CONVEX FOLIATIONS OF DEGREE 5 ON THE COMPLEX PROJECTIVE PLANE

by

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Abstract. — We show that up to automorphism of $\mathbb{P}^2_{\mathbb{C}}$ there are 14 homogeneous convex foliations of degree five on $\mathbb{P}^2_{\mathbb{C}}$. Using this result, we give a partial answer to a question posed in 2013 by D. MARÍN and J. PEREIRA about the classification of reduced convex foliations on $\mathbb{P}^2_{\mathbb{C}}$.
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Introduction

Following [10] a foliation on the complex projective plane is said to be *convex* if its leaves other than straight lines have no inflection points. Notice (*see* [11]) that if $\mathcal{F}$ is a foliation of degree $d \geq 1$ on $\mathbb{P}^2_{\mathbb{C}}$, then $\mathcal{F}$ can not have more than $3d$ (distinct) invariant lines. Moreover, if this bound is reached, then $\mathcal{F}$ is necessarily convex; in this case $\mathcal{F}$ is said to be *reduced convex*. To our knowledge the only reduced convex foliations known in the literature are those presented in [10, Table 1.1]: the FERMAT foliation $\mathcal{F}_0^d$ of degree d, the HESSE pencil $\mathcal{F}_H^4$ of degree 4, the HILBERT modular foliation $\mathcal{F}_H^5$ of degree 5 and the HILBERT modular foliation $\mathcal{F}_H^7$ of degree 7 defined in affine chart respectively by the 1-forms

$$\overline{\omega}_0^d = (x^d - x)\mathrm{d}y - (y^d - y)\mathrm{d}x,$$
$$\omega_H^4 = (2x^3 - y^3 - 1)y\mathrm{d}x + (2y^3 - x^3 - 1)x\mathrm{d}y,$$
$$\omega_H^5 = (y^2-1)(y^2-(\sqrt{5}-2)^2)(y+\sqrt{5}x)\mathrm{d}x - (x^2-1)(x^2-(\sqrt{5}-2)^2)(x+\sqrt{5}y)\mathrm{d}y,$$
$$\omega_H^7 = (y^3-1)(y^3+7x^3+1)y\mathrm{d}x - (x^3-1)(x^3+7y^3+1)x\mathrm{d}y.$$

D. MARÍN and J. PEREIRA [10, Problem 9.1] asked the following question: are there other reduced convex foliations? The answer in degree 2, resp. 3, resp. 4, to this question is negative, by [9, Proposition 7.4], resp. [3, Corollary 6.9], resp. [4, Theorem B]. In this paper we show that the answer in degree 5 to [10, Problem 9.1] is also negative. To do this, we follow the same approach as that described in degree 4 in [4].

Key words and phrases. — convex foliation, homogeneous foliation, singularity, inflection divisor.

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More precisely, we begin by establishing the following theorem classifying the convex foliations of degree 5 on $\mathbb{P}^2_{\mathbb{C}}$ which are *homogeneous*, *i.e.* which are invariant under homothety.

Theorem A. — *Up to automorphism of* $\mathbb{P}^2_{\mathbb{C}}$ *there are fourteen homogeneous convex foliations of degree five* $\mathcal{H}_1,\ldots,\mathcal{H}_{14}$ *on the complex projective plane. They are respectively described in affine chart by the following* 1*-forms*

1. $\omega_1 = y^5\mathrm{d}x - x^5\mathrm{d}y;$
2. $\omega_2 = y^2(10x^3+10x^2y+5xy^2+y^3)\mathrm{d}x - x^4(x+5y)\mathrm{d}y;$
3. $\omega_3 = y^3(10x^2+5xy+y^2)\mathrm{d}x - x^3(x^2+5xy+10y^2)\mathrm{d}y;$
4. $\omega_4 = y^4(5x-3y)\mathrm{d}x + x^4(3x-5y)\mathrm{d}y;$
5. $\omega_5 = y^3(5x^2-3y^2)\mathrm{d}x - 2x^5\mathrm{d}y;$
6. $\omega_6 = y^3(220x^2-165xy+36y^2)\mathrm{d}x - 121x^5\mathrm{d}y;$
7. $\omega_7 = y^4\Big(\big(5-\sqrt{5}\big)x-2y\Big)\mathrm{d}x + x^4\Big(\big(7-3\sqrt{5}\big)x-2\big(5-2\sqrt{5}\big)y\Big)\mathrm{d}y;$
8. $\omega_8 = y^4\Big(5\big(3-\sqrt{21}\big)x+6y\Big)\mathrm{d}x + x^4\Big(3\big(23-5\sqrt{21}\big)x-10\big(9-2\sqrt{21}\big)y\Big)\mathrm{d}y;$
9. $\omega_9 = y^3\Big(2\big(5+a\big)x^2-\big(15+a\big)xy+6y^2\Big)\mathrm{d}x - x^4\Big(\big(1-a\big)x+2ay\Big)\mathrm{d}y,$ *where* $a=\sqrt{5\big(4\sqrt{61}-31\big)};$
10. $\omega_{10} = y^3\Big(2\big(5+\mathrm{i}b\big)x^2-\big(15+\mathrm{i}b\big)xy+6y^2\Big)\mathrm{d}x - x^4\Big(\big(1-\mathrm{i}b\big)x+2\mathrm{i}by\Big)\mathrm{d}y,$ *where* $b=\sqrt{5\big(4\sqrt{61}+31\big)};$
11. $\omega_{11} = y^3(5x^2-y^2)\mathrm{d}x + x^3(x^2-5y^2)\mathrm{d}y;$
12. $\omega_{12} = y^3(20x^2-5xy-y^2)\mathrm{d}x + x^3(x^2+5xy-20y^2)\mathrm{d}y;$
13. $\omega_{13} = y^2(5x^3-10x^2y+10xy^2-4y^3)\mathrm{d}x - x^5\mathrm{d}y;$
14. $\omega_{14} = y^3\Big(u(\sigma)x^2+v(\sigma)xy+w(\sigma)y^2\Big)\mathrm{d}x + \sigma x^4\Big(2\sigma\big(\sigma^2-\sigma+1\big)x-\big(\sigma+1\big)\big(3\sigma^2-5\sigma+3\big)y\Big)\mathrm{d}y,$

   where $u(\sigma)=(\sigma^2-3\sigma+1)(\sigma^2+5\sigma+1),\;\; v(\sigma)=-2(\sigma+1)(\sigma^2-5\sigma+1),\;\; w(\sigma)=(\sigma^2-7\sigma+1),$ $\sigma=\rho+\mathrm{i}\sqrt{\tfrac{1}{6}-\tfrac{4}{3}\rho-\tfrac{1}{3}\rho^2}$ *and* ρ *is the unique real number satisfying* $8\rho^3-52\rho^2+134\rho-15=0.$

Then, using this classification, we prove the following theorem.

Theorem B. — *Up to automorphism of* $\mathbb{P}^2_{\mathbb{C}}$ *the* FERMAT *foliation* $\mathcal{F}_0^5$ *and the* HILBERT *modular foliation* $\mathcal{F}_H^5$ *are the only reduced convex foliations of degree five on* $\mathbb{P}^2_{\mathbb{C}}$.

1. Preliminaries

1.1. Singularities and inflection divisor of a foliation on the projective plane. — A degree d holomorphic foliation $\mathcal{F}$ on $\mathbb{P}^2_{\mathbb{C}}$ is defined in homogeneous coordinates $[x:y:z]$ by a 1-form

$$\omega = a(x,y,z)\mathrm{d}x + b(x,y,z)\mathrm{d}y + c(x,y,z)\mathrm{d}z,$$

where a, b and c are homogeneous polynomials of degree $d+1$ without common factor and satisfying the EULER condition $i_{\mathrm{R}}\omega = 0$, where $\mathrm{R} = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + z\frac{\partial}{\partial z}$ denotes the radial vector field and i_{R} is the interior product by R. The *singular locus* $\mathrm{Sing}\mathcal{F}$ of $\mathcal{F}$ is the projectivization of the singular locus of ω

$$\mathrm{Sing}\,\omega = \{(x,y,z) \in \mathbb{C}^3 \,|\, a(x,y,z) = b(x,y,z) = c(x,y,z) = 0\}.$$

Let us recall some local notions attached to the pair $(\mathcal{F}, s)$, where $s \in \mathrm{Sing}\mathcal{F}$. The germ of $\mathcal{F}$ at s is defined, up to multiplication by a unity in the local ring $\mathcal{O}_s$ at s, by a vector field $\mathrm{X} = A(\mathrm{u},\mathrm{v})\frac{\partial}{\partial \mathrm{u}} + B(\mathrm{u},\mathrm{v})\frac{\partial}{\partial \mathrm{v}}$. The *algebraic multiplicity* $\nu(\mathcal{F}, s)$ of $\mathcal{F}$ at s is given by

$$\nu(\mathcal{F}, s) = \min\{\nu(A,s), \nu(B,s)\},$$

where $\nu(g,s)$ denotes the algebraic multiplicity of the function g at s. The *tangency order* of $\mathcal{F}$ with a generic line passing through s is the integer

$$\tau(\mathcal{F}, s) = \min\{k \geq \nu(\mathcal{F}, s) \,:\, \det(J^k_s\mathrm{X}, \mathrm{R}_s) \neq 0\},$$

where $J^k_s\mathrm{X}$ denotes the k-jet of X at s and R_s is the radial vector field centered at s. The MILNOR *number* of $\mathcal{F}$ at s is the integer

$$\mu(\mathcal{F}, s) = \dim_{\mathbb{C}} \mathcal{O}_s/\langle A, B\rangle,$$

where $\langle A, B\rangle$ denotes the ideal of $\mathcal{O}_s$ generated by A and B.
The singularity s is called *radial of order* $n-1$ if $\nu(\mathcal{F}, s) = 1$ and $\tau(\mathcal{F}, s) = n$.
The singularity s is called *non-degenerate* if $\mu(\mathcal{F}, s) = 1$, or equivalently if the linear part $J^1_s\mathrm{X}$ of X possesses two non-zero eigenvalues λ, μ. In this case, the quantity $\mathrm{BB}(\mathcal{F}, s) = \frac{\lambda}{\mu} + \frac{\mu}{\lambda} + 2$ is called the BAUM-BOTT *invariant* of $\mathcal{F}$ at s (*see* [1]). By [6] there is at least a germ of curve $\mathcal{C}$ at s which is invariant by $\mathcal{F}$. Up to local diffeomorphism we can assume that $s = (0,0)$ $\mathrm{T}_s\mathcal{C} = \{\mathrm{v} = 0\}$ and $J^1_s\mathrm{X} = \lambda\mathrm{u}\frac{\partial}{\partial \mathrm{u}} + (\varepsilon\mathrm{u} + \mu\mathrm{v})\frac{\partial}{\partial \mathrm{v}}$, where we can take $\varepsilon = 0$ if $\lambda \neq \mu$. The quantity $\mathrm{CS}(\mathcal{F}, \mathcal{C}, s) = \frac{\lambda}{\mu}$ is called the CAMACHO-SAD *index* of $\mathcal{F}$ at s along $\mathcal{C}$.

Let us also recall the notion of inflection divisor of $\mathcal{F}$. Let $\mathrm{Z} = E\frac{\partial}{\partial x} + F\frac{\partial}{\partial y} + G\frac{\partial}{\partial z}$ be a homogeneous vector field of degree d on $\mathbb{C}^3$ non collinear to the radial vector field describing $\mathcal{F}$, *i.e.* such that $\omega = i_{\mathrm{R}}i_{\mathrm{Z}}\mathrm{d}x \wedge \mathrm{d}y \wedge \mathrm{d}z$. The *inflection divisor* of $\mathcal{F}$, denoted by $\mathrm{I}_{\mathcal{F}}$, is the divisor of $\mathbb{P}^2_{\mathbb{C}}$ defined by the homogeneous equation

$$\begin{vmatrix} x & E & \mathrm{Z}(E) \\ y & F & \mathrm{Z}(F) \\ z & G & \mathrm{Z}(G) \end{vmatrix} = 0.$$

This divisor has been studied in [11] in a more general context. In particular, the following properties has been proved.

1. On $\mathbb{P}^2_{\mathbb{C}} \smallsetminus \mathrm{Sing}\mathcal{F}$, $\mathrm{I}_{\mathcal{F}}$ coincides with the curve described by the inflection points of the leaves of $\mathcal{F}$;
2. If $\mathcal{C}$ is an irreducible algebraic curve invariant by $\mathcal{F}$ then $\mathcal{C} \subset \mathrm{I}_{\mathcal{F}}$ if and only if $\mathcal{C}$ is an invariant line;
3. $\mathrm{I}_{\mathcal{F}}$ can be decomposed into $\mathrm{I}_{\mathcal{F}} = \mathrm{I}^{\mathrm{inv}}_{\mathcal{F}} + \mathrm{I}^{\mathrm{tr}}_{\mathcal{F}}$, where the support of $\mathrm{I}^{\mathrm{inv}}_{\mathcal{F}}$ consists in the set of invariant lines of $\mathcal{F}$ and the support of $\mathrm{I}^{\mathrm{tr}}_{\mathcal{F}}$ is the closure of the isolated inflection points along the leaves of $\mathcal{F}$;
4. The degree of the divisor $\mathrm{I}_{\mathcal{F}}$ is $3d$.

The foliation $\mathcal{F}$ will be called *convex* if its inflection divisor $\mathrm{I}_{\mathcal{F}}$ is totally invariant by $\mathcal{F}$, *i.e.* if $\mathrm{I}_{\mathcal{F}}$ is a product of invariant lines.

1.2. Geometry of homogeneous foliations. — A foliation of degree d on $\mathbb{P}^2_{\mathbb{C}}$ is said to be *homogeneous* if there is an affine chart (x,y) of $\mathbb{P}^2_{\mathbb{C}}$ in which it is invariant under the action of the group of homotheties $(x,y) \longmapsto \lambda(x,y),\ \lambda \in \mathbb{C}^*$. Such a foliation $\mathcal{H}$ is then defined by a 1-form

$$\omega = A(x,y)\mathrm{d}x + B(x,y)\mathrm{d}y,$$

where A and B are homogeneous polynomials of degree d without common factor. This 1-form writes in homogeneous coordinates as

$$zA(x,y)\mathrm{d}x + zB(x,y)\mathrm{d}y - (xA(x,y) + yB(x,y))\,\mathrm{d}z.$$

Thus the foliation $\mathcal{H}$ has at most $d+2$ singularities whose origin O of the affine chart $z=1$ is the only singular point of $\mathcal{H}$ which is not situated on the line at infinity $L_\infty = \{z=0\}$; moreover $\nu(\mathcal{H},O) = d$.
In the sequel we will assume that d is greater than or equal to 2. In this case the point O is the only singularity of $\mathcal{H}$ having algebraic multiplicity d.
We know from [3] that the inflection divisor of $\mathcal{H}$ is given by $z\mathrm{C}_{\mathcal{H}}\mathrm{D}_{\mathcal{H}} = 0$, where $\mathrm{C}_{\mathcal{H}} = xA + yB \in \mathbb{C}[x,y]_{d+1}$ denotes the *tangent cone* of $\mathcal{H}$ at the origin O and $\mathrm{D}_{\mathcal{H}} = \frac{\partial A}{\partial x}\frac{\partial B}{\partial y} - \frac{\partial A}{\partial y}\frac{\partial B}{\partial x} \in \mathbb{C}[x,y]_{2d-2}$. From this we deduce that:

1. the support of the divisor $\mathrm{I}^{\mathrm{inv}}_{\mathcal{H}}$ consists of the lines of the tangent cone $\mathrm{C}_{\mathcal{H}} = 0$ and the line at infinity L_∞ ;
2. the divisor $\mathrm{I}^{\mathrm{tr}}_{\mathcal{H}}$ decomposes as $\mathrm{I}^{\mathrm{tr}}_{\mathcal{H}} = \prod_{i=1}^{n} T_i^{\rho_i - 1}$ for some number $n \leq \deg \mathrm{D}_{\mathcal{H}} = 2d-2$ of lines T_i passing through O, $\rho_i - 1$ being the inflection order of the line T_i.

Proposition 1.1 **([3],** Proposition 2.2). — *With the previous notations, for any point $s \in \mathrm{Sing}\mathcal{H} \cap L_\infty$, we have*
1. $\nu(\mathcal{H},s) = 1$;
2. *the line joining the origin O to the point s is invariant by $\mathcal{H}$ and it appears with multiplicity $\tau(\mathcal{H},s)-1$ in the divider $\mathrm{D}_{\mathcal{H}} = 0$, i.e.*

$$\mathrm{D}_{\mathcal{H}} = \mathrm{I}^{\mathrm{tr}}_{\mathcal{H}} \prod_{s \in \mathrm{Sing}\mathcal{H} \cap L_\infty} L_s^{\tau(\mathcal{H},s)-1}.$$

Definition 1.2 ([3]). — Let $\mathcal{H}$ be a homogeneous foliation of degree d on $\mathbb{P}^2_{\mathbb{C}}$ having a certain number $m \leq d+1$ of radial singularities s_i of order $\tau_i - 1$, $2 \leq \tau_i \leq d$ for $i = 1,2,\ldots,m$. The support of the divisor $\mathrm{I}^{\mathrm{tr}}_{\mathcal{H}}$ consists of a certain number $n \leq 2d-2$ of transverse inflection lines T_j of order $\rho_j - 1$, $2 \leq \rho_j \leq d$ for $j = 1,2,\ldots,n$. We define the *type of the foliation* $\mathcal{H}$ by

$$\mathcal{T}_{\mathcal{H}} = \sum_{i=1}^{m} \mathrm{R}_{\tau_i - 1} + \sum_{j=1}^{n} \mathrm{T}_{\rho_j - 1} = \sum_{k=1}^{d-1} (r_k \cdot \mathrm{R}_k + t_k \cdot \mathrm{T}_k) \in \mathbb{Z}\left[\mathrm{R}_1, \mathrm{R}_2, \ldots, \mathrm{R}_{d-1}, \mathrm{T}_1, \mathrm{T}_2, \ldots, \mathrm{T}_{d-1}\right].$$

Example 1.3. — Let us consider the homogeneous foliation $\mathcal{H}$ of degree 5 on $\mathbb{P}^2_{\mathbb{C}}$ defined by

$$\omega = y^5\mathrm{d}x + 2x^3(3x^2 - 5y^2)\mathrm{d}y.$$

A straightforward computation leads to

$$\mathrm{C}_{\mathcal{H}} = xy\left(6x^4 - 10x^2y^2 + y^4\right) \qquad \text{and} \qquad \mathrm{D}_{\mathcal{H}} = 150x^2y^4(x-y)(x+y).$$

We see that the set of radial singularities of $\mathcal{H}$ consists of the two points $s_1 = [0:1:0]$ and $s_2 = [1:0:0]$; their orders of radiality are equal to 2 and 4 respectively. Moreover the support of the divisor $\mathrm{I}^{\mathrm{tr}}_{\mathcal{H}}$ is the union of the two lines $x-y=0$ and $x+y=0$; they are transverse inflection lines of order 1. Therefore the foliation $\mathcal{H}$ is of type $\mathcal{T}_{\mathcal{H}} = 1\cdot\mathrm{R}_2 + 1\cdot\mathrm{R}_4 + 2\cdot\mathrm{T}_1$.

Following [3], to every homogeneous foliation $\mathcal{H}$ of degree d on $\mathbb{P}^2_{\mathbb{C}}$ we can associate a rational map $\underline{\mathcal{G}}_{\mathcal{H}} : \mathbb{P}^1_{\mathbb{C}} \to \mathbb{P}^1_{\mathbb{C}}$ in the following way: if $\mathcal{H}$ is described by $\omega = A(x,y)\mathrm{d}x + B(x,y)\mathrm{d}y$, with A and B being homogeneous polynomials of degree d without common factor, we define $\underline{\mathcal{G}}_{\mathcal{H}}$ by

$$\underline{\mathcal{G}}_{\mathcal{H}}([x:y]) = [-A(x,y) : B(x,y)];$$

it is clear that this definition does not depend on the choice of the homogeneous 1-form ω describing the foliation $\mathcal{H}$.

Conversely, every rational map $f : \mathbb{P}^1_{\mathbb{C}} \to \mathbb{P}^1_{\mathbb{C}}$ of degree d can be obtained in this way; indeed, if $f(z) = \dfrac{p(z)}{q(z)}$, with $p,q \in \mathbb{C}[z]$, $\mathrm{pgcd}(p,q) = 1$ and $\max(\deg p, \deg q) = d$, then $f = \underline{\mathcal{G}}_{\mathcal{H}_f}$, where $\mathcal{H}_f$ is the homogeneous foliation of degree d on $\mathbb{P}^2_{\mathbb{C}}$ defined by the 1-form

$$\omega_f = -x^d p\left(\frac{y}{x}\right)\mathrm{d}x + x^d q\left(\frac{y}{x}\right)\mathrm{d}y.$$

Let $\mathcal{H}$ be a homogeneous foliation of degree d on $\mathbb{P}^2_{\mathbb{C}}$. Notice (*see* [3]) that the map $\underline{\mathcal{G}}_{\mathcal{H}}$ has the following properties:

(i) the fixed points of $\underline{\mathcal{G}}_{\mathcal{H}}$ correspond to the singular points of $\mathcal{H}$ on the line at infinity (*i.e.* $[a:b] \in \mathbb{P}^1_{\mathbb{C}}$ is fixed by $\underline{\mathcal{G}}_{\mathcal{H}}$ if and only if the point $[b:a:0] \in L_\infty$ is singular for $\mathcal{H}$);

(ii) the point $[a:b] \in \mathbb{P}^1_{\mathbb{C}}$ is a fixed critical point of $\underline{\mathcal{G}}_{\mathcal{H}}$ if and only if the point $[b:a:0] \in L_\infty$ is a radial singularity of $\mathcal{H}$. The multiplicity of the critical point $[a:b]$ of $\underline{\mathcal{G}}_{\mathcal{H}}$ is exactly equal to the the radiality order of the singularity at infinity;

(iii) the point $[a:b] \in \mathbb{P}^1_{\mathbb{C}}$ is a non-fixed critical point of $\underline{\mathcal{G}}_{\mathcal{H}}$ if and only if the line $by - ax = 0$ is a transverse inflection line of $\mathcal{H}$. The multiplicity of the critical point $[a:b]$ of $\underline{\mathcal{G}}_{\mathcal{H}}$ is precisely equal to the inflection order of this line.

It follows, in particular, that a homogeneous foliation $\mathcal{H}$ on $\mathbb{P}^2_{\mathbb{C}}$ is convex if and only if its associated map $\underline{\mathcal{G}}_{\mathcal{H}}$ has only fixed critical points; more precisely, a homogeneous foliation $\mathcal{H}$ of degree d on $\mathbb{P}^2_{\mathbb{C}}$ is convex of type $\mathcal{T}_{\mathcal{H}} = \sum_{k=1}^{d-1} r_k \cdot \mathrm{R}_k$ if and only if the map $\underline{\mathcal{G}}_{\mathcal{H}}$ possesses r_1, resp. $r_2, \ldots,$ resp. r_{d-1} fixed critical points of multiplicity 1, resp. $2\ldots,$ resp. $d-1$, with $\sum_{k=1}^{d-1} k r_k = 2d-2$.

Remark 1.4. — Every homogeneous convex foliation of degree d on the complex projective plane has exactly $d+1$ singularities on the line at infinity, necessarily non-degenerate. This follows from remark (i) above and Theorem 4.3 of [8] which ensures that if a rational map f of degree d from the RIEMANN sphere to itself has only fixed critical points, then f admits $d+1$ distinct fixed points.

2. Proof of Theorems A and B

We need to know the numbers r_{ij} of radial singularities of order j of the homogeneous foliations $\mathcal{H}_i$, $i = 1, \ldots, 14$, $j = 1, \ldots, 4$, and the values of the CAMACHO-SAD indices $\mathrm{CS}(\mathcal{H}_i, L_\infty, s)$, $s \in \mathrm{Sing}\mathcal{H}_i \cap L_\infty$, $i = 1, \ldots, 14$. For this reason, we have computed, for each $i = 1, \ldots, 14$, the type $\mathcal{T}_{\mathcal{H}_i}$ of $\mathcal{H}_i$ and the following polynomial (called CAMACHO-SAD *polynomial of the homogeneous foliation* $\mathcal{H}_i$)

$$\mathrm{CS}_{\mathcal{H}_i}(\lambda) = \prod_{s \in \mathrm{Sing}\mathcal{H}_i \cap L_\infty} (\lambda - \mathrm{CS}(\mathcal{H}_i, L_\infty, s)).$$

The following table summarizes the types and the CAMACHO-SAD polynomials of the foliations $\mathcal{H}_1,\ldots,\mathcal{H}_{14}$.

i	$\mathcal{T}_{\mathcal{H}_i}$	$\mathrm{CS}_{\mathcal{H}_i}(\lambda)$
1	$2\cdot\mathrm{R}_4$	$(\lambda-1)^2(\lambda+\frac{1}{4})^4$
2	$1\cdot\mathrm{R}_1+1\cdot\mathrm{R}_3+1\cdot\mathrm{R}_4$	$\frac{1}{491}(\lambda-1)^3(491\lambda^3+982\lambda^2+463\lambda+64)$
3	$2\cdot\mathrm{R}_2+1\cdot\mathrm{R}_4$	$(\lambda-1)^3(\lambda+\frac{3}{7})^2(\lambda+\frac{8}{7})$
4	$1\cdot\mathrm{R}_2+2\cdot\mathrm{R}_3$	$(\lambda-1)^3(\lambda+\frac{9}{11})^2(\lambda+\frac{4}{11})$
5	$2\cdot\mathrm{R}_1+1\cdot\mathrm{R}_2+1\cdot\mathrm{R}_4$	$(\lambda-1)^4(\lambda+\frac{3}{2})^2$
6	$2\cdot\mathrm{R}_1+1\cdot\mathrm{R}_2+1\cdot\mathrm{R}_4$	$\frac{1}{59}(\lambda-1)^4(59\lambda^2+177\lambda+64)$
7	$2\cdot\mathrm{R}_1+2\cdot\mathrm{R}_3$	$(\lambda-1)^4(\lambda^2+3\lambda+1)$
8	$2\cdot\mathrm{R}_1+2\cdot\mathrm{R}_3$	$(\lambda-1)^4(\lambda+\frac{3}{2})^2$
9	$1\cdot\mathrm{R}_1+2\cdot\mathrm{R}_2+1\cdot\mathrm{R}_3$	$\frac{1}{197}(\lambda-1)^4(197\lambda^2+591\lambda+302-10\sqrt{61})$
10	$1\cdot\mathrm{R}_1+2\cdot\mathrm{R}_2+1\cdot\mathrm{R}_3$	$\frac{1}{197}(\lambda-1)^4(197\lambda^2+591\lambda+302+10\sqrt{61})$
11	$4\cdot\mathrm{R}_2$	$(\lambda-1)^4(\lambda+\frac{3}{2})^2$
12	$2\cdot\mathrm{R}_1+3\cdot\mathrm{R}_2$	$(\lambda-1)^5(\lambda+4)$
13	$4\cdot\mathrm{R}_1+1\cdot\mathrm{R}_4$	$(\lambda-1)^5(\lambda+4)$
14	$3\cdot\mathrm{R}_1+1\cdot\mathrm{R}_2+1\cdot\mathrm{R}_3$	$(\lambda-1)^5(\lambda+4)$

TABLE 1. Types and CAMACHO-SAD polynomials of the homogeneous foliations $\mathcal{H}_1,\ldots,\mathcal{H}_{14}$.

Before beginning the proof of Theorem A, let us recall the following result which follows from Propositions 4.1 and 4.2 of [3]:

Proposition 2.1 ([3]). — *Let $\mathcal{H}$ be a convex homogeneous foliation of degree $d\geq 3$ on $\mathbb{P}^2_{\mathbb{C}}$. Let ν be an integer between 1 and $d-2$. Then, $\mathcal{H}$ is of type*

$$\mathcal{T}_{\mathcal{H}}=2\cdot\mathrm{R}_{d-1},\qquad\textit{resp.}\ \mathcal{T}_{\mathcal{H}}=1\cdot\mathrm{R}_{\nu}+1\cdot\mathrm{R}_{d-\nu-1}+1\cdot\mathrm{R}_{d-1},$$

if and only if it is linearly conjugated to the foliation $\mathcal{H}_1^d$, resp. $\mathcal{H}_3^{d,\nu}$ given by

$$\omega_1^d=y^d\mathrm{d}x-x^d\mathrm{d}y,\qquad\textit{resp.}\ \omega_3^{d,\nu}=\sum_{i=\nu+1}^{d}\binom{d}{i}x^{d-i}y^i\mathrm{d}x-\sum_{i=0}^{\nu}\binom{d}{i}x^{d-i}y^i\mathrm{d}y.$$

Proof of Theorem A. — Let $\mathcal{H}$ be a convex homogeneous foliation of degree 5 on $\mathbb{P}^2_{\mathbb{C}}$, defined in the affine chart (x,y), by the 1-form

$$\omega = A(x,y)\mathrm{d}x + B(x,y)\mathrm{d}y, \quad A,B \in \mathbb{C}[x,y]_5, \ \ \gcd(A,B)=1.$$

By [5, Remark 2.5] the foliation $\mathcal{H}$ can not have $5+1=6$ distinct radial singularities; in other words it cannot be of one of the two types $5\cdot\mathrm{R}_1+1\cdot\mathrm{R}_3$ or $4\cdot\mathrm{R}_1+2\cdot\mathrm{R}_2$. We are then in one of the following situations:

$\mathcal{T}_{\mathcal{H}} = 2\cdot\mathrm{R}_4$; $\mathcal{T}_{\mathcal{H}} = 1\cdot\mathrm{R}_1+1\cdot\mathrm{R}_3+1\cdot\mathrm{R}_4$; $\mathcal{T}_{\mathcal{H}} = 2\cdot\mathrm{R}_2+1\cdot\mathrm{R}_4$;
$\mathcal{T}_{\mathcal{H}} = 1\cdot\mathrm{R}_2+2\cdot\mathrm{R}_3$; $\mathcal{T}_{\mathcal{H}} = 2\cdot\mathrm{R}_1+1\cdot\mathrm{R}_2+1\cdot\mathrm{R}_4$; $\mathcal{T}_{\mathcal{H}} = 2\cdot\mathrm{R}_1+2\cdot\mathrm{R}_3$;
$\mathcal{T}_{\mathcal{H}} = 1\cdot\mathrm{R}_1+2\cdot\mathrm{R}_2+1\cdot\mathrm{R}_3$; $\mathcal{T}_{\mathcal{H}} = 4\cdot\mathrm{R}_2$; $\mathcal{T}_{\mathcal{H}} = 2\cdot\mathrm{R}_1+3\cdot\mathrm{R}_2$;
$\mathcal{T}_{\mathcal{H}} = 4\cdot\mathrm{R}_1+1\cdot\mathrm{R}_4$; $\mathcal{T}_{\mathcal{H}} = 3\cdot\mathrm{R}_1+1\cdot\mathrm{R}_2+1\cdot\mathrm{R}_3$.

- If the foliation $\mathcal{H}$ is of type $\mathcal{T}_{\mathcal{H}} = 2\cdot\mathrm{R}_4$, resp. $\mathcal{T}_{\mathcal{H}} = 1\cdot\mathrm{R}_1+1\cdot\mathrm{R}_3+1\cdot\mathrm{R}_4$, resp. $\mathcal{T}_{\mathcal{H}} = 2\cdot\mathrm{R}_2+1\cdot\mathrm{R}_4$, then, by [3, Propositions 4.1, 4.2], the 1-form ω is linearly conjugated to

$$\omega_1^5 = y^5\mathrm{d}x - x^5\mathrm{d}y = \omega_1,$$

$$\begin{aligned}\text{resp. } \omega_3^{5,1} &= \sum_{i=2}^{5}\binom{5}{i}x^{5-i}y^i\mathrm{d}x - \sum_{i=0}^{1}\binom{5}{i}x^{5-i}y^i\mathrm{d}y\\ &= y^2(10x^3+10x^2y+5xy^2+y^3)\mathrm{d}x - x^4(x+5y)\mathrm{d}y\\ &= \omega_2,\end{aligned}$$

$$\begin{aligned}\text{resp. } \omega_3^{5,2} &= \sum_{i=3}^{5}\binom{5}{i}x^{5-i}y^i\mathrm{d}x - \sum_{i=0}^{2}\binom{5}{i}x^{5-i}y^i\mathrm{d}y\\ &= y^3(10x^2+5xy+y^2)\mathrm{d}x - x^3(x^2+5xy+10y^2)\mathrm{d}y\\ &= \omega_3.\end{aligned}$$

- Assume that $\mathcal{T}_{\mathcal{H}} = 1\cdot\mathrm{R}_2+2\cdot\mathrm{R}_3$. This means that the rational map $\underline{\mathcal{G}}_{\mathcal{H}}:\mathbb{P}^1_{\mathbb{C}}\to\mathbb{P}^1_{\mathbb{C}}$, $\underline{\mathcal{G}}_{\mathcal{H}}(z) = -\dfrac{A(1,z)}{B(1,z)}$, possesses three fixed critical points, one of multiplicity 2 and two of multiplicity 3. By [7, page 79], $\underline{\mathcal{G}}_{\mathcal{H}}$ is conjugated by a MÖBIUS transformation to $z\mapsto -\dfrac{z^4(3z-5)}{5z-3}$. As a result, ω is linearly conjugated to $\omega_4 = y^4(5x-3y)\mathrm{d}x + x^4(3x-5y)\mathrm{d}y$.
- Let us study the possibility $\mathcal{T}_{\mathcal{H}} = 2\cdot\mathrm{R}_1+1\cdot\mathrm{R}_2+1\cdot\mathrm{R}_4$. Up to linear conjugation we can assume that $\mathrm{D}_{\mathcal{H}} = cx^4y^2(y-x)(y-\alpha x)$ and $\mathrm{C}_{\mathcal{H}}(0,1) = \mathrm{C}_{\mathcal{H}}(1,0) = \mathrm{C}_{\mathcal{H}}(1,1) = \mathrm{C}_{\mathcal{H}}(1,\alpha) = 0$, for some $c,\alpha\in\mathbb{C}^*$, $\alpha\neq 1$. The points $\infty = [1:0]$, $[0:1]$, $[1:1]$, $[1:\alpha]\in\mathbb{P}^1_{\mathbb{C}}$ are then fixed and critical for $\underline{\mathcal{G}}_{\mathcal{H}}$, with respective multiplicities $4,2,1,1$. By [3, Lemma 3.9], there exist constants $a_0,a_2,b\in\mathbb{C}^*, a_1\in\mathbb{C}$ such that

$$B(x,y) = bx^5, \quad A(x,y) = (a_0x^2+a_1xy+a_2y^2)y^3, \quad (z-1)^2 \text{ divides } P(z), \quad (z-\alpha)^2 \text{ divides } Q(z),$$

where $P(z) := A(1,z)+B(1,z)$ and $Q(z) := A(1,z)+\alpha B(1,z)$.

Therefore we have

$$\begin{cases} P(1)=0 \\ P'(1)=0 \\ Q(\alpha)=0 \\ Q'(\alpha)=0 \end{cases} \Leftrightarrow \begin{cases} a_0+a_1+a_2+b=0 \\ 3a_0+4a_1+5a_2=0 \\ a_2\alpha^4+a_1\alpha^3+a_0\alpha^2+b=0 \\ 5a_2\alpha^2+4a_1\alpha+3a_0=0 \end{cases} \Leftrightarrow \begin{cases} a_0=\dfrac{5a_2\alpha}{3} \\ a_1=-\dfrac{5a_2(\alpha+1)}{4} \\ b=-\dfrac{a_2(5\alpha-3)}{12} \\ (\alpha+1)(3\alpha^2-5\alpha+3)=0 \end{cases}$$

Replacing ω by $\dfrac{12}{a_2}\omega$, we reduce it to

$$\omega = y^3(20\alpha x^2-15(\alpha+1)xy+12y^2)\mathrm{d}x-(5\alpha-3)x^5\mathrm{d}y, \qquad (\alpha+1)(3\alpha^2-5\alpha+3)=0.$$

This 1-form is linearly conjugated to one of the two 1-forms

$$\omega_5 = y^3(5x^2-3y^2)\mathrm{d}x-2x^5\mathrm{d}y \qquad \text{or} \qquad \omega_6 = y^3(220x^2-165xy+36y^2)\mathrm{d}x-121x^5\mathrm{d}y.$$

Indeed, on the one hand, if $\alpha=-1$, then $\omega_5=-\frac{1}{4}\omega$. On the other hand, if $3\alpha^2-5\alpha+3=0$, then

$$\omega_6 = \frac{121(15\alpha-16)}{81(3\alpha-8)^5}\varphi^*\omega, \quad \text{where } \varphi=\Big((3\alpha-8)x,-3y\Big).$$

- Assume that $\mathcal{T}_{\mathcal{H}}=2\cdot\mathrm{R}_1+2\cdot\mathrm{R}_3$. Then the rational map $\underline{\mathcal{G}}_{\mathcal{H}}$ admits four fixed critical points, two of multiplicity 1 and two of multiplicity 3. This implies, by [7, page 79], that up to conjugation by a MÖBIUS transformation, $\underline{\mathcal{G}}_{\mathcal{H}}$ writes as

$$z\mapsto -\frac{z^4(3z+4cz-5c-4)}{z+c},$$

where $c=-1/2\pm\sqrt{5}/10$ or $c=-3/10\pm\sqrt{21}/10$. Thus, up to linear conjugation

$$\omega = y^4(3y+4cy-5cx-4x)\mathrm{d}x+x^4(y+cx)\mathrm{d}y, \qquad c\in\left\{-\frac{1}{2}\pm\frac{\sqrt{5}}{10},\,-\frac{3}{10}\pm\frac{\sqrt{21}}{10}\right\}.$$

In the case where $c=-1/2\pm\sqrt{5}/10$, resp. $c=-3/10\pm\sqrt{21}/10$, the 1-form ω is linearly conjugated to

$$\omega_7 = y^4\Big((5-\sqrt{5})x-2y\Big)\mathrm{d}x+x^4\Big((7-3\sqrt{5})x-2\big(5-2\sqrt{5}\big)y\Big)\mathrm{d}y,$$

$$\text{resp. } \omega_8 = y^4\Big(5\big(3-\sqrt{21}\big)x+6y\Big)\mathrm{d}x+x^4\Big(3\big(23-5\sqrt{21}\big)x-10\big(9-2\sqrt{21}\big)y\Big)\mathrm{d}y.$$

Indeed, on the one hand, if $c=-1/2+\sqrt{5}/10$, resp. $c=-3/10+\sqrt{21}/10$, then $\omega_7=-2(5-2\sqrt{5})\omega$, resp. $\omega_8=-10(9-2\sqrt{21})\omega$. On the other hand, if $c=-1/2-\sqrt{5}/10$, resp. $c=-3/10-\sqrt{21}/10$, then

$$\omega_7 = -(25+11\sqrt{5})\varphi^*\omega, \quad \text{where } \varphi=\left(\tfrac{3-\sqrt{5}}{2}x,y\right),$$

$$\text{resp. } \omega_8 = 5(87+19\sqrt{21})\psi^*\omega, \quad \text{where } \psi=\left(\tfrac{\sqrt{21}-5}{2}x,y\right).$$

- By Table 1, we have on the one hand $\mathrm{CS}_{\mathcal{H}_9}\neq\mathrm{CS}_{\mathcal{H}_{10}}$, so that the foliations $\mathcal{H}_9$ and $\mathcal{H}_{10}$ are not linearly conjugated, and on the other hand $\mathcal{T}_{\mathcal{H}_9}=\mathcal{T}_{\mathcal{H}_{10}}=1\cdot\mathrm{R}_1+2\cdot\mathrm{R}_2+1\cdot\mathrm{R}_3$. Moreover, by [7, page 79], up to MÖBIUS transformation there are two rational maps of degree 5 from the RIEMANN sphere to itself having four distinct fixed critical points, one of multiplicity 1, two of multiplicity 2 and one of multiplicity 3; thus up to automorphism of $\mathbb{P}^2_{\mathbb{C}}$ there are two homogeneous convex foliations of degree 5 on $\mathbb{P}^2_{\mathbb{C}}$ having type $1\cdot\mathrm{R}_1+2\cdot\mathrm{R}_2+1\cdot\mathrm{R}_3$. As a result, we deduce that if the foliation $\mathcal{H}$ is of type $\mathcal{T}_{\mathcal{H}}=1\cdot\mathrm{R}_1+2\cdot\mathrm{R}_2+1\cdot\mathrm{R}_3$, then $\mathcal{H}$ is linearly conjugated to one of the two foliations $\mathcal{H}_9$ or $\mathcal{H}_{10}$.

- Assume that $\mathcal{T}_{\mathcal{H}}=4\cdot\mathrm{R}_2$. The rational map $\underline{\mathcal{G}}_{\mathcal{H}}$ has therefore four different fixed critical points of multiplicity 2. By [7, page 80], up to conjugation by a MÖBIUS transformation, $\underline{\mathcal{G}}_{\mathcal{H}}$ writes as
$$z\mapsto -\frac{z^3(z^2-5z+5)}{5z^2-10z+4}.$$
As a consequence, up to linear conjugation
$$\omega=y^3(5x^2-5xy+y^2)\mathrm{d}x+x^3(4x^2-10xy+5y^2)\mathrm{d}y.$$
This 1-form is linearly conjugated to $\omega_{11}=y^3(5x^2-y^2)\mathrm{d}x+x^3(x^2-5y^2)\mathrm{d}y$; indeed
$$\omega_{11}=\frac{1}{8}\varphi^*\omega,\quad \text{where } \varphi=(x+y,2y).$$
- Assume that $\mathcal{T}_{\mathcal{H}}=2\cdot\mathrm{R}_1+3\cdot\mathrm{R}_2$. Then the rational map $\underline{\mathcal{G}}_{\mathcal{H}}$ possesses five fixed critical points, two of multiplicity 1 and three of multiplicity 2. By [7, page 80], $\underline{\mathcal{G}}_{\mathcal{H}}$ is conjugated by a MÖBIUS transformation to $z\mapsto -\dfrac{z^3(z^2+5z-20)}{20z^2-5z-1}$, which implies that ω is linearly conjugated to
$$\omega_{12}=y^3(20x^2-5xy-y^2)\mathrm{d}x+x^3(x^2+5xy-20y^2)\mathrm{d}y.$$
- Let us consider the eventuality $\mathcal{T}_{\mathcal{H}}=4\cdot\mathrm{R}_1+1\cdot\mathrm{R}_4$. We can assume, up to linear conjugation, that $\mathrm{D}_{\mathcal{H}}=cx^4y(y-x)(y-\alpha x)(y-\beta x)$ and $\mathrm{C}_{\mathcal{H}}(0,1)=\mathrm{C}_{\mathcal{H}}(1,0)=\mathrm{C}_{\mathcal{H}}(1,1)=\mathrm{C}_{\mathcal{H}}(1,\alpha)=\mathrm{C}_{\mathcal{H}}(1,\beta)=0$, where $\alpha,\beta\in\mathbb{C}\setminus\{0,1\}, c\in\mathbb{C}^*$, with $\alpha\neq\beta$. The points $\infty=[1:0]$, $[0:1]$, $[1:1]$, $[1:\alpha]$, $[1:\beta]\in\mathbb{P}^1_{\mathbb{C}}$ are therefore fixed and critical for $\underline{\mathcal{G}}_{\mathcal{H}}$, with respective multiplicities $4,1,1,1,1$. By [3, Lemma 3.9], there exist constants $a_0,a_3,b\in\mathbb{C}^*, a_1,a_2\in\mathbb{C}$ such that
$$B(x,y)=bx^5,\qquad A(x,y)=(a_0x^3+a_1x^2y+a_2xy^2+a_3y^3)y^2,\qquad (z-1)^2 \text{ divides } P(z),$$
$$(z-\alpha)^2 \text{ divides } Q(z),\qquad (z-\beta)^2 \text{ divides } R(z),$$
where $P(z):=A(1,z)+B(1,z)$, $Q(z):=A(1,z)+\alpha B(1,z)$ and $R(z):=A(1,z)+\beta B(1,z)$. Then we have
$$\left\{\begin{array}{l} P(1)=0\\ P'(1)=0\\ Q(\alpha)=0\\ Q'(\alpha)=0\\ R(\beta)=0\\ R'(\beta)=0\end{array}\right. \Leftrightarrow \left\{\begin{array}{l} a_0+a_1+a_2+a_3+b=0\\ 2a_0+3a_1+4a_2+5a_3=0\\ a_3\alpha^4+a_2\alpha^3+a_1\alpha^2+a_0\alpha+b=0\\ 5a_3\alpha^3+4a_2\alpha^2+3a_1\alpha+2a_0=0\\ a_3\beta^4+a_2\beta^3+a_1\beta^2+a_0\beta+b=0\\ 5a_3\beta^3+4a_2\beta^2+3a_1\beta+2a_0=0\end{array}\right. \Leftrightarrow \left\{\begin{array}{l} a_0=-\dfrac{a_3\alpha(\alpha+1)(3\alpha^2-5\alpha+3)}{2(\alpha^2-\alpha+1)}\\ a_1=\dfrac{a_3(\alpha^4+2\alpha^3-3\alpha^2+2\alpha+1)}{\alpha^2-\alpha+1}\\ a_2=-\dfrac{a_3(\alpha+1)(4\alpha^2-5\alpha+4)}{2(\alpha^2-\alpha+1)}\\ b=\dfrac{a_3\alpha^2(\alpha-1)^2}{2(\alpha^2-\alpha+1)}\\ \beta=\dfrac{(\alpha+1)(3\alpha^2-5\alpha+3)}{5(\alpha^2-\alpha+1)}\\ (\alpha^2-2\alpha+2)(2\alpha^2-2\alpha+1)(\alpha^2+1)=0\end{array}\right.$$

Multiplying ω by $\dfrac{2}{a_3}(\alpha^2-\alpha+1)$, we reduce it to

$$\omega=-y^2\Big(\alpha(\alpha+1)(3\alpha^2-5\alpha+3)x^3+(\alpha+1)(4\alpha^2-5\alpha+4)xy^2-2(\alpha^2-\alpha+1)y^3\Big)\mathrm{d}x$$
$$+2(\alpha^4+2\alpha^3-3\alpha^2+2\alpha+1)x^2y^3\mathrm{d}x+\alpha^2(\alpha-1)^2x^5\mathrm{d}y,$$

with $(\alpha^2-2\alpha+2)(2\alpha^2-2\alpha+1)(\alpha^2+1)=0$. This 1-form ω is linearly conjugated to

$$\omega_{13}=y^2(5x^3-10x^2y+10xy^2-4y^3)\mathrm{d}x-x^5\mathrm{d}y.$$

Indeed, the fact that α satisfies $(\alpha^2-2\alpha+2)(2\alpha^2-2\alpha+1)(\alpha^2+1)=0$ implies that

$$\omega_{13}=-\frac{(\alpha+1)(3\alpha^2-5\alpha+3)}{5\alpha^3(\alpha-1)^4}\varphi^*\omega,\quad \text{where } \varphi=\left(x,\frac{5\alpha\,(\alpha-1)^2}{(\alpha+1)\,(3\alpha^2-5\alpha+3)}y\right).$$

- Finally let us examine the case $\mathcal{T}_{\mathcal{H}}=3\cdot\mathrm{R}_1+1\cdot\mathrm{R}_2+1\cdot\mathrm{R}_3$. Up to isomorphism, we can assume that $\mathrm{D}_{\mathcal{H}}=cx^3y^2(y-x)(y-\alpha x)(y-\beta x)$ and $\mathrm{C}_{\mathcal{H}}(0,1)=\mathrm{C}_{\mathcal{H}}(1,0)=\mathrm{C}_{\mathcal{H}}(1,1)=\mathrm{C}_{\mathcal{H}}(1,\alpha)=\mathrm{C}_{\mathcal{H}}(1,\beta)=0$, where $\alpha,\beta\in\mathbb{C}\setminus\{0,1\}, c\in\mathbb{C}^*$, with $\alpha\neq\beta$. A similar reasoning as in the previous case leads to

$$\omega=\omega(\alpha)=y^3\Big(\big(\alpha^2-3\alpha+1\big)\big(\alpha^2+5\alpha+1\big)x^2-2\big(\alpha+1\big)\big(\alpha^2-5\alpha+1\big)xy+\big(\alpha^2-7\alpha+1\big)y^2\Big)\mathrm{d}x$$
$$+\alpha x^4\Big(2\alpha\big(\alpha^2-\alpha+1\big)x-\big(\alpha+1\big)\big(3\alpha^2-5\alpha+3\big)y\Big)\mathrm{d}y,$$

with $P(\alpha)=0$ where $P(z):=3z^6-39z^5+194z^4-203z^3+194z^2-39z+3$. The 1-form ω is linearly conjugated to

$$\omega_{14}=y^3\Big(\big(\sigma^2-3\sigma+1\big)\big(\sigma^2+5\sigma+1\big)x^2-2\big(\sigma+1\big)\big(\sigma^2-5\sigma+1\big)xy+\big(\sigma^2-7\sigma+1\big)y^2\Big)\mathrm{d}x$$
$$+\sigma x^4\Big(2\sigma\big(\sigma^2-\sigma+1\big)x-\big(\sigma+1\big)\big(3\sigma^2-5\sigma+3\big)y\Big)\mathrm{d}y,$$

where $\sigma=\rho+\mathrm{i}\sqrt{\frac{1}{6}-\frac{4}{3}\rho-\frac{1}{3}\rho^2}$ and ρ is the unique real number satisfying $8\rho^3-52\rho^2+134\rho-15=0$. Indeed, on the one hand, it is easy to see that σ is a root of the polynomial P, so that $\omega_{14}=\omega(\sigma)$. On the other hand, a straightforward computation shows that if α_1 and α_2 are any two roots of P then

$$\omega(\alpha_2)=-\frac{\mu}{21600}\Big(13035\alpha_1^5-167802\alpha_1^4+821633\alpha_1^3-777667\alpha_1^2+743778\alpha_1-76185\Big)\varphi^*\big(\omega(\alpha_1)\big)$$

with $\mu=195\alpha_2^4-202\alpha_2^3+233\alpha_2^2-42\alpha_2+3,\ \varphi=\left(x,-\dfrac{\lambda}{43200}y\right)$ where

$$\lambda=\Big(39\alpha_2^5-501\alpha_2^4+2447\alpha_2^3-2293\alpha_2^2+2343\alpha_2-477\Big)\Big(24\alpha_1^5-309\alpha_1^4+1510\alpha_1^3-1415\alpha_1^2+1446\alpha_1-21\Big).$$

The foliations $\mathcal{H}_1,\ldots,\mathcal{H}_{14}$ are not linearly conjugated because for all $i,j\in\{1,\ldots,14\}$ with $i\neq j$ we have (*see* Table 1)

$$\mathcal{T}_{\mathcal{H}_i}\neq\mathcal{T}_{\mathcal{H}_j}\qquad\text{or}\qquad \mathrm{CS}_{\mathcal{H}_i}\neq\mathrm{CS}_{\mathcal{H}_j}.$$

This ends the proof of the theorem. □

An immediate consequence of Theorem A is the following.

Corollary 2.2. — *Up to* MÖBIUS *transformation there are fourteen rational maps of degree five from the* RIEMANN *sphere to itself having only fixed critical points, namely the maps* $\underline{\mathcal{G}}_{\mathcal{H}_1},\ldots,\underline{\mathcal{G}}_{\mathcal{H}_{14}}$.

Proof of Theorem B. — Let $\mathcal{F}$ be a reduced convex foliation of degree 5 on $\mathbb{P}^2_{\mathbb{C}}$. Let us denote by Σ the set of non radial singularities of $\mathcal{F}$. By [4, Lemma 3.4], Σ is nonempty. Since by hypothesis $\mathcal{F}$ is reduced convex, all its singularities have MILNOR number 1 ([3, Lemma 6.8]). The set Σ consists then of the singularities $s \in \mathrm{Sing}\mathcal{F}$ such that $\tau(\mathcal{F},s)=1$. Let m be a point of Σ; by [4, Lemma 3.1], through the point m pass exactly two $\mathcal{F}$-invariant lines $\ell_m^{(1)}$ and $\ell_m^{(2)}$.
On the other hand, according to [4, Proposition 3.2] or [3, Proposition 6.4], for any line ℓ invariant by $\mathcal{F}$, there exists a homogeneous convex foliation $\mathcal{H}_\ell$ of degree 5 on $\mathbb{P}^2_{\mathbb{C}}$ such that the line ℓ is $\mathcal{H}_\ell$-invariant. Therefore $\mathcal{H}_\ell$, and in particular each $\mathcal{H}_{\ell_m^{(i)}}$, is linearly conjugated to one of the fourteen homogeneous foliations given by Theorem A. Proposition 3.2 of [4] also ensures that

($\mathfrak{a}$) $\mathrm{Sing}\mathcal{F}\cap\ell=\mathrm{Sing}\mathcal{H}_\ell\cap\ell$;
($\mathfrak{b}$) $\forall\, s\in\mathrm{Sing}\mathcal{H}_\ell\cap\ell,\ \mu(\mathcal{H}_\ell,s)=1$;
($\mathfrak{c}$) $\forall\, s\in\mathrm{Sing}\mathcal{H}_\ell\cap\ell,\ \tau(\mathcal{H}_\ell,s)=\tau(\mathcal{F},s)$;
($\mathfrak{d}$) $\forall\, s\in\mathrm{Sing}\mathcal{H}_\ell\cap\ell,\ \mathrm{CS}(\mathcal{H}_\ell,\ell,s)=\mathrm{CS}(\mathcal{F},\ell,s)$.

Since $\mathrm{CS}(\mathcal{F},\ell_m^{(1)},m)\mathrm{CS}(\mathcal{F},\ell_m^{(2)},m)=1$, relation ($\mathfrak{d}$) implies that $\mathrm{CS}(\mathcal{H}_{\ell_m^{(1)}},\ell_m^{(1)},m)\mathrm{CS}(\mathcal{H}_{\ell_m^{(2)}},\ell_m^{(2)},m)=1$. This equality and Table 1 lead to

$$\Big\{\big\{\mathrm{CS}(\mathcal{H}_{\ell_m^{(1)}},\ell_m^{(1)},m),\,\mathrm{CS}(\mathcal{H}_{\ell_m^{(2)}},\ell_m^{(2)},m)\big\}\Big\}=\Big\{\big\{-4,-\tfrac{1}{4}\big\},\big\{-\tfrac{3}{2}+\tfrac{1}{2}\sqrt{5},-\tfrac{3}{2}-\tfrac{1}{2}\sqrt{5}\big\}\Big\}. \tag{2.1}$$

At first let us assume that it is possible to choose $m\in\Sigma$ so that

$$\{\mathrm{CS}(\mathcal{H}_{\ell_m^{(1)}},\ell_m^{(1)},m),\,\mathrm{CS}(\mathcal{H}_{\ell_m^{(2)}},\ell_m^{(2)},m)\}=\{-4,-\tfrac{1}{4}\}.$$

Up to renumbering the $\ell_m^{(i)}$ we can assume that $\mathrm{CS}(\mathcal{H}_{\ell_m^{(1)}},\ell_m^{(1)},m)=-\frac{1}{4}$ and $\mathrm{CS}(\mathcal{H}_{\ell_m^{(2)}},\ell_m^{(2)},m)=-4$. Consulting Table 1, we see that

$$\mathcal{T}_{\mathcal{H}_{\ell_m^{(1)}}}=2\cdot\mathrm{R}_4,\qquad\qquad \mathcal{T}_{\mathcal{H}_{\ell_m^{(2)}}}\in\Big\{2\cdot\mathrm{R}_1+3\cdot\mathrm{R}_2,4\cdot\mathrm{R}_1+1\cdot\mathrm{R}_4,3\cdot\mathrm{R}_1+1\cdot\mathrm{R}_2+1\cdot\mathrm{R}_3\Big\}.$$

Therefore, it follows from relations ($\mathfrak{a}$) and ($\mathfrak{c}$) that $\mathcal{F}$ possesses two radial singularities m_1,m_2 of order 4 on the line $\ell_m^{(1)}$ and a radial singularity m_3 of order 2 or 4 on the line $\ell_m^{(2)}$.
We will see that the radiality order of the singularity m_3 of $\mathcal{F}$ is necessarily 4, *i.e.* $\tau(\mathcal{F},m_3)=5$. By [2, Proposition 2, page 23], the fact that $\tau(\mathcal{F},m_1)+\tau(\mathcal{F},m_3)\geq 5+3>\deg\mathcal{F}$ implies the invariance by $\mathcal{F}$ of the line $\ell=(m_1m_3)$; if $\tau(\mathcal{F},m_3)$ were equal to 3, then relations ($\mathfrak{a}$), ($\mathfrak{b}$) and ($\mathfrak{c}$), combined with the convexity of the foliation $\mathcal{H}_\ell$, would imply that

$$\mathcal{T}_{\mathcal{H}_\ell}\in\Big\{2\cdot\mathrm{R}_2+1\cdot\mathrm{R}_4,2\cdot\mathrm{R}_1+1\cdot\mathrm{R}_2+1\cdot\mathrm{R}_4\Big\}$$

so that (*see* Table 1) $\mathcal{H}_\ell$ would possess a singularity m' on the line ℓ satisfying

$$\mathrm{CS}(\mathcal{H}_\ell,\ell,m')\in\Big\{\lambda\in\mathbb{C}\,:\,\big(\lambda+\tfrac{3}{7}\big)\big(\lambda+\tfrac{8}{7}\big)\big(\lambda+\tfrac{3}{2}\big)\big(59\lambda^2+177\lambda+64\big)=0\Big\}$$

which is not possible by (2.1).
By construction, the three points m_1, m_2 and m_3 are not aligned. We have thus shown that $\mathcal{F}$ admits three non-aligned radial singularities of order 4. By [3, Proposition 6.3] the foliation $\mathcal{F}$ is linearly conjugated to the FERMAT foliation $\mathcal{F}_0^5$.

Let us now consider the eventuality $\{\mathrm{CS}(\mathcal{H}_{\ell_m^{(1)}}, \ell_m^{(1)}, m), \mathrm{CS}(\mathcal{H}_{\ell_m^{(2)}}, \ell_m^{(2)}, m)\} = \{-\frac{3}{2}+\frac{1}{2}\sqrt{5}, -\frac{3}{2}-\frac{1}{2}\sqrt{5}\}$ for any choice of $m \in \Sigma$. In this case, Table 1 leads to $\mathcal{T}_{\mathcal{H}_{\ell_m^{(i)}}} = 2\cdot \mathrm{R}_1 + 2\cdot \mathrm{R}_3$ for $i=1,2$. Then, as before, by using relations $(\mathfrak{a})$, $(\mathfrak{b})$ and $(\mathfrak{c})$, we obtain that $\mathcal{F}$ possesses exactly four radial singularities on each of the lines $\ell_m^{(i)}$, two of order 1 and two of order 3. Moreover, every line joining a radial singularity of order 3 of $\mathcal{F}$ on $\ell_m^{(1)}$ and a radial singularity of order 3 of $\mathcal{F}$ on $\ell_m^{(2)}$ must necessarily contain two radial singularities of order 1 of $\mathcal{F}$. We can then choose a homogeneous coordinate system $[x:y:z] \in \mathbb{P}^2_{\mathbb{C}}$ in such a way that the points $m_1 = [0:0:1]$, $m_2 = [1:0:0]$ and $m_3 = [0:1:0]$ are radial singularities of order 3 of $\mathcal{F}$. Moreover, in this coordinate system the lines $x=0$, $y=0$, $z=0$ must be invariant by $\mathcal{F}$ and there exist $x_0, y_0, z_0, x_1, y_1, z_1 \in \mathbb{C}^*$, $x_1 \neq x_0, y_1 \neq y_0, z_1 \neq z_0$, such that the points $m_4 = [x_0:0:1]$, $m_5 = [1:y_0:0]$, $m_6 = [0:1:z_0]$, $m_7 = [x_1:0:1]$, $m_8 = [1:y_1:0]$, $m_9 = [0:1:z_1]$ are radial singularities of order 1 of $\mathcal{F}$. Let us set $\xi = \frac{x_1}{x_0}$, $\rho = \frac{y_1}{y_0}$, $\sigma = \frac{z_1}{z_0}$, $w_0 = x_0y_0z_0$; then $w_0 \in \mathbb{C}^*$, $\xi, \rho, \sigma \in \mathbb{C}\setminus\{0,1\}$ and, up to renumbering the x_i, y_i, z_i, we can assume that ξ, ρ and σ are all of modulus greater than or equal to 1. Let ω be a 1-form describing $\mathcal{F}$ in the affine chart $z=1$. By conjugating ω by the diagonal linear transformation (x_0x, x_0y_0y), we reduce ourselves to $m_4 = [1:0:1]$, $m_5 = [1:1:0]$, $m_6 = [0:1:w_0]$, $m_7 = [\xi:0:1]$, $m_8 = [1:\rho:0]$, $m_9 = [0:1:\sigma w_0]$. The equalities $\nu(\mathcal{F}, m_1) = 1$, $\tau(\mathcal{F}, m_1) = 4$ and the invariance of the line $z=0$ by $\mathcal{F}$ ensure that ω is of type

$$\begin{aligned}\omega = {} & (x\mathrm{d}y - y\mathrm{d}x)(\gamma + \lambda_0 x + \lambda_1 y + c_0x^2 + c_1xy + c_2y^2) + (\alpha_0x^4 + \alpha_1x^3y + \alpha_2x^2y^2 + \alpha_3xy^3 + \alpha_4y^4)\mathrm{d}x \\ & + (\beta_0x^4 + \beta_1x^3y + \beta_2x^2y^2 + \beta_3xy^3 + \beta_4y^4)\mathrm{d}y + (a_0x^5 + a_1x^4y + a_2x^3y^2 + a_3x^2y^3 + a_4xy^4 + a_5y^5)\mathrm{d}x \\ & + (b_0x^5 + b_1x^4y + b_2x^3y^2 + b_3x^2y^3 + b_4xy^4 + b_5y^5)\mathrm{d}y,\end{aligned}$$

where $a_i, b_i, c_j, \alpha_k, \beta_k, \lambda_l \in \mathbb{C}$ and $\gamma \in \mathbb{C}^*$.
In the affine chart $x=1$, resp. $y=1$, the foliation $\mathcal{F}$ is given by

$$\begin{aligned}\theta = {} & -(\beta_0z + \beta_1yz + \beta_2y^2z + \beta_3y^3z + \beta_4y^4z + b_0 + b_1y + b_2y^2 + b_3y^3 + b_4y^4 + b_5y^5)(y\mathrm{d}z - z\mathrm{d}y) \\ & -(\alpha_0z + \alpha_1yz + \alpha_2y^2z + \alpha_3y^3z + \alpha_4y^4z + a_0 + a_1y + a_2y^2 + a_3y^3 + a_4y^4 + a_5y^5)\mathrm{d}z \\ & + z^3(\gamma z^2 + \lambda_1yz + \lambda_0z + c_0 + c_1y + c_2y^2)\mathrm{d}y, \\ \text{resp. } \eta = {} & (\alpha_0x^4z + \alpha_1x^3z + \alpha_2x^2z + \alpha_3xz + \alpha_4z + a_0x^5 + a_1x^4 + a_2x^3 + a_3x^2 + a_4x + a_5)(z\mathrm{d}x - x\mathrm{d}z) \\ & -(\beta_0x^4z + \beta_1x^3z + \beta_2x^2z + \beta_3xz + \beta_4z + b_0x^5 + b_1x^4 + b_2x^3 + b_3x^2 + b_4x + b_5)\mathrm{d}z \\ & - z^3(\gamma z^2 + \lambda_0xz + \lambda_1z + c_0x^2 + c_1x + c_2)\mathrm{d}x.\end{aligned}$$

A straightforward computation shows that

$$\begin{aligned}&\Big(J^3_{(y,z)=(0,0)}\theta\Big)\wedge\Big(y\mathrm{d}z - z\mathrm{d}y\Big) = -zP(y,z)\mathrm{d}y\wedge\mathrm{d}z, && \Big(J^3_{(x,z)=(0,0)}\eta\Big)\wedge\Big(z\mathrm{d}x - x\mathrm{d}z\Big) = zQ(x,z)\mathrm{d}x\wedge\mathrm{d}z, \\ &\Big(J^1_{(x,y)=(1,0)}\omega\Big)\wedge\Big((x-1)\mathrm{d}y - y\mathrm{d}x\Big) = R(x,y)\mathrm{d}x\wedge\mathrm{d}y, && \Big(J^1_{(y,z)=(1,0)}\theta\Big)\wedge\Big((y-1)\mathrm{d}z - z\mathrm{d}y\Big) = -zS(y,z)\mathrm{d}y\wedge\mathrm{d}z, \\ &\Big(J^1_{(x,z)=(0,w_0)}\eta\Big)\wedge\Big((z-w_0)\mathrm{d}x - x\mathrm{d}z\Big) = T(x,z)\mathrm{d}x\wedge\mathrm{d}z, && \Big(J^1_{(x,y)=(\xi,0)}\omega\Big)\wedge\Big((x-\xi)\mathrm{d}y - y\mathrm{d}x\Big) = \xi U(x,y)\mathrm{d}x\wedge\mathrm{d}y, \\ &\Big(J^1_{(y,z)=(\rho,0)}\theta\Big)\wedge\Big((y-\rho)\mathrm{d}z - z\mathrm{d}y\Big) = -zV(y,z)\mathrm{d}y\wedge\mathrm{d}z, && \Big(J^1_{(x,z)=(0,\sigma w_0)}\eta\Big)\wedge\Big((z-\sigma w_0)\mathrm{d}x - x\mathrm{d}z\Big) = W(x,z)\mathrm{d}x\wedge\mathrm{d}z\end{aligned}$$

with

$$P(y,z)=a_0+a_1y+\alpha_0z+a_2y^2+\alpha_1yz+a_3y^3+\alpha_2y^2z-c_0yz^2,$$
$$Q(x,z)=b_5+b_4x+\beta_4z+b_3x^2+\beta_3xz+b_2x^3+\beta_2x^2z+c_2xz^2,$$
$$R(x,y)=4a_0+3\alpha_0-(9a_0+7\alpha_0)x+(\gamma-c_0-a_1-\alpha_1-4b_0-3\beta_0)y+(\lambda_0+2c_0+a_1+\alpha_1+5b_0+4\beta_0)xy$$
$$+(5a_0+4\alpha_0)x^2+(\lambda_1+c_1+b_1+\beta_1)y^2,$$
$$S(y,z)=(a_1+2a_2+3a_3+4a_4+5a_5+b_1+2b_2+3b_3+4b_4+5b_5)y+a_0-a_2-2a_3-3a_4-4a_5+b_0-b_2-2b_3$$
$$-3b_4-4b_5+(\alpha_0+\alpha_1+\alpha_2+\alpha_3+\alpha_4+\beta_0+\beta_1+\beta_2+\beta_3+\beta_4)z,$$
$$T(x,z)=-b_5w_0-w_0(4\gamma w_0^4+3\lambda_1w_0^3+2c_2w_0^2+\beta_3w_0+a_5+b_4)x-(\beta_4w_0-b_5)z+w_0(\lambda_0w_0^3+c_1w_0^2-\alpha_3w_0-a_4)x^2$$
$$+(5\gamma w_0^4+4\lambda_1w_0^3+3c_2w_0^2-\alpha_4w_0+\beta_3w_0+b_4)xz+\beta_4z^2,$$
$$U(x,y)=(4a_0\xi+3\alpha_0)\xi^4-\xi^3(9a_0\xi+7\alpha_0)x-(a_1\xi^4+4b_0\xi^4+\alpha_1\xi^3+3\beta_0\xi^3+c_0\xi^2-\gamma)y+\xi^2(5a_0\xi+4\alpha_0)x^2$$
$$+(a_1\xi^3+5b_0\xi^3+\alpha_1\xi^2+4\beta_0\xi^2+2c_0\xi+\lambda_0)xy+(b_1\xi^3+\beta_1\xi^2+c_1\xi+\lambda_1)y^2,$$
$$V(y,z)=(5b_5\rho^5+5a_5\rho^4+4b_4\rho^4+4a_4\rho^3+3b_3\rho^3+3a_3\rho^2+2b_2\rho^2+2a_2\rho+b_1\rho+a_1)y-4b_5\rho^6-(4a_5+3b_4)\rho^5$$
$$+(\beta_4\rho^5+\alpha_4\rho^4+\beta_3\rho^4+\alpha_3\rho^3+\beta_2\rho^3+\alpha_2\rho^2+\beta_1\rho^2+\alpha_1\rho+\beta_0\rho+\alpha_0)z-(3a_4+2b_3)\rho^4-(2a_3+b_2)\rho^3$$
$$-a_2\rho^2+b_0\rho+a_0,$$
$$W(x,z)=\sigma w_0(\lambda_0\sigma^3w_0^3+c_1\sigma^2w_0^2-\alpha_3\sigma w_0-a_4)x^2+(5\gamma\sigma^4w_0^4+4\lambda_1\sigma^3w_0^3+3c_2\sigma^2w_0^2-\alpha_4\sigma w_0+\beta_3\sigma w_0+b_4)xz$$
$$+\beta_4z^2-\sigma w_0(4\gamma\sigma^4w_0^4+3\lambda_1\sigma^3w_0^3+2c_2\sigma^2w_0^2+\beta_3\sigma w_0+a_5+b_4)x+(b_5-\beta_4\sigma w_0)z-b_5\sigma w_0,$$

so that the equality $\tau(\mathcal{F},m_2)=4$ (resp. $\tau(\mathcal{F},m_3)=4$, resp. $\tau(\mathcal{F},m_4)=2$, resp. $\tau(\mathcal{F},m_5)=2$, resp. $\tau(\mathcal{F},m_6)=2$, resp. $\tau(\mathcal{F},m_7)=2$, resp. $\tau(\mathcal{F},m_8)=2$, resp. $\tau(\mathcal{F},m_9)=2$) implies that the polynomial P (resp. Q, resp. R, resp. S, resp. T, resp. U, resp. V, resp. W) is identically zero. From $P=Q=0$ we obtain $a_0=a_1=a_2=a_3=\alpha_0=\alpha_1=\alpha_2=b_2=b_3=b_4=b_5=\beta_2=\beta_3=\beta_4=c_0=c_2=0$. Next, from the equalities $R=S=T=U=V=W=0$ we deduce that

$$\xi=\rho=\sigma=\frac{3}{2}+\frac{\sqrt{5}}{2},\qquad \gamma=\frac{47+21\sqrt{5}}{2}a_5,\qquad \lambda_0=-\frac{65+29\sqrt{5}}{2}a_5,$$
$$a_4=-\frac{5+\sqrt{5}}{2}a_5,\qquad b_1=(5+2\sqrt{5})a_5,\qquad \beta_0=\frac{25+11\sqrt{5}}{2}a_5,$$
$$\alpha_3=(9+4\sqrt{5})(5w_0+5-2\sqrt{5})a_5,\qquad \alpha_4=-\frac{25+11\sqrt{5}}{2}a_5w_0,\qquad \lambda_1=-(85+38\sqrt{5})a_5w_0,$$
$$\beta_1=-\frac{(65+29\sqrt{5})(w_0+5-2\sqrt{5})}{2}a_5,\qquad c_1=\frac{(47+21\sqrt{5})(5w_0+5-2\sqrt{5})}{2}a_5,\qquad w_0=\pm(\sqrt{5}-2)$$

with $a_5\neq0$. Thus ω is of type

$$\omega=\frac{a_5(47+21\sqrt{5})}{4}\Big(x\mathrm{d}y-y\mathrm{d}x\Big)\Big(2-\big(5-\sqrt{5}\big)x-w_0\big(5+\sqrt{5}\big)y+\big(10w_0+10-4\sqrt{5}\big)xy\Big)$$
$$+\frac{a_5}{2}y^3\Big(\big(9+4\sqrt{5}\big)\big(10w_0+10-4\sqrt{5}\big)x-w_0\big(25+11\sqrt{5}\big)y-\big(5+\sqrt{5}\big)xy+2y^2\Big)\mathrm{d}x$$
$$+\frac{a_5}{2}x^3\Big(\big(25+11\sqrt{5}\big)x-\big(65+29\sqrt{5}\big)\big(w_0+5-2\sqrt{5}\big)y-\big(7+3\sqrt{5}\big)x^2+\big(10+4\sqrt{5}\big)xy\Big)\mathrm{d}y,$$

where $w_0=\pm(\sqrt{5}-2)$ and $a_5\in\mathbb{C}^*$. The 1-form is linearly conjugated to

$$\omega_H^5=\big(y^2-1\big)\big(y^2-(\sqrt{5}-2)^2\big)\big(y+\sqrt{5}x\big)\mathrm{d}x-\big(x^2-1\big)\big(x^2-(\sqrt{5}-2)^2\big)\big(x+\sqrt{5}y\big)\mathrm{d}y.$$

Indeed, if $w_0=\sqrt{5}-2$, resp. $w_0=2-\sqrt{5}$, then

$$\omega_H^5=\frac{32(3571-1597\sqrt{5})}{a_5}\varphi_1^*\omega,\qquad \text{where }\ \varphi_1=\left(\frac{3+\sqrt{5}}{4}(x+1),-\frac{2+\sqrt{5}}{2}(y-1)\right),$$

$$\text{resp. }\ \omega_H^5=\frac{32(64079-28657\sqrt{5})}{a_5}\varphi_2^*\omega,\quad \text{where }\ \varphi_2=\left(\frac{2+\sqrt{5}}{2}(x+\sqrt{5}-2),-\frac{7+3\sqrt{5}}{4}(y+\sqrt{5}-2)\right).$$

□

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