CONVEX FOLIATIONS OF DEGREE 5 ON THE COMPLEX PROJECTIVE PLANE

by

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Abstract. — We show that up to automorphism of $\mathbb{P}^2_\mathbb{C}$ there are 14 homogeneous convex foliations of degree five on $\mathbb{P}^2_\mathbb{C}$. Using this result, we give a partial answer to a question posed in 2013 by D. Marín and J. Pereira about the classification of reduced convex foliations on $\mathbb{P}^2_\mathbb{C}$.

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Introduction

Following [10] a foliation on the complex projective plane is said to be convex if its leaves other than straight lines have no inflection points. Notice (see [11]) that if $\mathcal{F}$ is a foliation of degree $d \geq 1$ on $\mathbb{P}^2_\mathbb{C}$, then $\mathcal{F}$ can not have more than $3d$ (distinct) invariant lines. Moreover, if this bound is reached, then $\mathcal{F}$ is necessarily convex; in this case $\mathcal{F}$ is said to be reduced convex. To our knowledge the only reduced convex foliations known in the literature are those presented in [10, Table 1.1]: the FERMAT foliation $\mathcal{F}_d^0$ of degree $d$, the HESSE pencil $\mathcal{F}_H^4$ of degree 4, the HILBERT modular foliation $\mathcal{F}_H^5$ of degree 5 and the HILBERT modular foliation $\mathcal{F}_H^7$ of degree 7 defined in affine chart respectively by the 1-forms
\begin{align*}
\omega_0^d &= (x^d-x)dy - (y^d-y)dx, \\
\omega_H^4 &= (2x^3-y^3-1)dy + (2y^3-x^3-1)xdy, \\
\omega_H^5 &= (y^2-1)(y^2-(\sqrt{5}-2)^2)(y+\sqrt{5})dx - (x^2-1)(x^2-(\sqrt{5}-2)^2)(x+\sqrt{5})dy, \\
\omega_H^7 &= (y^3-1)(y^3+7x^3+1)dy - (x^3-1)(x^3+7y^3+1)xdy.
\end{align*}

D. Marín and J. Pereira [10, Problem 9.1] asked the following question: are there other reduced convex foliations? The answer in degree 2, resp. 3, resp. 4, to this question is negative, by [9, Proposition 7.4], resp. [3, Corollary 6.9], resp. [4, Theorem B]. In this paper we show that the answer in degree 5 to [10, Problem 9.1] is also negative. To do this, we follow the same approach as that described in degree 4 in [4].

Key words and phrases. — convex foliation, homogeneous foliation, singularity, inflection divisor.

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More precisely, we begin by establishing the following theorem classifying the convex foliations of degree 5 on \( \mathbb{P}^2 \) which are homogeneous, i.e. which are invariant under homothety.

**Theorem A.** — Up to automorphism of \( \mathbb{P}^2 \), there are fourteen homogeneous convex foliations of degree five \( \mathcal{H}_1, \ldots, \mathcal{H}_{14} \) on the complex projective plane. They are respectively described in affine chart by the following 1-forms

1. \( \omega_1 = y^5 dx - x^5 dy; \)
2. \( \omega_2 = y^2(10x^3 + 10x^2y + 5xy^2 + y^3)dx - x^4(x + 5y)dy; \)
3. \( \omega_3 = y^3(10x^2 + 5xy + y^2)dx - x^3(x^2 + 5xy + 10y^2)dy; \)
4. \( \omega_4 = y^3(5x - 3y)dx + x^4(3x - 5y)dy; \)
5. \( \omega_5 = y^3(5x^2 - 3y^2)dx - 2x^5 dy; \)
6. \( \omega_6 = y^3(220x^2 - 165xy + 36y^2)dx - 121x^5 dy; \)
7. \( \omega_7 = y^4((5 - \sqrt{5})x - 2y)dx + x^4((7 - 3\sqrt{5})x - 2(5 - 2\sqrt{5})y)dy; \)
8. \( \omega_8 = y^4(5(3 - \sqrt{21})x + 6y)dx + x^4(3(23 - 5\sqrt{21})x - 10(9 - 2\sqrt{21})y)dy; \)
9. \( \omega_9 = y^4\left(2(5 + a)x^2 - (15 + a)xy + 6y^2\right)dx - x^4\left((1 - a)x + 2ay\right)dy, \text{ where } a = \sqrt{5(4\sqrt{61} - 31)}; \)
10. \( \omega_{10} = y^4\left(2(5 + ib)x^2 - (15 + ib)xy + 6y^2\right)dx - x^4\left((1 - ib)x + 2iby\right)dy, \text{ where } b = \sqrt{5(4\sqrt{61} + 31)}; \)
11. \( \omega_{11} = y^3(5x^2 - y^2)dx + x^3(x^2 - 5y^2)dy; \)
12. \( \omega_{12} = y^3(20x^2 - 5xy - y^2)dx + x^3(x^2 + 5xy - 20y^2)dy; \)
13. \( \omega_{13} = y^3(5x^2 - 10x^2y + 10xy^2 - 4y^3)dx - x^5 dy; \)
14. \( \omega_{14} = y^3\left(u(\sigma)x^2 + v(\sigma)xy + w(\sigma)y^2\right)dx + x^4\left(2\sigma(\sigma^2 - \sigma + 1)x - (\sigma + 1)(3\sigma^2 - 5\sigma + 3)y\right)dy, \)

where \( u(\sigma) = (\sigma^2 - 3\sigma + 1)(\sigma^2 + 5\sigma + 1), \ v(\sigma) = -2(\sigma + 1)(\sigma^2 - 5\sigma + 1), \ w(\sigma) = (\sigma^2 - 7\sigma + 1), \)
\( \sigma = \rho + i\sqrt{\frac{1}{6} - \frac{4}{3}\rho - \frac{1}{2\rho^2}} \) and \( \rho \) is the unique real number satisfying \( 8\rho^3 - 52\rho^2 + 134\rho - 15 = 0. \)

Then, using this classification, we prove the following theorem.

**Theorem B.** — Up to automorphism of \( \mathbb{P}^2 \), the Fermat foliation \( \mathcal{F}_0^5 \) and the Hilbert modular foliation \( \mathcal{F}_H^5 \) are the only reduced convex foliations of degree five on \( \mathbb{P}^2 \).
1. Preliminaries

1.1. Singularities and inflection divisor of a foliation on the projective plane. — A degree $d$ holomorphic foliation $\mathcal{F}$ on $\mathbb{P}^2_{\mathbb{C}}$ is defined in homogeneous coordinates $[x : y : z]$ by a 1-form

$$\omega = a(x, y, z)dx + b(x, y, z)dy + c(x, y, z)dz,$$

where $a$, $b$ and $c$ are homogeneous polynomials of degree $d + 1$ without common factor and satisfying the Euler condition $i_R\omega = 0$, where $R = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + z\frac{\partial}{\partial z}$ denotes the radial vector field and $i_R$ is the interior product by $R$. The singular locus $\text{Sing}\, \mathcal{F}$ of $\mathcal{F}$ is the projectivization of the singular locus of $\omega$

$$\text{Sing}\, \omega = \{(x, y, z) \in \mathbb{C}^3 | a(x, y, z) = b(x, y, z) = c(x, y, z) = 0\}.$$

Let us recall some local notions attached to the pair $(\mathcal{F}, s)$, where $s \in \text{Sing}\, \mathcal{F}$. The germ of $\mathcal{F}$ at $s$ is defined, up to multiplication by a unity in the local ring $O_s$ at $s$, by a vector field $X = A(u, v)\frac{\partial}{\partial u} + B(u, v)\frac{\partial}{\partial v}$. The algebraic multiplicity $\nu(\mathcal{F}, s)$ of $\mathcal{F}$ at $s$ is given by

$$\nu(\mathcal{F}, s) = \min\{\nu(A, s), \nu(B, s)\},$$

where $\nu(g, s)$ denotes the algebraic multiplicity of the function $g$ at $s$. The tangency order of $\mathcal{F}$ with a generic line passing through $s$ is the integer

$$\tau(\mathcal{F}, s) = \min\{k \geq \nu(\mathcal{F}, s) : \det(J^k_s X, R_s) \neq 0\},$$

where $J^k_s X$ denotes the $k$-jet of $X$ at $s$ and $R_s$ is the radial vector field centered at $s$. The Milnor number of $\mathcal{F}$ at $s$ is the integer

$$\mu(\mathcal{F}, s) = \dim_{\mathbb{C}} O_s / \langle A, B \rangle,$$

where $\langle A, B \rangle$ denotes the ideal of $O_s$ generated by $A$ and $B$. The singularity $s$ is called radial of order $n - 1$ if $\nu(\mathcal{F}, s) = 1$ and $\tau(\mathcal{F}, s) = n$.

The singularity $s$ is called non-degenerate if $\mu(\mathcal{F}, s) = 1$, or equivalently if the linear part $J^1_s X$ of $X$ possesses two non-zero eigenvalues $\lambda, \mu$. In this case, the quantity $BB(\mathcal{F}, s) = \frac{\lambda}{\mu} + \frac{\mu}{\lambda} + 2$ is called the Baum-Bott invariant of $\mathcal{F}$ at $s$ (see [1]). By [6] there is at least a germ of curve $\mathcal{C}$ at $s$ which is invariant by $\mathcal{F}$. Up to local diffeomorphism we can assume that $s = (0, 0)$, $T_s \mathcal{C} = \{v = 0\}$ and $J^1_s X = \lambda u \frac{\partial}{\partial u} + (\mu u + \epsilon v) \frac{\partial}{\partial v}$, where we can take $\epsilon = 0$ if $\lambda \neq \mu$. The quantity $CS(\mathcal{F}, \mathcal{C}, s) = \frac{\lambda}{\mu}$ is called the Camacho-Sad index of $\mathcal{F}$ at $s$ along $\mathcal{C}$.

Let us also recall the notion of inflection divisor of $\mathcal{F}$. Let $Z = E\frac{\partial}{\partial x} + F\frac{\partial}{\partial y} + G\frac{\partial}{\partial z}$ be a homogeneous vector field of degree $d$ on $\mathbb{C}^3$ non-collinear to the radial vector field describing $\mathcal{F}$, i.e. such that $\omega = i_R(Z) = 0$. The inflection divisor of $\mathcal{F}$, denoted by $I_\mathcal{F}$, is the divisor of $\mathbb{P}^2_{\mathbb{C}}$ defined by the homogeneous equation

$$\begin{vmatrix}
  x & E & Z(E) \\
  y & F & Z(F) \\
  z & G & Z(G)
\end{vmatrix} = 0.$$

This divisor has been studied in [11] in a more general context. In particular, the following properties has been proved.

1. On $\mathbb{P}^2_{\mathbb{C}} \setminus \text{Sing}\, \mathcal{F}$, $I_\mathcal{F}$ coincides with the curve described by the inflection points of the leaves of $\mathcal{F}$;
2. If $\mathcal{C}$ is an irreducible algebraic curve invariant by $\mathcal{F}$ then $\mathcal{C} \subset I_\mathcal{F}$ if and only if $\mathcal{C}$ is an invariant line;
3. $I_\mathcal{F}$ can be decomposed into $I_\mathcal{F} = I_{\mathcal{F}}^{\text{inv}} + I_{\mathcal{F}}^{\text{n inv}}$, where the support of $I_{\mathcal{F}}^{\text{inv}}$ consists in the set of invariant lines of $\mathcal{F}$ and the support of $I_{\mathcal{F}}^{\text{n inv}}$ is the closure of the isolated inflection points along the leaves of $\mathcal{F}$;
4. The degree of the divisor $I_\mathcal{F}$ is $3d$. 


The foliation $\mathcal{F}$ will be called convex if its inflection divisor $I_{\mathcal{F}}$ is totally invariant by $\mathcal{F}$, i.e. if $I_{\mathcal{F}}$ is a product of invariant lines.

1.2. Geometry of homogeneous foliations. — A foliation of degree $d$ on $\mathbb{P}^2_\mathbb{C}$ is said to be homogeneous if there is an affine chart $(x, y)$ of $\mathbb{P}^2_\mathbb{C}$ in which it is invariant under the action of the group of homotheties $(x, y) \mapsto \lambda(x, y), \lambda \in \mathbb{C}^\times$. Such a foliation $\mathcal{H}$ is then defined by a 1-form

$$\omega = A(x, y)dx + B(x, y)dy,$$

where $A$ and $B$ are homogeneous polynomials of degree $d$ without common factor. This 1-form writes in homogeneous coordinates as

$$zA(x, y)dx + zB(x, y)dy - (xA(x, y) + yB(x, y))dz.$$

Thus the foliation $\mathcal{H}$ has at most $d + 2$ singularities whose origin $O$ of the affine chart $z = 1$ is the only singular point of $\mathcal{H}$ which is not situated on the line at infinity $L_\infty = \{z = 0\}$; moreover $\nu(\mathcal{H}, O) = d$. In the sequel we will assume that $d$ is greater than or equal to $2$. In this case the point $O$ is the only singularity of $\mathcal{H}$ having algebraic multiplicity $d$.

We know from [3] that the inflection divisor of $\mathcal{H}$ is given by $zC_{\mathcal{H}}D_{\mathcal{H}} = 0$, where $C_{\mathcal{H}} = xA + yB \in \mathbb{C}[x, y]_{d+1}$ denotes the tangent cone of $\mathcal{H}$ at the origin $O$ and $D_{\mathcal{H}} = \frac{\partial A}{\partial x} \frac{\partial B}{\partial y} - \frac{\partial A}{\partial y} \frac{\partial B}{\partial x} \in \mathbb{C}[x, y]_{2d-2}$. From this we deduce that:

1. the support of the divisor $I_{\mathcal{H}}^{\text{inv}}$ consists of the lines of the tangent cone $C_{\mathcal{H}} = 0$ and the line at infinity $L_\infty$; 

2. the divisor $I_{\mathcal{H}}^{\text{tr}}$ decomposes as $I_{\mathcal{H}}^{\text{tr}} = \prod_{i=1}^{d} T_i^{\tau(\mathcal{H}, T_i)}$ for some number $n \leq \deg D_{\mathcal{H}} = 2d - 2$ of lines $T_i$ passing through $O$, $\rho_i - 1$ being the inflection order of the line $T_i$.

Proposition 1.1 ([3], Proposition 2.2). — With the previous notations, for any point $s \in \text{Sing}\mathcal{H} \cap L_\infty$, we have $\nu(\mathcal{H}, s) = 1$; the line joining the point $O$ to the point $s$ is invariant by $\mathcal{H}$ and it appears with multiplicity $\tau(\mathcal{H}, s) - 1$ in the divisor $D_{\mathcal{H}} = 0$, i.e.

$$D_{\mathcal{H}} = I_{\mathcal{H}}^{\text{tr}} \prod_{s \in \text{Sing}\mathcal{H} \cap L_\infty} L_s^{\tau(\mathcal{H}, s)-1}.$$

Definition 1.2 ([3]). — Let $\mathcal{H}$ be a homogeneous foliation of degree $d$ on $\mathbb{P}^2_\mathbb{C}$ having a certain number $m \leq d + 1$ of radial singularities $s_i$ of order $\tau_i - 1$, $2 \leq \tau_i \leq d$ for $i = 1, 2, \ldots, m$. The support of the divisor $I_{\mathcal{H}}^{\text{tr}}$ consists of a certain number $n \leq 2d - 2$ of transverse inflection lines $T_j$ of order $\rho_j - 1$, $2 \leq \rho_j \leq d$ for $j = 1, 2, \ldots, n$. We define the type of the foliation $\mathcal{H}$ by

$$T_{\mathcal{H}} = \sum_{i=1}^{m} R_{\tau_i-1} + \sum_{j=1}^{n} T_{\rho_j-1} = \sum_{k=1}^{d-1} (r_k \cdot R_k + t_k \cdot T_k) \in \mathbb{Z}[R_1, R_2, \ldots, R_{d-1}, T_1, T_2, \ldots, T_{d-1}].$$

Example 1.3. — Let us consider the homogeneous foliation $\mathcal{H}$ of degree $5$ on $\mathbb{P}^2_\mathbb{C}$ defined by

$$\omega = y^5 dx + 2x^3 (3x^2 - 5y^2) dy.$$

A straightforward computation leads to

$$C_{\mathcal{H}} = xy (6x^4 - 10x^2 y^2 + y^4) \quad \text{and} \quad D_{\mathcal{H}} = 150x^2 y^4 (x - y)(x + y).$$

We see that the set of radial singularities of $\mathcal{H}$ consists of the two points $s_1 = [0 : 1 : 0]$ and $s_2 = [1 : 0 : 0]$; their orders of radiality are equal to 2 and 4 respectively. Moreover the support of the divisor $I_{\mathcal{H}}^{\text{tr}}$ is the union of the two lines $x - y = 0$ and $x + y = 0$; they are transverse inflection lines of order 1. Therefore the foliation $\mathcal{H}$ is of type $T_{\mathcal{H}} = 1 \cdot R_2 + 1 \cdot R_4 + 2 \cdot T_1$. 

Following [3], to every homogeneous foliation $\mathcal{H}$ of degree $d$ on $\mathbb{P}^2_{\mathbb{C}}$ we can associate a rational map $\mathcal{G}_{\mathcal{H}} : \mathbb{P}^1_{\mathbb{C}} \rightarrow \mathbb{P}^1_{\mathbb{C}}$ in the following way: if $\mathcal{H}$ is described by $\omega = A(x,y)dx + B(x,y)dy$, with $A$ and $B$ being homogeneous polynomials of degree $d$ without common factor, we define $\mathcal{G}_{\mathcal{H}}$ by

$$\mathcal{G}_{\mathcal{H}}([x:y]) = [-A(x,y) : B(x,y)].$$

it is clear that this definition does not depend on the choice of the homogeneous 1-form $\omega$ describing the foliation $\mathcal{H}$.

Conversely, every rational map $f : \mathbb{P}^1_{\mathbb{C}} \rightarrow \mathbb{P}^1_{\mathbb{C}}$ of degree $d$ can be obtained in this way; indeed, if $f(z) = \frac{p(z)}{q(z)}$, with $p, q \in \mathbb{C}[z]$, $\operatorname{pgcd}(p, q) = 1$ and $\max(\deg p, \deg q) = d$, then $f = \mathcal{G}_{\mathcal{H}_f}$, where $\mathcal{H}_f$ is the homogeneous foliation of degree $d$ on $\mathbb{P}^2_{\mathbb{C}}$ defined by the 1-form

$$\omega_f = -x^d p \left( \frac{y}{x} \right) dx + x^d q \left( \frac{y}{x} \right) dy.$$

Let $\mathcal{H}$ be a homogeneous foliation of degree $d$ on $\mathbb{P}^2_{\mathbb{C}}$. Notice (see [3]) that the map $\mathcal{G}_{\mathcal{H}}$ has the following properties:

(i) the fixed points of $\mathcal{G}_{\mathcal{H}}$ correspond to the singular points of $\mathcal{H}$ on the line at infinity (i.e. $[a : b] \in \mathbb{P}^1_{\mathbb{C}}$ is fixed by $\mathcal{G}_{\mathcal{H}}$ if and only if the point $[b : a : 0] \in \mathbb{L}_\infty$ is singular for $\mathcal{H}$);

(ii) the point $[a : b] \in \mathbb{P}^1_{\mathbb{C}}$ is a fixed critical point of $\mathcal{G}_{\mathcal{H}}$ if and only if the point $[b : a : 0] \in \mathbb{L}_\infty$ is a radial singularity of $\mathcal{H}$. The multiplicity of the critical point $[a : b]$ of $\mathcal{G}_{\mathcal{H}}$ is exactly equal to the the radiality order of the singularity at infinity;

(iii) the point $[a : b] \in \mathbb{P}^1_{\mathbb{C}}$ is a non-fixed critical point of $\mathcal{G}_{\mathcal{H}}$ if and only if the line $by - ax = 0$ is a transverse inflection line of $\mathcal{H}$. The multiplicity of the critical point $[a : b]$ of $\mathcal{G}_{\mathcal{H}}$ is precisely equal to the inflection order of this line.

It follows, in particular, that a homogeneous foliation $\mathcal{H}$ on $\mathbb{P}^2_{\mathbb{C}}$ is convex if and only if its associated map $\mathcal{G}_{\mathcal{H}}$ has only fixed critical points; more precisely, a homogeneous foliation $\mathcal{H}$ of degree $d$ on $\mathbb{P}^2_{\mathbb{C}}$ is convex of type $\mathcal{T}_{\mathcal{H}} = \sum_{k=1}^{d-1} r_k \cdot \mathbb{R}_k$ if and only if the map $\mathcal{G}_{\mathcal{H}}$ possesses $r_1$, resp. $r_2$, ..., resp. $r_{d-1}$ fixed critical points of multiplicity 1, resp. 2, ..., resp. $d-1$, with $\sum_{k=1}^{d-1} kr_k = 2d - 2$.

**Remark 1.4.** — Every homogeneous convex foliation of degree $d$ on the complex projective plane has exactly $d + 1$ singularities on the line at infinity, necessarily non-degenerate. This follows from remark (i) above and Theorem 4.3 of [8] which ensures that if a rational map $f$ of degree $d$ from the RIEMANN sphere to itself has only fixed critical points, then $f$ admits $d + 1$ distinct fixed points.

### 2. Proof of Theorems A and B

We need to know the numbers $r_{ij}$ of radial singularities of order $j$ of the homogeneous foliations $\mathcal{H}_i$, $i = 1, \ldots, 14$, $j = 1, \ldots, 4$, and the values of the CAMACHO-SAD indices $\operatorname{CS}(\mathcal{H}_i, \mathbb{L}_\infty, s)$, $s \in \operatorname{Sing}\mathcal{H}_i \cap \mathbb{L}_\infty$, $i = 1, \ldots, 14$. For this reason, we have computed, for each $i = 1, \ldots, 14$, the type $\mathcal{T}_{\mathcal{H}_i}$ of $\mathcal{H}_i$ and the following polynomial (called CAMACHO-SAD polynomial of the homogeneous foliation $\mathcal{H}_i$)

$$\operatorname{CS}_{\mathcal{H}_i}(\lambda) = \prod_{s \in \operatorname{Sing}\mathcal{H}_i \cap \mathbb{L}_\infty} (\lambda - \operatorname{CS}(\mathcal{H}_i, \mathbb{L}_\infty, s)).$$


The following table summarizes the types and the CAMACHO-SAD polynomials of the foliations $\mathcal{F}_1, \ldots, \mathcal{F}_{14}$.

<table>
<thead>
<tr>
<th>$i$</th>
<th>$T_{\mathcal{F}_i}$</th>
<th>$\text{CS}_{\mathcal{F}_i}(\lambda)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$2 \cdot R_4$</td>
<td>$(\lambda - 1)^2(\lambda + \frac{1}{4})^4$</td>
</tr>
<tr>
<td>2</td>
<td>$1 \cdot R_1 + 1 \cdot R_3 + 1 \cdot R_4$</td>
<td>$\frac{1}{12} (\lambda - 1)^3(491\lambda^3 + 982\lambda^2 + 463\lambda + 64)$</td>
</tr>
<tr>
<td>3</td>
<td>$2 \cdot R_2 + 1 \cdot R_4$</td>
<td>$(\lambda - 1)^3(\lambda + \frac{3}{2})^2(\lambda + \frac{9}{4})$</td>
</tr>
<tr>
<td>4</td>
<td>$1 \cdot R_2 + 2 \cdot R_3$</td>
<td>$(\lambda - 1)^3(\lambda + \frac{9}{11})^2(\lambda + \frac{4}{11})$</td>
</tr>
<tr>
<td>5</td>
<td>$2 \cdot R_1 + 1 \cdot R_2 + 1 \cdot R_4$</td>
<td>$(\lambda - 1)^4(\lambda + \frac{5}{2})^2$</td>
</tr>
<tr>
<td>6</td>
<td>$2 \cdot R_1 + 1 \cdot R_2 + 1 \cdot R_4$</td>
<td>$\frac{1}{30} (\lambda - 1)^4(591\lambda^2 + 177\lambda + 64)$</td>
</tr>
<tr>
<td>7</td>
<td>$2 \cdot R_1 + 2 \cdot R_3$</td>
<td>$(\lambda - 1)^4(\lambda^2 + 3\lambda + 1)$</td>
</tr>
<tr>
<td>8</td>
<td>$2 \cdot R_1 + 2 \cdot R_3$</td>
<td>$(\lambda - 1)^4(\lambda + \frac{3}{2})^2$</td>
</tr>
<tr>
<td>9</td>
<td>$1 \cdot R_1 + 2 \cdot R_2 + 1 \cdot R_3$</td>
<td>$\frac{1}{12} (\lambda - 1)^4(197\lambda^2 + 591\lambda + 302 - 10\sqrt{61})$</td>
</tr>
<tr>
<td>10</td>
<td>$1 \cdot R_1 + 2 \cdot R_2 + 1 \cdot R_3$</td>
<td>$\frac{1}{12} (\lambda - 1)^4(197\lambda^2 + 591\lambda + 302 + 10\sqrt{61})$</td>
</tr>
<tr>
<td>11</td>
<td>$4 \cdot R_2$</td>
<td>$(\lambda - 1)^4(\lambda + \frac{3}{2})^2$</td>
</tr>
<tr>
<td>12</td>
<td>$2 \cdot R_1 + 3 \cdot R_2$</td>
<td>$(\lambda - 1)^5(\lambda + 4)$</td>
</tr>
<tr>
<td>13</td>
<td>$4 \cdot R_1 + 1 \cdot R_4$</td>
<td>$(\lambda - 1)^5(\lambda + 4)$</td>
</tr>
<tr>
<td>14</td>
<td>$3 \cdot R_1 + 1 \cdot R_2 + 1 \cdot R_3$</td>
<td>$(\lambda - 1)^5(\lambda + 4)$</td>
</tr>
</tbody>
</table>

**Table 1.** Types and CAMACHO-SAD polynomials of the homogeneous foliations $\mathcal{F}_1, \ldots, \mathcal{F}_{14}$.

Before beginning the proof of Theorem A, let us recall the following result which follows from Propositions 4.1 and 4.2 of [3]:

**Proposition 2.1 ([3]).** — Let $\mathcal{F}$ be a convex homogeneous foliation of degree $d \geq 3$ on $\mathbb{P}^2_C$. Let $\nu$ be an integer between 1 and $d - 2$. Then, $\mathcal{F}$ is of type

$$T_{\mathcal{F}} = 2 \cdot R_{d-1},$$

resp. $T_{\mathcal{F}} = 1 \cdot R_{\nu} + 1 \cdot R_{d-\nu-1} + 1 \cdot R_{d-1},$

if and only if it is linearly conjugated to the foliation $\mathcal{F}_{1}^{d}$, resp. $\mathcal{F}_{2}^{d, \nu}$ given by

$$\omega_{1}^{d} = x^d dx - x^d dy,$$

resp. $\omega_{2}^{d, \nu} = \sum_{i=\nu+1}^{d} \binom{d}{i} x^{d-i} y^i dx - \sum_{i=0}^{\nu} \binom{d}{i} x^{d-i} y^i dy.$
Proof of Theorem A. — Let \( \mathcal{H} \) be a convex homogeneous foliation of degree 5 on \( \mathbb{P}^2 \mathbb{C} \), defined in the affine chart \((x,y)\), by the 1-form
\[
\omega = A(x,y)dx + B(x,y)dy, \quad A, B \in \mathbb{C}[x,y]_5, \quad \gcd(A,B) = 1.
\]

By [5, Remark 2.5] the foliation \( \mathcal{H} \) can not have \( 5 + 1 = 6 \) distinct radial singularities; in other words it cannot be of one of the two types \( 5 \cdot R_1 + 1 \cdot R_3 \) or \( 4 \cdot R_1 + 2 \cdot R_2 \). We are then in one of the following situations:

\[
\begin{align*}
\mathcal{T}_g & = 2 \cdot R_4; & \mathcal{T}_g & = 1 \cdot R_1 + 1 \cdot R_3 + 1 \cdot R_4; & \mathcal{T}_g & = 2 \cdot R_2 + 1 \cdot R_4; \\
\mathcal{T}_g & = 1 \cdot R_2 + 2 \cdot R_3; & \mathcal{T}_g & = 2 \cdot R_1 + 1 \cdot R_2 + 1 \cdot R_4; & \mathcal{T}_g & = 2 \cdot R_1 + 2 \cdot R_3; \\
\mathcal{T}_g & = 1 \cdot R_1 + 2 \cdot R_2 + 1 \cdot R_3; & \mathcal{T}_g & = 4 \cdot R_2; & \mathcal{T}_g & = 2 \cdot R_1 + 3 \cdot R_2; \\
\mathcal{T}_g & = 4 \cdot R_1 + 1 \cdot R_4; & \mathcal{T}_g & = 3 \cdot R_1 + 1 \cdot R_2 + 1 \cdot R_3.
\end{align*}
\]

- If the foliation \( \mathcal{H} \) is of type \( \mathcal{T}_g = 2 \cdot R_4 \), resp. \( \mathcal{T}_g = 1 \cdot R_1 + 1 \cdot R_3 + 1 \cdot R_4 \), resp. \( \mathcal{T}_g = 2 \cdot R_2 + 1 \cdot R_4 \), then, by [3, Propositions 4.1, 4.2], the 1-form \( \omega \) is linearly conjugated to
\[
\omega_1 = y^5 dx - x^5 dy = \omega_1,
\]
resp. \( \omega_3 = \sum_{i=1}^{5} \begin{pmatrix} 5 \\ i \end{pmatrix} x^{5-i} y^i dx - \sum_{i=0}^{1} \begin{pmatrix} 5 \\ i \end{pmatrix} x^{5-i} y^i dy = y^2(10x^3 + 10xy^2 + y^3) dx - x^4(x + 5y) dy = \omega_2,
\]
resp. \( \omega_3 = \sum_{i=3}^{5} \begin{pmatrix} 5 \\ i \end{pmatrix} x^{5-i} y^i dx - \sum_{i=1}^{2} \begin{pmatrix} 5 \\ i \end{pmatrix} x^{5-i} y^i dy = y^3(10x^2 + 5xy + y^2) dx - x^4(x^2 + 5xy + 10y^2) dy = \omega_3.
\]

- Assume that \( \mathcal{T}_g = 1 \cdot R_2 + 2 \cdot R_3 \). This means that the rational map \( \mathcal{G}_{\mathcal{H}} : \mathbb{P}^1 \mathbb{C} \to \mathbb{P}^1 \mathbb{C} \), \( \mathcal{G}_{\mathcal{H}}(z) = \frac{A(1,z)}{B(1,z)} \), possesses three fixed critical points, one of multiplicity 2 and two of multiplicity 3. By [7, page 79], \( \mathcal{G}_{\mathcal{H}} \)
is conjugated by a Möbius transformation to \( z \mapsto -\frac{z^3(3z^2 - 5)}{5z - 3} \). As a result, \( \omega \) is linearly conjugated to
\[
\omega_4 = y^4(5x - 3y) dx + x^4(3x - 5y) dy.
\]

- Let us study the possibility \( \mathcal{T}_g = 2 \cdot R_1 + 1 \cdot R_2 + 1 \cdot R_4 \). Up to linear conjugation we can assume that \( D_\mathcal{H} = cx^2y^2(y-x)(y-\alpha x) \) and \( C_\mathcal{H}(0,1) = C_\mathcal{H}(1,0) = C_\mathcal{H}(1,1) = C_\mathcal{H}(1,\alpha) = 0 \), for some \( c, \alpha \in \mathbb{C}^* \), \( \alpha \neq 1 \). The points \( \infty = [1 : 0], [0 : 1], [1 : 1], [1 : \alpha] \in \mathbb{P}^1 \mathbb{C} \) are then fixed and critical for \( \mathcal{G}_{\mathcal{H}} \), with respective multiplicities 4, 2, 1, 1. By [3, Lemma 3.9], there exist constants \( a_0, a_2, b \in \mathbb{C}^*, a_1 \in \mathbb{C} \) such that
\[
B(x,y) = bx^5, \quad A(x,y) = (a_0x^2 + a_1xy + a_2y^2)y^3, \quad (z - 1)^2 \text{ divides } P(z), \quad (z - \alpha)^2 \text{ divides } Q(z),
\]
where \( P(z) := A(1,z) + B(1,z) \) and \( Q(z) := A(1,z) + \alpha B(1,z) \).
Therefore we have
\[
\begin{align*}
P(1) &= 0 \quad \Leftrightarrow \quad a_0 + a_1 + a_2 + b = 0 \\
P'(1) &= 0 \quad \Leftrightarrow \quad 3a_0 + 4a_1 + 5a_2 = 0 \\
Q(\alpha) &= 0 \quad \Leftrightarrow \quad a_2\alpha^4 + a_1\alpha^3 + a_0\alpha^2 + b = 0 \\
Q'(\alpha) &= 0 \quad \Leftrightarrow \quad 5a_2\alpha^2 + 4a_1\alpha + 3a_0 = 0 \\
\end{align*}
\]

Replacing \( \frac{12}{d_2} \omega \), we reduce it to
\[
\omega = y^3(20\alpha x^2 - 15(\alpha + 1)xy + 12y^2)dx - (5\alpha - 3)x^5dy, \quad (\alpha + 1)(3\alpha^2 - 5\alpha + 3) = 0.
\]

This 1-form is linearly conjugated to one of the two 1-forms
\[
\omega_5 = y^3(5x^2 - 3y^2)dx - 2x^5dy \quad \text{or} \quad \omega_6 = y^3(220x^2 - 165xy + 36y^2)dx - 121x^5dy.
\]

Indeed, on the one hand, if \( \alpha = -1 \), then \( \omega_5 = -\frac{1}{4}\omega \). On the other hand, if \( 3\alpha^2 - 5\alpha + 3 = 0 \), then
\[
\omega_6 = \frac{121(15\alpha - 16)}{81(3\alpha - 8)^2}\phi^\ast\omega, \quad \text{where} \quad \phi = \left(3\alpha - 8\right)x - 3y.
\]

Assume that \( \mathcal{T}_{\mathcal{H}} = 2 \cdot \mathcal{R}_1 + 2 \cdot \mathcal{R}_3 \). Then the rational map \( \mathcal{G}_{\mathcal{H}} \) admits four fixed critical points, two of multiplicity 1 and two of multiplicity 3. This implies, by [7, page 79], that up to conjugation by a Möbius transformation, \( \mathcal{G}_{\mathcal{H}} \) writes as
\[
z \mapsto \frac{\varepsilon^4(3z + 4cz - 5c - 4)}{z + c},
\]

where \( c = -1/2 \pm \sqrt{5}/10 \) or \( c = -3/10 \pm \sqrt{21}/10 \). Thus, up to linear conjugation
\[
\omega = y^4(3y + 4cy - 5cx - 4x)dx + x^4(y + cx)dy, \quad c \in \left\{-\frac{1}{2} \pm \frac{\sqrt{5}}{10}, \frac{3}{10} \pm \frac{\sqrt{21}}{10}\right\}.
\]

In the case where \( c = -1/2 \pm \sqrt{5}/10 \), resp. \( c = -3/10 \pm \sqrt{21}/10 \), the 1-form \( \omega \) is linearly conjugated to
\[
\omega_7 = y^4\left(5 - \sqrt{5}\right)x - 2y)dx + x^4\left(7 - 3\sqrt{5}\right)x - 2(5 - 2\sqrt{5})y)dy,
\]

resp. \( \omega_8 = y^4\left(5(3 - \sqrt{21})x + 6y\right)dx + x^4\left(3(23 - 5\sqrt{21})x - 10(9 - 2\sqrt{21})y\right)dy.
\]

Indeed, on the one hand, if \( c = -1/2 + \sqrt{5}/10 \), resp. \( c = -3/10 + \sqrt{21}/10 \), then \( \omega_7 = -2(5 - 2\sqrt{5})\omega \), resp. \( \omega_8 = -10(9 - 2\sqrt{21})\omega \). On the other hand, if \( c = -1/2 - \sqrt{5}/10 \), resp. \( c = -3/10 - \sqrt{21}/10 \), then
\[
\omega_7 = -\left(25 + 11\sqrt{5}\right)\varphi^\ast\omega, \quad \text{where} \quad \varphi = \left(\frac{1 - \sqrt{5}}{2}x, y\right),
\]

resp. \( \omega_8 = 5(87 + 19\sqrt{21})\psi^\ast\omega, \quad \text{where} \quad \psi = \left(\frac{\sqrt{21} - 5}{2}x, y\right).
\]

By Table 1, we have on the one hand \( \text{CS}_{\mathcal{H}_9} \neq \text{CS}_{\mathcal{H}_{10}} \), so that the foliations \( \mathcal{H}_9 \) and \( \mathcal{H}_{10} \) are not linearly conjugated, and on the other hand \( \mathcal{T}_{\mathcal{H}_9} = \mathcal{T}_{\mathcal{H}_{10}} = 1 \cdot \mathcal{R}_1 + 2 \cdot \mathcal{R}_2 + 1 \cdot \mathcal{R}_3 \). Moreover, by [7, page 79], up to Möbius transformation there are two rational maps of degree 5 from the Riemann sphere to itself having four distinct fixed critical points, one of multiplicity 1, two of multiplicity 2 and one of
multiplicity 3; thus up to automorphism of $\mathbb{P}^2_C$ there are two homogeneous convex foliations of degree 5 on $\mathbb{P}^2_C$ having type $1 \cdot R_1 + 2 \cdot R_2 + 1 \cdot R_3$. As a result, we deduce that if the foliation $\mathcal{F}$ is of type $\mathcal{I}_{5\mathcal{F}} = 1 \cdot R_1 + 2 \cdot R_2 + 1 \cdot R_3$, then $\mathcal{F}$ is linearly conjugated to one of the two foliations $\mathcal{F}_0$ or $\mathcal{F}_{10}$.

- Assume that $\mathcal{I}_{5\mathcal{F}} = 4 \cdot R_2$. The rational map $\mathcal{G}_{\mathcal{F}}$ has therefore four different fixed critical points of multiplicity 2. By [7, page 80], up to conjugation by a MÖBIUS transformation, $\mathcal{G}_{\mathcal{F}}$ writes as

$$z \mapsto -\frac{z^3(z^2 - 5z + 5)}{5z^2 - 10z + 4}.$$ 

As a consequence, up to linear conjugation

$$\omega = y^3(5x^2 - 5xy + y^2)dx + x^3(4x^2 - 10xy + 5y^2)dy.$$ 

This 1-form is linearly conjugated to $\omega_{11} = y^3(5x^2 - y^2)dx + x^3(x^2 - 5y^2)dy$; indeed

$$\omega_{11} = \frac{1}{8}\varphi^*\omega, \quad \text{where } \varphi = (x, y, 2y).$$

- Assume that $\mathcal{I}_{5\mathcal{F}} = 2 \cdot R_1 + 3 \cdot R_2$. Then the rational map $\mathcal{G}_{\mathcal{F}}$ possesses five fixed critical points, two of multiplicity 1 and three of multiplicity 2. By [7, page 80], $\mathcal{G}_{\mathcal{F}}$ is conjugated by a MÖBIUS transformation to $z \mapsto -\frac{z^3(z^2 + 5z - 20)}{20z^2 - 5z - 1}$, which implies that $\omega$ is linearly conjugated to

$$\omega_{12} = y^3(20x^2 - 5xy - y^2)dx + x^3(x^2 + 5xy - 20y^2)dy.$$ 

- Let us consider the eventuality $\mathcal{I}_{5\mathcal{F}} = 4 \cdot R_1 + 1 \cdot R_4$. We can assume, up to linear conjugation, that $D_{5\mathcal{F}} = cx^4(y - x)(y - \alpha x)(y - \beta x)$ and $C_{5\mathcal{F}}(0, 1) = C_{5\mathcal{F}}(1, 0) = C_{5\mathcal{F}}(1, 1) = C_{5\mathcal{F}}(1, \alpha) = C_{5\mathcal{F}}(1, \beta) = 0$, where $\alpha, \beta \in \mathbb{C} \setminus \{0, 1\}, c \in \mathbb{C}^\ast$, with $\alpha \neq \beta$. The points $\omega = [1 : 0], [0 : 1], [1 : 1], [1 : \alpha], [1 : \beta] \in \mathbb{P}_1^1$ are therefore fixed and critical for $\mathcal{G}_{\mathcal{F}}$, with respective multiplicities 4, 1, 1, 1, 1, 1. By [3, Lemma 3.9], there exist constants $a_0, a_3, b \in \mathbb{C}^\ast, a_1, a_2 \in \mathbb{C}$ such that

$$B(x,y) = bx^5, \quad A(x,y) = (a_0x^3 + a_1x^2y + a_2xy^2 + a_3y^3)y^2, \quad (z - \alpha)^2 \text{ divides } P(z), \quad (z - \beta)^2 \text{ divides } Q(z),$$

where $P(z) := A(1,z) + B(1,z), Q(z) := A(1,z) + \alpha B(1,z)$ and $R(z) := A(1,z) + \beta B(1,z)$. Then we have

$$\begin{align*}
P(1) &= 0 & a_0 + a_1 + a_2 + a_3 + b &= 0 & a_0 &= -\frac{a_3a_\alpha(\alpha + 1)(3\alpha^2 - 5\alpha + 3)}{2(\alpha^2 - \alpha + 1)} \\
P'(1) &= 0 & 2a_0 + 3a_1 + 4a_2 + 5a_3 &= 0 & a_1 &= \frac{a_3(\alpha^4 + 2\alpha^3 - 3\alpha^2 + 2\alpha + 1)}{\alpha^2 - \alpha + 1} \\
Q(\alpha) &= 0 & a_3\alpha^4 + a_2\alpha^3 + a_1\alpha^2 + a_0\alpha + b &= 0 & a_2 &= \frac{a_3(\alpha + 1)(4\alpha^4 - 5\alpha^2 + 4\alpha + 4)}{2(\alpha^2 - \alpha + 1)} \\
Q'(\alpha) &= 0 & 5a_3\alpha^3 + 4a_2\alpha^2 + 3a_1\alpha + 2a_0 &= 0 & b &= \frac{a_3\alpha^2(\alpha - 1)^2}{2(\alpha^2 - \alpha + 1)} \\
R(\beta) &= 0 & a_3\beta^4 + a_2\beta^3 + a_1\beta^2 + a_0\beta + b &= 0 & \beta &= \frac{(\alpha + 1)(3\alpha^2 - 5\alpha + 3)}{5(\alpha^2 - \alpha + 1)} \\
R'(\beta) &= 0 & 5a_3\beta^3 + 4a_2\beta^2 + 3a_1\beta + 2a_0 &= 0 & (\alpha^2 - 2\alpha + 2)(2\alpha^2 - 2\alpha + 1)(\alpha^2 + 1) &= 0
\end{align*}$$

CONVEX FOLIATIONS OF DEGREE 5 ON THE COMPLEX PROJECTIVE PLANE
Multiplying $\omega$ by $\frac{2}{\alpha^3} (\alpha^2 - \alpha + 1)$, we reduce it to

$$\omega = -y^2 \left( \alpha (\alpha + 1) (3\alpha^2 - 5\alpha + 3)x^3 + (\alpha + 1) (4\alpha^2 - 5\alpha + 4)xy^2 - 2(\alpha^2 - \alpha + 1)y^3 \right) dx$$

$$+ 2(\alpha^4 + 2\alpha^3 - 3\alpha^2 + 2\alpha + 1)x^2 y dx + \alpha^2 (\alpha - 1)^2 x^5 dy,$$

with $(\alpha^2 - 2\alpha + 2)(2\alpha^2 - 2\alpha + 1)(\alpha^2 + 1) = 0$. This 1-form $\omega$ is linearly conjugated to

$$\omega_{13} = y^2 (5x^3 - 10x^2 y + 10xy^2 - 4y^3) dx - x^5 dy.$$ 

Indeed, the fact that $\alpha$ satisfies $(\alpha^2 - 2\alpha + 2)(2\alpha^2 - 2\alpha + 1)(\alpha^2 + 1) = 0$ implies that

$$\omega_{13} = -\frac{(\alpha + 1)(3\alpha^2 - 5\alpha + 3)}{5\alpha^2 (\alpha - 1)^4} \varphi^* \omega,$$

where $\varphi = \left( x, \frac{5\alpha (\alpha - 1)^2}{(\alpha + 1)(3\alpha^2 - 5\alpha + 3)} y \right)$.

Finally let us examine the case $T_{\gamma} = 3 \cdot R_1 + 1 \cdot R_2 + 1 \cdot R_3$. Up to isomorphism, we can assume that $D_{\gamma} = cx^3 y^3 (y - x)(y - \alpha x)(y - \beta x)$ and $C_{\gamma}(0, 1) = C_{\gamma}(1, 0) = C_{\gamma}(1, 1) = C_{\gamma}(1, \alpha) = C_{\gamma}(1, \beta) = 0$, where $\alpha, \beta \in \mathbb{C} \setminus \{0, 1\}, c \in \mathbb{C}^*$, with $\alpha \neq \beta$. A similar reasoning as in the previous case leads to

$$\omega = \omega(\alpha) = y^3 \left( (\alpha^2 - 3\alpha + 1)(\alpha^2 + 5\alpha + 1)x^2 - 2(\alpha + 1)(\alpha^2 - 5\alpha + 1)xy + (\alpha^2 - 7\alpha + 1)y^2 \right) dx$$

$$+ \alpha x^4 \left( 2\alpha(\alpha^2 - \alpha + 1)x - (\alpha + 1)(3\alpha^2 - 5\alpha + 3)y \right) dy,$$

with $P(\alpha) = 0$ where $P(z) := 3z^6 - 39z^5 + 194z^4 - 203z^3 + 194z^2 - 39z + 3$. The 1-form $\omega$ is linearly conjugated to

$$\omega_{14} = y^3 \left( (\sigma^2 - 3\sigma + 1)(\sigma^2 + 5\sigma + 1)x^2 - 2(\sigma + 1)(\sigma^2 - 5\sigma + 1)xy + (\sigma^2 - 7\sigma + 1)y^2 \right) dx$$

$$+ \sigma x^4 \left( 2\sigma(\sigma^2 - \sigma + 1)x - (\sigma + 1)(3\sigma^2 - 5\sigma + 3)y \right) dy,$$

where $\sigma = \rho + i\sqrt{\frac{\sqrt{5} - 4}{6}}\rho - \frac{1}{4}\rho^2$ and $\rho$ is the unique real number satisfying $8\rho^3 - 52\rho^2 + 134\rho - 15 = 0$. Indeed, on the one hand, it is easy to see that $\sigma$ is a root of the polynomial $P$, so that $\omega_{14} = \omega(\sigma)$. On the other hand, a straightforward computation shows that if $\alpha_1$ and $\alpha_2$ are any two roots of $P$ then

$$\omega(\alpha_2) = -\frac{\mu}{21600} \left( 13035\alpha_2^4 - 167802\alpha_2^3 + 821633\alpha_2^2 - 777666\alpha_2 + 743778\alpha_1 - 76185 \right) \varphi^* (\omega(\alpha_1))$$

with $\mu = 195\alpha_2^4 - 202\alpha_2^3 + 233\alpha_2^2 - 42\alpha_2 + 3$, $\varphi = \left( x, -\frac{\lambda}{43200} y \right)$ where

$$\lambda = \left( 39\alpha_2^5 - 501\alpha_2^4 + 2447\alpha_2^3 - 2293\alpha_2^2 + 2343\alpha_2 - 477 \right) \left( 124\alpha_1^5 - 309\alpha_1^4 + 1510\alpha_1^3 - 1415\alpha_1^2 + 1446\alpha_1 - 21 \right).$$

The foliations $\mathcal{F}_1, \ldots, \mathcal{F}_{14}$ are not linearly conjugated because for all $i, j \in \{1, \ldots, 14\}$ with $i \neq j$ we have (see Table 1)

$$T_{\mathcal{F}_i} \neq T_{\mathcal{F}_j} \quad \text{or} \quad \text{CS}_{\mathcal{F}_i} \neq \text{CS}_{\mathcal{F}_j}.$$ 

This ends the proof of the theorem. $\square$

An immediate consequence of Theorem A is the following.

**Corollary 2.2.** — Up to Möbius transformation there are fourteen rational maps of degree five from the Riemann sphere to itself having only fixed critical points, namely the maps $G_{\mathcal{F}_1}, \ldots, G_{\mathcal{F}_{14}}$. 

Proof of Theorem B. — Let $\mathcal{F}$ be a reduced convex foliation of degree 5 on $\mathbb{P}^2\mathbb{C}$. Let us denote by $\Sigma$ the set of non radial singularities of $\mathcal{F}$. By [4, Lemma 3.4], $\Sigma$ is nonempty. Since by hypothesis $\mathcal{F}$ is reduced convex, all its singularities have MILNOR number 1 ([3, Lemma 6.8]). The set $\Sigma$ consists then of the singularities $s \in \text{Sing}\mathcal{F}$ such that $\tau(\mathcal{F}, s) = 1$. Let $m$ be a point of $\Sigma$; by [4, Lemma 3.1], through the point $m$ pass exactly two $\mathcal{F}$-invariant lines $\ell_m^{(1)}$ and $\ell_m^{(2)}$.

On the other hand, according to [4, Proposition 3.2] or [3, Proposition 6.4], for any line $\ell$ invariant by $\mathcal{F}$, there exists a homogeneous convex foliation $\mathcal{H}_\ell$ of degree 5 on $\mathbb{P}^2\mathbb{C}$ such that the line $\ell$ is $\mathcal{H}_\ell$-invariant. Therefore $\mathcal{H}_\ell$, and in particular each $\mathcal{H}_{\ell_m}$, is linearly conjugated to one of the fourteen homogeneous foliations given by Theorem A. Proposition 3.2 of [4] also ensures that

(a) $\text{Sing}\mathcal{F} \cap \ell = \text{Sing}\mathcal{H}_\ell \cap \ell$;

(b) $\forall s \in \text{Sing}\mathcal{H}_\ell \cap \ell$, $\mu(\mathcal{H}_\ell, s) = 1$;

(c) $\forall s \in \text{Sing}\mathcal{H}_\ell \cap \ell$, $\tau(\mathcal{H}_\ell, s) = \tau(\mathcal{F}, s)$;

(d) $\forall s \in \text{Sing}\mathcal{H}_\ell \cap \ell$, $\text{CS}(\mathcal{H}_\ell, \ell, s) = \text{CS}(\mathcal{F}, \ell, s)$.

Since $\text{CS}(\mathcal{F}, \ell_m^{(1)}, m)\text{CS}(\mathcal{F}, \ell_m^{(2)}, m) = 1$, relation (d) implies that $\text{CS}(\mathcal{H}_{\ell_m}, \ell_m^{(1)}, m)\text{CS}(\mathcal{H}_{\ell_m}, \ell_m^{(2)}, m) = 1$. This equality and Table 1 lead to

$$\text{(2.1)} \quad \left\{ \text{CS}(\mathcal{H}_{\ell_m}, \ell_m^{(1)}, m), \text{CS}(\mathcal{H}_{\ell_m}, \ell_m^{(2)}, m) \right\} = \left\{ -4, -\frac{1}{4} \right\}.$$

At first let us assume that it is possible to choose $m \in \Sigma$ so that

$$\{\text{CS}(\mathcal{H}_{\ell_m}, \ell_m^{(1)}, m), \text{CS}(\mathcal{H}_{\ell_m}, \ell_m^{(2)}, m)\} = \{-4, -\frac{1}{4}\}.$$

Up to renumbering the $\ell_m^{(i)}$ we can assume that $\text{CS}(\mathcal{H}_{\ell_m}, \ell_m^{(1)}, m) = -\frac{1}{4}$ and $\text{CS}(\mathcal{H}_{\ell_m}, \ell_m^{(2)}, m) = -4$. Consulting Table 1, we see that

$$\mathcal{T}_{\mathcal{H}_{\ell_m}} = 2 \cdot R_4, \quad \mathcal{T}_{\mathcal{H}_{\ell_m}} = \left\{ 2 \cdot R_1 + 3 \cdot R_2, 4 \cdot R_1 + 1 \cdot R_4, 3 \cdot R_1 + 1 \cdot R_2 + 1 \cdot R_3 \right\}.$$

Therefore, it follows from relations (a) and (c) that $\mathcal{F}$ possesses two radial singularities $m_1, m_2$ of order 4 on the line $\ell_m^{(1)}$ and a radial singularity $m_3$ of order 2 or 4 on the line $\ell_m^{(2)}$.

We will see that the radicial order of the singularity $m_3$ of $\mathcal{F}$ is necessarily 4, i.e. $\tau(\mathcal{F}, m_3) = 5$. By [2, Proposition 2, page 23], the fact that $\tau(\mathcal{F}, m_1) + \tau(\mathcal{F}, m_3) \geq 5 + 3 \geq \deg \mathcal{F}$ implies the invariance by $\mathcal{F}$ of the line $\ell = (m_1, m_3)$; if $\tau(\mathcal{F}, m_3)$ were equal to 3, then relations (a), (b) and (c), combined with the convexity of the foliation $\mathcal{H}_\ell$, would imply that

$$\mathcal{T}_{\mathcal{H}_\ell} \in \left\{ 2 \cdot R_2 + 1 \cdot R_4, 2 \cdot R_1 + 1 \cdot R_2 + 1 \cdot R_4 \right\}$$

so that (see Table 1) $\mathcal{H}_\ell$ would possess a singularity $m'$ on the line $\ell$ satisfying

$$\text{CS}(\mathcal{H}_\ell, \ell, m') \in \left\{ \lambda \in \mathbb{C} : (\lambda + \frac{3}{2}) (\lambda + \frac{1}{2}) (59\lambda^2 + 177\lambda + 64) = 0 \right\}$$

which is not possible by (2.1).

By construction, the three points $m_1, m_2$ and $m_3$ are not aligned. We have thus shown that $\mathcal{F}$ admits three non-aligned radial singularities of order 4. By [3, Proposition 6.3] the foliation $\mathcal{F}$ is linearly conjugated to the FERMAT foliation $\mathcal{F}_0^5$. 
Let us now consider the eventuality \( \{CS(\ell^{(1)}_m, \ell^{(1)}_m), CS(\ell^{(2)}_m, \ell^{(2)}_m)\} = \{-\frac{3}{2} + \frac{1}{2}\sqrt{5}, -\frac{3}{2} - \frac{1}{2}\sqrt{5}\} \) for any choice of \( m \in \Sigma \). In this case, Table 1 leads to \( T_{\ell^{(1)}_m} = 2 \cdot R_1 + 2 \cdot R_3 \) for \( i = 1, 2 \). Then, as before, by using relations (a), (b) and (c), we obtain that \( \mathcal{F} \) possesses exactly four radial singularities on each of the lines \( \ell^{(1)}_m \), two of order 1 and two of order 3. Moreover, every line joining a radial singularity of order 3 of \( \mathcal{F} \) on \( \ell^{(1)}_m \) and a radial singularity of order 3 of \( \mathcal{F} \) on \( \ell^{(2)}_m \) must necessarily contain two radial singularities of order 1 of \( \mathcal{F} \). We can then choose a homogeneous coordinate system \( [x : y : z] \in \mathbb{P}^2 \mathbb{C} \) in such a way that the points \( m_1 = [0 : 0 : 1] \), \( m_2 = [1 : 0 : 0] \) and \( m_3 = [0 : 1 : 0] \) are radial singularities of order 3 of \( \mathcal{F} \). Moreover, in this coordinate system the lines \( x = 0, y = 0, z = 0 \) must be invariant by \( \mathcal{F} \) and there exist \( x_0, y_0, z_0, x_1, y_1, z_1 \in \mathbb{C}^*, x_1 \neq x_0, y_1 \neq y_0, z_1 \neq z_0 \), such that the points \( m_4 = [x_0 : 0 : 1] \), \( m_5 = [1 : y_0 : 0] \), \( m_6 = [0 : 1 : z_0] \), \( m_7 = [x_1 : 0 : 1] \), \( m_8 = [1 : y_1 : 0] \), \( m_9 = [0 : 1 : z_1] \) are radial singularities of order 1 of \( \mathcal{F} \). Let us set \( \xi = \frac{x_1}{x_0}, \rho = \frac{y_1}{y_0}, \sigma = \frac{z_1}{z_0} \); then \( w_0 \in \mathbb{C}^*, \xi, \rho, \sigma \in \mathbb{C} \setminus \{0, 1\} \) and, up to renumbering the \( x_i, y_i, z_i \), we can assume that \( \xi, \rho \) and \( \sigma \) are all of modulus greater than or equal to 1. Let \( \omega \) be a 1-form describing \( \mathcal{F} \) in the affine chart \( z = 1 \). By conjugating \( \omega \) by the diagonal linear transformation \( (x_0, x_0y_0), \) we reduce ourselves to \( m_4 = [1 : 0 : 1] \), \( m_5 = [1 : 1 : 0] \), \( m_6 = [0 : 1 : w_0] \), \( m_7 = [\xi : 0 : 1] \), \( m_8 = [1 : \rho : 0] \), \( m_9 = [0 : 1 : \sigma w_0] \). The equalities \( \nu(\mathcal{F}, m_1) = 1 \), \( \tau(\mathcal{F}, m_1) = 4 \) and the invariance of the line \( z = 0 \) by \( \mathcal{F} \) ensure that \( \omega \) is of type

\[
\omega = (dx - y dx)(y + \lambda_0 x + \lambda_1 y + c_0 x^2 + c_1 x y + c_2 y^2) + (\alpha_0 x^4 + \alpha_1 x^3 y + \alpha_2 x^2 y^2 + \alpha_3 x y^3 + \alpha_4 y^4) dx
\]

\[
+ (\beta_0 x^4 + \beta_1 x^3 y + \beta_2 x^2 y^2 + \beta_3 x y^3 + \beta_4 y^4) dy + (a_0 x^5 + a_1 x^4 y + a_2 x^3 y^2 + a_3 x^2 y^3 + a_4 x y^4 + a_5 y^5) dx
\]

\[
+ (b_0 x^5 + b_1 x^4 y + b_2 x^3 y^2 + b_3 x^2 y^3 + b_4 x y^4 + b_5 y^5) dy,
\]

where \( a_i, b_i, c_i, \alpha_k, \beta_k, \lambda_k \in \mathbb{C} \) and \( \gamma \in \mathbb{C}^* \).

In the affine chart \( x = 1 \), resp. \( y = 1 \), the foliation \( \mathcal{F} \) is given by

\[
\theta = -(\beta_0 z + \beta_1 y z + \beta_2 y^2 z + \beta_3 y^3 z + \beta_4 y^4 z + b_0 + b_1 y + b_2 y^2 + b_3 y^3 + b_4 y^4 + b_5 y^5)(y dz - z dy)
\]

\[
- (\alpha_0 z + \alpha_1 y z + \alpha_2 y^2 z + \alpha_3 y^3 z + \alpha_4 y^4 z + a_0 + a_1 y + a_2 y^2 + a_3 y^3 + a_4 y^4 + a_5 y^5) dz
\]

\[
+ z^3(\gamma_2 z + \lambda_0 z + \lambda_1 z + c_0 + c_1 y + c_2 y^2) dy,
\]

resp. \( \eta = (\alpha_0 x^4 z + \alpha_1 x^3 z + \alpha_2 x^2 z + \alpha_3 x z + \alpha_4 z + a_0 x^5 + a_1 x^4 z + a_2 x^3 z + a_3 x^2 z + a_4 x z + a_5 z)(z dx - x dz)
\]

\[
- (\beta_0 x^4 z + \beta_1 x^3 z + \beta_2 x^2 z + \beta_3 x z + \beta_4 z + b_0 x^5 + b_1 x^4 z + b_2 x^3 z + b_3 x^2 z + b_4 x z + b_5 z) dz
\]

\[
- z^3(\gamma_2 z + \lambda_0 x z + \lambda_1 z + c_0 x^2 + c_1 x + c_2) dx.
\]

A straightforward computation shows that

\[
(J_1^{(3)} y = 0, 0) \theta \wedge (y dz - z dy) = -z P(y, z) dy \wedge dz,
\]

\[
(J_1^{(3)} y = 0, 0) \eta \wedge (z dx - x dz) = z Q(x, z) dx \wedge dz,
\]

\[
(J_1^{(1)} y = 1, 0) \theta \wedge ((x - 1) dy - y dx) = R(x, y) dx \wedge dy,
\]

\[
(J_1^{(1)} y = 1, 0) \eta \wedge (y - 1) dz - z dy = -z S(y, z) dy \wedge dz,
\]

\[
(J_1^{(1)} y = 0, w_0) \theta \wedge ((z - w_0) dx - z dx) = T(x, z) dx \wedge dz,
\]

\[
(J_1^{(1)} y = 0, w_0) \eta \wedge ((x - \xi) dy - y dx) = \xi U(x, y) dx \wedge dy,
\]

\[
(J_1^{(1)} y = (\sigma w_0) \theta \wedge (y - \rho) dz - z dy) = -z V(y, z) dy \wedge dz,
\]

\[
(J_1^{(1)} y = (\sigma w_0) \eta \wedge ((z - \sigma w_0) dx - x dz) = W(x, z) dx \wedge dz.
\]
with

\[ P(y, z) = a_0 + a_1 y + a_2 y^2 + a_3 y^3 + a_4 y^4 - c_0 y^2, \]
\[ Q(x, z) = b_1 + b_2 x + b_3 x^2 + b_4 x^3 + b_5 x^4 + b_6 x^5 + c_0 x^2, \]
\[ R(x, y) = 4 a_0 + 3 a_0 - (9 a_0 + 7 a_0)x - (\gamma - c_0 - a_1 - a_2 - 4 b_0 - 3 b_0) y + (a_0 + 2 a_1 + a_1 + a_3 + 5 b_0 + 4 b_0) x y + (5 a_0 + 4 a_1) y^2 + (b_1 + b_1) y^2, \]
\[ S(y, z) = (a_1 + a_2 + a_3 + a_4 + a_5 + b_1 + b_2 + b_3 + b_4 + b_5 + 5 b_5) y + a_0 - a_2 - a_3 - a_4 - a_5 - b_0 - b_2 - 2 b_3 - 3 b_4 - 4 b_5 + (a_0 - a_1 + a_2 + a_3 + a_4 + b_0 + b_1 + b_2 + b_3 + b_4) z, \]
\[ T(x, z) = b_3 w_0 - w_0 (4 \rho w_0^4 + 3 \lambda_1 w_0^3 + 2 c_2 w_0 + \lambda_3 w_0 + a_5 + b_4) x - (\beta_4 w_4 - b_4) z + w_0 (\lambda_0 w_0^3 + c_1 w_0^2 - a_3 w_0 - a_4) x^2 + (5 w_0^4 + 4 \lambda_1 w_0^3 + 3 c_2 w_0^2 - a_4 w_0 + b_3 w_0 + b_4) x z + b_4 z^2, \]
\[ U(x, y) = (4 a_0 \xi^3 + 3 a_0 \xi^4 - \xi \eta (9 a_0 \xi + 7 a_0) x - (a_1 \xi^5 + 4 b_0 \xi^4 + 3 \beta_0 \xi^3 + c_0 \xi^2 - \gamma y + \xi^2 (5 a_0 \xi + 4 a_0) x^2 + (a_1 \xi^3 + 5 b_0 \xi^2 + a_1 \xi^2 + 4 b_0 \xi^2 + 2 c_0 \xi + \lambda_0) x y - (b_1 \xi^3 + b_1 \xi^2 + c_1 \xi + \lambda_1) y^2, \]
\[ V(y, z) = (5 b_3 p^5 + 5 a_5 p^4 + 4 b_4 p^4 + 4 a_5 p^3 + 3 b_3 p^3 + 3 b_2 p^2 + 2 a_2 p + b_1 p + a_1) y - 4 b_3 p^6 - (4 a_5 + 3 b_3) p^5 + (b_3 p^5 + a_5 p^4 + a_5 p^3 + a_3 p^2 + a_2 p^2 + a_5 p + b_1 p + a_1) y - 4 b_3 p^6 - (4 a_5 + 3 b_3) p^5 - a_2 p^2 + b_0 p + a_0, \]
\[ W(x, z) = a_5 w_0 (\lambda_0 \sigma_0 w_0^3 + c_1 \sigma_0^2 w_0^2 - a_3 \sigma w_0 - a_4) x^2 + (5 \gamma \sigma_0 w_0^4 + 4 \lambda_1 \sigma_0 w_0^3 + 3 \rho \sigma_0 w_0^2 - a_4 \sigma w_0 + b_3 \sigma w_0 + b_4) x z + b_4 z^2 - a_4 \sigma w_0 (4 \gamma \sigma_0 w_0^4 + 3 \lambda_1 \sigma_0 w_0^3 + 2 c_2 \sigma_0 w_0^2 + \lambda_3 \sigma w_0 + a_5 + b_4) x + (b_5 - b_4 \sigma w_0) z, \]

so that the equality \( \tau(f, m_2) = 4 \) (resp. \( \tau(f, m_3) = 4 \), resp. \( \tau(f, m_4) = 2 \), resp. \( \tau(f, m_5) = 2 \), resp. \( \tau(f, m_6) = 2 \), resp. \( \tau(f, m_7) = 2 \), resp. \( \tau(f, m_8) = 2 \), resp. \( \tau(f, m_9) = 2 \)) implies that the polynomial \( P \) (resp. \( Q \), resp. \( R \), resp. \( S \), resp. \( U \), resp. \( V \), resp. \( W \)) is identically zero. From \( P = Q = 0 \) we obtain \( a_0 = a_1 = a_2 = a_3 = a_0 = a_1 = a_2 = b_2 = b_3 = b_4 = b_5 = b_2 = b_3 = b_4 = c_0 = c_1 = 0 = 0 \). Next, from the equalities \( R = S = T = U = V = W = 0 \) we deduce that

\[ \xi = \rho = \sigma = \frac{3}{2} + \frac{\sqrt{5}}{2}, \]
\[ \gamma = \frac{47 + 21 \sqrt{5}}{2} \frac{a_3}{a_3}, \]
\[ \lambda_0 = -\frac{65 + 29 \sqrt{5}}{2} \frac{a_7}{a_7}, \]
\[ a_4 = \frac{5 + \sqrt{5}}{2} \frac{a_3}{a_3}, \]
\[ b_1 = (5 + 2 \sqrt{5}) a_5, \]
\[ b_0 = \frac{25 + 11 \sqrt{5}}{2} a_5, \]
\[ a_5 = (9 + 4 \sqrt{5})(5 w_0 + 5 - 2 \sqrt{5}) a_5, \]
\[ a_4 = \frac{25 + 11 \sqrt{5}}{2} a_5, \]
\[ \beta_1 = \frac{65 + 29 \sqrt{5}}{2} a_5, c_1 = \frac{a_5}{a_5} \frac{47 + 21 \sqrt{5}}{2} \frac{5 + 2 \sqrt{5}}{a_5}, \]
\[ w_0 = \pm(\sqrt{5} - 2) \]

with \( a_5 \neq 0 \). Thus \( \omega \) is of type

\[ \omega = a_5 \frac{47 + 21 \sqrt{5}}{4} \left( x dy - y dx \left( 2 - (5 - \sqrt{5}) x - w_0 (5 + \sqrt{5}) y + (10 w_0 + 10 - 4 \sqrt{5}) x y \right) + a_5 \frac{3}{2} \left( 9 + 4 \sqrt{5} \right) (10 w_0 + 10 - 4 \sqrt{5}) x - w_0 (25 + 11 \sqrt{5}) y - (5 + \sqrt{5}) x y + 2 y^2 \right) dx \]
\[ + a_5 \frac{3}{2} \left( 25 + 11 \sqrt{5} \right) x - (65 + 29 \sqrt{5}) (w_0 + 5 + 2 \sqrt{5}) y - (7 + 3 \sqrt{5}) x^2 + (10 + 4 \sqrt{5}) x y \right) dy, \]

where \( w_0 = \pm(\sqrt{5} - 2) \) and \( a_5 \in \mathbb{C}^* \). The 1-form is linearly conjugated to

\[ \omega_0 = \left( y^2 - 1 \right) \left( y^2 - (\sqrt{5} - 2)^2 \right) \left( y + \sqrt{5} y \right) dx - (x^2 - 1) \left( x^2 - (\sqrt{5} - 2)^2 \right) \left( x + \sqrt{5} x \right) dy. \]
Indeed, if $w_0 = \sqrt{5} - 2$, resp. $w_0 = 2 - \sqrt{5}$, then

\[
\omega_5^5 = \frac{32(3571 - 1597\sqrt{5})}{a_5} \varphi_1 \omega, \quad \text{where} \quad \varphi_1 = \left( \frac{3 + \sqrt{5}}{4} (x + 1), -\frac{2 + \sqrt{5}}{2} (y - 1) \right),
\]

resp.

\[
\omega_5^5 = \frac{32(64079 - 28657\sqrt{5})}{a_5} \varphi_2 \omega, \quad \text{where} \quad \varphi_2 = \left( \frac{2 + \sqrt{5}}{2} (x + \sqrt{5} - 2), -\frac{7 + 3\sqrt{5}}{4} (y + \sqrt{5} - 2) \right).
\]

\[\Box\]

References