SIMPLICITY OF FUSION SYSTEMS OF FINITE SIMPLE GROUPS

BOB OLIVER AND ALBERT RUIZ

Abstract. We determine for which known finite simple groups G and which primes p the p-fusion system of G is simple. This means first collecting together the results that were already known (and correcting two errors made in an earlier study of this question), and then handling the remaining cases. At the same time, we develop some new tools to use when determining $O^{p'}(\mathcal{F})$ for arbitrary saturated fusion systems.

For a prime p and a finite group G, the *fusion system* of G over a Sylow p-subgroup S of G is the category $\mathcal{F}_S(G)$ whose objects are the subgroups of S, and whose morphisms are those homomorphisms between subgroups induced by conjugation by elements of G. The fusion system of G thus encodes its local structure: the conjugacy relations among its p-subgroups and p-elements. Motivated by connections with modular representation theory, Puig in the 1990s defined the concept of abstract fusion systems (published later in [Pg]): an (abstract) fusion system $\mathcal{F}$ over a finite p-group S is a category whose objects are the subgroups of S, and whose morphisms are injective homomorphisms satisfying certain conditions motivated by properties of finite groups such as the Sylow theorems. (See Section 1 for more detail.)

Normal fusion subsystems and simple fusion systems are defined by analogy to those of finite groups. Our main result here is to determine for exactly which known finite simple groups G and which primes p the p-fusion system of G is simple. This question was studied and answered in most cases by Aschbacher in Chapter 16 of [A2], but a few cases (all involving groups of Lie type in cross characteristic) were left open, and it is on those that we focus here. We also correct two errors found among Aschbacher's conclusions (explained in the proof of Proposition 4.1(d)). One problem of particular interest is that of for which G and p, the p-fusion system of G contains a normal subsystem of index prime to p that is exotic, and we are also able to describe this situation quite precisely.

Our detailed group-by-group results are stated in Propositions 4.1 (alternating and sporadic groups and groups of Lie type in characteristic p), 4.2 (groups of Lie type in odd characteristic when $p = 2$), 4.5 (classical groups in characteristic different from p when p is odd), and 4.6 (exceptional groups of Lie type in characteristic different from p for p odd). They are then summarized by the following theorem, which will be restated with more details as Theorem 4.8.

Theorem A. *Fix a prime p, a known finite simple group G, and $S \in \mathrm{Syl}_p(G)$. Set $\mathcal{F} = \mathcal{F}_S(G)$, and assume that $S \ntrianglelefteq \mathcal{F}$. Then either*

(a) *$p \geq 5$, G is one of the classical groups $PSL_n^{\pm}(q)$, $PSp_{2n}(q)$, $\Omega_{2n+1}(q)$, or $P\Omega_{2n+2}^{\pm}(q)$ where $n \geq 2$ and $q \not\equiv 0, \pm 1 \pmod p$, in which case $O^{p'}(\mathcal{F})$ is simple and exotic; or*

(b) *$p = 3$ and $G \cong G_2(q)$ for some $q \equiv \pm 1 \pmod 9$, in which case $|O_3(\mathcal{F})| = 3$, and $O^{3'}(\mathcal{F})$ is realized by $SL_3^{\pm}(q)$; or*

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(c) *$O^{p'}(\mathcal{F})$ is simple, and it is realized by a known finite simple group G^*.*

Moreover, in case (a)*, there is a subsystem $\mathcal{F}_0 \trianglelefteq \mathcal{F}$ of index at most 2 in $\mathcal{F}$ with the property that for each saturated fusion system $\mathcal{E}$ over S such that $O^{p'}(\mathcal{E}) = O^{p'}(\mathcal{F})$, $\mathcal{E}$ is realizable if and only if it contains $\mathcal{F}_0$.*

The statements in Theorem A that certain fusion systems are exotic are proved using the classification of finite simple groups. It has been formulated so that the other statements are independent of this.

Here, by analogy with the notation used for finite groups, $O^{p'}(\mathcal{F})$ denotes the smallest (saturated) fusion subsystem of index prime to p (see Definition 2.1 and Proposition 2.4(b)). Also, a fusion system is called *realizable* if it is isomorphic to the fusion system of some finite group, and is called *exotic* otherwise.

Among the tools developed to determine $O^{p'}(\mathcal{F})$ in these cases, we note in particular the following, which reduce the information needed in two different directions:

- Although $O^{p'}(\mathcal{F})$ is characterized by its restriction to $\mathcal{F}$-centric subgroups, we can, in fact, get the information we need from the smaller family of subgroups that are $\mathcal{F}$-centric and $\mathcal{F}$-radical, as described in Proposition 2.8.
- Although subsystems of index prime to p are determined by their automizers on S, it is sometimes more convenient to deal with the automizer of some subgroup $A \trianglelefteq S$ that is $\mathcal{F}$-centric and weakly closed in $\mathcal{F}$ (such as the p-power torsion in a maximal torus in a group of Lie type). Tools for doing this are developed in Section 3.

This paper was originally motivated by our work (still continuing) with Carles Broto and Jesper Møller on tameness of realizable fusion systems (see [AKO, Definition III.6.3]). While trying to determine whether or not all realizable fusion systems are tame, we found that it is first necessary to understand more precisely the normal fusion subsystems of fusion systems of simple groups. Independently of that, some of our results are used in recent work of Radha Kessar, Gunter Malle, and Jason Semeraro to calculate weights (in the context of the Alperin weight conjecture in modular representation theory) attached to exotic fusions arising from homotopy fixed points of p-compact groups.

We begin the paper with a brief introduction to fusion systems in Section 1, and an introduction to (normal) subsystems of index prime to p in Section 2. We then develop tools, in the last part of Section 2 and in Section 3, to help determine $O^{p'}(\mathcal{F})$ for a given fusion system $\mathcal{F}$. Finally, in Section 4, we go through the list of all fusion systems of known finite simple groups at primes dividing their order, starting with a survey of the cases already handled by Aschbacher in [A2, Chapter 16], to determine which of them are simple.

Notation: As usual, for a finite group G, $O_p(G)$ and $O_{p'}(G)$ denote the largest normal subgroups of p-power order and of order prime to p, respectively, while $O^p(G)$ and $O^{p'}(G)$ are the smallest normal subgroups of p-power index and of index prime to p. We follow the standard conventions by setting $(P)SL_n^+(q) = (P)SL_n(q)$ and $(P)SL_n^-(q) = (P)SU_n(q)$, as well as $E_6^+(q) = E_6(q)$ and $E_6^-(q) = {}^2E_6(q)$.

1. Definitions and earlier results

We begin by listing some of the basic definitions and properties of fusion systems. We use [AKO, Part I] as our main reference, but most or all of the following definitions are due originally to Puig: first in unpublished notes and then in [Pg].

For a prime p, a *fusion system* over a finite p-group S is a category whose objects are the subgroups of S, and whose morphisms are injective homomorphisms between subgroups such that for each $P, Q \leq S$:

- $\mathrm{Hom}_{\mathcal{F}}(P,Q) \supseteq \mathrm{Hom}_S(P,Q) \stackrel{\text{def}}{=} \{c_g = (x \mapsto {}^g x) \,|\, g \in S,\ {}^g P \leq Q\}$; and
- for each $\varphi \in \mathrm{Hom}_{\mathcal{F}}(P,Q)$, $\varphi^{-1} \in \mathrm{Hom}_{\mathcal{F}}(\varphi(P), P)$.

Thus $\mathrm{Hom}_{\mathcal{F}}(P,Q)$ denotes the set of morphisms in $\mathcal{F}$ from P to Q. We also write $\mathrm{Iso}_{\mathcal{F}}(P,Q)$ for the set of isomorphisms, $\mathrm{Aut}_{\mathcal{F}}(P) = \mathrm{Iso}_{\mathcal{F}}(P,P)$, and $\mathrm{Out}_{\mathcal{F}}(P) = \mathrm{Aut}_{\mathcal{F}}(P)/\mathrm{Inn}(P)$. For $P \leq S$ and $g \in S$, we set

$$P^{\mathcal{F}} = \{\varphi(P) \,|\, \varphi \in \mathrm{Hom}_{\mathcal{F}}(P,S)\} \qquad \text{and} \qquad g^{\mathcal{F}} = \{\varphi(g) \,|\, \varphi \in \mathrm{Hom}_{\mathcal{F}}(\langle g \rangle, S)\}$$

(the sets of subgroups and elements $\mathcal{F}$-*conjugate* to P and to g).

Since we will need to refer to the Sylow and extension axioms on several occasions, the following version of the definition of a saturated fusion system seems the most convenient one to give here. (See Definitions I.2.2 and I.2.4 and Proposition I.2.5 in [AKO].)

Definition 1.1. *Let $\mathcal{F}$ be a fusion system over a finite p-group S.*

(a) *A subgroup $P \leq S$ is* fully normalized (fully centralized) *in $\mathcal{F}$ if $|N_S(P)| \geq |N_S(Q)|$ ($|C_S(P)| \geq |C_S(Q)|$) for each $Q \in P^{\mathcal{F}}$.*

(b) *The fusion system $\mathcal{F}$ is* saturated *if it satisfies the following two conditions:*
   (I) (Sylow axiom) *For each subgroup $P \leq S$ fully normalized in $\mathcal{F}$, P is fully centralized and $\mathrm{Aut}_S(P) \in \mathrm{Syl}_p(\mathrm{Aut}_{\mathcal{F}}(P))$.*
   (II) (extension axiom) *For each isomorphism $\varphi \in \mathrm{Iso}_{\mathcal{F}}(P,Q)$ in $\mathcal{F}$ such that Q is fully centralized in $\mathcal{F}$, φ extends to a morphism $\overline{\varphi} \in \mathrm{Hom}_{\mathcal{F}}(N_\varphi, S)$ where*
$$N_\varphi = \{g \in N_S(P) \,|\, \varphi c_g \varphi^{-1} \in \mathrm{Aut}_S(Q)\}.$$

If G is a finite group and $P, Q \leq G$, then $\mathrm{Hom}_G(P,Q) \subseteq \mathrm{Hom}(P,Q)$ denotes the set of (injective) homomorphisms induced by conjugation in G. If $S \in \mathrm{Syl}_p(G)$, then $\mathcal{F}_S(G)$ denotes the fusion system over S where $\mathrm{Hom}_{\mathcal{F}_S(G)}(P,Q) = \mathrm{Hom}_G(P,Q)$ for $P, Q \leq S$, and $\mathcal{F}_S(G)$ is saturated by [AKO, Theorem I.2.3]. A saturated fusion system $\mathcal{F}$ over S will be called *realizable* if $\mathcal{F} = \mathcal{F}_S(G)$ for some finite group G with $S \in \mathrm{Syl}_p(G)$, and will be called *exotic* otherwise.

We next list some of the terminology used to describe certain subgroups in a fusion system.

Definition 1.2. *Let $\mathcal{F}$ be a fusion system over a finite p-group S, and let $P \leq S$ be a subgroup.*

(a) *P is $\mathcal{F}$*-centric *if $C_S(Q) \leq Q$ for each $Q \in P^{\mathcal{F}}$.*

(b) *P is $\mathcal{F}$*-radical *if $O_p(\mathrm{Out}_{\mathcal{F}}(P)) = 1$.*

(c) *$\mathcal{F}^c \supseteq \mathcal{F}^{rc}$ denote the sets of $\mathcal{F}$-centric, and $\mathcal{F}$-centric $\mathcal{F}$-radical, subgroups of S, and also (depending on context) the full subcategories of $\mathcal{F}$ with those objects.*

(d) *P is* weakly closed *in $\mathcal{F}$ if $P^{\mathcal{F}} = \{P\}$.*

(e) *P is* strongly closed *in $\mathcal{F}$ if for each $x \in P$, $x^{\mathcal{F}} \subseteq P$.*

(f) *P is* normal *in $\mathcal{F}$ ($P \trianglelefteq \mathcal{F}$) if each $\varphi \in \mathrm{Hom}_{\mathcal{F}}(Q,R)$ (for $Q, R \leq S$) extends to a morphism $\overline{\varphi} \in \mathrm{Hom}_{\mathcal{F}}(PQ, PR)$ such that $\overline{\varphi}(P) = P$.*

(g) *P is* central *in $\mathcal{F}$ if each $\varphi \in \mathrm{Hom}_{\mathcal{F}}(Q,R)$ (for $Q, R \leq S$) extends to a morphism $\overline{\varphi} \in \mathrm{Hom}_{\mathcal{F}}(PQ, PR)$ such that $\overline{\varphi}|_P = \mathrm{Id}_P$.*

(h) $O_p(\mathcal{F}) \geq Z(\mathcal{F})$ *denote the (unique) largest normal and central subgroups, respectively, in* $\mathcal{F}$.

When X is a set of homomorphisms between subgroups of a given finite p-group S, we let $\langle X\rangle$ denote the smallest fusion system over S (not necessarily saturated) that contains X among its morphisms. In other words, $\langle X\rangle$ is the category whose objects are the subgroups of S, and whose morphisms are composites of restrictions of homomorphisms in $X \cup \mathrm{Inn}(S)$. Similarly, when $\mathcal{E}$ is a fusion system over S or a subgroup of S, and X is a set of homomorphisms between subgroups of S, we set $\langle \mathcal{E}, X\rangle = \langle \mathrm{Mor}(\mathcal{E}) \cup X\rangle$: the smallest fusion system over S containing $\mathcal{E}$ and X.

The following version of Alperin's fusion theorem for fusion systems will suffice for our purposes here. We refer to Theorem I.3.5 and Proposition I.3.3(a) in [AKO].

Theorem 1.3. *For each saturated fusion system* $\mathcal{F}$ *over a finite* p*-group* S,

$$\mathcal{F} = \bigl\langle \mathrm{Aut}_{\mathcal{F}}(P) \bigm| P \in \mathcal{F}^{rc} \text{ and is fully normalized in } \mathcal{F} \bigr\rangle.$$

Equivalently, each morphism in $\mathcal{F}$ *is a composite of restrictions of* $\mathcal{F}$*-automorphisms of subgroups that are centric and radical in* $\mathcal{F}$.

We finish the section with a much more specialized lemma: one which will be useful in the last section when showing the nonexistence of certain fusion systems.

Lemma 1.4. *Let* p *be an odd prime, and let* $\mathcal{F}$ *be a saturated fusion system over a finite* p*-group* S*. Assume that there is an abelian subgroup* $A \trianglelefteq S$ *such that* $|S/A| = p$, $|[S,A]| \geq p^2$, *and* $A \ntrianglelefteq \mathcal{F}$*. Set* $G = \mathrm{Aut}_{\mathcal{F}}(A)$ *and* $U = \mathrm{Aut}_S(A) \in \mathrm{Syl}_p(G)$*; then* $|\mathrm{Aut}_G(U)| = p-1$.

Proof. Since $|[S,A]| > p$, A is the unique abelian subgroup of index p in S. By Lemmas 2.2(a,c) and 2.3(a,b) in [COS] and since $A \ntrianglelefteq \mathcal{F}$, there is a subgroup $A \neq P < S$ in one of the classes $\mathcal{H}$ or $\mathcal{B}$ of subgroups defined in [COS, Notation 2.1], maximal among all $\mathcal{F}$-centric $\mathcal{F}$-radical subgroups properly contained in S, and with the properties that $|N_S(P)/P| = p$ and $O^{p'}(\mathrm{Out}_{\mathcal{F}}(P)) \cong SL_2(p)$. Choose $\alpha \in O^{p'}(\mathrm{Aut}_{\mathcal{F}}(P))$ of order prime to p whose class in $O^{p'}(\mathrm{Out}_{\mathcal{F}}(P))$ normalizes $\mathrm{Out}_S(P) \cong C_p$ and has order $p-1$. Then α extends to an element of $\mathrm{Aut}_{\mathcal{F}}(N_S(P))$ by the extension axiom (Definition 1.1(b)), and hence by the maximality of P and Theorem 1.3 (Alperin's fusion theorem) to some $\overline{\alpha} \in \mathrm{Aut}_{\mathcal{F}}(S)$. Then $\overline{\alpha}|_A$ normalizes U, and its class in $\mathrm{Aut}_G(U)$ has order $p-1$. □

2. Subsystems of index prime to p

Subsystems of fusion systems of index prime to p were first studied by Puig [Pg, § 6.5], and later in [5a2, § 3,5]. We begin the section with some basic results, where for simplicity we give references only in [AKO] when possible. After covering the basic properties, we develop some tools which will be used in the next section to determine $O^{p'}(\mathcal{F})$ and its index in $\mathcal{F}$ in specific cases.

Note in the following definition that the analogy for a group G is "subgroups of G that contain $O^{p'}(G)$", not arbitrary subgroups of index prime to p in G. For example, the fusion system of S is not, in general, of index prime to p in an arbitrary fusion system $\mathcal{F}$ over S.

Definition 2.1. *Let* $\mathcal{F}$ *be a saturated fusion system over a finite* p*-group* S.

(a) [AKO, Theorem I.7.7.d] *Let* $O^{p'}_*(\mathcal{F}) \subseteq \mathcal{F}$ *be the fusion subsystem (not necessarily saturated) generated by the groups* $O^{p'}(\mathrm{Aut}_{\mathcal{F}}(P))$ *for all* $P \le S$*, and set*

$$\mathrm{Aut}^0_{\mathcal{F}}(S) = \big\langle \alpha \in \mathrm{Aut}_{\mathcal{F}}(S) \,\big|\, \alpha|_P \in \mathrm{Hom}_{O^{p'}_*(\mathcal{F})}(P,S),\ \text{some } P \in \mathcal{F}^c \big\rangle$$
$$\Gamma_{p'}(\mathcal{F}) = \mathrm{Aut}_{\mathcal{F}}(S)/\mathrm{Aut}^0_{\mathcal{F}}(S).$$

(b) *A fusion subsystem* $\mathcal{E} \le \mathcal{F}$ *has* index prime to p *in* $\mathcal{F}$ *if* $\mathcal{E}$ *contains* $O^{p'}_*(\mathcal{F})$*; equivalently, if* $\mathcal{E}$ *is a fusion system over* S *and* $\mathrm{Aut}_{\mathcal{E}}(P) \ge O^{p'}(\mathrm{Aut}_{\mathcal{F}}(P))$ *for each* $P \le S$*. For such a subsystem* $\mathcal{E} \le \mathcal{F}$*, the* index *of* $\mathcal{E}$ *in* $\mathcal{F}$ *is defined to be the index of* $\mathrm{Aut}_{\mathcal{E}}(S)$ *in* $\mathrm{Aut}_{\mathcal{F}}(S)$*.*

Note that $O^{p'}_*(\mathcal{F})$ is denoted $\mathcal{E}_0$ in [AKO, Theorem I.7.7(d)].

Lemma 2.2. *Let* $\mathcal{F}$ *be a saturated fusion system over a finite* p*-group* S*, and let* $\mathcal{E} \subseteq \mathcal{F}$ *be a fusion subsystem (not necessarily saturated) of index prime to* p*. Then*

(a) $\mathcal{E}^c = \mathcal{F}^c$ *and* $\mathcal{E}^{cr} = \mathcal{F}^{cr}$*;*

(b) *a subgroup of* S *is fully normalized (fully centralized) in* $\mathcal{E}$ *if and only if it is fully normalized (fully centralized) in* $\mathcal{F}$*; and*

(c) (Frattini condition) *for each* $P, Q \le S$ *and* $\varphi \in \mathrm{Hom}_{\mathcal{F}}(P,Q)$*, there are*

$$\alpha_1, \alpha_2 \in \mathrm{Aut}_{\mathcal{F}}(S), \quad \varphi_1 \in \mathrm{Hom}_{\mathcal{E}}(P, \alpha_1^{-1}(Q)), \quad \text{and} \quad \varphi_2 \in \mathrm{Hom}_{\mathcal{E}}(\alpha_2(P), Q)$$

such that $\varphi = (\alpha_1|_{\alpha_1^{-1}(Q)}) \circ \varphi_1 = \varphi_2 \circ (\alpha_2|_P)$*.*

Proof. Point (c), and the equality $\mathcal{E}^c = \mathcal{F}^c$, are shown in [AKO, Lemma I.7.6(a)]. For all $P \le S$, we have $O^{p'}(\mathrm{Aut}_{\mathcal{F}}(P)) \le \mathrm{Aut}_{\mathcal{E}}(P) \le \mathrm{Aut}_{\mathcal{F}}(P)$, and hence

$$O_p(\mathrm{Aut}_{\mathcal{F}}(P)) \le O_p(\mathrm{Aut}_{\mathcal{E}}(P)) \le O_p(O^{p'}(\mathrm{Aut}_{\mathcal{F}}(P))) \le O_p(\mathrm{Aut}_{\mathcal{F}}(P)),$$

where the last inclusion holds because $O_p(G)$ is characteristic in G for all G. Thus P is $\mathcal{F}$-radical if and only if it is $\mathcal{E}$-radical, and so $\mathcal{E}^{cr} = \mathcal{F}^{cr}$, proving (a).

By (c), for each pair $P \le S$ and $Q \in P^{\mathcal{F}}$, there is $Q^* \in P^{\mathcal{E}}$ such that Q^* is $\mathrm{Aut}_{\mathcal{F}}(S)$-conjugate to Q. Thus $N_S(Q^*) \cong N_S(Q)$, so P is fully normalized in $\mathcal{F}$ if and only if it is fully normalized in $\mathcal{E}$, and similarly for fully centralized. □

The following terminology will be useful when working with fusion subsystems of index prime to p.

Definition 2.3. *Let* $\mathcal{F}$ *be a saturated fusion system over a finite* p*-group* S*, and let* $\mathcal{F}_0 \subseteq \mathcal{F}$ *be a full subcategory with* $S \in \mathrm{Ob}(\mathcal{F}_0)$*. A map* $\theta\colon \mathrm{Mor}(\mathcal{F}_0) \longrightarrow \Gamma$ *to a group* Γ *will be called* multiplicative *if it sends composites in* $\mathcal{F}_0$ *to products in* Γ *and sends inclusions to the identity. It will be called* strongly multiplicative *if in addition,* $\theta(O^{p'}(\mathrm{Aut}_{\mathcal{F}}(P))) = 1$ *for each* $P \in \mathrm{Ob}(\mathcal{F}_0)$*.*

Of course, the condition that θ send composites to products is equivalent to saying that it extends to a functor from $\mathcal{F}_0$ to $\mathcal{B}(\Gamma)$, where $\mathcal{B}(\Gamma)$ is the category with one object and morphism group Γ. In these terms, the basic properties of the group $\Gamma_{p'}(\mathcal{F})$ and of saturated fusion subsystems of index prime to p are summarized as follows.

Proposition 2.4 ([AKO, Theorem I.7.7])**.** *The following hold for each saturated fusion system* $\mathcal{F}$ *over a finite* p*-group* S*.*

(a) *There is a unique multiplicative map* $\theta_{\mathcal{F}}\colon \mathrm{Mor}(\mathcal{F}^c) \longrightarrow \Gamma_{p'}(\mathcal{F})$ *whose restriction to* $\mathrm{Aut}_{\mathcal{F}}(S)$ *is the natural surjection. Also,* $\theta_{\mathcal{F}}$ *is strongly multiplicative, and is universal in the following sense: if* Γ *is a group and* $\theta\colon \mathrm{Mor}(\mathcal{F}^c) \longrightarrow \Gamma$ *is strongly multiplicative, then there is a unique homomorphism* $f\colon \Gamma_{p'}(\mathcal{F}) \longrightarrow \Gamma$ *such that* $\theta = f \circ \theta_{\mathcal{F}}$*.*

(b) *There is a bijection*

$$\big\{\text{subgroups of } \Gamma_{p'}(\mathcal{F})\big\} \xrightarrow{\;\;\cong\;\;} \left\{\begin{matrix}\text{saturated fusion subsystems}\\ \mathcal{E}\le\mathcal{F} \text{ of index prime to } p\end{matrix}\right\}$$

that sends $H \le \Gamma_{p'}(\mathcal{F})$ *to the fusion subsystem* $\mathcal{E}_H = \langle \theta_{\mathcal{F}}^{-1}(H)\rangle$. *This subsystem has the properties that* $(\mathcal{E}_H)^c = \mathcal{F}^c$ *as sets and* $(\mathcal{E}_H)^c = \theta_{\mathcal{F}}^{-1}(H)$ *as categories. In particular, there is a unique smallest fusion subsystem* $O^{p'}(\mathcal{F}) = \langle\theta_{\mathcal{F}}^{-1}(1)\rangle$ *of index prime to p.*

Proof. Point (a) follows from [AKO, Theorem I.7.7(d)] (where $\Gamma_{p'}(\mathcal{F})$ is defined to be the target of the universal strongly multiplicative map on $\mathrm{Mor}(\mathcal{F}^c)$). Point (b) is shown in [AKO, Theorem I.7.7(b,c)]. □

By definition, $\Gamma_{p'}(\mathcal{F}) = 1$ if $\mathrm{Aut}_{\mathcal{F}}(S)$ is generated by elements $\alpha \in \mathrm{Aut}_{\mathcal{F}}(S)$ such that $\alpha|_P \in O^{p'}(\mathrm{Aut}_{\mathcal{F}}(P))$ for some $P \in \mathcal{F}^c$, and in this case, $O^{p'}(\mathcal{F}) = \mathcal{F}$ by Proposition 2.4. In particular, this always applies if $\mathrm{Out}_{\mathcal{F}}(S) = 1$.

There is also a geometric interpretation of universal multiplicative maps. For each fusion system $\mathcal{F}$ over a p-group S, and each full subcategory $\mathcal{F}_0 \subseteq \mathcal{F}$ containing S, let $\theta\colon \mathrm{Mor}(\mathcal{F}_0) \longrightarrow \pi_1(|\mathcal{F}_0|, *)$ be the map that sends $\varphi \in \mathrm{Hom}_{\mathcal{F}}(P,Q)$ to the loop (based at $* = [S]$) formed by the edges corresponding to morphisms incl_P^S, φ, and incl_Q^S. Then θ is the universal multiplicative map on $\mathrm{Mor}(\mathcal{F}_0)$ (see, e.g., [AKO, Proposition III.2.8]). By [AKO, Theorem III.4.19], $\theta_{\mathcal{F}}$ is universal among all multiplicative maps defined on $\mathrm{Mor}(\mathcal{F}^c)$ (not only those that are strongly multiplicative), and hence $\Gamma_{p'}(\mathcal{F}) \cong \pi_1(|\mathcal{F}^c|)$.

We next recall some standard notation for isomorphisms and automorphisms of fusion systems. If $\beta\colon S \xrightarrow{\cong} T$ is an isomorphism between finite p-groups, and $\mathcal{F}$ is a fusion system over S, then we set ${}^{\beta}\mathcal{F} = \beta\mathcal{F}\beta^{-1}$: the fusion system over T for which $\mathrm{Hom}_{{}^{\beta}\mathcal{F}}(\beta(P),\beta(Q))$ (for $P,Q \le S$) is the set of all composites $(\beta|_Q)\circ\varphi\circ(\beta|_P)^{-1}$ for $\varphi \in \mathrm{Hom}_{\mathcal{F}}(P,Q)$. When $\mathcal{E}$ and $\mathcal{F}$ are fusion systems over finite p-groups T and S, respectively, $\mathcal{E} \cong \mathcal{F}$ means that $\mathcal{E} = {}^{\beta}\mathcal{F}$ for some $\beta \in \mathrm{Iso}(S,T)$. Using this notation, we define, for a fusion system $\mathcal{F}$ over S,

$$\mathrm{Aut}(\mathcal{F}) = \{\beta \in \mathrm{Aut}(S) \mid {}^{\beta}\mathcal{F} = \mathcal{F}\} \qquad \text{and} \qquad \mathrm{Out}(\mathcal{F}) = \mathrm{Aut}(\mathcal{F})/\mathrm{Aut}_{\mathcal{F}}(S).$$

This notation is useful when classifying extensions of index prime to p.

Rather than define normal fusion subsystems here, we note for the purpose of the following lemma that if $\mathcal{E} \le \mathcal{F}$ has index prime to p, where both are saturated fusion systems over S, then $\mathcal{E} \trianglelefteq \mathcal{F}$ if and only if ${}^{\alpha}\mathcal{E} = \mathcal{E}$ for each $\alpha \in \mathrm{Aut}_{\mathcal{F}}(S)$. This follows immediately from [AKO, Definition I.6.1] and Lemma 2.2(c) (the Frattini condition).

Lemma 2.5. *Let* $\mathcal{F}_0$ *be a saturated fusion system over a finite p-group S. Then the map*

$$\Psi\colon \left\{\begin{matrix}\textit{saturated fusion systems } \mathcal{F}\\ \textit{over } S \textit{ with } \mathcal{F}_0 \trianglelefteq \mathcal{F} \textit{ normal}\\ \textit{of index prime to } p\end{matrix}\right\} \xrightarrow{\;\;\cong\;\;} \left\{\begin{matrix}\textit{subgroups of } \mathrm{Out}(\mathcal{F}_0)\\ \textit{of order prime to } p\end{matrix}\right\}$$

which sends $\mathcal{F}$ *to* $\mathrm{Aut}_{\mathcal{F}}(S)/\mathrm{Aut}_{\mathcal{F}_0}(S) \le \mathrm{Out}(\mathcal{F}_0)$ *is a bijection.*

Proof. If $\mathcal{F}_1$ and $\mathcal{F}_2$ are two such extensions of $\mathcal{F}_0$, and $\mathrm{Aut}_{\mathcal{F}_1}(S) = \mathrm{Aut}_{\mathcal{F}_2}(S)$, then $\mathcal{F}_1 = \mathcal{F}_2$ since $\mathcal{F}_i = \langle \mathcal{F}_0, \mathrm{Aut}_{\mathcal{F}_i}(S)\rangle$ (the Frattini condition holds for $\mathcal{F}_0 \le \mathcal{F}_i$ by Lemma 2.2(c)). Thus Ψ is injective.

If $\pi = \widetilde{\pi}/\mathrm{Aut}_{\mathcal{F}_0}(S) \le \mathrm{Out}(\mathcal{F}_0)$ has order prime to p, then by [5a2, Theorem 5.7(a)], the fusion system $\mathcal{F} \overset{\text{def}}{=} \langle \mathcal{F}_0, \widetilde{\pi}\rangle$ is saturated and contains $\mathcal{F}_0$ as a normal subsystem, and $\Psi(\mathcal{F}) = \Gamma/\mathrm{Aut}_{\mathcal{F}_0}(S)$. So Ψ is onto. □

The injectivity of Ψ in Lemma 2.5 implies that if $\mathcal{F}_1$ and $\mathcal{F}_2$ are two saturated fusion systems over the same p-group S, then $\mathcal{F}_1=\mathcal{F}_2$ if $O^{p'}(\mathcal{F}_1)=O^{p'}(\mathcal{F}_2)$ and $\mathrm{Aut}_{\mathcal{F}_1}(S)=\mathrm{Aut}_{\mathcal{F}_2}(S)$. Later, in Lemma 3.4, we will replace this by a more general hypothesis.

If $\mathcal{F}_1$ and $\mathcal{F}_2$ are fusion systems over finite p-groups S_1 and S_2, respectively, then

$$\mathcal{F}_1\times\mathcal{F}_2=\big\langle(\alpha_1,\alpha_2)\in\mathrm{Hom}(P_1\times P_2,Q_1\times Q_2)\,\big|\,\alpha_i\in\mathrm{Hom}_{\mathcal{F}_i}(P_i,Q_i)\text{ for }i=1,2\big\rangle.$$

By [AKO, Theorem I.6.6], $\mathcal{F}_1\times\mathcal{F}_2$ is saturated if $\mathcal{F}_1$ and $\mathcal{F}_2$ are both saturated.

Lemma 2.6. (a) *If G_1 and G_2 are finite groups with $S_i\in\mathrm{Syl}_p(G_i)$ for $i=1,2$, then $\mathcal{F}_{S_1\times S_2}(G_1\times G_2)=\mathcal{F}_{S_1}(G_1)\times\mathcal{F}_{S_2}(G_2)$.*

(b) *For each pair of fusion systems $\mathcal{F}_1$ and $\mathcal{F}_2$ over S_1 and S_2,*

$$O^{p'}(\mathcal{F}_1\times\mathcal{F}_2)=O^{p'}(\mathcal{F}_1)\times O^{p'}(\mathcal{F}_2)\quad\text{and hence}\quad\Gamma_{p'}(\mathcal{F}_1\times\mathcal{F}_2)=\Gamma_{p'}(\mathcal{F}_1)\times\Gamma_{p'}(\mathcal{F}_2).$$

Proof. We leave the proof of (a) as an exercise.

To prove (b), let $\rho_1\colon\mathrm{Mor}(\mathcal{F}_1^c)\longrightarrow\mathrm{Mor}((\mathcal{F}_1\times\mathcal{F}_2)^c)$ be the map that sends a morphism φ to the product $\varphi\times\mathrm{Id}_{S_2}$, and similarly for ρ_2. Then ρ_1 and ρ_2 send composites to composites and inclusions to inclusions, and hence by Proposition 2.4(a) induce a commutative diagram of groups and homomorphisms

$$\begin{tikzcd}
\mathrm{Aut}_{\mathcal{F}_1}(S_1) \arrow[r, "\rho_1^0"] \arrow[d, two heads, "\theta^0_{\mathcal{F}_1}"'] & \mathrm{Aut}_{\mathcal{F}_1\times\mathcal{F}_2}(S_1\times S_2) \arrow[d, two heads, "\theta^0_{\mathcal{F}_1\times\mathcal{F}_2}"'] & \mathrm{Aut}_{\mathcal{F}_2}(S_2) \arrow[l, "\rho_2^0"'] \arrow[d, two heads, "\theta^0_{\mathcal{F}_2}"'] \\
\Gamma_{p'}(\mathcal{F}_1) \arrow[r, "\widehat{\rho}_1"] & \Gamma_{p'}(\mathcal{F}_1\times\mathcal{F}_2) & \Gamma_{p'}(\mathcal{F}_2)\,, \arrow[l, "\widehat{\rho}_2"']
\end{tikzcd}$$

where $(-)^0$ means restriction to the given automorphism group. Thus

$$\mathrm{Aut}_{O^{p'}(\mathcal{F}_1)}(S_1)\times\mathrm{Aut}_{O^{p'}(\mathcal{F}_2)}(S_2)=\mathrm{Ker}(\theta^0_{\mathcal{F}_1}\times\theta^0_{\mathcal{F}_2})\le\mathrm{Ker}(\theta^0_{\mathcal{F}_1\times\mathcal{F}_2})=\mathrm{Aut}_{O^{p'}(\mathcal{F}_1\times\mathcal{F}_2)}(S_1\times S_2)$$

and hence $O^{p'}(\mathcal{F}_1)\times O^{p'}(\mathcal{F}_2)\le O^{p'}(\mathcal{F}_1\times\mathcal{F}_2)$ by Proposition 2.4(b). The opposite inclusion is clear. $\square$

The next lemma says that when checking whether a fusion subsystem of $\mathcal{F}$ has index prime to p, it suffices to look at $\mathcal{F}$-centric $\mathcal{F}$-radical subgroups.

Lemma 2.7. *Let $\mathcal{F}$ be a saturated fusion system over a finite p-group S, and assume $\mathcal{F}_0\subseteq\mathcal{F}$ is a fusion subsystem over S (not necessarily saturated) such that $\mathcal{F}\subseteq\langle\mathcal{F}_0,\mathrm{Aut}(\mathcal{F}_0)\rangle$. Then $\mathcal{F}_0\supseteq O^{p'}_*(\mathcal{F})$ and thus has index prime to p in $\mathcal{F}$. Thus*

$$O^{p'}_*(\mathcal{F})=\big\langle O^{p'}(\mathrm{Aut}_{\mathcal{F}}(R))\,\big|\,R\in\mathcal{F}^{cr}\big\rangle.$$

Proof. Assume $\mathcal{F}_0\subseteq\mathcal{F}$ is such that $\mathcal{F}\subseteq\langle\mathcal{F}_0,\mathrm{Aut}(\mathcal{F}_0)\rangle$. We claim that for each $P,Q\le S$ and each $\varphi\in\mathrm{Hom}_{\mathcal{F}}(P,Q)$, conjugation by φ sends $\mathrm{Aut}_{\mathcal{F}_0}(P)$ onto $\mathrm{Aut}_{\mathcal{F}_0}(\varphi(P))$. This is clear when $\varphi\in\mathrm{Hom}_{\mathcal{F}_0}(P,Q)$, and holds by definition when φ is the restriction of an element of $\mathrm{Aut}(\mathcal{F}_0)\le\mathrm{Aut}(S)$. So it holds for all $\varphi\in\mathrm{Hom}_{\mathcal{F}}(P,Q)$, since φ is a composite of such morphisms.

In particular, $\mathrm{Aut}_{\mathcal{F}_0}(P)\trianglelefteq\mathrm{Aut}_{\mathcal{F}}(P)$. If P is fully normalized in $\mathcal{F}$, then $\mathrm{Aut}_{\mathcal{F}_0}(P)\ge\mathrm{Aut}_S(P)\in\mathrm{Syl}_p(\mathrm{Aut}_{\mathcal{F}}(P))$ since $\mathrm{Aut}_{\mathcal{F}_0}(S)\ge\mathrm{Inn}(S)$, and so $\mathrm{Aut}_{\mathcal{F}_0}(P)$ contains the normal closure $O^{p'}(\mathrm{Aut}_{\mathcal{F}}(P))$ of $\mathrm{Aut}_S(P)$ in $\mathrm{Aut}_{\mathcal{F}}(P)$. We just saw that $\mathrm{Aut}_{\mathcal{F}_0}(Q)$ is isomorphic to $\mathrm{Aut}_{\mathcal{F}_0}(P)$ whenever $Q\in P^{\mathcal{F}}$, and hence $\mathrm{Aut}_{\mathcal{F}_0}(P)\ge O^{p'}(\mathrm{Aut}_{\mathcal{F}}(P))$ for all $P\le S$. Thus $\mathcal{F}_0\supseteq O^{p'}_*(\mathcal{F})$, and has index prime to p in $\mathcal{F}$.

We now apply this to $\mathcal{F}_0 \overset{\text{def}}{=} \langle O^{p'}(\mathrm{Aut}_{\mathcal{F}}(R)) \,|\, R \in \mathcal{F}^{cr} \rangle$. In this case,

$$\begin{aligned}\mathcal{F} &= \bigl\langle \mathrm{Aut}_{\mathcal{F}}(P) \bigm| P \in \mathcal{F}^{cr} \text{ fully normalized in } \mathcal{F} \bigr\rangle \\ &= \bigl\langle O^{p'}(\mathrm{Aut}_{\mathcal{F}}(P)) \cdot N_{\mathrm{Aut}_{\mathcal{F}}(P)}(\mathrm{Aut}_S(P)) \bigm| P \in \mathcal{F}^{cr} \text{ fully normalized in } \mathcal{F} \bigr\rangle \\ &= \bigl\langle O^{p'}(\mathrm{Aut}_{\mathcal{F}}(P)),\ \mathrm{Aut}_{\mathcal{F}}(N_S(P)) \bigm| P \in \mathcal{F}^{cr} \text{ fully normalized in } \mathcal{F} \bigr\rangle = \langle \mathcal{F}_0, \mathrm{Aut}_{\mathcal{F}}(S) \rangle :\end{aligned}$$

the first equality by Theorem 1.3 (Alperin's fusion theorem), the second by the Frattini argument and the Sylow axiom (Definition 1.1), the third by the extension axiom, and the fourth by a downward induction on $|P|$ (and since $N_S(P) > P$ for $P < S$). Also, $\mathrm{Aut}_{\mathcal{F}}(S) \le \mathrm{Aut}(\mathcal{F}_0)$ (i.e., $\mathrm{Aut}_{\mathcal{F}}(S)$ normalizes $\mathcal{F}_0$) since each $\alpha \in \mathrm{Aut}_{\mathcal{F}}(S)$ permutes the members of $\mathcal{F}^{cr}$. So $\mathcal{F}_0 \supseteq O^{p'}_*(\mathcal{F})$, and the opposite inclusion is clear. □

Note that by using the stronger form of Alperin's fusion theorem stated in [AKO, Theorem I.3.5], we could replace $\mathcal{F}^{cr}$ by the set consisting of essential subgroups and S.

The following proposition says that restrictions of $\theta_{\mathcal{F}}$ to certain full subcategories are also universal among *strongly* multiplicative maps.

Proposition 2.8. *Let $\mathcal{F}$ be a saturated fusion system over a finite p-group S, let $\mathscr{P} \subseteq \mathcal{F}^c$ be a family of subgroups closed under $\mathcal{F}$-conjugacy and containing $\mathcal{F}^{cr}$, and let $\mathcal{F}^{\mathscr{P}} \subseteq \mathcal{F}^c$ be the full subcategory with object set $\mathscr{P}$. Then*

$$\theta_{\mathcal{F}}^{\mathscr{P}} = \theta_{\mathcal{F}}|_{\mathrm{Mor}(\mathcal{F}^{\mathscr{P}})} \colon \mathrm{Mor}(\mathcal{F}^{\mathscr{P}}) \xrightarrow{\quad\quad} \Gamma_{p'}(\mathcal{F})$$

is universal among all strongly multiplicative maps $\mathrm{Mor}(\mathcal{F}^{\mathscr{P}}) \longrightarrow \Gamma$.

Proof. Let $\theta\colon \mathrm{Mor}(\mathcal{F}^{\mathscr{P}}) \longrightarrow \Gamma$ be the universal strongly multiplicative map, and let $f\colon \Gamma \longrightarrow \Gamma_{p'}(\mathcal{F})$ be such that $f \circ \theta = \theta_{\mathcal{F}}^{\mathscr{P}}$. In particular, $\theta(O^{p'}(\mathrm{Aut}_{\mathcal{F}}(P))) = 1$ for all $P \in \mathscr{P}$ and $\Gamma = \langle \theta(\mathrm{Mor}(\mathcal{F}^{\mathscr{P}})) \rangle$. Also, f is onto since $\theta_{\mathcal{F}}^{\mathscr{P}}$ is onto.

Set $\mathcal{E} = \langle \theta^{-1}(1) \rangle$: the smallest fusion system over S containing all morphisms in $\theta^{-1}(1)$. Thus $\mathcal{E} \supseteq O^{p'}_*(\mathcal{F})$ by Lemma 2.7 and since $\mathscr{P} \supseteq \mathcal{F}^{cr}$, and so $\mathcal{E}$ has index prime to p in $\mathcal{F}$. By Lemma 2.2(b), $\mathcal{E}$ and $\mathcal{F}$ have the same fully normalized and fully centralized subgroups. By Lemma 2.2(c), the Frattini condition holds: for each $P \le S$ and each $\varphi \in \mathrm{Hom}_{\mathcal{F}}(P,S)$, there are $\alpha \in \mathrm{Aut}_{\mathcal{F}}(S)$ and $\varphi_0 \in \mathrm{Hom}_{\mathcal{E}}(P,S)$ such that $\varphi = \alpha \circ \varphi_0$.

By the Frattini condition, each $\varphi \in \mathrm{Mor}(\mathcal{F}^{\mathscr{P}})$ is a composite of restrictions of morphisms in $\theta^{-1}(1)$ and elements of $\mathrm{Aut}_{\mathcal{F}}(S)$. Hence $\theta(\mathrm{Aut}_{\mathcal{F}}(S)) = \Gamma$. Since f is onto, to show that it is an isomorphism and hence that $\theta_{\mathcal{F}}^{\mathscr{P}}$ is universal, it remains to show that $\mathrm{Ker}(\theta|_{\mathrm{Aut}_{\mathcal{F}}(S)}) \ge \mathrm{Aut}^0_{\mathcal{F}}(S)$. We do this by first proving that $\mathcal{E}$ is saturated.

By construction, $\mathcal{E}$ is $\mathscr{P}$-generated in the sense of [5a1, Definition 2.1]. We claim that $\mathcal{E}$ is also $\mathscr{P}$-saturated: that the Sylow and extension axioms hold for members of $\mathscr{P}$. If $P \in \mathscr{P}$ is fully normalized in $\mathcal{E}$, hence in $\mathcal{F}$, then $\mathrm{Aut}_S(P) \in \mathrm{Syl}_p(\mathrm{Aut}_{\mathcal{E}}(P))$ since $\mathrm{Aut}_S(P) \in \mathrm{Syl}_p(\mathrm{Aut}_{\mathcal{F}}(P))$. Thus the Sylow axiom holds in $\mathcal{E}$ for members of $\mathscr{P}$.

Fix $\varphi \in \mathrm{Hom}_{\mathcal{E}}(P,Q)$, where $P,Q \in \mathscr{P}$ and $\varphi(P)$ is fully centralized in $\mathcal{E}$ (hence in $\mathcal{F}$), and let $N_\varphi \le N_S(P)$ be as in Definition 1.1(b). (Note that N_φ is the same, whether we are working in $\mathcal{E}$ or in $\mathcal{F}$.) By the extension axiom for $\mathcal{F}$, φ extends to some $\overline{\varphi} \in \mathrm{Hom}_{\mathcal{F}}(N_\varphi, S)$. By the Frattini condition, $\overline{\varphi} = \alpha \circ \varphi_0$, where $\alpha \in \mathrm{Aut}_{\mathcal{F}}(S)$ and $\varphi_0 \in \mathrm{Hom}_{\mathcal{E}}(P,S)$. Thus

$$1 = \theta(\varphi) = \theta(\alpha \circ \varphi_0|_P) = \theta(\alpha) \cdot \theta(\varphi_0|_P) = \theta(\alpha)$$

where $\theta(\varphi) = \varphi(\varphi_0|_P) = 1$ since φ and $\varphi_0|_P$ are both in $\mathcal{E} = \langle \theta^{-1}(1) \rangle$. Hence $\alpha \in \mathrm{Aut}_{\mathcal{E}}(S)$, and so $\overline{\varphi} \in \mathrm{Hom}_{\mathcal{E}}(N_\varphi, S)$. The extension axiom for $\mathcal{E}$ thus holds for members of $\mathscr{P}$, and so $\mathcal{E}$ is $\mathscr{P}$-saturated.

If $P \in \mathcal{F}^c \smallsetminus \mathscr{P}$ is fully normalized in $\mathcal{F}$, then

$$\mathrm{Aut}_S(P) \cap O_p(\mathrm{Aut}_{\mathcal{E}}(P)) = \mathrm{Aut}_S(P) \cap O_p(\mathrm{Aut}_{\mathcal{F}}(P)) = O_p(\mathrm{Aut}_{\mathcal{F}}(P)) > \mathrm{Inn}(P)$$

since P is not $\mathcal{F}$-radical. Since each $\mathcal{E}$-conjugacy class in $\mathcal{F}^c \smallsetminus \mathscr{P}$ contains such subgroups, $\mathcal{E}$ satisfies condition $(*)$ in [5a1, Theorem 2.2]. Since $\mathcal{E}$ is $\mathscr{P}$-generated and $\mathscr{P}$-saturated, $\mathcal{E}$ is saturated by that theorem. So $\mathcal{E} \geq O^{p'}(\mathcal{F})$, and hence

$$\mathrm{Ker}(\theta|_{\mathrm{Aut}_{\mathcal{F}}(S)}) = \mathrm{Aut}_{\mathcal{E}}(S) \geq \mathrm{Aut}_{O^{p'}(\mathcal{F})}(S) = \mathrm{Aut}^0_{\mathcal{F}}(S). \qquad \square$$

Proposition 2.8 is similar to a theorem of Grodal [Gr, Theorem 7.5.1], though that theorem is stated only when $\mathcal{F}$ is the fusion system of a group (and for a slightly different definition of radical subgroups).

The following very elementary family of examples shows that $\theta_{\mathcal{F}}|_{\mathcal{F}^{cr}}$ need *not*, in general, be universal among all multiplicative maps on $\mathrm{Mor}(\mathcal{F}^{cr})$: only among those that are strongly multiplicative. Recall [AKO, Definition I.4.8] that a saturated fusion system $\mathcal{F}$ over a finite p-group S is *constrained* if $O_p(\mathcal{F})$ is $\mathcal{F}$-centric.

Example 2.9. *Let $\mathcal{F}$ be a constrained saturated fusion system over S, and set $Q = O_p(\mathcal{F})$. Then $Q \in \mathcal{F}^{cr}$ and is contained in all other members of $\mathcal{F}^{cr}$. The universal group for multiplicative maps defined on $\mathrm{Mor}(\mathcal{F}^{cr})$ is isomorphic to $\mathrm{Aut}_{\mathcal{F}}(Q)$, while that for strongly multiplicative maps is isomorphic to $\Gamma_{p'}(\mathcal{F}) \cong \mathrm{Aut}_{\mathcal{F}}(Q)/O^{p'}(\mathrm{Aut}_{\mathcal{F}}(Q))$.*

In fact, the universal group for multiplicative maps on $\mathrm{Mor}(\mathcal{F}^{cr})$ need not even be finite.

Example 2.10. *Set $G = SL_2(9)$ and $p = 2$, and choose $S \in \mathrm{Syl}_2(G)$. Thus $S \cong Q_{16}$ and contains two subgroups $Q_1, Q_2 \leq S$ that are quaternion of order 8. Set $\mathcal{F} = \mathcal{F}_S(G)$; then $\mathcal{F}^{cr} = \{S, Q_1, Q_2\}$, and hence the category $\mathcal{F}^{cr}$ is the union of the two categories $N_{\mathcal{F}}(Q_1)^{cr}$ and $N_{\mathcal{F}}(Q_2)^{cr}$, with intersection $N_{\mathcal{F}}(S)^{cr}$. Since these three normalizers are all constrained, we see, using Example 2.9, that the universal group for multiplicative maps on $\mathrm{Mor}(\mathcal{F}^{cr})$ is isomorphic to*

$$\mathrm{Aut}_{\mathcal{F}}(Q_1) *_{\mathrm{Aut}_{\mathcal{F}}(S)} \mathrm{Aut}_{\mathcal{F}}(Q_2) \cong \Sigma_4 *_{D_8} \Sigma_4.$$

We now list some more tools that will be useful when working with fusion subsystems of index prime to p.

When $\mathcal{E}$ is a saturated fusion subsystem of $\mathcal{F}$, $O^{p'}(\mathcal{E})$ is not, in general, a fusion subsystem of $O^{p'}(\mathcal{F})$. The next two lemmas describe some cases when this is, in fact, true.

Lemma 2.11. *Let $\mathcal{F}$ be a saturated fusion system over a finite p-group S, and let $\mathcal{E} \leq \mathcal{F}$ be a saturated fusion subsystem over $T \leq S$.*

(a) *If $\mathcal{E}^{cr} \subseteq \mathcal{F}^c$ as sets, then $O^{p'}(\mathcal{E}) \leq O^{p'}(\mathcal{F})$.*

(b) *If $\mathcal{E} = C_{\mathcal{F}}(U)$ for some abelian subgroup $1 \neq U \leq S$ fully centralized in $\mathcal{F}$, then $\mathcal{E}^c \subseteq \mathcal{F}^c$ as sets, and hence $O^{p'}(\mathcal{E}) \leq O^{p'}(\mathcal{F})$.*

Proof. **(a)** Since $\mathcal{E}^{cr} \subseteq \mathcal{F}^c$ as categories, $\theta_{\mathcal{F}}$ restricts to a strongly multiplicative map $\mathrm{Mor}(\mathcal{E}^{cr}) \longrightarrow \Gamma_{p'}(\mathcal{F})$. By the universal property of $\theta_{\mathcal{E}}|_{\mathrm{Mor}(\mathcal{E}^{cr})}$ (Proposition 2.8), there is a (unique) homomorphism $\omega\colon \Gamma_{p'}(\mathcal{E}) \longrightarrow \Gamma_{p'}(\mathcal{F})$ such that $\omega \circ \theta_{\mathcal{E}}|_{\mathrm{Mor}(\mathcal{E}^{cr})} = \theta_{\mathcal{F}}|_{\mathrm{Mor}(\mathcal{E}^{cr})}$. Hence

$$\mathrm{Mor}(O^{p'}(\mathcal{E})^{cr}) = (\theta_{\mathcal{E}}|_{\mathrm{Mor}(\mathcal{E}^{cr})})^{-1}(1) \leq \theta_{\mathcal{F}}^{-1}(1) = \mathrm{Mor}(O^{p'}(\mathcal{F})^c)$$

where the first equality holds by Lemma 2.2(a) ($O^{p'}(\mathcal{E})^{cr}$ and $\mathcal{E}^{cr}$ have the same objects). Since each morphism in $O^{p'}(\mathcal{E})$ is a composite of restrictions of morphisms in $O^{p'}(\mathcal{E})^{cr}$, we conclude that $O^{p'}(\mathcal{E}) \leq O^{p'}(\mathcal{F})$.

(b) Assume $\mathcal{E} = C_{\mathcal{F}}(U)$ and hence $T = C_S(U)$, where $U \neq 1$ is abelian and fully centralized in $\mathcal{F}$. Fix $P \in \mathcal{E}^c$; we must show that $P \in \mathcal{F}^c$. Note that $P \geq C_T(P) \geq U$.

Choose $P^* \in P^{\mathcal{F}}$ fully centralized in $\mathcal{F}$ and $\varphi \in \mathrm{Iso}_{\mathcal{F}}(P, P^*)$, and set $U^* = \varphi(U)$. Since U is fully centralized in $\mathcal{F}$, there is $\psi \in \mathrm{Hom}_{\mathcal{F}}(C_S(U^*), T)$ (recall $T = C_S(U)$) such that $\psi|_{U^*} = \varphi^{-1}|_{U^*} \in \mathrm{Iso}_{\mathcal{F}}(U^*, U)$. Also, $P^* \leq C_S(U^*)$ since $P \leq C_S(U)$. Set $R = \psi(P^*)$; then $\psi\varphi \in \mathrm{Iso}_{\mathcal{E}}(P, R)$ since its restriction to U is the identity. So $R \in P^{\mathcal{E}} \subseteq \mathcal{E}^c$, hence $C_S(R) \leq C_S(U) = T$, and thus $C_S(R) = C_T(R) \leq R$. But then $C_S(P^*) \leq P^*$ since $\psi(C_S(P^*)) \leq C_S(R)$, and $P, P^* \in \mathcal{F}^c$ since $P^* \in P^{\mathcal{F}}$ is fully centralized in $\mathcal{F}$. □

We will also need to deal with "subsystems of p-power index" in a few cases.

Definition 2.12 ([5a2, Definitions 2.1 & 3.1])**.** *Let $\mathcal{F}$ be a saturated fusion system over a finite p-group S.*

(a) *The* hyperfocal subgroup *of $\mathcal{F}$ is defined as follows:*

$$\mathfrak{hyp}(\mathcal{F}) = \big\langle g^{-1}\alpha(g) \,\big|\, g \in P \leq S,\ \alpha \in O^p(\mathrm{Aut}_{\mathcal{F}}(P)) \big\rangle.$$

(b) *A fusion subsystem $\mathcal{E} \leq \mathcal{F}$ over $T \leq S$ has p*-power index *in $\mathcal{F}$ if $T \geq \mathfrak{hyp}(\mathcal{F})$, and for all $P \leq T$, $\mathrm{Aut}_{\mathcal{E}}(P) \geq O^p(\mathrm{Aut}_{\mathcal{F}}(P))$.*

By [5a2, Theorem 4.3] or [AKO, Theorem I.7.4], for each saturated fusion system $\mathcal{F}$ over S and each $T \leq S$ containing $\mathfrak{hyp}(\mathcal{F})$, there is a unique saturated fusion subsystem $\mathcal{F}_T$ over T of p-power index in $\mathcal{F}$, and $\mathcal{F}_T = \langle \mathrm{Inn}(S), O^p(\mathrm{Aut}_{\mathcal{F}}(P)) \,|\, P \leq T \rangle$ by the second reference.

Lemma 2.13. *Let $\mathcal{E} \leq \mathcal{F}$ be saturated fusion systems over finite p-groups $T \leq S$, where $\mathcal{E}$ has p-power index in $\mathcal{F}$ and T is normal in S. Then $O^{p'}(\mathcal{E}) \leq O^{p'}(\mathcal{F})$, and $\Gamma_{p'}(\mathcal{F})$ is isomorphic to a subquotient of $\Gamma_{p'}(\mathcal{E})$.*

Proof. Set $\mathcal{F}_0 = O^{p'}(\mathcal{F})$. Then $T \geq \mathfrak{hyp}(\mathcal{F}) \geq \mathfrak{hyp}(\mathcal{F}_0)$, so by [AKO, Theorem I.7.4], there is a unique saturated fusion subsystem

$$\mathcal{E}_0 = \big\langle \mathrm{Inn}(T), O^p(\mathrm{Aut}_{\mathcal{F}_0}(P)) \,\big|\, P \leq T \big\rangle \leq \mathcal{F}_0$$

over T of p-power index in $\mathcal{F}_0$. Also, $\mathcal{E}_0 \leq \mathcal{E}$ since $\mathcal{E} = \langle \mathrm{Inn}(T), O^p(\mathrm{Aut}_{\mathcal{F}}(P)) \,|\, P \leq T \rangle$.

We first prove that $O^{p'}(\mathcal{E}) \leq O^{p'}(\mathcal{F})$. By [5a2, Lemma 3.4(b)],

$$\mathcal{F} = \langle O^p_*(\mathcal{F}), \mathrm{Aut}_{\mathcal{F}}(S) \rangle = \langle O^p_*(\mathcal{F}), \mathrm{Inn}(S) \rangle \quad \text{where} \quad O^p_*(\mathcal{F}) = \langle O^p(\mathrm{Aut}_{\mathcal{F}}(P)) \,|\, P \leq S \rangle$$

(as fusion systems over S), where the second equality holds since $\mathrm{Inn}(S) \in \mathrm{Syl}_p(\mathrm{Aut}_{\mathcal{F}}(S))$. Also, T is strongly closed in $O^p_*(\mathcal{F})$ by definition of $\mathfrak{hyp}(\mathcal{F})$ and since $T \geq \mathfrak{hyp}(\mathcal{F})$. So T is also strongly closed in $\mathcal{F} = \langle O^p_*(\mathcal{F}), \mathrm{Inn}(S) \rangle$ (recall $T \trianglelefteq S$).

Let $\mathcal{F}_0|_T \subseteq \mathcal{F}_0$ and $\mathcal{F}|_T \subseteq \mathcal{F}$ be the full subcategories with objects the subgroups of T. Also, $\mathcal{F} = \langle \mathcal{F}_0, \mathrm{Aut}_{\mathcal{F}}(S) \rangle$ by Lemma 2.2(c) and $\mathcal{F}_0|_T = \langle \mathcal{E}_0, \mathrm{Aut}_{\mathcal{F}_0}(T) \rangle$ by [5a2, Lemma 3.4(b)] again, so

$$\mathcal{F}|_T = \big\langle \mathcal{F}_0|_T, \alpha|_T \,\big|\, \alpha \in \mathrm{Aut}_{\mathcal{F}}(S) \big\rangle = \langle \mathcal{E}_0, \mathrm{Aut}_{\mathcal{F}}(T) \rangle$$

since T is strongly closed. Also, $\mathrm{Aut}_{\mathcal{F}}(T)$ normalizes $\mathcal{F}_0|_T$ since $\mathrm{Aut}_{\mathcal{F}_0}(T)$ and $\mathrm{Aut}_{\mathcal{F}}(S)$ both normalize it, and hence $\mathrm{Aut}_{\mathcal{F}}(T)$ also normalizes $\mathcal{E}_0$ by definition of $\mathcal{E}_0$. Thus $\mathrm{Aut}_{\mathcal{F}}(T) \leq \mathrm{Aut}(\mathcal{E}_0)$, so $\mathcal{E} \leq \mathcal{F}|_T \leq \langle \mathcal{E}_0, \mathrm{Aut}(\mathcal{E}_0) \rangle$, and $\mathcal{E}_0$ has index prime to p in $\mathcal{E}$ by Lemma 2.7. Since $\mathcal{E}_0$ is saturated, $O^{p'}(\mathcal{E}) \leq \mathcal{E}_0 \leq \mathcal{F}_0 = O^{p'}(\mathcal{F})$.

It remains to show that $\Gamma_{p'}(\mathcal{F})$ is isomorphic to a subquotient of $\Gamma_{p'}(\mathcal{E})$. Consider the homomorphisms

$$\Gamma_{p'}(\mathcal{F}) = \mathrm{Aut}_{\mathcal{F}}(S)/\mathrm{Aut}_{\mathcal{F}_0}(S) \xrightarrow{f_1} \mathrm{Aut}_{\mathcal{F}}(T)/\mathrm{Aut}_{\mathcal{F}_0}(T) \cong \mathrm{Out}_{\mathcal{F}}(T)/\mathrm{Out}_{\mathcal{F}_0}(T)$$

$$\Gamma_{p'}(\mathcal{E}) = \mathrm{Out}_{\mathcal{E}}(T)/\mathrm{Out}_{O^{p'}(\mathcal{E})}(T) \xrightarrow{f_2} \mathrm{Out}_{\mathcal{E}}(T)/\mathrm{Out}_{\mathcal{E}_0}(T) \xrightarrow[\cong]{f_3} \mathrm{Out}_{\mathcal{F}}(T)/\mathrm{Out}_{\mathcal{F}_0}(T)$$

where f_1 is induced by restriction to T (recall that T is strongly closed in $\mathcal{F}$), f_2 is the natural surjection, and f_3 is induced by the inclusion $\mathrm{Out}_{\mathcal{E}}(T) \le \mathrm{Out}_{\mathcal{F}}(T)$. Here, f_3 is an isomorphism since $\mathrm{Out}_{\mathcal{E}}(T) = O^p(\mathrm{Out}_{\mathcal{F}}(T))$ and $\mathrm{Out}_{\mathcal{E}_0}(T) = O^p(\mathrm{Out}_{\mathcal{F}_0}(T))$ by the above descriptions of $\mathcal{E}$ and $\mathcal{E}_0$, both have order prime to p, and $\mathrm{Out}_{\mathcal{F}_0}(T) \ge O^{p'}(\mathrm{Out}_{\mathcal{F}}(T))$ since $\mathcal{F}_0 = O^{p'}(\mathcal{F}) \supseteq O^{p'}_*(\mathcal{F})$.

If $\alpha \in \mathrm{Aut}_{\mathcal{F}}(S)$ is such that $\alpha|_T \in \mathrm{Aut}_{\mathcal{F}_0}(T)$, then by the extension axiom (Definition 1.1(b)), there is $\beta \in \mathrm{Aut}_{\mathcal{F}_0}(S)$ such that $\beta|_T = \alpha|_T$, so $\beta^{-1}\alpha$ is the identity on T, induces an automorphism of p-power order on S/T, and hence has p-power order and lies in $\mathrm{Inn}(S)$. So f_1 is injective, and $\Gamma_{p'}(\mathcal{F})$ is isomorphic to a subquotient of $\Gamma_{p'}(\mathcal{E})$. □

The next lemma is useful when comparing subsystems of index prime to p in a simple group with those in its quasisimple coverings.

Lemma 2.14. *Let $\mathcal{F}$ be a saturated fusion system over a finite p-group S. Then for each central subgroup $Z \le Z(\mathcal{F})$, we have $O^{p'}(\mathcal{F}/Z) = O^{p'}(\mathcal{F})/Z$ and $\Gamma_{p'}(\mathcal{F}/Z) \cong \Gamma_{p'}(\mathcal{F})$. Thus if $\mathcal{F} = \mathcal{F}_S(G)$ for a finite group G, then for $Z \le Z(G)$, we have $\Gamma_{p'}(\mathcal{F}_{SZ/Z}(G/Z)) \cong \Gamma_{p'}(\mathcal{F})$.*

Proof. Assume $P \in \mathcal{F}^{cr}$, and consider the subgroup $\Delta = \{\alpha \in \mathrm{Aut}_{\mathcal{F}}(P) \,|\, [\alpha, P] \le Z\}$. All elements of $\mathrm{Aut}_{\mathcal{F}}(P)$ restrict to the identity on Z, so $\Delta \trianglelefteq \mathrm{Aut}_{\mathcal{F}}(P)$, and each $\alpha \in \Delta$ induces the identity on Z and on S/Z and hence has p-power order [G, Corollary 5.3.3]. Thus $\Delta \le O_p(\mathrm{Aut}_{\mathcal{F}}(P)) = \mathrm{Inn}(P)$ since P is $\mathcal{F}$-radical. So if $gZ \in C_{S/Z}(P/Z)$, then $c_g \in \mathrm{Inn}(P)$, and $g \in C_S(P) \le P$. Since this applies to all members of $P^{\mathcal{F}}$, we see that $P/Z \in (\mathcal{F}/Z)^c$.

Consider the following diagram:

$$\begin{tikzcd} \mathrm{Mor}(\mathcal{F}^{cr}) \arrow[r, two heads, "\theta_{\mathcal{F}}^{cr}"] \arrow[d, "P\mapsto P/Z"'] & \Gamma_{p'}(\mathcal{F}) \arrow[d, dashed, two heads, "f"] \\ \mathrm{Mor}((\mathcal{F}/Z)^c) \arrow[r, two heads, "\theta_{\mathcal{F}/Z}"] & \Gamma_{p'}(\mathcal{F}/Z) \end{tikzcd}$$

where $\theta_{\mathcal{F}}^{cr}$ is the restriction of $\theta_{\mathcal{F}}$, and where f exists (making the square commute) by the universality of $\theta_{\mathcal{F}}^{cr}$ (Proposition 2.8). This commutativity implies that $(O^{p'}(\mathcal{F}))^{cr}/Z \le O^{p'}(\mathcal{F}/Z)$, and hence that $O^{p'}(\mathcal{F})/Z \le O^{p'}(\mathcal{F}/Z)$ by Alperin's fusion theorem (Theorem 1.3). The opposite inclusion holds since $O^{p'}(\mathcal{F})/Z$ is normal of index prime to p in $\mathcal{F}/Z$.

We next show that f is an isomorphism. Let $\alpha \in \mathrm{Aut}_{\mathcal{F}}(S)$ be such that $\theta_{\mathcal{F}}(\alpha) \in \mathrm{Ker}(f)$, and let $\alpha' \in \mathrm{Aut}_{\mathcal{F}/Z}(S/Z)$ be the induced automorphism of S/Z. Thus

$$\alpha' \in \theta_{\mathcal{F}/Z}^{-1}(1) = \mathrm{Aut}_{O^{p'}(\mathcal{F}/Z)}(S/Z) = \mathrm{Aut}_{O^{p'}(\mathcal{F})/Z}(S/Z),$$

so there is $\beta \in \mathrm{Aut}_{O^{p'}(\mathcal{F})}(S)$ such that $\beta^{-1}\alpha$ induces the identity on S/Z. But $\beta^{-1}\alpha$ also induces the identity on $Z \le Z(\mathcal{F})$, so it has p-power order, and $\theta_{\mathcal{F}}(\alpha) = \theta_{\mathcal{F}}(\beta) = 1$ since $\Gamma_{p'}(\mathcal{F})$ has order prime to p. This proves that $\mathrm{Ker}(f) = 1$, and hence that f is an isomorphism.

The last statement is now immediate. □

The next lemma deals with normal subgroups of subsystems of index prime to p.

Lemma 2.15. *Assume $\mathcal{E} \le \mathcal{F}$ are saturated fusion systems over S, where $\mathcal{E}$ has index prime to p in $\mathcal{F}$. Then for $Q \trianglelefteq S$, $Q \trianglelefteq \mathcal{F}$ if and only if $Q \trianglelefteq \mathcal{E}$ and Q is weakly closed in $\mathcal{F}$.*

Proof. Assume $Q \trianglelefteq \mathcal{E}$ and Q is weakly closed in $\mathcal{F}$. By the Frattini condition (Lemma 2.2(c)), for each $\varphi \in \mathrm{Hom}_{\mathcal{F}}(P,R)$, there are $\alpha \in \mathrm{Aut}_{\mathcal{F}}(S)$ and $\varphi_0 \in \mathrm{Hom}_{\mathcal{E}}(P,\alpha^{-1}(R))$ such that $\varphi = (\alpha|_{\alpha^{-1}(R)}) \circ \varphi_0$. Then $\alpha(Q) = Q$ since Q is weakly closed, and φ_0 extends to a morphism $\overline{\varphi}_0 \in \mathrm{Hom}_{\mathcal{E}}(PQ,\alpha^{-1}(RQ))$ such that $\overline{\varphi}_0(Q) = Q$ since $Q \trianglelefteq \mathcal{E}$. So $(\alpha|_{\alpha^{-1}(RQ)}) \circ \overline{\varphi}_0 \in \mathrm{Hom}_{\mathcal{F}}(PQ,RQ)$ extends φ. This proves that $Q \trianglelefteq \mathcal{F}$, and the converse is clear. □

The next lemma, which gives one criterion for proving that $O^{p'}(\mathcal{F})$ is simple, will be applied in Section 4.

Lemma 2.16. *Fix a prime p and a finite p-group S, let $\mathcal{F}$ be a saturated fusion system over S, and assume that no nontrivial proper subgroup of S is strongly closed in $\mathcal{F}$. Then either $O^{p'}(\mathcal{F})$ is simple; or there are subgroups $T_1,\dots,T_k < S$ (for $2 \le k \le |\Gamma_{p'}(\mathcal{F})|$) that are strongly closed in $O^{p'}(\mathcal{F})$ and $\mathrm{Aut}_{\mathcal{F}}(S)$-conjugate to each other, and such that $S = T_1 \times \cdots \times T_k$. In particular, $\mathcal{F}$ is simple if $\mathcal{F} = O^{p'}(\mathcal{F})$.*

Proof. Assume $O^{p'}(\mathcal{F})$ is not simple, and let $1 \ne \mathcal{E} \trianglelefteq O^{p'}(\mathcal{F})$ be a nontrivial proper normal subsystem over $1 \ne T \trianglelefteq S$. In particular, T is strongly closed in $O^{p'}(\mathcal{F})$ (one of the conditions for $\mathcal{E}$ to be normal). If $T = S$, then $\mathcal{E}$ has index prime to p in $O^{p'}(\mathcal{F})$ by [AOV, Lemma 1.26], which is impossible since $\mathcal{E} < O^{p'}(\mathcal{F})$ and $O^{p'}(O^{p'}(\mathcal{F})) = O^{p'}(\mathcal{F})$. So $T < S$.

Thus there are proper nontrivial subgroups of S strongly closed in $O^{p'}(\mathcal{F})$, and so $O^{p'}(\mathcal{F}) < \mathcal{F}$ by assumption. Let $1 \ne U < S$ be minimal among all such subgroups. Set $m = |\Gamma_{p'}(\mathcal{F})| = |\mathrm{Aut}_{\mathcal{F}}(S)/\mathrm{Aut}_{O^{p'}(\mathcal{F})}(S)|$ (Proposition 2.4), choose coset representatives $\alpha_1,\dots,\alpha_m \in \mathrm{Aut}_{\mathcal{F}}(S)$ for $\mathrm{Aut}_{O^{p'}(\mathcal{F})}(S)$, and set $U_i = \alpha_i(U)$. Then each U_i is a minimal strongly closed subgroup in $O^{p'}(\mathcal{F})$, and $\widehat{U} = U_1 \cdots U_m$ is strongly closed in $O^{p'}(\mathcal{F})$ by [A2, Theorem 2] or [Cr, Theorem 5.22]. Also, $\widehat{U}$ is normalized by $\mathrm{Aut}_{\mathcal{F}}(S)$ by construction. Since each morphism in $\mathcal{F}$ is the composite of a morphism in $O^{p'}(\mathcal{F})$ followed by the restriction of some element of $\mathrm{Aut}_{\mathcal{F}}(S)$, $\widehat{U}$ is strongly closed in $\mathcal{F}$. So $\widehat{U} = S$ by assumption.

For each $1 \le i \le m-1$, $U_1 \cdots U_i$ is strongly closed in $O^{p'}(\mathcal{F})$ by [A2, Theorem 2] again, so $(U_1 \cdots U_i) \cap U_{i+1}$ is also strongly closed. So by the minimality of U among strongly closed subgroups, either $(U_1 \cdots U_i) \cap U_{i+1} = 1$ or $U_{i+1} \le U_1 \cdots U_i$. Hence there is a subset $I \subseteq \{1,\dots,m\}$ such that S is the direct product of the U_i for $i \in I$, and $|I| \ge 2$ since $U < S$. □

3. Fusion systems with centric and weakly closed subgroups

We saw in the last section that when $\mathcal{F}$ is a saturated fusion system over a finite p-group S, its subsystems and extensions of index prime to p are determined by their automizers on S. In this section, we show that in fact, it suffices to consider the automizers of some subgroup $A \trianglelefteq S$ that is $\mathcal{F}$-centric and weakly closed in $\mathcal{F}$. In particular, this works in many cases when A is the p-power torsion in the maximal torus of a finite group of Lie type in defining characteristic different from p, and allows us to work with $\mathrm{Aut}_{\mathcal{F}}(A)$ rather than $\mathrm{Aut}_{\mathcal{F}}(S)$.

We begin with three lemmas that describe the relationship between $\mathrm{Aut}_{\mathcal{F}}(A)$ and $\Gamma_{p'}(\mathcal{F})$ for such a subgroup $A \trianglelefteq S$, and show how this can be used to get upper and lower bounds for $|\Gamma_{p'}(\mathcal{F})|$.

Lemma 3.1. *Let $\mathcal{F}$ be a saturated fusion system over a finite p-group S, and assume $A \trianglelefteq S$ is a subgroup that is $\mathcal{F}$-centric and weakly closed in $\mathcal{F}$. Let*

$$\theta_{\mathcal{F}}^{(A)} \colon \mathrm{Aut}_{\mathcal{F}}(A) \longrightarrow \Gamma_{p'}(\mathcal{F})$$

be the restriction of $\theta_{\mathcal{F}} \colon \mathrm{Mor}(\mathcal{F}^c) \longrightarrow \Gamma_{p'}(\mathcal{F})$ to $\mathrm{Aut}_{\mathcal{F}}(A)$. Then

(a) *$\theta_{\mathcal{F}}^{(A)}$ is surjective and $\mathrm{Ker}(\theta_{\mathcal{F}}^{(A)}) = \mathrm{Aut}_{O^{p'}(\mathcal{F})}(A)$;*

(b) *$\Gamma_{p'}(\mathcal{F}) = 1$ if $O^{p'}(\mathrm{Aut}_{\mathcal{F}}(A)) = \mathrm{Aut}_{\mathcal{F}}(A)$; and*

(c) *for $\mathcal{E} \leq \mathcal{F}$ of index prime to p, $\mathcal{E} = \mathcal{F}$ if and only if $\mathrm{Aut}_{\mathcal{E}}(A) = \mathrm{Aut}_{\mathcal{F}}(A)$.*

Proof. Since A is weakly closed, each $\alpha \in \mathrm{Aut}_{\mathcal{F}}(S)$ restricts to an element of $\mathrm{Aut}_{\mathcal{F}}(A)$, and so the surjectivity of $\theta_{\mathcal{F}}^{(A)}$ follows from that of $\theta_{\mathcal{F}}|_{\mathrm{Aut}_{\mathcal{F}}(S)}$ (Proposition 2.4(a)). Point (b) is then immediate.

Set $\mathcal{E}_H = \langle \theta_{\mathcal{F}}^{-1}(H) \rangle \leq \mathcal{F}$ for each subgroup $H \leq \Gamma_{p'}(\mathcal{F})$. Then $\mathrm{Aut}_{\mathcal{E}_H}(A) = (\theta_{\mathcal{F}}^{(A)})^{-1}(H)$, since each $\alpha \in \mathrm{Aut}_{\mathcal{E}_H}(A)$ is a composite of restrictions of morphisms in $\theta_{\mathcal{F}}^{-1}(H)$ (and inclusions are sent to the identity). In particular, $\mathrm{Aut}_{O^{p'}(\mathcal{F})}(A) = \mathrm{Ker}(\theta_{\mathcal{F}}^{(A)})$ (the case $H = 1$), finishing the proof of (a). For each $\mathcal{E} \leq \mathcal{F}$ of index prime to p, $\mathcal{E} = \mathcal{E}_H$ for some $H \leq \Gamma_{p'}(\mathcal{F})$ by Proposition 2.4(b), and from the surjectivity of $\theta_{\mathcal{F}}^{(A)}$, we see that $\mathrm{Aut}_{\mathcal{E}}(A) = \mathrm{Aut}_{\mathcal{F}}(A)$ implies $H = \Gamma_{p'}(\mathcal{F})$, and hence $\mathcal{E} = \mathcal{F}$. □

One immediate consequence of Lemma 3.1 is the following upper bound for $|\Gamma_{p'}(\mathcal{F})|$.

Lemma 3.2. *Let $\mathcal{F}$ be a saturated fusion system over a finite p-group S, and let $\mathcal{E} \leq \mathcal{F}$ be a saturated fusion subsystem over $T \leq S$. Assume that $O^{p'}(\mathcal{E}) \leq O^{p'}(\mathcal{F})$, that $A \trianglelefteq S$ is $\mathcal{F}$-centric and weakly closed in $\mathcal{F}$, and also that $A \leq T$. Then $\mathrm{Aut}_{O^{p'}(\mathcal{E})}(A)$ is contained in the kernel of the homomorphism $\theta_{\mathcal{F}}^{(A)}$ from $\mathrm{Aut}_{\mathcal{F}}(A)$ onto $\Gamma_{p'}(\mathcal{F})$, and hence*

$$|\Gamma_{p'}(\mathcal{F})| \leq \bigl|\mathrm{Aut}_{\mathcal{F}}(A) : O^{p'}(\mathrm{Aut}_{\mathcal{F}}(A)) \cdot \mathrm{Aut}_{O^{p'}(\mathcal{E})}(A)\bigr|.$$

In particular, $O^{p'}(\mathcal{F}) = \mathcal{F}$ if $\mathrm{Aut}_{\mathcal{F}}(A) = O^{p'}(\mathrm{Aut}_{\mathcal{F}}(A)) \cdot \mathrm{Aut}_{O^{p'}(\mathcal{E})}(A)$.

Proof. Since $O^{p'}(\mathcal{E}) \leq O^{p'}(\mathcal{F})$, we have $\mathrm{Aut}_{O^{p'}(\mathcal{E})}(A) \leq \mathrm{Aut}_{O^{p'}(\mathcal{F})}(A) = \mathrm{Ker}(\theta_{\mathcal{F}}^{(A)})$ by Lemma 3.1(a). The other claims now follow from the surjectivity of $\theta_{\mathcal{F}}^{(A)}$ and since $p \nmid |\Gamma_{p'}(\mathcal{F})|$. □

If A is an *abelian* subgroup that is $\mathcal{F}$-centric and weakly closed, then we can say more. The next lemma will be our main tool for getting lower bounds on $|\Gamma_{p'}(\mathcal{F})|$ when it is nontrivial.

Lemma 3.3. *Let $\mathcal{F}$ be a saturated fusion system over the p-group S, and let $A \trianglelefteq S$ be an abelian subgroup which is $\mathcal{F}$-centric and weakly closed in $\mathcal{F}$. Set*

$$X = \{t \in A \,|\, t^{\mathcal{F}} \subseteq A\},$$

and assume $X \cap Z(S) \neq \{1\}$. Choose $1 \neq Z \leq \langle X \cap Z(S) \rangle$, set $\mathcal{E} = C_{\mathcal{F}}(Z)$ and $\mathcal{E}_0 = O^{p'}(\mathcal{E})$, and let $N_0 \leq N \trianglelefteq \mathrm{Aut}_{\mathcal{F}}(A)$ be the normal closures of $\mathrm{Aut}_{\mathcal{E}_0}(A)$ and $\mathrm{Aut}_{\mathcal{E}}(A)$ in $\mathrm{Aut}_{\mathcal{F}}(A)$. Then the following hold.

(a) *There is a surjective homomorphism $f \colon \Gamma_{p'}(\mathcal{F}) \longrightarrow \mathrm{Aut}_{\mathcal{F}}(A)/N$ with the following property: for each morphism $\varphi \in \mathrm{Hom}_{\mathcal{F}}(P,Q)$ in $\mathcal{F}^c$, there is $\alpha \in \mathrm{Aut}_{\mathcal{F}}(A)$ such that $\alpha|_Z = \varphi|_Z$, and for each such α, $f(\theta_{\mathcal{F}}(\varphi)) = \alpha N$.*

(b) *We have $O^{p'}(\mathrm{Aut}_{\mathcal{F}}(A))N_0 \leq \mathrm{Ker}(\theta_{\mathcal{F}}^{(A)}) \leq N$.*

Proof. Since $\mathcal{E}_0 \leq O^{p'}(\mathcal{F})$ by Lemma 2.11(b), $\mathrm{Aut}_{\mathcal{E}_0}(A) \leq \mathrm{Aut}_{O^{p'}(\mathcal{F})}(A) = \mathrm{Ker}(\theta_{\mathcal{F}}^{(A)})$, and hence $N_0 \leq \mathrm{Ker}(\theta_{\mathcal{F}}^{(A)})$. Since $\mathrm{Im}(\theta_{\mathcal{F}}^{(A)})$ has order prime to p, this proves the first inclusion in (b).

Before proving the other claims, we check that

$$\text{for each } P \leq A \text{ and } \varphi \in \mathrm{Hom}_{\mathcal{F}}(P, A),\ \varphi \text{ extends to some } \overline{\varphi} \in \mathrm{Aut}_{\mathcal{F}}(A). \tag{3.1}$$

To see this, set $Q = \varphi(P) \leq A$, choose $R \in P^{\mathcal{F}} = Q^{\mathcal{F}}$ that is fully centralized in $\mathcal{F}$, and fix $\chi \in \mathrm{Iso}_{\mathcal{F}}(Q, R)$. By the extension axiom, χ extends to $\overline{\chi} \in \mathrm{Hom}_{\mathcal{F}}(C_S(Q), S)$ and $\chi\varphi$ extends to $\overline{\varphi} \in \mathrm{Hom}_{\mathcal{F}}(C_S(P), S)$, and $\overline{\chi}(A) = A = \overline{\varphi}(A)$ since A is weakly closed. Then $\overline{\chi}^{-1}\overline{\varphi}|_A \in \mathrm{Aut}_{\mathcal{F}}(A)$, and $(\overline{\chi}^{-1}\overline{\varphi})|_P = \chi^{-1}(\chi\varphi) = \varphi$.

Let $\theta_0 : \mathrm{Aut}_{\mathcal{F}}(A) \longrightarrow \mathrm{Aut}_{\mathcal{F}}(A)/N$ be the natural map. We first extend θ_0 to a strongly multiplicative map defined on $\mathrm{Mor}(\mathcal{F}^c)$ (see Definition 2.3).

Fix $\varphi \in \mathrm{Mor}_{\mathcal{F}}(P, Q)$, where $P, Q \in \mathcal{F}^c$. In particular, $P, Q \geq Z(S) \geq Z$, and $\varphi(Z) \subseteq \varphi(X \cap Z(S)) \subseteq A$ by definition of X. By (3.1), $\varphi|_Z$ is the restriction of some automorphism $\alpha \in \mathrm{Aut}_{\mathcal{F}}(A)$. If $\alpha' \in \mathrm{Aut}_{\mathcal{F}}(A)$ is another automorphism for which $\alpha'|_Z = \varphi|_Z$, then $\alpha^{-1}\alpha' \in C_{\mathrm{Aut}_{\mathcal{F}}(A)}(Z) \leq N$, and so we can define $\theta(\varphi) = \theta_0(\alpha) = \theta_0(\alpha')$. We thus have a well defined map of sets $\theta\colon \mathrm{Mor}(\mathcal{F}^c) \longrightarrow \mathrm{Aut}_{\mathcal{F}}(A)/N$, and $\theta|_{\mathrm{Aut}_{\mathcal{F}}(A)} = \theta_0$. Clearly, θ sends all inclusions to the identity, and it is surjective since θ_0 is surjective. Also, $\mathrm{Aut}_{\mathcal{F}}(A)/N$ has order prime to p since $N \geq \mathrm{Aut}_S(A) \in \mathrm{Syl}_p(\mathrm{Aut}_{\mathcal{F}}(A))$, and hence $\theta(O^{p'}(\mathrm{Aut}_{\mathcal{F}}(P))) = 1$ for each $P \in \mathcal{F}^c$.

It remains to check that θ sends composites in $\mathrm{Mor}(\mathcal{F}^c)$ to products in $\mathrm{Aut}_{\mathcal{F}}(A)/N$. Assume $\varphi \in \mathrm{Hom}_{\mathcal{F}}(P, Q)$ and $\psi \in \mathrm{Hom}_{\mathcal{F}}(Q, R)$, where $P, Q, R \in \mathcal{F}^c$, and set $P^* = \langle P \cap X \rangle$ and $Q^* = \langle Q \cap X \rangle$. Then $\varphi(P \cap X) \subseteq Q \cap X$, so $\varphi(P^*) \leq Q^*$. By (3.1), there are $\alpha, \beta \in \mathrm{Aut}_{\mathcal{F}}(A)$ which are extensions of $\varphi|_{P^*}$ and $\psi|_{Q^*}$, respectively. Then $\beta \circ \alpha$ is an extension of $(\psi \circ \varphi)|_{P^*}$, hence of $(\psi \circ \varphi)|_Z$, and thus $\theta(\psi \circ \varphi) = \beta \circ \alpha = \theta(\psi) \circ \theta(\varphi)$.

Since $\theta_{\mathcal{F}}$ is universal among strongly multiplicative maps (Proposition 2.4(a)), there is a unique homomorphism $f\colon \Gamma_{p'}(\mathcal{F}) \longrightarrow \mathrm{Aut}_{\mathcal{F}}(A)/N$ such that $\theta = f \circ \theta_{\mathcal{F}}$, and f is surjective since θ is surjective. This proves (a). Also, $N = \mathrm{Ker}(\theta_0) \geq \mathrm{Ker}(\theta_{\mathcal{F}}^{(A)})$, which proves the second inclusion in (b). $\square$

We next note that two extensions of index prime to p of the same fusion system over S are equal, or at least isomorphic, if they have the same automizer on some centric and weakly closed subgroup $A \trianglelefteq S$.

Lemma 3.4. *Let $\mathcal{F}_1$ and $\mathcal{F}_2$ be saturated fusion systems over the same finite p-group S. Assume that $A \trianglelefteq S$ is $\mathcal{F}_i$-centric and weakly closed in $\mathcal{F}_i$ for $i = 1, 2$, and also that*

$$O^{p'}(\mathcal{F}_1) = O^{p'}(\mathcal{F}_2) \qquad \text{and} \qquad \mathrm{Aut}_{\mathcal{F}_1}(A) = \mathrm{Aut}_{\mathcal{F}_2}(A).$$

Then there is $\beta \in \mathrm{Aut}(O^{p'}(\mathcal{F}_2))$ such that $\beta|_A = \mathrm{Id}_A$ and $\mathcal{F}_1 = {}^{\beta}\mathcal{F}_2$. If in addition, $H^1(\mathrm{Out}_{O^{p'}(\mathcal{F}_1)}(A); Z(A)) = 0$, then $\mathcal{F}_1 = \mathcal{F}_2$.

Proof. Set $\mathcal{F}_0 = O^{p'}(\mathcal{F}_1) = O^{p'}(\mathcal{F}_2)$, and consider the subgroup

$$H = \bigl\{\beta \in \mathrm{Aut}(S) \bigm| \beta|_A = \mathrm{Id}\bigr\}.$$

Each $\beta \in H$ induces the identity on S/A since $C_S(A) \leq A$, so H is a p-group by, e.g., [G, Corollary 5.3.3]. Each $\alpha \in \mathrm{Aut}_{\mathcal{F}_i}(S)$ (for $i = 1, 2$) normalizes A since A is weakly closed, and hence also normalizes H. For each $\alpha \in \mathrm{Aut}_{\mathcal{F}_1}(S)$, $\alpha|_A \in \mathrm{Aut}_{\mathcal{F}_1}(A) = \mathrm{Aut}_{\mathcal{F}_2}(A)$ and normalizes $\mathrm{Aut}_S(A)$, and hence extends to some $\alpha' \in \mathrm{Aut}_{\mathcal{F}_2}(S)$ by the extension axiom (Definition 1.1(b)). Then $\alpha' \in \alpha H$, and this together with a similar argument in the opposite direction shows that $\mathrm{Aut}_{\mathcal{F}_1}(S)H = \mathrm{Aut}_{\mathcal{F}_2}(S)H$. Set $G = \mathrm{Aut}_{\mathcal{F}_1}(S)H = \mathrm{Aut}_{\mathcal{F}_2}(S)H$.

For $i=1,2$, $\mathrm{Inn}(S)H \trianglelefteq \mathrm{Aut}_{\mathcal{F}_i}(S)H$ since $\mathrm{Inn}(S) \trianglelefteq \mathrm{Aut}_{\mathcal{F}_i}(S)$ and both normalize H, and $\mathrm{Aut}_{\mathcal{F}_i}(S)H/\mathrm{Inn}(S)H \cong \mathrm{Aut}_{\mathcal{F}_i}(S)/\mathrm{Inn}(S)$ has order prime to p by the Sylow axiom (Definition 1.1(b) again). So by the Schur-Zassenhaus theorem (see [G, Theorem 6.2.1(i)]), all splittings of the extension

$$1 \longrightarrow \mathrm{Inn}(S)H/\mathrm{Inn}(S) \longrightarrow G/\mathrm{Inn}(S) \longrightarrow G/\mathrm{Inn}(S)H \longrightarrow 1$$

are conjugate. Since $\mathrm{Aut}_{\mathcal{F}_1}(S)/\mathrm{Inn}(S)$ and $\mathrm{Aut}_{\mathcal{F}_2}(S)/\mathrm{Inn}(S)$ are two such splittings (since $\mathrm{Aut}_{\mathcal{F}_i}(S) \cap H \le \mathrm{Inn}(S)$ by the Sylow axiom), there is $\beta \in H$ such that ${}^{\beta}(\mathrm{Aut}_{\mathcal{F}_2}(S)) = \mathrm{Aut}_{\mathcal{F}_1}(S)$. Thus $\beta|_A = \mathrm{Id}_A$ and

$$\mathrm{Aut}_{\mathcal{F}_1}(S) = {}^{\beta}(\mathrm{Aut}_{\mathcal{F}_2}(S)) = \mathrm{Aut}_{{}^{\beta}\mathcal{F}_2}(S),$$

and so $\mathcal{F}_1 = {}^{\beta}\mathcal{F}_2$ by Lemma 2.5 (and since $O^{p'}(\mathcal{F}_1) = \mathcal{F}_0 = O^{p'}(\mathcal{F}_2)$).

Now assume that $H^1(\mathrm{Out}_{\mathcal{F}_0}(A); Z(A)) = 0$, and let G be a model for $N_{\mathcal{F}_0}(A)$ (see [AKO, Theorem I.4.9]). Thus $A \trianglelefteq G$, $S \in \mathrm{Syl}_p(G)$, $O_{p'}(G) = 1$, and $\mathcal{F}_S(G) = N_{\mathcal{F}_0}(A)$. Then $C_G(A) \le A$, so $G/A \cong \mathrm{Out}_{\mathcal{F}_0}(A)$. Since $\beta \in \mathrm{Aut}(\mathcal{F}_0) \le \mathrm{Aut}(S)$ and $\beta|_A = \mathrm{Id}_A$, β is also an automorphism of $N_{\mathcal{F}_0}(A)$ and extends to an automorphism $\widehat{\beta} \in \mathrm{Aut}(G)$ by the uniqueness condition in the model theorem [AKO, Lemma II.4.3(a)]. Also, $\widehat{\beta}$ induces the identity on G/A since $\widehat{\beta}|_A = \mathrm{Id}$ and $C_G(A) \le A$, and $\widehat{\beta}$ is conjugation by some $a \in Z(A)$ since $H^1(\mathrm{Out}_{\mathcal{F}_0}(A); Z(A)) = 0$ (see, e.g., [OV, Lemma 1.2]). But then $\beta \in \mathrm{Inn}(S) \le \mathrm{Aut}(\mathcal{F}_2)$, and so $\mathcal{F}_1 = {}^{\beta}\mathcal{F}_2 = \mathcal{F}_2$. □

The last lemma implies as a special case that when G is a finite group of Lie type in characteristic different from p, adding diagonal automorphisms of order prime to p doesn't change its p-fusion system.

Lemma 3.5. *Let p be a prime, and let $G \trianglelefteq \widehat{G}$ be finite groups such that $p \nmid |O^p(\widehat{G}/G)|$. Choose $\widehat{S} \in \mathrm{Syl}_p(\widehat{G})$, set $S = \widehat{S} \cap G \in \mathrm{Syl}_p(G)$, and set $\mathcal{F} = \mathcal{F}_S(G)$ and $\widehat{\mathcal{F}} = \mathcal{F}_{\widehat{S}}(\widehat{G})$. Assume $A \trianglelefteq S$ is $\mathcal{F}$-centric and weakly closed in $\mathcal{F}$, and is such that $\widehat{G} = GC_{\widehat{G}}(A)$. Then $\mathcal{F}$ is normal of p-power index in $\widehat{\mathcal{F}}$, and $\mathcal{F} = \widehat{\mathcal{F}}$ if $p \nmid |\widehat{G}/G|$.*

Proof. Set $\overline{G} = G \cdot O^p(\widehat{G})$. Then $S \in \mathrm{Syl}_p(\overline{G})$ since $p \nmid |\overline{G}/G|$, and $\mathcal{F}_S(\overline{G})$ is normal of p-power index in $\widehat{\mathcal{F}}$. It remains to show that $\mathcal{F} = \mathcal{F}_S(\overline{G})$. Since $\mathcal{F}$ is normal of index prime to p in $\mathcal{F}_S(\overline{G})$, it suffices by Lemma 3.1(c) to show that $\mathrm{Aut}_{\mathcal{F}}(A) = \mathrm{Aut}_{\overline{G}}(A)$, and this holds since $N_{\overline{G}}(A) = N_G(A)C_{\overline{G}}(A)$. □

4. Simplicity of fusion systems of known simple groups

We are now ready to explicitly describe $O^{p'}(\mathcal{F})$ and $\Gamma_{p'}(\mathcal{F})$, when $\mathcal{F}$ is the fusion system of a known simple group and $S \ntrianglelefteq \mathcal{F}$.

When r is a prime, let $\mathfrak{Lie}(r)$ denote the class of finite groups of Lie type in defining characteristic r. Since this will be used only when restricting attention to simple groups, we needn't go into details as to exactly which groups are included in this class.

In most cases, the simplicity of $\mathcal{F}$ was determined by Aschbacher in [A2, Chapter 16]. His conclusions are summarized in the following proposition.

Proposition 4.1. *Fix a prime p, a known simple group G, and $S \in \mathrm{Syl}_p(G)$. Set $\mathcal{F} = \mathcal{F}_S(G)$, and assume that $S \ntrianglelefteq \mathcal{F}$.*

(a) *If $G \cong A_n$, then $O^{p'}(\mathcal{F}) = \mathcal{F}$ if $n \equiv 0, 1 \pmod p$, and $|\Gamma_{p'}(\mathcal{F})| = 2$ otherwise. In all cases, $O^{p'}(\mathcal{F})$ is simple, and is realized by $A_{n'}$, where $n' = p \cdot [\frac{n}{p}]$.*

(b) *If $G \in \mathfrak{Lie}(p)$, or if $p = 2$ and $G \cong {}^2F_4(2)'$, then G has Lie rank at least two, $O^{p'}(\mathcal{F}) = \mathcal{F}$, and $\mathcal{F}$ is simple.*

(c) *If G is a sporadic simple group, then either $\mathcal{F}$ is simple, or $O^{p'}(\mathcal{F})$ has index 2 in $\mathcal{F}$ and is simple, and is realized by a known simple group H. See Table 4.1 for details in the individual cases.*

G	$p=2$	$p=3$	$p=5$	$p=7$	$p=11$	$p=13$
M_{11}	simple	ab	ab		ab	
M_{12}	simple	$\boxed{\text{simple}}$	ab		ab	
M_{22}	simple	ab	ab	ab	ab	
M_{23}	simple	ab	ab	ab	ab	
M_{24}	simple	$\boxed{M_{12} : 2}$	ab	ab	ab	
J_1	ab	ab	ab	ab	ab	
J_2	simple	$\boxed{S \trianglelefteq \mathcal{F}}$	ab	ab		
J_3	simple	$S \trianglelefteq \mathcal{F}$	ab			
J_4	simple	$\boxed{{}^2F_4(2)' : 2}$	ab	ab	$\boxed{S \trianglelefteq \mathcal{F}}$	
Co_3	simple	simple	$\boxed{S \trianglelefteq \mathcal{F}}$	ab	ab	
Co_2	simple	simple	$\boxed{S \trianglelefteq \mathcal{F}}$	ab	ab	
Co_1	simple	simple	simple	ab	ab	
HS	simple	ab	$\boxed{S \trianglelefteq \mathcal{F}}$	ab	ab	
McL	simple	simple	$\boxed{S \trianglelefteq \mathcal{F}}$	ab	ab	
Suz	simple	simple	ab	ab	ab	
He	simple	$\boxed{M_{12} : 2}$	ab	$\boxed{\text{simple}}$		
Ly	simple	simple	simple	ab	ab	
Ru	simple	$\boxed{{}^2F_4(2)' : 2}$	$\boxed{SL_3(5) : 2}$	ab		ab
$O'N$	simple	ab	ab	$\boxed{\text{simple}}$	ab	
Fi_{22}	simple	simple	ab	ab	ab	ab
Fi_{23}	simple	simple	ab	ab	ab	ab
Fi_{24}'	simple	simple	ab	$\boxed{\text{simple}}$	ab	ab
F_5	simple	simple	simple	ab	ab	
F_3	simple	simple	$\boxed{\text{simple}}$	ab		ab
F_2	simple	simple	simple	ab	ab	ab
F_1	simple	simple	simple	simple	ab	$\boxed{\text{simple}}$

TABLE 4.1. In all cases, G is a sporadic simple group, $S \in \mathrm{Syl}_p(G)$, and $\mathcal{F} = \mathcal{F}_S(G)$. An entry "$H : 2$" means that $O^{p'}(\mathcal{F})$ is simple, has index 2 in $\mathcal{F}$, and is realized by the simple group H. A blank entry means that $p \nmid |G|$, and "ab" means that S is abelian. A box indicates that S is extraspecial of order p^3 and exponent p.

Proof. **(a)** See [A2, 16.5] or [AOV, Proposition 4.9].

(b) Assume $G \in \mathfrak{Lie}(p)$, or $p = 2$ and $G \cong {}^2F_4(2)'$. Then G has Lie rank at least 2 by [A2, Theorem 15.6.b], and so $\mathcal{F}$ is simple by [A2, 16.3].

(c) Assume G is sporadic. If $p=2$, then $G \not\cong J_1$ since $S \ntrianglelefteq \mathcal{F}$, and hence $\mathcal{F}$ is simple in all cases by [A2, 16.8]. If p is odd, then since $S \ntrianglelefteq \mathcal{F}$, G is not "p-Goldschmidt" in the terminology of [A2], and hence is not one of the groups listed in [A2, Theorem 15.6]. By [A2, 16.10], either $\mathcal{F}$ is simple, or (G,p) is one of the pairs listed there (and marked as such in Table 4.1), and $\mathcal{F}$ is realized by the almost simple group given in the same reference. In particular, when $\mathcal{F}$ is not simple, $O^{p'}(\mathcal{F})$ is realized by one of the simple groups M_{12} or ${}^2F_4(2)'$ (when $p=3$) or $SL_3(5)$ (when $p=5$), and hence is simple by [A2, 16.10] again or by (b), respectively.

Note, however, the following two errors in the statement of [A2, Theorem 15.6]. When $p=3$ and $G \cong He$, $\mathcal{F}$ is isomorphic to the fusion system of M_{24} and hence that of $\mathrm{Aut}(M_{12})$: this follows from the tables on pp. 47–48 of [RV]. The error in the proof in [A2] came from mis-identifying $D = N_G(S)$ (see line 3 on p. 102 in [A2]).

When $p=5$ and $G \cong Co_1$, $\mathcal{F}$ is simple by Theorem 2.8 and Table 2.2 in [OR]. In this case, the error in the proof of [A2, 16.9.5] occurred because the $\mathcal{F}$-radical subgroups of order 5^2 were overlooked.

In addition, when $p=3$ and $G \cong Fi_{24}'$, argument (ii) on p. 101 in [A2] cannot be used to prove that $O^{3'}(\mathcal{F}) = \mathcal{F}$, since there is no maximal subgroup $H \leq G$ such that $O_3(H) = F^*(H)$, $H = O^{3'}(H)$, and $H \geq N_G(S)$. Instead, argument (iii) (on the same page) can be applied. $\square$

It remains to consider the cases where $G \in \mathfrak{Lie}(q)$ for $q \neq p$, using Lemmas 2.11, and 3.2, and 3.3. We first look at the case $p=2$.

Proposition 4.2. *Let G be a finite simple group of Lie type in odd characteristic, fix $S \in \mathrm{Syl}_2(G)$, and set $\mathcal{F} = \mathcal{F}_S(G)$. Assume $S \ntrianglelefteq \mathcal{F}$. Then $O^{2'}(\mathcal{F}) = \mathcal{F}$, and $\mathcal{F}$ is simple.*

Proof. We will prove that $O^{p'}(\mathcal{F}) = \mathcal{F}$. Once we know this, then since no proper nontrivial subgroup of S is strongly closed with respect to G by [FF, Theorem 1.1], it follows by Lemma 2.16 that $\mathcal{F}$ is simple.

By [BMO2, Proposition 6.2], we can assume that G satisfies Case (III.1) of [BMO2, Hypotheses 5.1]. Hence $G \cong \mathbb{G}(q)$ for $q \equiv 1 \pmod 4$ (i.e., a Chevalley group), or $G \cong {}^2\mathbb{G}(q)$ for $\mathbb{G} = A_n$, D_n, or E_6 and $q \equiv 1 \pmod 4$. So by [BMO2, Lemma 5.3], there is a centric abelian subgroup $A \trianglelefteq S$ such that $\mathrm{Aut}_G(A)$ is generated by reflections; in particular, such that $O^{2'}(\mathrm{Aut}_G(A)) = \mathrm{Aut}_G(A)$. (These results in [BMO2] are stated for the groups of universal type, but they carry over easily to the adjoint case.) By [BMO2, Proposition 5.13], in all cases except when $G \cong Sp_{2n}(q)$ for some $n \geq 2$ and $q \equiv 5 \pmod 8$, A is weakly closed in $\mathcal{F}$, and hence $\Gamma_{p'}(\mathcal{F}) = 1$ and $O^{2'}(\mathcal{F}) = \mathcal{F}$ by Lemma 3.1(b).

Now assume that $G \cong PSp_{2n}(q)$, where $n \geq 2$ and $q \equiv 5 \pmod 8$, and set $\widetilde{G} = Sp_{2n}(q)$. More generally, we use tildes for subgroups and elements of $\widetilde{G}$, and the same symbol without tilde for their images in G. Thus $\widetilde{S} \in \mathrm{Syl}_2(\widetilde{G})$ and also $\widetilde{\mathcal{F}} = \mathcal{F}_{\widetilde{S}}(\widetilde{G})$. Let $(V, \mathfrak{b})$ be the symplectic space on which $\widetilde{G}$ acts as the group of isometries, choose an orthogonal decomposition $V = V_1 \perp \cdots \perp V_n$ where $\dim_{\mathbb{F}_q}(V_i) = 2$ for each i, and let $\widetilde{z}_i \in \widetilde{G}$ act via $-\mathrm{Id}$ on V_i and the identity on the other summands. Set $\widetilde{Z} = \langle \widetilde{z}_1, \ldots, \widetilde{z}_n \rangle$: then $C_{\widetilde{G}}(\widetilde{Z})$ is a product of n copies of $Sp_2(q) \cong SL_2(q)$ and $N_{\widetilde{G}}(\widetilde{Z}) \cong Sp_2(q) \wr \Sigma_n$. We can assume that $S \in \mathrm{Syl}_2(G)$ was chosen so that $\widetilde{S} \leq N_{\widetilde{G}}(\widetilde{Z})$ (hence so that $\widetilde{Z} \leq \widetilde{S}$).

Set $\widetilde{B} = C_{\widetilde{S}}(\widetilde{Z}) \cong (Q_8)^n$. Each $g \in \widetilde{z}_1^{\widetilde{\mathcal{F}}}$ acts on V with 2-dimensional (-1)-eigenspace, hence centralizes the z_i or permutes them via a 2-cycle, and no element that permutes them via a 2-cycle can be a square in $\widetilde{S}$. So each $\varphi \in \operatorname{Hom}_{\widetilde{\mathcal{F}}}(\widetilde{B}, \widetilde{S})$ sends $\widetilde{Z}$ to itself, and thus sends $\widetilde{B} = C_{\widetilde{S}}(\widetilde{Z})$ to itself. This proves that $\widetilde{B}$ is weakly closed in $\widetilde{\mathcal{F}}$.

Set $\widetilde{W} = \langle \widetilde{z}_1\widetilde{z}_2 \rangle$. We can assume $\widetilde{S}$ (or the ordering of the $\widetilde{z}_i$) was chosen so that $\widetilde{W}$ is fully centralized in $\widetilde{\mathcal{F}}$. Set $\widetilde{\mathcal{E}} = C_{\widetilde{\mathcal{F}}}(\widetilde{W})$: the fusion system of $C_{\widetilde{G}}(\widetilde{W}) \cong Sp_4(q) \times Sp_{2n-4}(q)$. By Lemma 2.6, $\widetilde{\mathcal{E}} = \widetilde{\mathcal{E}}_1 \times \widetilde{\mathcal{E}}_2$ and $O^{2'}(\widetilde{\mathcal{E}}) = O^{2'}(\widetilde{\mathcal{E}}_1) \times O^{2'}(\widetilde{\mathcal{E}}_2)$, where $\widetilde{\mathcal{E}}_1$ and $\widetilde{\mathcal{E}}_2$ are the fusion systems of $Sp_4(q)$ and $Sp_{2n-4}(q)$, respectively. The fusion system of $PSp_4(q)$ is simple by [O2, Proposition 5.1], so $O^{2'}(\widetilde{\mathcal{E}}_1) = \widetilde{\mathcal{E}}_1$ by Lemma 2.14, and hence $\operatorname{Aut}_{O^{2'}(\widetilde{\mathcal{E}})}(\widetilde{B})$ contains $(C_3 \wr \Sigma_2) \times 1$ as a subgroup of $\operatorname{Aut}_{\widetilde{\mathcal{F}}}(\widetilde{B}) \cong C_3 \wr \Sigma_n$. Thus $\operatorname{Aut}_{O^{2'}(\widetilde{\mathcal{E}})}(\widetilde{B})$ and $O^{2'}(\operatorname{Aut}_{\widetilde{\mathcal{F}}}(\widetilde{B}))$ generate $\operatorname{Aut}_{\widetilde{\mathcal{F}}}(\widetilde{B})$, so $\Gamma_{p'}(\widetilde{\mathcal{F}}) = 1$ by Lemma 3.2, and hence $\Gamma_{p'}(\mathcal{F}) = 1$ by Lemma 2.14 again. □

When handling the cases where p is odd, the following list of isomorphisms between fusion systems of groups of Lie type will be helpful. It is convenient to write $G \sim_p H$ to mean that G and H are finite groups whose p-fusion systems are isomorphic.

For each prime p and each q prime to p, $\operatorname{ord}_p(q)$ denotes the multiplicative order of q modulo p.

Lemma 4.3. *Fix an odd prime p and a prime power q such that $p \nmid q$. Then each of the following inclusions of finite groups induces an isomorphism of p-fusion systems:*

(a) $GL_n(q) \le GL_{n+1}(q)$ *when $n \ge 2$ and $\operatorname{ord}_p(q) \nmid (n+1)$;*

(b) $Sp_{2n}(q) \le GL_{2n}(q)$ *(all $n \ge 1$) when $\operatorname{ord}_p(q)$ is even;*

(c) $SO_{2n+1}(q) \le GL_{2n+1}(q)$ *(all $n \ge 1$) when $\operatorname{ord}_p(q)$ is even;*

(d) $GO^{\varepsilon}_{2n}(q) \le GL_{2n}(q)$ *when $\varepsilon = \pm 1$, $n \ge 1$, $\operatorname{ord}_p(q)$ is even, and $q^n \not\equiv -\varepsilon$ (mod p); and*

(e) $SO_{2n-1}(q) \le SO^{\varepsilon}_{2n}(q)$ *when $\varepsilon = \pm 1$, $n \ge 2$, and $q^n \not\equiv \varepsilon$ (mod p).*

Proof. By [BMO1, Lemma A.2], for each of the inclusions (a,b,c,d,e), the p-fusion system of the subgroup is a full subcategory of the p-fusion system of the larger group. So it remains only to check that each subgroup has index prime to p in the larger group, and this holds since by the well known formulas for the orders of these groups,

$$\begin{aligned} v_p\big(|GL_{n+1}(q) : GL_n(q)|\big) &= v_p(q^{n+1} - 1) \\ v_p\big(|GL_{2n}(q) : Sp_{2n}(q)|\big) &= \textstyle\sum_{i=1}^{n} v_p(q^{2i-1} - 1) \\ v_p\big(|GL_{2n+1}(q) : SO_{2n+1}(q)|\big) &= \textstyle\sum_{i=1}^{n+1} v_p(q^{2i-1} - 1) \\ v_p\big(|GL_{2n}(q) : GO^{\varepsilon}_{2n}(q)|\big) &= v_p(q^n + \varepsilon) \cdot \textstyle\sum_{i=1}^{n} v_p(q^{2i-1} - 1) \\ v_p\big(|SO^{\varepsilon}_{2n}(q) : SO_{2n-1}(q)|\big) &= v_p(q^n - \varepsilon) \end{aligned}$$

(see, e.g., [Ta, pp. 19, 70, 141]). □

Lemma 4.4. *Fix an odd prime p and a prime power q such that $p \nmid q$. Let $q^\vee$ be a prime power such that $q^\vee \equiv -q$ (mod p) and $v_p((q^\vee)^{p-1} - 1) = v_p(q^{p-1} - 1)$. Then*

(a) $PSU_n(q) \sim_p PSL_n(q^\vee)$ *for all n;*

(b) $\Omega^{\varepsilon}_{2n}(q) \sim_p \Omega^{(-1)^n\varepsilon}_{2n}(q^\vee)$ *for all $\varepsilon \in \{\pm 1\}$ and all n;*

(c) $E_6(q) \sim_p {}^2E_6(q^\vee) \sim_p F_4(q^\vee)$*; and*

(d) $\mathbb{G}(q) \sim_p \mathbb{G}(q^\vee)$ *when $\mathbb{G} = \Omega_{2n+1}$ or Sp_{2n} for $n \ge 2$, or $\mathbb{G} = G_2$, F_4, E_7, or E_8.*

Proof. These are all special cases of [BMO1, Theorem A(c,d)], together with [BMO1, Example 4.4(b)] (for the second equivalence in (c)). □

We now look at the classical groups when p is odd. We use the following, standard notation to describe certain automizers. For a given prime p, $k \mid m \mid (p-1)$ and $n \geq 1$, and a commutative ring R whose group of m-th roots of unity is cyclic of order m, define $G(m,k,n) \leq GL_n(R)$ as follows:

$$G(m,k,n) = \left\{ \mathrm{diag}(u_1,\ldots,u_n) \in GL_n(R) \,\middle|\, u_i^m = 1 \ \forall\, i,\ (u_1 \cdots u_n)^{m/k} = 1 \right\} \cdot \mathfrak{Perm}(n)$$

where $\mathfrak{Perm}(n) \cong \Sigma_n$ is the group of permutation matrices: the matrices that permute the canonical basis. Thus, for example, $G(m,1,n) \cong C_m \wr \Sigma_n$, and $G(m,m,n) \cong (C_m)^{n-1} \rtimes \Sigma_n$. This notation will be used in particular when $R = \mathbb{Z}/p^\ell$ (some $\ell \geq 1$) and $m \mid (p-1)$.

Proposition 4.5. *Fix an odd prime p, and let G be a finite simple linear, unitary, symplectic, or orthogonal group in defining characteristic different from p. Fix $S \in \mathrm{Syl}_p(G)$, assume S is nonabelian, and set $\mathcal{F} = \mathcal{F}_S(G)$. Then $O^{p'}(\mathcal{F})$ is simple, and $\Gamma_{p'}(\mathcal{F}) \cong \mathrm{Aut}_{\mathcal{F}}(S)/\mathrm{Aut}_{O^{p'}(\mathcal{F})}(S)$ is as follows:*

(a) *If $G \cong PSL_n^\varepsilon(q)$ for some $\varepsilon \in \{\pm 1\}$ and $n \geq 2$, then $\Gamma_{p'}(\mathcal{F})$ is cyclic of order $\mathrm{ord}_p(\varepsilon q)$.*

(b) *If $G \cong PSp_{2n}(q)$ or $\Omega_{2n+1}(q)$, then $\Gamma_{p'}(\mathcal{F})$ is cyclic of order $\mathrm{lcm}(2, \mathrm{ord}_p(q))$.*

(c) *If $G \cong P\Omega_{2n}^\varepsilon(q)$ where $\varepsilon = \pm 1$ and $q^n \not\equiv \varepsilon$ (mod p), then $\Gamma_{p'}(\mathcal{F})$ is cyclic of order $\mathrm{lcm}(2, \mathrm{ord}_p(q))$.*

(d) *If $G \cong P\Omega_{2n}^\varepsilon(q)$ where $\varepsilon = \pm 1$ and $q^n \equiv \varepsilon$ (mod p), then $\Gamma_{p'}(\mathcal{F})$ is cyclic of order $\frac{1}{2} \cdot \mathrm{lcm}(2, \mathrm{ord}_p(q))$. If, in addition, $\mathrm{ord}_p(q)$ is even, then $\mathcal{F}$ is isomorphic to a normal subsystem of index 2 in the p-fusion system of $SL_n(q)$.*

In each of the above cases, there is an abelian subgroup $A \trianglelefteq S$ that is $\mathcal{F}$-centric and weakly closed in $\mathcal{F}$. More precisely, if we define m, μ, θ, and κ as in Table 4.2 and set $\ell = v_p(q^\mu - \theta)$, then A has exponent p^ℓ, and A, $\mathrm{Aut}_{\mathcal{F}}(A)$, and $\mathrm{Aut}_{O^{p'}(\mathcal{F})}(A)$ are as described in Table 4.2.

case	m	μ	θ	κ	$\mathrm{Aut}_{\mathcal{F}}(A)$	$\mathrm{Aut}_{O^{p'}(\mathcal{F})}(A)$
(a)	$\mathrm{ord}_p(\varepsilon q)$	m	1	$[n/\mu]$	$C_\mu \wr \Sigma_\kappa$	$G(\mu,\mu,\kappa)$
(b)	$\mathrm{ord}_p(q)$	$\mathrm{lcm}(2,m)$	$(-1)^{m+1}$	$[2n/\mu]$	$C_\mu \wr \Sigma_\kappa$	$G(\mu,\mu,\kappa)$
(c)	$\mathrm{ord}_p(q)$	$\mathrm{lcm}(2,m)$	$(-1)^{m+1}$	$[2(n-1)/\mu]$	$C_\mu \wr \Sigma_\kappa$	$G(\mu,\mu,\kappa)$
(d)	$\mathrm{ord}_p(q)$	$\mathrm{lcm}(2,m)$	$(-1)^{m+1}$	$[2n/\mu]$	$G(\mu,2,\kappa)$	$G(\mu,\mu,\kappa)$

TABLE 4.2. In all cases except (a) when $p \mid (q-\varepsilon)$, $A \cong (C_{p^\ell})^\kappa$.

Proof. We first show that $O^{p'}(\mathcal{F})$ is simple, assuming the other claims hold. If $G = PSL_n^\varepsilon(q)$ where $\varepsilon = \pm 1$ and $p \mid (q-\varepsilon)$, then no proper nontrivial subgroup of S is strongly closed in $\mathcal{F}$ by [FF, Theorem 1.2] and since S is nonabelian. So once we know that $O^{p'}(\mathcal{F}) = \mathcal{F}$, then Lemma 2.16 implies that $\mathcal{F}$ is simple.

In all other cases, by Table 4.2, $\Omega_1(A)$ is a simple $\mathbb{F}_p\mathrm{Aut}_{O^{p'}(\mathcal{F})}(A)$-module, and $|A|^2 > |S|$ since $|A| > |S/A|$. If $S = T_1 \times \cdots \times T_k$, where $k \geq 2$, $T_i \cong T_j$ for each i,j, and each T_i is strongly closed in $\mathrm{Aut}_{O^{p'}(\mathcal{F})}(A)$, then either A is contained in one of the factors, or it is sent by the projection injectively into each factor. But this is impossible since $|A|^2 > |S|$. So by

Lemma 2.16 and since no proper nontrivial subgroup of S is strongly closed in $\mathcal{F}$ by [FF, Theorem 1.2] again, $O^{p'}(\mathcal{F})$ is simple.

(a) By Lemma 4.4(a), it suffices to prove this when $G \cong PSL_n(q)$: the case handled by Ruiz in [Ru]. We sketch a slightly different argument, based on Lemmas 3.2 and 3.3. Recall that $m = \mu = \mathrm{ord}_p(q)$.

Case a.1: Assume $m = 1$; i.e., $p \mid (q-1)$. Set $\overline{G} = SL_n(q)$, and let $A \le \overline{G}$ be the subgroup of diagonal matrices of p-power order and determinant 1. Thus $A \cong (C_{p^\ell})^{n-1}$ (recall that $\ell = v_p(q-1)$). Choose $\overline{S} \in \mathrm{Syl}_p(\overline{G})$ such that $A \le \overline{S} \le N_{\overline{G}}(A)$, and set $\overline{\mathcal{F}} = \mathcal{F}_{\overline{S}}(\overline{G})$. By Lemma 2.14, $\Gamma_{p'}(\mathcal{F}) \cong \Gamma_{p'}(\overline{\mathcal{F}})$, and hence it suffices to show that $\Gamma_{p'}(\overline{\mathcal{F}}) = 1$. Set

$$\Gamma = \mathrm{Aut}_{\overline{\mathcal{F}}}(A) \cong \Sigma_n \qquad \text{and} \qquad \Gamma_0 = \mathrm{Aut}_{O^{p'}(\overline{\mathcal{F}})}(A).$$

If $\Gamma_{p'}(\overline{\mathcal{F}}) \ne 1$, then by Lemma 3.1(b), $\Gamma_0 < \Gamma$ and hence $\Gamma_0 \cong A_n$ (the alternating group).

If $p \le n \le 2p-1$, then $|S/A| = p$, and $|[A,S]| \ge p^2$. (Note that if $p = n = 3$, then $\ell > 1$ since $S \in \mathrm{Syl}_p(PSL_n(q))$ is assumed to be nonabelian.) If $\Gamma_{p'}(\overline{\mathcal{F}}) \ne 1$, then $\Gamma_0 \cong A_p$, so that $|\mathrm{Aut}_{\Gamma_0}(U)| = (p-1)/2$ for $U \in \mathrm{Syl}_p(\Gamma_0)$. So $A \trianglelefteq O^{p'}(\overline{\mathcal{F}})$ by Lemma 1.4, hence $A \trianglelefteq \overline{\mathcal{F}}$, which is impossible since each element of $\overline{G}$ of p-power order is diagonalizable.

Now assume $n \ge 2p$. Let $A_0 \le A$ be the subgroup of all diagonal matrices $\mathrm{diag}(a_1,\dots,a_n)$ of p-power order such that $a_1 = a_2 = \cdots = a_{p+1}$. Thus $C_{\overline{G}}(A_0) \cong \overline{G} \cap (GL_{p+1}(q) \times (\mathbb{F}_q^\times)^{n-p-1})$. Set $G_0 = O^{p'}(O^p(C_{\overline{G}}(A_0))) \cong SL_{p+1}(q)$, so that $G_0 \trianglelefteq C_{\overline{G}}(A_0)$ and $C_{\overline{G}}(A_0)/G_0 \cong (\mathbb{F}_q^\times)^{n-p-1}$.

Set $S_0 = G_0 \cap C_{\overline{S}}(A_0) \in \mathrm{Syl}_p(G_0)$ and $\mathcal{F}_0 = \mathcal{F}_{S_0}(G_0)$. By Lemma 3.5, $\mathcal{F}_0$ is normal of p-power index in $C_{\overline{\mathcal{F}}}(A_0)$. So by Lemmas 2.13 and 2.11(b),

$$O^{p'}(\mathcal{F}_0) \le O^{p'}(C_{\overline{\mathcal{F}}}(A_0)) \le O^{p'}(\overline{\mathcal{F}}).$$

We already saw that $\mathrm{Aut}_{O^{p'}(\mathcal{F}_0)}(A) \cong \Sigma_{p+1}$, so $\mathrm{Aut}_{\overline{\mathcal{F}}}(A) = O^{p'}(\mathrm{Aut}_{\overline{\mathcal{F}}}(A)) \cdot \mathrm{Aut}_{O^{p'}(\mathcal{F}_0)}(A)$, and hence $\Gamma_{p'}(\overline{\mathcal{F}}) = 1$ by Lemma 3.2.

Case a.2: Now assume $m > 1$. Fix a vector space $V \cong (\mathbb{F}_q)^n$, and set $\Gamma = \mathrm{Aut}_{\mathbb{F}_q}(V) \cong GL_n(q)$ and $\Gamma_0 = O^{p'}(\Gamma) \cong SL_n(q)$. Note that $\mathcal{F} = \mathcal{F}_S(G)$ is isomorphic to those of Γ_0 and of Γ (Lemmas 2.14 and 3.5).

Fix an irreducible polynomial $f \mid (X^p - 1)$ of degree m in $\mathbb{F}_q[X]$, and identify $\mathbb{F}_{q^m} \cong \mathbb{F}_q[X]/(f)$. Choose a decomposition $V = V_1 \oplus \cdots \oplus V_\kappa \oplus W$, where $\dim(V_i) = m$ for each i and $\dim(W) = n - m\kappa < m$ (recall $\kappa = [n/m]$). Thus $p \nmid |\mathrm{Aut}_{\mathbb{F}_q}(W)|$. Set $\widehat{V} = V_1 \oplus \cdots \oplus V_\kappa$. Choose $z_i \in \mathrm{Aut}_{\mathbb{F}_q}(V_i)$ with characteristic polynomial f, and set $z = z_1 \oplus \cdots \oplus z_\kappa \oplus \mathrm{Id}_W \in C_\Gamma(W)$. Thus z determines an $\mathbb{F}_{q^m}$-vector space structure on $\widehat{V}$, and we set

$$H = C_\Gamma(z) = \mathrm{Aut}_{\mathbb{F}_{q^m}}(\widehat{V}) \times \mathrm{Aut}_{\mathbb{F}_q}(W) \cong GL_\kappa(q^m) \times GL_{n-m\kappa}(q)$$
$$H_0 = O^{p'}(H) \cong SL_\kappa(q^m) \rtimes C_{p^\ell}.$$

Now set

$$A = O_p\big(\mathrm{Aut}_{\mathbb{F}_{q^m}}(V_1) \times \cdots \times \mathrm{Aut}_{\mathbb{F}_{q^m}}(V_\kappa)\big) \cong O_p\big((\mathbb{F}_{q^m}^\times)^\kappa\big) \cong (C_{p^\ell})^\kappa.$$

Thus $\Omega_1(A) = \langle z_1, \dots, z_\kappa \rangle$. Each element of $N_\Gamma(A)$ permutes the subgroups $\langle z_i \rangle$, and so

$$N_\Gamma(A) \cong (\mathbb{F}_{q^m}^\times \rtimes C_m) \wr \Sigma_\kappa, \quad \mathrm{Aut}_\Gamma(A) = \mathrm{Aut}_{\Gamma_0}(A) \cong C_m \wr \Sigma_\kappa, \quad \text{and} \quad \mathrm{Aut}_H(A) \cong \Sigma_\kappa.$$

Choose $S_0 \in \mathrm{Syl}_p(\Sigma_\kappa)$, and set $S = AS_0$. Then $S \in \mathrm{Syl}_p(N_\Gamma(A))$, and from the formula for $|\Gamma| = |GL_n(q)|$, we see that

$$v_p(|\Gamma|) = \sum_{i=1}^{n} v_p(q^i - 1) = \sum_{i=1}^{\kappa} v_p(q^{mi} - 1) = \sum_{i=1}^{\kappa} \bigl(v_p(q^m - 1) + v_p(i)\bigr) = \kappa\ell + v_p(\kappa!) = v_p(|S|)$$

and hence that $S \in \mathrm{Syl}_p(\Gamma) = \mathrm{Syl}_p(\Gamma_0)$. We can thus identify $\mathcal{F} = \mathcal{F}_S(\Gamma_0)$.

Set $X = \{t \in A \,|\, t^{\mathcal{F}} \subseteq A\}$. Each element in $S \smallsetminus A$ acts on the set $\{V_i\}$ via a permutation of order p^i for some $i \geq 1$, and hence acts on V with at least p distinct eigenvalues. In particular, each element of A with fewer than p distinct eigenvalues lies in X. Since $\Omega_1(A)$ is generated by elements $z_1^{q^{i_1}} \oplus \cdots \oplus z_\kappa^{q^{i_\kappa}}$ for $i_1, \ldots, i_\kappa \in \mathbb{Z}$, each of which has eigenvalues the m roots of f in $\mathbb{F}_{q^m}$, and $m \leq (p-1)$, this shows that $\Omega_1(A) = \langle X \rangle$. So $\Omega_1(A)$, and $A = C_S(\Omega_1(A))$ are weakly closed in $\mathcal{F}$. In particular, $z \in X \cap Z(S)$ since it has exactly m distinct eigenvalues.

Consider the surjective homomorphism

$$\theta_{\mathcal{F}}^{(A)} \colon \mathrm{Aut}_{\mathcal{F}}(A) = \mathrm{Aut}_\Gamma(A) \twoheadrightarrow \Gamma_{p'}(\mathcal{F})$$

of Lemma 3.1. Set $\mathcal{E} = C_{\mathcal{F}}(z) = \mathcal{F}_S(H)$: the p-fusion system of $C_\Gamma(z)$ and hence of $GL_\kappa(q^m)$. We showed in Case a.1 that $O^{p'}(\mathcal{E})$ contains the fusion system of $SL_\kappa(q^m)$, and hence $\mathrm{Aut}_{O^{p'}(\mathcal{E})}(A) = \mathrm{Aut}_{\mathcal{E}}(A) = \mathrm{Aut}_H(A) \cong \Sigma_\kappa$. So $\mathrm{Ker}(\theta_{\mathcal{F}}^{(A)}) = O^{p'}(\mathrm{Aut}_{\mathcal{F}}(A))\mathrm{Aut}_H(A) \cong G(m, m, \kappa)$ by Lemma 3.3(b) applied with $\langle z \rangle$ in the role of Z, and thus $\Gamma_{p'}(\mathcal{F}) \cong \mathrm{Aut}_{\mathcal{F}}(A)/\mathrm{Ker}(\theta_{\mathcal{F}}^{(A)}) \cong C_m$.

(b,c,d) If $m = \mathrm{ord}_p(q)$ is odd, then by Lemma 4.4(b,d) (and Dirichlet's theorem), there is a prime power $q^\vee \equiv -q \pmod{p}$ such that for each n, $Sp_{2n}(q) \sim_p Sp_{2n}(q^\vee)$, $\Omega_{2n+1}(q) \sim_p \Omega_{2n+1}(q^\vee)$, and $\Omega_{2n}^{\varepsilon}(q) \sim_p \Omega_{2n}^{(-1)^n\varepsilon}(q^\vee)$ (for $\varepsilon = \pm 1$). Also, $\mathrm{ord}_p(q^\vee) = 2m$ is even, and $q^n \equiv \varepsilon \pmod{p}$ if and only if $(q^\vee)^n \equiv (-1)^n\varepsilon \pmod{p}$. Thus the claims in (b), (c), and (d) and in Table 4.2 all hold when $\mathrm{ord}_p(q)$ is odd if they hold when $\mathrm{ord}_p(q)$ is even.

(b) Assume $\mathrm{ord}_p(q)$ is even. If $G \cong PSp_{2n}(q)$, then $G \sim_p Sp_{2n}(q) \sim_p GL_{2n}(q) \sim_p SL_{2n}(q)$ by Lemmas 4.3(b) and 3.5, and hence $\Gamma_{p'}(\mathcal{F})$ is cyclic of order $\mathrm{ord}_p(q)$ by (a). If $G \cong \Omega_{2n+1}(q)$, then $G \sim_p SO_{2n+1}(q) \sim_p GL_{2n+1}(q) \sim_p SL_{2n+1}(q)$ by Lemmas 4.3(c) and 3.5, and hence $\Gamma_{p'}(\mathcal{F})$ is cyclic of order $\mathrm{ord}_p(q)$. In both cases, the claims in Table 4.2 about A, $\mathrm{Aut}_{\mathcal{F}}(A)$, and $\mathrm{Aut}_{O^{p'}(\mathcal{F})}(A)$ all follow from (a).

(c) Assume $G \cong P\Omega_{2n}^{\varepsilon}(q)$, where $\varepsilon = \pm 1$ and $q^n \not\equiv \varepsilon \pmod{p}$. Then $G \sim_p SO_{2n}^{\varepsilon}(q) \sim_p SO_{2n-1}(q)$ by Lemmas 3.5 and 4.3(e). So by (b), $\Gamma_{p'}(\mathcal{F})$ is cyclic of order $\mathrm{lcm}(2, \mathrm{ord}_p(q))$, and the claims in Table 4.2 about A, $\mathrm{Aut}_{\mathcal{F}}(A)$, and $\mathrm{Aut}_{O^{p'}(\mathcal{F})}(A)$ all hold.

(d) Assume $G \cong P\Omega_{2n}^{\varepsilon}(q)$, where $\varepsilon = \pm 1$ and $q^n \equiv \varepsilon \pmod{p}$, and where $m = \mathrm{ord}_p(q)$ is even. Set $\overline{G} = GO_{2n}^{\varepsilon}(q)$ and let $\overline{\mathcal{F}}$ be the fusion system of $\overline{G}$; thus $\overline{\mathcal{F}}$ contains $\mathcal{F}$ as a normal subsystem of index 1 or 2. Then $\overline{G} \sim_p GL_{2n}(q) \sim_p SL_{2n}(q)$ by Lemmas 4.3(d) and 3.5, so $\Gamma_{p'}(\overline{\mathcal{F}})$ is cyclic of order m, and $\Gamma_{p'}(\mathcal{F})$ is cyclic of order $m/2$ or m.

Recall that $\mu = \mathrm{lcm}(2, m) = m$ and $\kappa = [2n/\mu]$. Set $\lambda = m/2 = \mu/2$, so that $\kappa = [n/\lambda]$. Since $q^n \equiv \varepsilon = \pm 1$ by assumption, $\lambda \mid n$, and so $n = \kappa\lambda$. Let $(x \mapsto \overline{x})$ be the field automorphism of order 2 in $\mathbb{F}_{q^m}$; thus $\overline{x} = x^{q^\lambda}$. Set $V = (\mathbb{F}_{q^m})^\kappa$ as an $\mathbb{F}_q$-vector space, and define $\mathfrak{q}_0 \colon \mathbb{F}_{q^m} \longrightarrow \mathbb{F}_q$ and $\mathfrak{q} \colon V \longrightarrow \mathbb{F}_q$ by

$$\mathfrak{q}_0(x) = \mathrm{Tr}(x\overline{x}) \qquad \text{and} \qquad \mathfrak{q}(x_1, \ldots, x_\kappa) = \textstyle\sum_{i=1}^{\kappa} \mathfrak{q}_0(x_i) = \mathrm{Tr}(x_1\overline{x}_1 + \cdots + x_\kappa\overline{x}_\kappa),$$

where $\mathrm{Tr} \colon \mathbb{F}_{q^\lambda} \longrightarrow \mathbb{F}_q$ is the trace for the field extension. Then $|\mathfrak{q}_0^{-1}(0)| = (q^\lambda + 1)(q^{\lambda-1} - 1) + 1$, and by comparison with [Ta, Theorem 11.5], we see that $\mathfrak{q}_0$ has minus type as a quadratic form over $\mathbb{F}_q$. So $\mathfrak{q}$ has $(-1)^\kappa$ type, where $(-1)^\kappa \equiv (q^\lambda)^\kappa \equiv \varepsilon \pmod p$, and $GO(V,\mathfrak{q}) \cong GO_{2\lambda}^{\varepsilon}(q) = \overline{G}$.

We claim, for $u_1,\dots,u_\kappa \in \mathbb{F}_{q^m}^\times$, that

$$\mathrm{diag}(u_1,\dots,u_\kappa) \text{ is orthogonal } \iff u_i\bar{u}_i = 1 \text{ for each } i. \tag{4.1}$$

This means showing, for each $u \in \mathbb{F}_{q^m}^\times$, that $\mathfrak{q}_0(ux) = \mathfrak{q}_0(x)$ for all $x \in \mathbb{F}_{q^m}$, or equivalently $\mathrm{Tr}(u\bar{u}x\bar{x}) = \mathrm{Tr}(x\bar{x})$, if and only if $u\bar{u} = 1$. Since $u \mapsto u\bar{u} = u^{1+q^\lambda}$ sends $\mathbb{F}_{q^m} = \mathbb{F}_{q^{2\lambda}}$ surjectively onto $\mathbb{F}_{q^\lambda}$, we must prove, for all $v \in \mathbb{F}_{p^\lambda}^\times$, that $\mathrm{Tr}((v-1)\mathbb{F}_{q^\lambda}) = 0$ only if $v = 1$, and this is clear since $v \neq 1$ implies $(v-1)\mathbb{F}_{q^\lambda} = \mathbb{F}_{q^\lambda}$.

Thus the subgroup

$$A = \bigl\{\mathrm{diag}(u_1,\dots,u_\kappa) \bigm| u_i^{p^\ell} = 1 \text{ for all } i\bigr\} \cong (C_{p^\ell})^\kappa$$

is contained in $GO(V,\mathfrak{q})$ (recall $\ell = v_p(q^m-1) = v_p(q^\lambda+1)$). We saw in the proof of (a) that $N_{GL(V)}(A) \cong (\mathbb{F}_{q^m}^\times \rtimes C_m) \wr \Sigma_\kappa$. A diagonal matrix is orthogonal only if it has order dividing $q^\lambda + 1$, and the field automorphisms and permutations are clearly orthogonal. So

$$N_{\overline{G}}(A) = N_{GO(V,\mathfrak{q})}(A) \cong (C_{q^\lambda+1} \rtimes C_m) \wr \Sigma_\kappa$$

and hence $\mathrm{Aut}_{\overline{G}}(A) \cong C_m \wr \Sigma_\kappa$ as expected (recall $\overline{G} \sim_p GL_{2n}(q) \sim_p SL_{2n}(q)$).

If q is odd, then the diagonal matrices of order dividing $q^\lambda + 1$ have determinant 1 (over $\mathbb{F}_q$), and the permutations of coordinates have determinant 1 since m is even. By the normal basis theorem (see [Ja, § 4.14]), the field automorphism $(x \mapsto x^q)$ acting on just one factor $\mathbb{F}_{q^m}$ permutes cyclically an $\mathbb{F}_q$-basis for $\mathbb{F}_{q^m}$, and hence has determinant -1. It follows that $\mathrm{Aut}_G(A) \cong G(\mu,2,\kappa)$ has index 2 in $\mathrm{Aut}_{\overline{G}}(A)$, and hence that $\Gamma_{p'}(\mathcal{F})$ is cyclic of order $\lambda = \mu/2$.

If q is a power of 2, then a similar argument applies upon replacing the determinant by the Dickson invariant $D(\alpha) = \dim([V,\alpha]) \pmod 2$ for $\alpha \in GO(V,\mathfrak{q})$ (see, e.g., [Ta, Theorem 11.43]). The diagonal matrices and the permutations all lie in $\mathrm{Ker}(D) = \Omega(V,\mathfrak{q})$ since $\dim([V,\alpha]) \in m\mathbb{Z} \le 2\mathbb{Z}$ for all such α, while if α is the field automorphism $(x \mapsto x^q)$ acting on one coordinate, then $\dim([V,\alpha]) = m-1$ is odd. □

It remains to consider the exceptional groups of Lie type.

Proposition 4.6. *Fix an odd prime p, and let G be a finite simple group of exceptional Lie type in defining characteristic different from p. Fix $S \in \mathrm{Syl}_p(G)$, assume $S \ntrianglelefteq \mathcal{F}$, and set $\mathcal{F} = \mathcal{F}_S(G)$. Then either*

(a) *$p = 3$ and $G \cong {}^3D_4(q)$ where $3 \nmid q$, ${}^2F_4(q)$ where $q = 2^{2k+1}$ for $k \ge 1$, or ${}^2F_4(2)'$; or*

(b) *$G \cong F_4(q)$, $E_6(q)$, ${}^2E_6(q)$, $E_7(q)$, or $E_8(q)$, where $q \equiv \pm1 \pmod p$ and (G,p) is one of the pairs listed in Table 4.3; or*

(c) *$p = 5$ and $G \cong E_8(q)$, where $q \equiv \pm2 \pmod 5$; or*

(d) *$p = 3$, $G \cong G_2(q)$ where $q \equiv \pm1 \pmod 9$, $|O_3(\mathcal{F})| = 3$, and $O^{3'}(\mathcal{F})$ has index 2 in $\mathcal{F}$ and is realized by $SL_3^{\pm}(q)$.*

In all cases except when $p = 3$ and $G \cong {}^2F_4(2)'$ (and $S \cong 3_+^{1+2}$), there is a unique abelian subgroup $A \trianglelefteq S$ of maximal order, and A and $\mathrm{Aut}_G(A)$ are as described in Table 4.3. In all of the cases (a), (b), and (c), $\mathcal{F}$ is simple, and in particular, $O^{p'}(\mathcal{F}) = \mathcal{F}$.

Proof. Assume p is an odd prime and G is a finite simple group of exceptional Lie type in defining characteristic different from p. Choose $S \in \mathrm{Syl}_p(G)$, and set $\mathcal{F} = \mathcal{F}_S(G)$.

case	p	G	q	A	$\mathrm{Aut}_{\mathcal{F}}(A)$
(a)	3	${}^2F_4(q)$	$q = 2^{2n+1} \geq 8$	$C_{3^\ell} \times C_{3^\ell}$	$GL_2(3)$
(a)	3	${}^3D_4(q)$	$3 \nmid q$	$C_{3^{\ell+1}} \times C_{3^\ell}$	$\Sigma_3 \times C_2$
(b)	3	$F_4(q)$	$q \equiv \pm 1 \pmod p$	$(C_{p^\ell})^4$	$W(F_4)$
(b)	3	$E_6^\varepsilon(q)$	$q \equiv \varepsilon \pmod p$	$(C_{3^\ell})^5 \times C_{3^{\ell-1}}$	$W(E_6)$
(b)	3	$E_6^\varepsilon(q)$	$q \equiv -\varepsilon \pmod p$	$(C_{p^\ell})^4$	$W(F_4)$
(b)	5	$E_6^\varepsilon(q)$	$q \equiv \varepsilon \pmod p$	$(C_{5^\ell})^6$	$W(E_6)$
(b)	$3, 5, 7$	$E_n(q)$ $(n = 7, 8)$	$q \equiv \pm 1 \pmod p$	$(C_{p^\ell})^n$	$W(E_n)$
(c)	5	$E_8(q)$	$q \equiv \pm 2 \pmod p$	$(C_{p^\ell})^4$	$(C_4 \circ 2^{1+4}).\Sigma_6$
(d)	3	$G_2(q)$	$q \equiv \pm 1 \pmod 9$	$(C_{3^\ell})^2$	$\Sigma_3 \times C_2$

TABLE 4.3. Here, $\ell = v_p(q^2 - 1)$, except in case (c) where $\ell = v_5(q^2 + 1)$, and $\varepsilon = \pm 1$.

- If G is a Suzuki or Ree group or $G \cong {}^3D_4(q)$, then either S is abelian and hence $S \trianglelefteq \mathcal{F}$, or (G, p) is one of the pairs appearing in (a).
- If $G \cong F_4(q)$, $E_n(q)$, or ${}^2E_6(q)$, where $p \nmid q$, then by [GLS3, Theorem 4.10.2(a,b,c)] or [GL, (10-1(3))], either S is abelian, or for some m such that $p \cdot \mathrm{ord}_p(q) \mid m$, the cyclotomic polynomial $\Phi_m(q)$ appears in the formula for $|G|$. This occurs only when (G, p) is one of the pairs listed in Table 4.3, case (b) or (c).
- If $G \cong G_2(q)$ where $p \nmid q$, then either $p \geq 5$ and S is abelian; or $p = 3$, $v_3(q^2 - 1) = 1$, and $S \trianglelefteq \mathcal{F}$ by [A2, 16.11.4]; or $q \equiv \pm 1 \pmod 9$ and (G, p) is as in (d).

Thus whenever $S \ntrianglelefteq \mathcal{F}$, we are in the situation of one of the cases (a)–(d), and in fact, of one of the cases listed in Table 4.3.

In cases (a), (b), and (c), we will prove below that $O^{p'}(\mathcal{F}) = \mathcal{F}$. Once this has been shown, then $\mathcal{F}$ is simple by Lemma 2.16, together with [FF, Theorem 1.2] which says that no proper nontrivial subgroup of S is strongly closed in $\mathcal{F}$.

(a) Assume $p = 3$. If $G = {}^2F_4(2)'$ and $S \in \mathrm{Syl}_3(G)$, then by [RV, Lemma 4.13], S is extraspecial of order 3^3 and exponent 3, $\mathrm{Out}_G(S) \cong D_8$, and all subgroups of order 9 in S are $\mathcal{F}$-radical. A normal subsystem of index 2 would have the same radical subgroups, and so there is no such subsystem by the same lemma.

If $G = {}^2F_4(q)$ where $q = 2^{2k+1}$ for $k \geq 1$, then by the main theorem in [Ma], G contains a maximal torus $H \cong C_{q+1} \times C_{q+1}$ such that $\mathrm{Aut}_G(H) \cong GL_2(3)$ and $N_G(H) \geq S$. Set $A = O_p(H) \cong C_{3^\ell} \times C_{3^\ell}$, where $\ell = v_3(q+1) = v_3(q^2-1)$: then $|S/A| = 3$ and $\mathrm{Aut}_G(A) \cong GL_2(3)$. By Lemma 3.1(a,b), if $\Gamma_{p'}(\mathcal{F}) \neq 1$, then it has order 2 and $\mathrm{Aut}_{O^{3'}(\mathcal{F})}(A) = O^{3'}(\mathrm{Aut}_{\mathcal{F}}(A)) \cong SL_2(3) \cong 2A_4$. But this contradicts Lemma 1.4.

If $G = {}^3D_4(q)$ where $3 \nmid q$, then set $\ell = v_3(q^2 - 1)$, so that $v_3(|G|) = 2(\ell + 1)$. By the main theorem in [Kl], S contains a subgroup $A \cong C_{3^{\ell+1}} \times C_{3^\ell}$ of index 3, and $\mathrm{Aut}_G(A) \cong \Sigma_3 \times C_2 \cong D_{12}$. (If $q \equiv \varepsilon \pmod 3$ for $\varepsilon \in \{\pm 1\}$, then $N_G(A)$ is contained inside the maximal subgroup $(C_{q^2+\varepsilon q+1} \times_{C_3} SL_3^\varepsilon(q)) \rtimes \Sigma_3$ listed in [Kl].) By the conditions listed in [O1, Theorem 2.8(a)], there can be no normal subsystem of index 2 in $\mathcal{F} = \mathcal{F}_S(G)$.

Alternatively, the result for $G = {}^2F_4(q)$ or ${}^3D_4(q)$ follows from Tables 2 and 4 in [DRV]: when $m = v_3(|G|)$, then $S \cong B(3, m; 0, 0, 0)$ in the notation of [DRV], and those two tables list all saturated fusion systems over these groups.

(b) If $q \equiv \varepsilon \pmod{p}$ where $\varepsilon = \pm 1$, and $q^\vee \equiv -\varepsilon \pmod{p}$ is another prime power such that $v_p(q-\varepsilon) = v_p(q^\vee + \varepsilon)$, then $E_6(q) \sim_p {}^2E_6(q^\vee) \sim_p F_4(q^\vee)$ by Lemma 4.4(c). This, together with similar applications of Lemma 4.4(d), shows that it suffices to prove (b) when $G = \mathbb{G}(q)$ for $q \equiv 1 \pmod{p}$ and $\mathbb{G} = F_4$ or E_n. In these cases, A is homocyclic of rank $\mathrm{rk}(\mathbb{G})$ and exponent p^ℓ (where $\ell = v_p(q-1)$), except when $\mathbb{G} = E_6$ and $p = 3$, and $\mathrm{Aut}_{\mathcal{F}}(A) \cong W(\mathbb{G})$. (The description of A can be found in [GL, 10-1(2)] or [GLS3, Theorem 4.10.2(c)], and more details on $\mathrm{Aut}_G(A)$ are given in [BMO2, Lemmas 6.1 & 5.3].) Recall that we are working with the simple groups, hence the adjoint forms, which is why the description of A is slightly different when $(G, p) = (E_6(q), 3)$.

It remains to show that $O^{p'}(\mathcal{F}) = \mathcal{F}$, still assuming $G = \mathbb{G}(q)$ for $\mathbb{G} = F_4$ or E_n and $q \equiv 1 \pmod{p}$ (and S is nonabelian). Since $\mathrm{Aut}_{\mathcal{F}}(A) \cong W(\mathbb{G})$ is the usual Weyl group for G, we have

$$\mathrm{Aut}_{\mathcal{F}}(A)/O^{p'}(\mathrm{Aut}_{\mathcal{F}}(A)) \cong \begin{cases} C_2 \times C_2 & \text{if } \mathbb{G} = F_4 \text{ and } p = 3 \\ C_2 & \text{if } \mathbb{G} = E_6, E_7, E_8 \text{ and } 3 \le p \mid |W(\mathbb{G})|. \end{cases} \tag{4.2}$$

When $p = 3$, these follow from the "standard presentation" of $W(\mathbb{G})$ as a group of reflections acting on a vector space, with a fundamental system of reflections as generating set [Ca, Theorem 2.4.1]. When $p = 5, 7$, the result follows from the fact that $W(E_n)$ contains a quasisimple subgroup of index 2. In all cases, we set $\ell = v_p(q-1) \ge 1$.

We consider two different cases.

Case (b.1) Assume $|S/A| = p$; equivalently, $v_p(|W(\mathbb{G})|) = 1$. Then either $p = 5$ and $\mathbb{G} = E_6$ or E_7, or $p = 7$ and $\mathbb{G} = E_7$ or E_8, and we are in the situation of [COS, Section 2]. We refer to Notation 2.4 in that paper. By [COS, Lemma 2.6(a)] and since $A \ntrianglelefteq \mathcal{F}$, $\mathrm{Aut}^\vee_{O^{p'}(\mathcal{F})}(A)$ must contain one of the subgroups Δ_0 or Δ_{-1}, where for $t \in \mathbb{Z}$,

$$\Delta_t = \{(r, r^t) \mid r \in (\mathbb{Z}/p)^\times\} \le \Delta \overset{\text{def}}{=} (\mathbb{Z}/p)^\times \times (\mathbb{Z}/p)^\times.$$

By Table 4.1 in [COS], this would not be the case if $\mathrm{Aut}_{O^{p'}(\mathcal{F})}(A)$ were strictly contained in $\mathrm{Aut}_{\mathcal{F}}(A) \cong W(\mathbb{G})$. Thus $\mathrm{Aut}_{O^{p'}(\mathcal{F})}(A) = \mathrm{Aut}_{\mathcal{F}}(A)$, and so $O^{p'}(\mathcal{F}) = \mathcal{F}$ by Lemma 3.1(c).

Case (b.2) Assume $|S/A| \ge p^2$; equivalently, $p^2 \mid |W(\mathbb{G})|$. Then either $p = 3$ and $\mathbb{G} = F_4$ or E_n, or $p = 5$ and $\mathbb{G} = E_8$. In each case, there is an element $z \in Z(S)$ whose centralizer $H = C_G(z)$ is as described in the Table 4.4. The normalizer $N_G(z)$ is described in [LSS, Table 5.1], and the descriptions of $O^p(H)/\langle z \rangle$ and $H/O^p(H)$ are based on that. A comparison of orders shows that $p \nmid [G:H]$ in each case, and hence that we can choose $z \in Z(S)$.

$\mathbb{G}$	p	$O^p(H)/\langle z\rangle = O^p(H/\langle z\rangle)$	$H/O^p(H)$	$\overline{H}$
F_4	3	$PSL_3(q) \times PSL_3(q)$	C_3	$SL_3(\overline{\mathbb{F}}_q) \circ SL_3(\overline{\mathbb{F}}_q)$
E_6	3	$PSL_3(q) \times PSL_3(q) \times PSL_3(q)$	$C_3 \times C_3$	$SL_3(\overline{\mathbb{F}}_q) \circ SL_3(\overline{\mathbb{F}}_q) \circ SL_3(\overline{\mathbb{F}}_q)$
E_7	3	$E_6(q) \times C_{(q-1)/3^\ell}$	C_{3^ℓ}	$3E_6(\overline{\mathbb{F}}_q) \circ \overline{\mathbb{F}}_q^\times$
E_8	3	$E_6(q) \times PSL_3(q)$	C_3	$3E_6(\overline{\mathbb{F}}_q) \circ SL_3(\overline{\mathbb{F}}_q)$
E_8	5	$PSL_5(q) \times PSL_5(q)$	C_5	$SL_5(\overline{\mathbb{F}}_q) \circ SL_5(\overline{\mathbb{F}}_q)$

TABLE 4.4. In all cases, $G = \mathbb{G}(q)$ where $\ell = v_p(q-1) \ge 1$, and $\overline{H} = C_{\mathbb{G}(\overline{\mathbb{F}}_q)}(z)$ for a certain $z \in Z(S)$ of order p. Also, $E_6(-)$ means the simple group and $3E_6(-)$ its universal form, and "$X \circ Y$" denotes a central product of X and Y over C_p.

The description of the centralizer of z in the algebraic group $\mathbb{G}(\overline{\mathbb{F}}_q)$ in the last column of the table (taken from [Gr, Table VI]), is included to better explain the structure of H. For example, when $p = 3$ and $G = F_4(q)$, H is an extension of $SL_3(q) \circ SL_3(q)$ by C_3 acting via diagonal automorphisms of order 3 on each factor.

Set $\mathcal{E} = C_{\mathcal{F}}(z) = \mathcal{F}_S(H)$. If $\ell = v_p(q-1) \geq 2$ or $p = 5$, then the p-fusion system of $PSL_p(q)$ is simple by Proposition 4.5(a), and $O^{p'}(O^p(\mathcal{E})/\langle z\rangle)$ is the fusion system of $O^{p'}(O^p(H)/\langle z\rangle)$ by the table. By Lemmas 2.13 and 2.14, $O^{p'}(\mathcal{E}) = \mathcal{E}$, except when $(\mathbb{G}, p) = (E_7, 3)$, in which case $O^{p'}(\mathcal{E}) = \mathcal{F}_S(O^{p'}(H))$. So by (4.2), $\mathrm{Aut}_{\mathcal{F}}(A)$ is generated by $O^{p'}(\mathrm{Aut}_{\mathcal{F}}(A))$ and $\mathrm{Aut}_{O^{p'}(\mathcal{E})}(A)$, and $O^{p'}(\mathcal{F}) = \mathcal{F}$ by Lemma 3.2.

Now assume $\ell = 1$ and $p = 3$. Set $K = PSL_3(q)$ and fix $U \in \mathrm{Syl}_3(K)$. Then $U \cong C_3 \times C_3$, so $\mathcal{F}_U(K) = \mathcal{F}_U(N_K(U)) \cong \mathcal{F}_U(U \rtimes Q_8)$. If $\mathbb{G} = F_4$, then $\mathcal{E}/\langle z\rangle$ is the fusion system of an extension of $(C_3)^4 \rtimes (Q_8 \times Q_8)$ by C_3 acting faithfully on each factor Q_8, so $O^{3'}(\mathcal{E}/\langle z\rangle) = \mathcal{E}/\langle z\rangle$, and $O^{3'}(\mathcal{E}) = \mathcal{E}$ by Lemma 2.14. A similar argument shows that $O^{3'}(\mathcal{E}) = \mathcal{E}$ when $\mathbb{G} = E_6$. This, together with the descriptions in Table 4.4, show that $\mathrm{Aut}_{\mathcal{F}}(A)$ is generated by $O^{p'}(\mathrm{Aut}_{\mathcal{F}}(A))$ and $\mathrm{Aut}_{O^{p'}(\mathcal{E})}(A)$ in all four cases (also when $\mathbb{G} = E_7$ or E_8) and hence that $O^{p'}(\mathcal{F}) = \mathcal{F}$ by Lemma 3.2 again.

(c) Now assume $p = 5$, and $G = E_8(q)$ where $q \equiv \pm 2 \pmod 5$. Set $\ell = v_5(q^4-1) = v_5(q^2+1)$. By [BMO2, Lemma 6.7], there is a unique $A = (C_{5^\ell})^4$ of index 5 in S, and $\mathrm{Aut}_{\mathcal{F}}(A) \cong (C_4 \circ 2^{1+4}).\Sigma_6$ (where "$\circ$" denotes the central product over C_2). We are thus in the situation of [COS, Section 2]. Since $\mathrm{rk}(A) \leq p$, we have $|Z(S)| = p$ and hence $\mathrm{Aut}^{\vee}_{\mathcal{F}}(S) = \mathrm{Aut}_{\mathcal{F}}(S)$ as defined in [COS, Notation 2.4]. So $O^{p'}(\mathcal{F}) = \mathcal{F}$ by Table 2.2 and Lemma 2.7(c.i) in [COS].

(d) This case, where $G = G_2(q)$ and $p = 3$, was handled by Aschbacher [A2, 16.11]. In particular, he showed there that $S \trianglelefteq \mathcal{F}$ if $q \equiv \pm 4, \pm 5 \pmod 9$, while $O_3(\mathcal{F}) = Z(S)$ and $O^{3'}(\mathcal{F}) = C_{\mathcal{F}}(Z(S))$ has index 2 in $\mathcal{F}$ and is realized by $SL_3^{\pm}(q)$ if $q \equiv \pm 1 \pmod 9$. □

With the help of Propositions 4.5 and 4.6 and Tables 4.2 and 4.3, we can now check that certain fusion systems are exotic.

Lemma 4.7. *Assume the classification of finite simple groups. Let $\mathcal{F}$ be a saturated fusion system over a finite p-group S, for some prime $p \geq 5$, and assume $A \trianglelefteq S$ is abelian and $\mathcal{F}$-centric. Assume also, for some $\ell \geq 1$, $\kappa \geq p$, and $2 < m \mid (p-1)$, that A is homocyclic of rank κ and exponent p^ℓ, and that with respect to some basis $\{a_1, \ldots, a_\kappa\}$ for A as a $\mathbb{Z}/p^\ell$-module,* $\mathrm{Aut}_{\mathcal{F}}(A)$ *contains $G(m, m, \kappa)$ with index prime to p, and*

$$\mathrm{Aut}_{\mathcal{F}}(A) \cap G(m, 1, \kappa) = G(m, r, \kappa) \leq GL_\kappa(\mathbb{Z}/p^\ell) \qquad \text{for some } 2 < r \mid m.$$

Then either $A \trianglelefteq \mathcal{F}$ or $\mathcal{F}$ is exotic.

Proof. Assume otherwise: assume $A \ntrianglelefteq \mathcal{F}$ and $\mathcal{F}$ is realized by the finite group G. Assume also that $|G|$ is minimal among all such counterexamples $(A, \mathcal{F}, G)$ to the lemma. In particular, $|G|$ is minimal among all orders of finite groups realizing $\mathcal{F}$, and so $O_{p'}(G) = 1$.

If $\ell > 1$ and $\Omega_1(A) \trianglelefteq \mathcal{F}$, then $\Omega_1(A) \trianglelefteq G$ by the minimality assumption ($\mathcal{F}$ is realized by $N_G(\Omega_1(A))$). So in this case, $A/\Omega_1(A) \ntrianglelefteq \mathcal{F}/\Omega_1(A)$ where $\mathcal{F}/\Omega_1(A)$ is realized by $G/\Omega_1(A)$, which again contradicts our minimality assumption on $|G|$. Thus $\Omega_1(A) \ntrianglelefteq \mathcal{F}$, and hence $\Omega_k(A) \ntrianglelefteq \mathcal{F}$ for each $1 \leq k \leq \ell$.

Since $\mathrm{Aut}_{\mathcal{F}}(A)$ contains $G(m, r, \kappa)$ with index prime to p, some subgroup conjugate to $\mathrm{Aut}_S(A)$ is contained in $G(1, 1, \kappa) \cong \Sigma_\kappa$, and hence $\mathrm{Aut}_S(A)$ permutes some basis of $\Omega_1(A)$. Also, $G(m, r, \kappa)$ acts faithfully on $\Omega_1(A)$, as does each subgroup of $\mathrm{Aut}_{\mathcal{F}}(A)$ of order prime to p (see [G, Theorem 5.2.4]). So by the assumptions on $\mathrm{Aut}_{\mathcal{F}}(A)$,

$$\begin{array}{l}\mathrm{Aut}_{\mathcal{F}}(A) \text{ acts faithfully on } \Omega_1(A),\ \mathrm{Aut}_S(A) \text{ permutes} \\ \text{a basis of } \Omega_1(A), \text{ and } C_S(\Omega_1(A)) = A.\end{array} \tag{4.3}$$

We claim that

$$\Omega_1(A) \text{ is the only elementary abelian subgroup of } S \text{ of rank } \kappa. \tag{4.4}$$

This is well known, but the proof is simple enough that we give it here. Set $V = \Omega_1(A)$ for short, let $W \le S$ be another elementary abelian subgroup, and set $\overline{W} = \mathrm{Aut}_W(V)$ and $r = \mathrm{rk}(\overline{W})$. Then $r = \mathrm{rk}(W/(V \cap W))$ since $C_W(V) = W \cap A = W \cap V$ by (4.3). Let $\mathcal{B}$ be a basis for V permuted by $\overline{W}$, and assume $\overline{W}$ acts on $\mathcal{B}$ with s orbits (including fixed orbits) of lengths $p^{m_1}, \dots, p^{m_s}$. Then $m_1 + \cdots + m_s \ge r = \mathrm{rk}(\overline{W})$, and hence

$$\mathrm{rk}(W) = \mathrm{rk}(\overline{W}) + \mathrm{rk}(W \cap V) \le r + \mathrm{rk}(C_V(\overline{W})) = r + s \le \sum_{i=1}^{s}(1 + m_i) < \sum_{i=1}^{s} p^{m_i} = \mathrm{rk}(V),$$

proving (4.4). In particular, $\Omega_1(A)$ and $A = C_S(\Omega_1(A))$ are weakly closed in $\mathcal{F}$.

Set $G_0 = O^{p'}(G)$ and $\mathcal{F}_0 = \mathcal{F}_S(G_0) \trianglelefteq \mathcal{F}$, and let $\Gamma = \mathrm{Aut}_{\mathcal{F}}(A)$ and $\Gamma_0 = \mathrm{Aut}_{\mathcal{F}_0}(A)$ for short. By Lemma 2.15 and since $\Omega_1(A)$ is weakly closed in $\mathcal{F}$, we have $\Omega_1(A) \ntrianglelefteq \mathcal{F}_0$.

Step 1: We first claim that G_0 is simple. Assume otherwise: let $H \trianglelefteq G_0$ be a proper nontrivial normal subgroup, and set $T = H \cap S \in \mathrm{Syl}_p(H)$. Thus $1 \ne T < S$ since $O_{p'}(G_0) = 1$ and $O^{p'}(G_0) = G_0$, and T is strongly closed in $\mathcal{F}_0$.

Now, $\mathrm{Aut}_H(A) \trianglelefteq \Gamma_0$ since $H \trianglelefteq G_0$, and by assumption, Γ_0 contains $G(m,m,\kappa)$ with index prime to p. So $\mathrm{Aut}_H(A) \cap G(m,m,\kappa)$ is normal in $G(m,m,\kappa) \cong (C_m)^{\kappa-1} \rtimes \Sigma_\kappa$, where $\kappa \ge p \ge 5$. Consequently, either $\mathrm{Aut}_H(A) \cap G(m,m,\kappa)$ and hence $\mathrm{Aut}_H(A)$ have order prime to p, or $\mathrm{Aut}_H(A)$ contains $O^{p'}(G(m,m,\kappa))$. In the latter case, $H \ge [N_H(A), A] \ge [O^{p'}(G(m,m,\kappa)), A] = A$, and hence $N_H(A)$ has index prime to p in $N_G(A)$. But then $T = S$, a contradiction.

Thus $p \nmid |\mathrm{Aut}_H(A)|$, and so $T = O^{p'}(N_H(A)) \le A$. Also, T is normalized by $\Gamma_0 \ge O^{p'}(G(m,m,\kappa))$, and so $T = \Omega_k(A)$ for some $1 \le k \le \ell$. But then T is abelian and strongly closed in $\mathcal{F}$, so $T \trianglelefteq \mathcal{F}_0$ by [AKO, Corollary I.4.7(a)], hence $\Omega_1(A) = \Omega_1(T) \trianglelefteq \mathcal{F}_0$, which we already showed is impossible. So there is no such T, and G_0 must be simple.

Step 2: It remains to show that G_0 cannot be any known simple group. Note that A is a radical p-subgroup of G_0, since $O_p(\mathrm{Aut}_{G_0}(A)) = 1$ and $p \nmid |C_{G_0}(A)/A|$ (i.e., A is $\mathcal{F}_0$-centric). Although we do not know $\mathrm{Aut}_{G_0}(A)$ precisely, we know that it is contained in $\mathrm{Aut}_{\mathcal{F}}(A)$ and contains $O^{p'}(G(m,1,\kappa)) \cong (C_m)^{\kappa-1} \rtimes A_\kappa$.

Since $p \ge 5$ and $\mathrm{rk}_p(G_0) \ge p$, G_0 cannot be a sporadic group by [GLS3, Table 5.6.1].

By [AF, § 2], for each abelian radical p-subgroup $B \le \Sigma_\kappa$, $\mathrm{Aut}_{\Sigma_\kappa}(B)$ is a product of wreath products of the form $GL_c(p) \wr \Sigma_\kappa$ for $c \ge 1$ and $\kappa \ge 1$. Thus $\mathrm{Aut}_{A_\kappa}(B)$ can have index 2 in $C_{p-1} \wr \Sigma_\kappa$ for some κ, but not index larger than 2. So G_0 cannot be an alternating group.

If $G_0 \in \mathfrak{Lie}(p)$, then $N_{G_0}(A)$ is a parabolic subgroup by the Borel-Tits theorem [GLS3, Corollary 3.1.5] and since A is centric and radical. So in the notation of [GLS3, § 2.6], $A = U_J$ and $N_{G_0}(A) = P_J$ (up to conjugacy) for some set J of primitive roots for G_0. Hence by [GLS3, Theorem 2.6.5(f,g)], $O^{p'}(N_{G_0}(A)/A) \cong O^{p'}(L_J)$ is a central product of groups in $\mathfrak{Lie}(p)$, contradicting the assumption that $O^{p'}(\mathrm{Aut}_G(A)) \cong G(m,r,\kappa)$.

Now assume that $G_0 \in \mathfrak{Lie}(q_0)$ for some prime $q_0 \ne p$. By [GL, 10-2] (and since $p \ge 5$), $S \in \mathrm{Syl}_p(G_0)$ contains a unique elementary abelian p-subgroup of maximal rank, and by

(4.4), it must be equal to $\Omega_1(A)$. Hence $\mathrm{Aut}_{\mathcal{F}}(A)$ must be as in one of the entries in Table 4.2 or 4.3.

• If G_0 is a classical group and hence $\mathrm{Aut}_{\mathcal{F}}(A) \cong G(\widehat{m}, \widehat{r}, \kappa)$ for $\widehat{m} = \mu$ or 2μ and $\widehat{r} \leq 2$ (see the next-to-last column in Table 4.2, and recall that $G(\widehat{m}, 1, \kappa) \cong C_{\widehat{m}} \wr \Sigma_\kappa$), then the identifications $\mathrm{Aut}_{\mathcal{F}}(A) \cong G(\widehat{m}, \widehat{r}, \kappa)$ and $\mathrm{Aut}_{\mathcal{F}}(A) \geq G(m, r, \kappa)$ are based on the same decompositions of A as a direct sum of cyclic subgroups, and hence we have $m \mid \widehat{m}$ and $r \leq 2$, contradicting our original assumption.

• If G_0 is an exceptional group, then by Table 4.3, either $\kappa = \mathrm{rk}(A) < p$, or $p = 3$, or (in case (b)) $m^{\kappa-1} \cdot \kappa!$ does not divide $|\mathrm{Aut}_{\mathcal{F}}(A)|$ for any $m > 2$ and hence $\mathrm{Aut}_{\mathcal{F}}(A)$ cannot contain any such $G(m, r, \kappa)$. □

The results in this section are now summarized in the following theorem, in which the statements that certain fusion systems are exotic depend on the classification of finite simple groups. Recall that a fusion subsystem of $\mathcal{F}$ is *characteristic* in $\mathcal{F}$ if it gets sent to itself by each $\alpha \in \mathrm{Aut}(\mathcal{F})$.

Theorem 4.8. *Fix a prime p, a known finite simple group G, and $S \in \mathrm{Syl}_p(G)$. Set $\mathcal{F} = \mathcal{F}_S(G)$, and assume that $S \ntrianglelefteq \mathcal{F}$. Then $\Gamma_{p'}(\mathcal{F})$ is cyclic, and either*

(a) *$p \geq 5$, G is one of the classical groups $PSL^{\pm}_n(q)$, $PSp_{2n}(q)$, $\Omega_{2n+1}(q)$, or $P\Omega^{\pm}_{2n+2}(q)$ where $n \geq 2$ and $q \not\equiv 0, \pm 1 \pmod p$, in which case $O^{p'}(\mathcal{F})$ is simple and exotic; or*

(b) *$p = 3$ and $G \cong G_2(q)$ for some $q \equiv \pm 1 \pmod 9$, in which case $|O_3(\mathcal{F})| = 3$, $|\Gamma_{p'}(\mathcal{F})| = 2$, and $O^{3'}(\mathcal{F})$ is realized by $SL^{\pm}_3(q)$; or*

(c) *$|\Gamma_{p'}(\mathcal{F})| \leq 2$, $O^{p'}(\mathcal{F})$ is simple, and it is realized by a known finite simple group G^*.*

Moreover, the following hold in case (a).

(a.1) *There are a subsystem $\mathcal{F}_0 \trianglelefteq \mathcal{F}$ of index at most 2 in $\mathcal{F}$, and a finite group G_0 realizing $\mathcal{F}_0$, with the following properties. For each saturated fusion system $\mathcal{E}$ over S such that $O^{p'}(\mathcal{E}) = O^{p'}(\mathcal{F})$, either*
- *$\mathcal{E} \geq \mathcal{F}_0$, and $\mathcal{E} = \mathcal{F}_S(H)$ for some group H with $G_0 \trianglelefteq H$ and $p \nmid |H/G_0|$; or*
- *$\mathcal{E} \ngeq \mathcal{F}_0$, and $\mathcal{E}$ is exotic.*

(a.2) *If $\mathcal{E} \geq \mathcal{F}$ is an extension of index prime to p (i.e., $\mathcal{E}$ is saturated and $\mathcal{F}$ is normal of index prime to p in $\mathcal{E}$), then $\mathcal{F}$ is characteristic in $\mathcal{E}$.*

Proof. By Propositions 4.1, 4.2, and 4.6, point (b) or (c) holds except possibly when p is odd and G is one of the classical groups $PSL^{\pm}_n(q)$, $PSp_{2n}(q)$, $P\Omega_{2n+1}(q)$, or $P\Omega^{\pm}_{2n}(q)$ for $p \nmid q$. So assume we are in one of those cases. In particular, $O^{p'}(\mathcal{F})$ is simple by Proposition 4.5.

If $G \cong PSU_n(q)$ for some n and q, then by Lemma 4.4(a), $G \sim_p PSL_n(q^\vee)$ for some prime power $q^\vee \equiv -q \pmod p$. So we can assume G is linear, symplectic, or orthogonal.

Case 1: Assume $q \equiv \pm 1 \pmod p$. If $G = PSL_n(q)$ and $q \equiv 1 \pmod p$, then $O^{p'}(\mathcal{F}) = \mathcal{F}$ by Proposition 4.5(a). If $G = PSL_n(q)$ and $q \equiv -1 \pmod p$, so that $\mathrm{ord}_p(q) = 2$, then set $k = [n/2]$. We have

$$G \sim_p GL_n(q) \sim_p GL_{2k}(q) \sim_p GO^{\varepsilon}_{2k}(q)$$

for $\varepsilon = (-1)^k$ by Lemma 4.3(a,d). Also, $O^{p'}(\mathcal{F})$ has index 2 in $\mathcal{F}$ by Proposition 4.5(a,d), and it is realized by the simple group $P\Omega^{\varepsilon}_{2n}(q)$ by Table 4.2 and Lemma 3.1(c).

If G is one of the groups $PSp_{2n}(q)$ or $\Omega_{2n+1}(q)$ for $q \equiv \pm 1 \pmod p$, or $P\Omega^{\varepsilon}_{2n}(q)$ for $q \equiv \pm 1$ and $q^n \not\equiv \varepsilon \pmod p$, then a similar argument shows that $O^{p'}(\mathcal{F})$ has index 2 in $\mathcal{F}$ and is realizable by a simple orthogonal group. If $G = P\Omega^{\varepsilon}_{2n}(q)$ where $q \equiv \pm 1$ and $q^n \equiv \varepsilon \pmod p$, then $O^{p'}(\mathcal{F}) = \mathcal{F}$ by Proposition 4.5(c).

Thus $O^{p'}(\mathcal{F})$ is realizable (and we are in the situation of (c)) whenever G is one of the classical groups defined over $\mathbb{F}_q$ and $q \equiv \pm 1 \pmod p$.

Case 2: Assume $q \not\equiv 0, \pm 1 \pmod p$, where G is again one of the classical groups $PSL_n(q)$, $PSp_{2n}(q)$, $P\Omega_{2n+1}(q)$, or $P\Omega^{\pm}_{2n}(q)$. In particular, $p \geq 5$. Set $m = \mathrm{ord}_p(q)$ and $\ell = v_p(q^m - 1)$; thus $\ell \geq 1$ and $2 < m \mid (p-1)$. By Table 4.2, there are $\kappa \geq 1$ and $A \trianglelefteq S$ weakly closed in $\mathcal{F}$ such that $A \cong (C_{p^\ell})^\kappa$ and $\mathrm{Aut}_{O^{p'}(\mathcal{F})}(A) \cong G(m, m, \kappa)$. So $O^{p'}(\mathcal{F})$ is exotic by Lemma 4.7.

Proof of (a.1): Assume $\mathcal{E}$ is a saturated fusion system over S such that $O^{p'}(\mathcal{E}) = O^{p'}(\mathcal{F})$. Then $\mathrm{Aut}_{O^{p'}(\mathcal{F})}(A) \trianglelefteq \mathrm{Aut}_{\mathcal{E}}(A)$. Let $\{a_1, \dots, a_\kappa\}$ be a basis for A as a $\mathbb{Z}/p^\ell$-module such that with respect to this basis, $\mathrm{Aut}_{O^{p'}(\mathcal{F})}(A) = G(m, m, \kappa)$: the group of monomial matrices generated by the permutation matrices, and the diagonal matrices of order dividing m and of determinant 1.

Now, $\mathrm{Aut}_{\mathcal{E}}(A) \leq N_{\mathrm{Aut}(A)}(\mathrm{Aut}_{O^{p'}(\mathcal{F})}(A)) = G(m, 1, \kappa)\langle(\mathbb{Z}/p^\ell)^\times \cdot \mathrm{Id}_A\rangle$: each $\alpha \in \mathrm{Aut}(A)$ that normalizes $G(m, m, \kappa)$ permutes the subgroups $\langle a_i \rangle$ and thus has a monomial matrix whose nonzero entries differ pairwise by m-th roots of unity. Furthermore, since $\mathrm{Aut}_{O^{p'}(\mathcal{F})}(A)$ has index prime to p in $\mathrm{Aut}_{\mathcal{E}}(A)$, we have

$$\mathrm{Aut}_{\mathcal{E}}(A) \leq G(m, 1, \kappa)\langle \omega \cdot \mathrm{Id}_A \rangle, \tag{4.5}$$

for $\omega \in \mathbb{Z}$ whose class in $(\mathbb{Z}/p^\ell)^\times$ has order exactly $p - 1$.

By Dirichlet's theorem, we can choose a prime $q_0 \equiv \omega \pmod{p^\ell}$ such that $v_p(q_0^{p-1} - 1) = \ell$. Set $q^* = q_0^a$, where a is chosen so that $p \nmid a$ and $q^* \equiv q \pmod p$. Upon replacing q by q^* and $G = \mathbb{G}^{\pm}(q)$ by $\mathbb{G}^{\pm}(q^*)$, we can arrange that $q = q_0^a$ without changing the fusion system $\mathcal{F}$ (see [BMO1, Theorem A(a)]).

Let $G_0 \trianglelefteq G_1$ be as defined in Table 4.5, and set $\mathcal{F}_0 = \mathcal{F}_S(G_0)$. In the first two cases shown in the table, the isomorphism in the third column follows from Table 4.2, and $G_0 = G$ and $\mathcal{F}_0 = \mathcal{F}$. In the third case, $G \sim_p PGO^{\varepsilon}_{2n'}(q)$ by Lemma 4.3 (and Table 4.2), and we identify S as a subgroup of $G_0 = P\Omega^{\varepsilon}_{2n'}(q)$ in such a way that $\mathcal{F}_0 \leq \mathcal{F}$. Then $\mathrm{Aut}_{\mathcal{F}_0}(A) = G(m, 2, \kappa)$ and hence $\mathcal{F}_0$ has index 2 in $\mathcal{F}$. In all cases, ψ_{q_0} denotes the field automorphism $(x \mapsto x^{q_0})$, this acts on A via $(a \mapsto a^{q_0} = a^\omega)$, and so $\mathrm{Aut}_{G_1}(A) = G(m, 1, \kappa)\langle \omega \cdot \mathrm{Id} \rangle$.

m	$\mathrm{Aut}_{\mathcal{F}}(A)$	G	G_0	G_1
odd	$G(m, 1, \kappa)$	$G \cong PSL_n(q)$	$PSL_n(q)$	$PSL_n(q)\langle \psi_{q_0} \rangle$
even	$G(m, 2, \kappa)$	$G \cong P\Omega^{\varepsilon}_{2n'}(q)$	$P\Omega^{\varepsilon}_{2n'}(q)$	$PGO^{\varepsilon}_{2n'}(q)\langle \psi_{q_0} \rangle$
even	$G(m, 1, \kappa)$	$G \sim_p PGO^{\varepsilon}_{2n'}(q)$	$P\Omega^{\varepsilon}_{2n'}(q)$	$PGO^{\varepsilon}_{2n'}(q)\langle \psi_{q_0} \rangle$

TABLE 4.5. In the last two cases, $n' \geq p$ and $\varepsilon = \pm 1$ are such that $q^{n'} \equiv \varepsilon \pmod p$.

Thus in all three cases,

$$G(m, m, \kappa) = \mathrm{Aut}_{O^{p'}(\mathcal{F})}(A) \leq \mathrm{Aut}_{\mathcal{E}}(A) \leq G(m, 1, \kappa)\langle \omega \cdot \mathrm{Id} \rangle = \mathrm{Aut}_{G_1}(A), \tag{4.6}$$

where the second inequality follows from (4.5). Hence

$$\mathrm{Aut}_{\mathcal{E}}(A) \cap G(m, 1, \kappa) = G(m, r, \kappa) \qquad \text{for some } r \mid m. \tag{4.7}$$

If $r > 2$, then $\mathcal{E}$ is exotic by Lemma 4.7, and $\mathcal{E} \ngeq \mathcal{F}_0$ since $\mathrm{Aut}_{\mathcal{F}_0}(A) = G(m, \gcd(m, 2), \kappa)$.

Now assume $r \leq 2$, and hence that $\mathrm{Aut}_{\mathcal{E}}(A) \geq \mathrm{Aut}_{\mathcal{F}_0}(A)$. By (4.6), there is $H \leq G_1$ such that $G_0 \trianglelefteq H$ and $\mathrm{Aut}_H(A) = \mathrm{Aut}_{\mathcal{E}}(A)$. Since $O^{p'}(\mathcal{F}_S(H)) = O^{p'}(\mathcal{F}_0) = O^{p'}(\mathcal{E})$, we have $\mathcal{E} \cong \mathcal{F}_S(H)$ by Lemma 3.4.

We claim that this is, in fact, an equality (and hence that $\mathcal{E} \geq \mathcal{F}_0$). Set $\Gamma = \mathrm{Aut}_{O^{p'}(\mathcal{F})}(A) = G(m,m,\kappa)$ and $\Gamma_0 = O_{p'}(\Gamma) \cong (C_m)^{\kappa-1}$ for short. The five-term exact sequence for the cohomology of Γ as an extension of Γ_0 by Γ/Γ_0 (see [Be2, p. 110]) restricts to an exact sequence

$$0 \longrightarrow H^1(\Gamma/\Gamma_0; \underbrace{H^0(\Gamma_0; A)}_{=0}) \longrightarrow H^1(\Gamma; A) \longrightarrow H^0(\Gamma/\Gamma_0; \underbrace{H^1(\Gamma_0; A)}_{=0}) \tag{4.8}$$

in which the first term is zero since $H^0(\Gamma_0; A) = C_A(\Gamma_0) = 1$, and the last term is zero since $\gcd(|\Gamma_0|, |A|) = 1$. Thus $H^1(\Gamma; A) = 0$, and so $\mathcal{E} = \mathcal{F}_S(H)$ by Lemma 3.4 again. In particular, $\mathcal{E} \geq \mathcal{F}_0$ in this case, and this finishes the proof of (a.1).

Proof of (a.2): Now assume $\mathcal{E} \geq \mathcal{F}$ is an extension of index prime to p, and let α be an automorphism of $\mathcal{E}$. Thus $\alpha \in \mathrm{Aut}(S)$ is such that ${}^{\alpha}\mathcal{E} = \mathcal{E}$, and in particular, ${}^{\alpha}(O^{p'}(\mathcal{F})) = O^{p'}(\mathcal{F})$, $\alpha(A) = A$, and ${}^{\alpha}(\mathrm{Aut}_{\mathcal{E}}(A)) = \mathrm{Aut}_{\mathcal{E}}(A)$. Since by (4.5), $\mathrm{Aut}_{\mathcal{F}}(A)$ is generated by $\mathrm{Aut}_{O^{p'}(\mathcal{F})}(A)$ together with either the m-torsion in $O_{p'}(\mathrm{Aut}_{\mathcal{E}}(A))$ or the unique normal subgroup of index 2 in that subgroup, we also have ${}^{\alpha}(\mathrm{Aut}_{\mathcal{F}}(A)) = \mathrm{Aut}_{\mathcal{F}}(A)$. Since $H^1(\mathrm{Aut}_{O^{p'}(\mathcal{F})}(A); A) = 0$ by (4.8), we have ${}^{\alpha}\mathcal{F} = \mathcal{F}$ by Lemma 3.4. Since $\alpha \in \mathrm{Aut}(\mathcal{E})$ was arbitary, $\mathcal{F}$ is characteristic in $\mathcal{E}$. □

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Université Sorbonne Paris Nord, LAGA, UMR 7539 du CNRS, 99, Av. J.-B. Clément, 93430 Villetaneuse, France.

Email address: `bobol@math.univ-paris13.fr`

Departament de Matemàtiques, Universitat Autònoma de Barcelona, E–08193 Bellaterra, Spain

Email address: `albert@mat.uab.cat`