]. Since A and $p(A \otimes \mathcal{K})p$ are stably isomorphic, they have isomorphic Cuntz semigroups. Thus, $\dim(\mathrm{Cu}(p(A \otimes \mathcal{K})p)) = 0$, and we deduce from Corollary 5.8 that $p(A \otimes \mathcal{K})p$ has real rank zero. By [BP91, Corollary 2.8 and 3.3], a C^*-algebra has real rank zero if and only if its stabiliation does. Thus, A has real rank zero.

To show the backward implication of the second statement, assume that A has real rank zero or is stably projectionless. We need to show that $\dim(\mathrm{Cu}(A)) = 0$. If A has real rank zero, this follow from Proposition 5.4. Let us consider the case that A is stably projectionless. Then $\mathrm{Cu}(A)$ contains no nonzero compact elements by [BC09]. Thus, $\mathrm{Cu}(A)$ is soft and has Z-multiplication, which by [APT18, Theorem 7.5.4] implies that $\mathrm{Cu}(A)$ has $[0,\infty]$-multiplication. Hence, $\dim(\mathrm{Cu}(A)) = 0$ by Proposition 3.20. □

6. Thin boundary and complementable elements

In this section, we study soft elements in simple Cu-semigroups that behave very similar to compact elements: the elements with thin boundary (Definition 6.3), and the complementable elements (Definition 6.9). If S is a simple, stably finite, soft Cu-semigroup satisfying (O5) and (O6) (for example, the Cuntz semigroup of a simple, stably projectionless C^*-algebra; see Proposition 6.2), then every element with thin boundary is complementable; see Corollary 6.11. The converse holds if S is also weakly cancellative (for example, the Cuntz semigroup of a simple, stably projectionless C^*-algebra of stable rank one); see Theorem 6.12.

In Section 7, we will show that zero-dimensionality of certain simple Cu-semigroups is characterized by sup-denseness of the elements with thin boundary.

6.1. We say that a simple Cu-semigroup S is *stably finite* if for all $x, z \in S$, we have that $x + z \ll z$ implies $x = 0$. Using that S is simple, one can show that this definition is equivalent to the one given in [APT18, Paragraph 5.2.2]. We note that every simple, weakly cancellative Cu-semigroup is stably finite.

Let S be a simple, stably finite Cu-semigroup satisfying (O5). Recall that an element $x \in S$ is *compact* if $x \ll x$. We say that $x \in S$ is *soft* if $x = 0$ or if $x \neq 0$ and for every $x' \in S$ satisfying $x' \ll x$ there exists a nonzero $t \in S$ such that $x' + t \ll x$. (Using [APT18, Proposition 5.3.8], one sees that this is equivalent to the original definition.) We say that S is *soft* if every element in S is soft.

We let S_c and S_{soft} denote the set of compact and soft elements in S, respectively. We also set $S_{\mathrm{soft}}^\times := S_{\mathrm{soft}} \setminus \{0\}$. It is easy to see that S_c, S_{soft} and $S_{\mathrm{soft}}^\times$ are submonoids of S. Further, $S_{\mathrm{soft}}^\times$ is absorbing in the sense that $x+y$ belongs to $S_{\mathrm{soft}}^\times$ whenever x or y does; see [APT18, Theorem 5.3.11].

By [APT18, Proposition 5.3.16], every element in S is either compact, or nonzero and soft. Hence, S can be decomposed as $S = S_{\mathrm{soft}}^\times \sqcup S_c$.

Proposition 6.2. *Let A be a simple, stably projectionless C^*-algebra. Then* $\mathrm{Cu}(A)$ *is a simple, stably finite, soft* Cu*-semigroup satisfying (O5) and (O6).*

Proof. The Cuntz semigroup $\mathrm{Cu}(A)$ is simple and satisfies (O5) and (O6) since it is the Cuntz semigroup of a simple C^*-algebra (see, for example, [APT18, Corollary 5.1.12]). As A is stably projectionless, $\mathrm{Cu}(A)$ has no nonzero compact elements by [BC09].

It is easy to check that a simple Cu-semigroup is stably finite if and only if ∞ is not compact or if S is zero. Therefore, the Cuntz semigroup of a stably projectionless C^*-algebra is always stably finite.

By [APT18, Proposition 5.3.16] we have $\mathrm{Cu}(A)^\times = \mathrm{Cu}(A)_{\mathrm{soft}}^\times$ as desired. □

Definition 6.3. Let S be a simple Cu-semigroup. We say that an element $x \in S$ has *thin boundary* if $x \ll x + t$ for every nonzero $t \in S$. We let S_{tb} denote the set of elements in S with thin boundary.

Note that every compact element has thin boundary, but the converse is not true: In $[0,\infty]$ every element has thin boundary, but only 0 is compact.

We will repeatedly use the following result.

Lemma 6.4. *Let S be a simple, nonelementary* Cu*-semigroup satisfying (O5) and (O6). Let $u_0, u_1 \in S$ be nonzero. Then there exists a nonzero $w \in S$ such that $2w \ll u_0, u_1$.*

Proof. This follows by combining [APT18, Lemma 5.1.18] and [Rob13, Proposition 5.2.1]. For the convenience of the reader, we include the simple argument.

First, choose nonzero elements $u_0'', u_0' \in S$ such that $u_0'' \ll u_0' \ll u_0$. Since S is simple and $u_1 \neq 0$, we have $u_0' \ll u_0 \leq \infty = \infty u_1$, which allows us to choose $n \geq 1$ such that $u_0' \leq n u_1$.

Applying (O6) to $u_0'' \ll u_0' \leq u_1 + \ldots_n + u_1$, we obtain $z_1, \ldots, z_n \in S$ such that

$$u_0'' \ll z_1 + \ldots + z_n, \quad \text{and} \quad z_1, \ldots, z_n \ll u_0', u_1.$$

Since u_0'' is nonzero, there is $j \in \{1, \ldots, n\}$ such that $v := z_j$ is nonzero. Then $v \ll u_0, u_1$.

Since S is nonelementary, v is not a minimal nonzero element. Thus, we can choose a nonzero $v' \in S$ with $v' \leq v$ and $v' \neq v$. Choose a nonzero $v'' \in S$ with $v'' \ll v'$. Applying (O5) to $v'' \ll v' \leq v$, we obtain $c \in S$ such that

$$v'' + c \leq v \leq v' + c.$$

Since $v' \neq v$, we have $c \neq 0$. Applying the first part of the argument to the nonzero elements v'' and c, we obtain $w \in S$ such that $0 \neq w \ll v'', c$. Then w has the desired properties. □

Lemma 6.5. *Let S be a simple* Cu*-semigroup satisfying (O5) and (O6). Then S_{tb} is a submonoid.*

Proof. This is clear if S is elementary, since then every element in S way-below another is compact and therefore $S_{\mathrm{tb}} = S$; see [APT18, Proposition 5.1.19].

We now assume that S is nonelementary. Let $x, y \in S_{\mathrm{tb}}$. To verify that $x + y$ has thin boundary, let $t \in S$ be nonzero. By Lemma 6.4, there is a nonzero element s such that $2s \leq t$. This implies

$$x + y \ll x + s + y + s \leq x + y + t,$$

as required. □

Lemma 6.6. *Let S be a simple, weakly cancellative* Cu*-semigroup satisfying (O5). Let $x, y, z \in S$ satisfy $x + z \leq y + z$. Assume that x, y are soft, and that z has thin boundary. Then $x \leq y$.*

Proof. If $x = 0$ the result is trivial, so we may assume otherwise.

Let $x' \in S$ satisfy $x' \ll x$. Choose $x'' \in S$ such that $x' \ll x'' \ll x$. Since x is nonzero and soft, there exists a nonzero $t \in S$ with $x'' + t \leq x$. Hence,

$$x' + z \ll x'' + (z + t) \leq x + z \leq y + z.$$

Using weak cancellation, we get $x' \ll y$.

Since this holds for every x' way-below x, we get $x \leq y$. □

Lemma 6.7. *Let S be a simple, weakly cancellative* Cu*-semigroup. Let $x, y \in S$ such that $x + y$ has thin boundary. Then x and y have thin boundary.*

Proof. To show that x has thin boundary, let $t \in S$ be nonzero. Then

$$x + y \ll (x + y) + t = (x + t) + y,$$

which, by weak cancellation, implies that $x \ll x + t$, as desired. Analogously, one shows that y has thin boundary. □

Lemma 6.8. *Let S be a simple, stably finite* Cu*-semigroup satisfying (O5). Let $x \in S$ have thin boundary, and let $s, t \in S$ satisfy $s \ll t$. Assume that t is nonzero and soft. Then $x + s \ll x + t$.*

Proof. Choose $t' \in S$ such that $s \ll t' \ll t$. Since t is nonzero and soft, there exists a nonzero $c \in S$ such that $t' + c \le t$. Then

$$x + s \ll (x + c) + t' \le x + t,$$

as desired. □

Definition 6.9. Let S be a simple, soft Cu-semigroup. We say that $x \in S$ is *complementable* if for every $y \in S$ satisfying $x \ll y$ there exists $z \in S$ such that $x + z = y$.

The next result implies that elements with thin boundary are complementable; see Corollary 6.11.

Proposition 6.10. *Let S be a simple, stably finite* Cu*-semigroup satisfying (O5) and (O6). Let $x, y \in S$ satisfy $x \ll y$. Assume that x has thin boundary and that y is soft. Then there exists $z \in S$ such that $x + z = y$.*

Proof. Applying [APT18, Proposition 5.1.19], the result is clear if S is elementary. Thus, we may assume that S is nonelementary. The result is also clear if $x = 0$, so we may assume that $x \neq 0$.

Step 1: *We construct an increasing sequence $(y_n)_n$ with supremum y and $x \le y_0$, and a sequence $(s_n)_n$ of nonzero elements such that*

$$y_n + s_n \ll y_{n+1}$$

for every $n \in \mathbb{N}$.

First, let $(\bar{y}_n)_n$ be any $\ll$-increasing sequence in S with supremum y. Set $y_0 := x$. Since S is simple and stably finite, it follows from [APT18, Proposition 5.3.18] that there exists a soft element y' such that $y_0 \ll y' \ll y$. Since y' is nonzero and soft, one can find a non-zero element s_0 such that $y_0 + s_0 \le y'$.

Using that $y_0 + s_0$ and $\bar{y}_1$ are way-below y, choose y_1 such that

$$y_0 + s_0 \ll y_1, \quad \bar{y}_1 \ll y_1, \quad \text{and} \quad y_1 \ll y.$$

Then $y_1 \ll y$, and we can apply the previous argument once again to obtain $s_1 \neq 0$ such that $y_1 + s_1 \ll y$. Using that $y_1 + s_1$ and $\bar{y}_2$ are way-below y, we obtain y_2 such that

$$y_0 + s_0 \ll y_1, \quad \bar{y}_1 \ll y_1, \quad \text{and} \quad y_1 \ll y.$$

Continuing this way, we obtain the desired sequences $(y_n)_n$ and $(s_n)_n$.

Step 2: *We construct a sequence $(r_n)_n$ of nonzero elements such that*

$$2r_{n+1} \ll r_n, s_{n+1}, \quad \textit{and} \quad y_n + r_n + r_{n+1} \ll y_{n+1} \tag{6.1}$$

for every $n \in \mathbb{N}$.

Applying Lemma 6.4 for s_0, we obtain a nonzero $r_0 \in S$ such that $2r_0 \ll s_0$. Then, applying Lemma 6.4 for r_0 and s_1, we obtain a nonzero $r_1 \in S$ such that $2r_1 \ll r_0, s_1$. Continuing this way, we obtain a sequence $(r_n)_n$ such that $2r_{n+1} \ll r_n, s_{n+1}$ for every $n \in \mathbb{N}$.

For each $n \in \mathbb{N}$, we have

$$y_n + r_n + r_{n+1} \le y_n + 2r_n \le y_n + s_n \ll y_{n+1},$$

which shows that $(r_n)_n$ has the desired properties.

Step 3: *We construct an $\ll$-increasing sequence $(w_n)_n$ and a sequence $(v_n)_n$ such that*

$$x + r_{n+1} + v_n \le y_n \le x + r_n + v_n, \quad w_n \ll r_n + v_n, v_{n+1}, \quad \textit{and} \quad y_{n-1} \le x + w_n$$

for every $n \geq 1$.

To start, using Lemma 6.8 at the first step, we have

$$x + r_2 \ll x + r_1 \leq y_0 + s_0 \leq y_1.$$

Applying (O5), we obtain $v_1 \in S$ such that

$$x + r_2 + v_1 \leq y_1 \leq x + r_1 + v_1.$$

Using that $y_0 \ll y_1$, we can choose $w_1 \in S$ such that

$$y_0 \leq x + w_1, \quad \text{and} \quad w_1 \ll r_1 + v_1.$$

Next, let $n \geq 1$, and assume that we have chosen v_n and w_n. Using for the first inequality that $x + r_{n+1} + v_n \leq y_n$ and (6.1), we have

$$x + r_{n+1} + r_n + v_n \leq y_{n+1}, \quad x + r_{n+2} \ll x + r_{n+1}, \quad \text{and} \quad w_n \ll r_n + v_n.$$

Applying (O5), we obtain $v_{n+1} \in S$ such that

$$x + r_{n+2} + v_{n+1} \leq y_{n+1} \leq x + r_{n+1} + v_{n+1}, \quad \text{and} \quad w_n \ll v_{n+1}.$$

Using that $y_n \ll y_{n+1}$ and $w_n \ll v_{n+1} \leq r_{n+1} + v_{n+1}$, we obtain $w_{n+1} \in S$ such that

$$y_n \leq x + w_{n+1}, \quad \text{and} \quad w_n \ll w_{n+1} \ll r_{n+1} + v_{n+1}.$$

Now, the sequence $(w_n)_n$ is increasing, which allows us to set $z := \sup_n w_n$. For every $n \geq 1$, we have

$$x + w_n \leq x + v_{n+1} \leq y_{n+2} \leq y$$

and therefore $x + z \leq y$. Further, for every $n \geq 1$, we have

$$y_n \leq x + w_{n+1} \leq x + z$$

and therefore $y \leq x + z$. This implies $x + z = y$. □

Corollary 6.11. *Let S be a simple, soft, stably finite* Cu*-semigroup satisfying (O5) and (O6). Then every element in S with thin boundary is complementable.*

If we additionally assume that S is weakly cancellative, then the converse of Corollary 6.11 also holds:

Theorem 6.12. *Let S be a simple, soft, weakly cancellative* Cu*-semigroup satisfying (O5) and (O6), and let $x \in S$ satisfy $x \ll \infty$. Then x has thin boundary if and only if x is complementable.*

Proof. The forwards implication follows from Corollary 6.11. To show the backwards implication, assume that x is complementable. To verify that x has thin boundary, let $t \in S$ be nonzero. Choose a nonzero element $t' \in S$ with $t' \ll t$. Then $x \ll \infty = \infty t'$, which allows us to choose $n \geq 1$ such that $x \leq nt'$. Choose $t_1, \ldots, t_n \in S$ such that

$$t' \ll t_1 \ll t_2 \ll \ldots \ll t_n \ll t.$$

Set $y := t_1 + \ldots + t_n$. Then $x \leq nt' \ll y$. Since x is complementable, we obtain $z \in S$ such that $x + z = y$.

Note that

$$y = t_1 + \ldots + t_n \ll t_2 + \ldots + t \leq y + t,$$

and therefore

$$x + z = y \ll y + t = x + z + t.$$

By weak cancellation, we obtain $x \ll x + t$, as desired. □

Theorem 6.13. *Let S be a simple, soft, weakly cancellative* Cu*-semigroup satisfying (O5) and (O6). Then S_{tb} is a cancellative monoid. Further, $x, y \in S_{\mathrm{tb}}$ satisfy $x \ll y$ if and only if there exists $z \in S_{\mathrm{tb}}^\times$ with $x + z = y$.*

Proof. By Lemma 6.5 and 6.6, S_{tb} is a cancellative monoid. Let $x, y \in S_{\mathrm{tb}}$. If $x \ll y$, then by Theorem 6.12 there exists $z \in S$ such that $x + z = y$. Since y is not compact, we have $z \neq 0$. Further, by Lemma 6.7, we have $z \in S_{\mathrm{tb}}$. Conversely, if $z \in S$ is nonzero such that $x + z = y$, then $x \ll x + z = y$ by definition. □

7. Simple, zero-dimensional Cuntz semigroups

In this section, we study countably based, simple, weakly cancellative Cu-semigroups S that satisfy (O5) and (O6) (for example the Cuntz semigroups of separable, simple C^*-algebras of stable rank one). First, we prove a dichotomy: If S is zero-dimensional, then S is either algebraic or soft; see Lemma 7.1. Conversely, if S is algebraic, then S is automatically zero-dimensional by Corollary 5.3. On the other hand, if S is soft, then S is zero-dimensional if and only if the elements with thin boundary are sup-dense; see Theorem 7.8. We deduce that S is zero dimensional if and only if S is the retract of a simple, algebraic Cu-semigroup; see Theorem 7.10.

This should be compared with Corollary 5.9, where we showed that a separable, simple, $\mathcal{Z}$-stable C^*-algebra has zero-dimensional Cuntz semigroup if and only if A has real rank zero or A is stably projectionless.

Lemma 7.1. *Let S be a simple, weakly cancellative* Cu*-semigroup satisfying (O5). Assume that* $\dim(S) = 0$ *and* $S \neq \{0\}$*. Then, S is either algebraic or soft.*

Proof. Assume that S is not soft. Then there exists a nonzero compact element in S, which by Lemma 5.5 implies that S is algebraic. □

Lemma 7.2. *Let S be a simple, weakly cancellative* Cu*-semigroup satisfying (O5) and (O6). Assume that S_{tb} is sup-dense. Let $x, y, z \in S$ satisfy $x \ll y + z$, and assume that x has thin boundary and z is soft. Then there exist $v, w \in S_{\mathrm{tb}}$ such that*

$$x = v + w, \quad v \ll y, \quad \textit{and} \quad w \ll z.$$

Proof. If S is elementary, then it follows from [APT18, Proposition 5.1.19] that $S = \{0\}$. Thus, we may assume that S is nonelementary. We may also assume that z is nonzero, since otherwise $v = x$ and $w = 0$ trivially satisfy the required conditions.

Choose $z' \in S$ such that

$$x \ll y + z', \quad \text{and} \quad z' \ll z.$$

Using that z is nonzero and soft, we obtain a nonzero $t \in S$ such that $z' + t \ll z$. Since x has thin boundary, we have $x \ll x + t$, which allows us to choose $x' \in S$ such that

$$x' \ll x \ll x' + t.$$

Since S_{tb} is sup-dense, we may assume that x' has thin boundary.

Applying (O6) to $x' \ll x \ll y + z'$, we obtain $e, f \in S$ such that

$$x' \ll e + f, \quad e \ll x, y, \quad \text{and} \quad f \ll x, z'.$$

Since S_{tb} is sup-dense, we may assume that e has thin boundary.

By Corollary 6.11, e is complementable. Thus, we obtain $c \in S$ such that $e + c = x$. Then

$$e + c = x \ll x' + t \leq e + f + t.$$

By weak cancellation, we get $c \ll f + t$ and therefore

$$c \ll f + t \leq z' + t \ll z.$$

By Lemma 6.7, e and c have thin boundary. Hence, $v := e$ and $w := c$ have the desired properties. □

Proposition 7.3. *Let S be a simple, soft, weakly cancellative* Cu*-semigroup satisfying (O5) and (O6). Assume that S_{tb} is sup-dense. Then S_{tb} is a simple, cancellative refinement monoid and* $\dim(S) = 0$.

Proof. By Theorem 6.13, S_{tb} is a cancellative monoid such that $x, y \in S_{\mathrm{tb}}$ satisfy $x \ll y$ if and only if there exists $z \in S_{\mathrm{tb}}^{\times}$ with $x + z = y$. This implies that $x, y \in S_{\mathrm{tb}}$ satisfy $x \leq_{\mathrm{alg}} y$ if and only if $x = y$ or $x \ll y$.

It follows from Lemma 7.2 that S_{tb} satisfies the Riesz decomposition property for the pre-order induced by $\ll$. Hence, $\dim(S) = 0$ by Proposition 5.2.

Since S_{tb} is a cancellative monoid, to show that it is a refinement monoid it suffices to show that it satisfies the Riesz decomposition property for the algebraic partial order $\leq_{\mathrm{alg}}$. Let $x, y, z \in S_{\mathrm{tb}}$ satisfy $x \leq_{\mathrm{alg}} y + z$. We need to find $y', z' \in S_{\mathrm{tb}}$ such that $x = y' + z'$, $y' \leq_{\mathrm{alg}} y$ and $z' \leq_{\mathrm{alg}} z$. We either have $x = y + z$ or $x \ll y + z$. In the first case, $y' := y$ and $z' := z$ have the desired properties. In the second case, we apply Lemma 7.2 to obtain $y', z' \in S_{\mathrm{tb}}$ such that $x = y' + z'$, $y' \ll y$ and $z' \ll z$. Then $y' \leq_{\mathrm{alg}} y$ and $z' \leq_{\mathrm{alg}} z$, which shows that y' and z' have the desired properties. Using that S is simple, it easily follows that S_{tb} is a simple monoid. □

Example 7.4. Let Z be the Cuntz semigroup of the Jiang-Su algebra $\mathcal{Z}$. Then every element of Z has thin boundary, yet Z is neither algebraic nor soft, and therefore Z is not zero-dimensional. (We have $\dim(Z) = 1$ by Example 3.19.)

This show that Proposition 7.3 does not hold without assuming that S is soft.

Next, we prove the converse of Proposition 7.3: Zero-dimensionality implies that S_{tb} is sup-dense. We start with a crucial technical result.

Lemma 7.5. *Let S be a weakly cancellative* Cu*-semigroup satisfying (O5), and let $x', x'', x, e, t \in S$ satisfy*

$$x' \ll x'', \quad \text{and} \quad x'' + t \leq x \leq e \ll e + t.$$

Assume that $\dim(S) = 0$. *Then there exists y such that*

$$x' \ll y \ll x, \quad \text{and} \quad y \ll y + t.$$

Proof. Applying (O5) to $x' \ll x'' \leq e$, we obtain $c \in S$ such that

$$x' + c \leq e \leq x'' + c.$$

Then

$$e \ll e + t \leq x'' + c + t.$$

Using that $\dim(S) = 0$, we obtain $u, v \in S$ such that

$$u \ll x'', \quad v \ll c + t, \quad \text{and} \quad e \ll u + v \ll e + t.$$

Then

$$x' + c \leq e \ll u + v \leq u + c + t,$$

Using weak cancellation, we get $x' \ll u + t$. Further, we have

$$u + v \ll e + t \leq u + v + t$$

and therefore $u \ll u + t$ by weak cancellation.

Choose $t' \in S$ such that

$$t' \ll t, \quad x' \ll u + t', \quad \text{and} \quad u \ll u + t'.$$

Set $y := u + t'$. Then

$$x' \ll u + t' = y, \quad \text{and} \quad y = u + t' \ll x'' + t \leq x.$$

Using that $u \ll u + t'$ and $t' \ll t$, we get

$$y = u + t' \ll u + t' + t = y + t,$$

which shows that y has the desired properties. □

Lemma 7.6. *Let S be a countably based, simple, soft, weakly cancellative* Cu*-semigroup satisfying (O5) and (O6).*

Assume that for every $x', x, t \in S$ satisfying $x' \ll x$ and $t \neq 0$ there exists $y \in S$ such that

$$x' \ll y \ll x, \quad y \ll y + t.$$

Then for every $x', x \in S$ satisfying $x' \ll x$ there exists $y \in S$ with thin boundary such that $x' \ll y \ll x$.

Proof. Using that S is countably based, we can choose a sequence $(t_n)_{n\in\mathbb{N}}$ of nonzero elements such that for every nonzero $t \in S$ there exists n with $t_n \leq t$.

To prove the statement, let $x', x \in S$ satisfy $x' \ll x$. By assumption, we can choose $y_0 \in S$ with $x' \ll y_0 \ll x$ and $y_0 \ll y_0 + t_0$. Choose y_0' such that

$$x' \ll y_0' \ll y_0 \ll x, \quad y_0 \ll y_0' + t_0.$$

Next, applying the assumption for y_0', y_0, t_1, we obtain y_1 such that $y_0' \ll y_1 \ll y_0$ and $y_1 \ll y_1 + t_1$. Then choose y_1' such that

$$y_0' \ll y_1' \ll y_1 \ll y_0, \quad y_1 \ll y_1' + t_1.$$

Inductively, choose y_n' and y_n such that

$$x' \ll y_0' \ll \ldots \ll y_n' \ll y_n \ll \ldots \ll y_0 \ll x, \quad y_n \ll y_n' + t_n.$$

Set $y := \sup_n y_n'$. Then $x' \ll y_0' \leq y \leq y_0 \ll x$. To show that y has thin boundary, let $t \in S$ be nonzero. By choice of $(t_n)_n$, there exists n such that $t_n \leq t$. Then

$$y \leq y_n \ll y_n' + t_n \leq y + t_n \leq y + t,$$

as desired. □

Proposition 7.7. *Let S be a countably based, simple, soft, weakly cancellative* Cu*-semigroup satisfying (O5) and (O6). Assume that* $\dim(S) = 0$*. Then S_{tb} is sup-dense, that is, the elements with thin boundary form a basis.*

Proof. We verify the assumption of Lemma 7.6, which then proves the statement. Let $x', x, t \in S$ satisfy $x' \ll x$ and $t \neq 0$. We need to find $y \in S$ such that

$$x' \ll y \ll x, \quad y \ll y + t.$$

If $x' = 0$, then set $y := 0$. Thus, we may assume from now on that x' is nonzero.

Choose $x'', u \in S$ such that

$$x' \ll x'' \ll u \ll x.$$

Since u is nonzero and soft, we obtain a nonzero element $s \in S$ such that $x'' + s \ll u$. By Lemma 6.4, there exists a nonzero $r \in S$ with $r \leq s, t$.

Choose a nonzero $r' \in S$ such that $r' \ll r$. Then $u \ll \infty = \infty r'$, which allows us to choose $n \geq 1$ such that $u \leq nr'$. Choose $r_1, \ldots, r_n \in S$ such that

$$r' \ll r_1 \ll r_2 \ll \ldots \ll r_n \ll r.$$

Set $e := r_1 + \ldots + r_n$. As in the proof of Theorem 6.12, we obtain $e \ll e + r$, and consequently $e \ll e + t$. Further, we have

$$x' \ll x'', \quad x'' + r \leq x'' + s \ll u \leq nr' \leq e \ll e + t.$$

Applying Lemma 7.5, we obtain $y \in S$ such that

$$x' \ll y \ll u, \quad \text{and} \quad y \ll y + t,$$

Now y has the desired properties. □

Theorem 7.8. *Let S be a countably based, simple, soft, weakly cancellative* Cu*-semigroup satisfying (O5) and (O6). Then the following are equivalent:*

(1) $\dim(S) = 0$*;*
(2) the elements with thin boundary are sup-dense;
(3) there exists a countably based, simple, algebraic, weakly cancellative Cu-*semigroup* T *satisfying (O5) and (O6) such that* $S \cong T_{\mathrm{soft}}$.

Proof. By Proposition 7.7, (1) implies (2). Conversely, (2) implies (1) by Proposition 7.3. To show that (3) implies (1), let T be as in (3) such that $S \cong T_{\mathrm{soft}}$. Using Proposition 3.17 at the second step, and using Corollary 5.3 at the last step, we get

$$\dim(S) = \dim(T_{\mathrm{soft}}) \leq \dim(T) = 0.$$

Finally, assuming (2) let us verify (3). By Proposition 7.3, S_{tb} is a simple, cancellative refinement monoid. Using that S is countably based and that S_{tb} is sup-dense, we can choose a countable subset $M_0 \subseteq S_{\mathrm{tb}}$ that is sup-dense.

By successively adding elements to M_0 we can construct a countable refinement submonoid $M \subseteq S_{\mathrm{tb}}$ such that the algebraic order on M agrees with the restriction of the algebraic order on S_{tb} to M, that is, $(M, \leq_{\mathrm{alg}}) \to (S_{\mathrm{tb}}, \leq_{\mathrm{alg}})$ is an order-embedding. Set $T := \mathrm{Cu}(M, \leq_{\mathrm{alg}})$, the sequential round ideal completion of M with respect to the algebraic partial order; see [APT18, Section 5.5]. Then T is a countably based, algebraic Cu-semigroup. Using that M is a cancellative monoid that is algebraically ordered and that satisfies the Riesz decomposition property, it follows from [APT18, Proposition 5.5.8] that T is weakly cancelaltive and satisfies (O5) and (O6). Using that M is a simple monoid, it follows that T is simple.

Recall that a subset $I \subseteq M$ is an interval if I is downward hereditary and upward directed. Since M is countable, we can identify T with the set of intervals in M, ordered by inclusion. The compact elements in T are precisely the intervals $\{y \in M : y \leq_{\mathrm{alg}} x\}$ for $x \in M$. Thus, the nonzero soft elements in T are precisely the intervals that do not contain a largest element. Using that every upward directed set in a countably based Cu-semigroup has a supremum, we can define $\alpha\colon T_{\mathrm{soft}} \to S$ by

$$\alpha(I) := \sup I,$$

for every (soft) interval $I \subseteq M$. It is now straightforward to verify that α is an isomorphism. ☐

Remark 7.9. There is no canonical choice for the algebraic Cu-semigroup T in Theorem 7.8(3). Take for example $S = [0, \infty]$. For every supernatural number q satisfying $q = q^2 \neq 1$, we consider the UHF-algebra M_q of infinite type, and set $R_q := \mathrm{Cu}(M_q)$; see [APT18, Section 7.4]. Then R_q is a countably based, simple, algebraic, weakly cancellative Cu-semigroup satisfying (O5) and (O6), and $(R_q)_{\mathrm{soft}} \cong [0, \infty]$.

Given a countably based, simple, soft, weakly cancellative Cu-semigroup S satisfying (O5) and (O6), we can consider $T := \mathrm{Cu}(S_{\mathrm{soft}}, \leq_{\mathrm{alg}})$, which is a simple, algebraic, weakly cancellative Cu-semigroup satisfying (O5) and (O6) such that $S \cong T_{\mathrm{soft}}$. However, T is not countably based since every basis of T contains all compact elements of T and so has at least the cardinality of S_{tb}.

Recall the notion of a retract from Definition 3.14.

Theorem 7.10. *Let* S *be a countably based, simple, weakly cancellative* Cu-*semigroup satisfying (O5) and (O6). Then* S *is zero-dimensional if and only if* S *is a retract of a countably based, simple, algebraic, weakly cancellative* Cu-*semigroup satisfying (O5) and (O6).*

Proof. By Lemma 7.1, S is either algebraic or soft. In the first case, we consider S as a retract of itself. In the second case, the result follows from Theorem 7.8 and Proposition 3.16. ☐

Question 7.11. Is every zero-dimensional, weakly cancellative Cu-semigroup satisfying (O5) a retract of a weakly cancellative, algebraic Cu-semigroup satisfying (O5) and (O6)?

Recall that a partially ordered set M has the *Riesz interpolation property* if for all $x_0, x_1, y_0, y_1 \in M$ satisfying $x_j \leq y_k$ for all $j, k \in \{0, 1\}$, there exists $z \in M$ such that $x_j \leq z \leq y_k$ for all $j, k \in \{0, 1\}$. By [APRT18, Theorem 3.5], Cuntz semigroups of stable rank one C^*-algebras have the Riesz interpolation property.

Recall that a Cu-semigroup S is said to be *almost divisible* if for all $n \in \mathbb{N}$ and $x', x \in S$ satisfying $x' \ll x$ there exists $y \in S$ such that $ny \leq x$ and $x' \leq (n+1)y$; see [APT18, Definition 7.3.4].

Lemma 7.12. *Let S be a retract of a* Cu*-semigroup T. Then, if T is almost divisible, so is S. Further, if T has the Riesz interpolation property, then so does S.*

Proof. Let $\iota\colon S \to T$ be a Cu-morphism, and let $\sigma\colon T \to S$ be a generalized Cu-morphism with $\sigma \circ \iota = \mathrm{id}_S$.

First, assume that T has the Riesz interpolation property. Let $x_0, x_1, y_0, y_1 \in S$ satisfy $x_j \leq y_k$ for all $j, k \in \{0, 1\}$. Then $\iota(x_j) \leq \iota(y_k)$ in T for all $j, k \in \{0, 1\}$. By assumption, there is $z \in T$ such that $\iota(x_j) \leq z \leq \iota(y_k)$ and thus $x_j \leq \sigma(z) \leq y_k$ for all $j, k \in \{0, 1\}$. Thus, $\sigma(z)$ has the desired properties.

Next, assume that T is almost divisible. Let $n \in \mathbb{N}$ and let $x', x \in S$ satisfy $x' \ll x$. Then $\iota(x') \ll \iota(x)$ in T. By assumption, there exists $y \in T$ such that $ny \leq \iota(x)$ and $\iota(x') \leq (n+1)y$. Then $n\sigma(y) \leq x$ and $x' \leq (n+1)\sigma(y)$. □

Proposition 7.13. *Let S be a zero-dimensional, countably based, simple, weakly cancellative, nonelementary* Cu*-semigroup satisfying (O5). Then S satisfies the Riesz interpolation property and is almost divisible.*

Proof. By Theorem 7.10, there exists a countably based, simple, algebraic, weakly cancellative Cu-semigroup T satisfying (O5) and (O6) such that S is a retract of T. Then T_c is a simple, cancellative refinement monoid, and therefore T_c has the Riesz interpolation property. Hence, T has the Riesz interpolation property by [APT18, Proposition 5.5.8(3)].

Since S is nonelementary, it follows from [APGPSM10, Theorem 6.7] that T_c is weakly divisible, that is, for every $x \in T_c$ there exist $y, z \in T_c$ such that $x = 2y + 3z$. This implies that T is almost divisible.

Now the result follows from Lemma 7.12. □

Question 7.14. Let S be a countably based, simple, weakly cancellative Cu-semigroup satisfying (O5) and (O6). Assume that S is almost divisible and has the Riesz interpolation property. Is S zero-dimensional?

Question 7.15. Let S be a zero-dimensional, weakly cancellative Cu-semigroup satisfying (O5). Does S have the Riesz interpolation property? Assuming also that S has no elementary quotients, is S almost divisible?

Note that a positive answer to Question 7.11 entails a positive answer to Question 7.15.

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