PHASE PORTRAITS OF QUADRATIC SYSTEMS WHICH IMPLY THE EXISTENCE OF A NILPOTENT OR INTRICATE INFINITE SINGULARITY

JOAN C. ARTÉS[1], JAUME LLIBRE[1], DANA SCHLOMIUK[2] AND NICOLAE VULPE[3]

Abstract. In this paper we study the phase portraits of all the families of real planar quadratic differential systems with a nilpotent or intricate singularity at infinity. This paper is part of a series of papers devised to produce the complete set of topologically distinct phase portrait of quadratic differential systems.

1. Introduction and statement of main results

We consider here differential systems of the form

$$\frac{dx}{dt} = p(x, y), \qquad \frac{dy}{dt} = q(x, y), \tag{1}$$

where p, $q \in \mathbb{R}[x, y]$, i.e. p, q are polynomials in x, y over $\mathbb{R}$. We call *degree* of a system (1) the integer $m = \max(\deg p, \deg q)$. In particular we call *quadratic* a differential system (1) with $m = 2$. We denote here by **QS** the whole class of real quadratic differential systems.

After the geometrical classifications of configurations of singularities given in [3] (there are 1764 due to a second recounting), after the topological classification of configurations of singularities given in [4] (there are 208), after the classification of the structurally stable quadratic systems modulo limit cycle [1] (there are 44), after the classification of the structurally unstable quadratic systems of codimension one modulo limit cycle [2] (there are 202 plus 7 conjectured empty) and after hundreds of papers classifying different classes of quadratic systems [12], we are close to a complete classification of topologically different phase portraits of quadratic systems modulo limit cycle.

In this paper we are going to advance one step more by doing a systematic study of all phase portraits of non-degenerate quadratic systems with a nilpotent or intricate infinite singularity, which is not topologically equivalent to an elemental or semi-elemental singularity. In the process we collect information from the families already studied about some quadratic systems with such kind of singularities.

Most of the phase portraits we will find, have appeared previously in several other papers, but the most relevant are the articles [7, 10, 11, 13]. In these papers it turns out difficult to grasp how many phase portraits are obtained and how many of them are topologically different. Moreover there are some missed phase portraits as well as repeated ones. The authors of these papers not always give a clear label to each phase portrait. With our classification all these problems are solved.

Our main result is:

Theorem 1. *A quadratic system with a infinite nilpotent or intricate singularities (which is not topologically equivalent to an elemental or semi-elemental singularity) at least 155 topologically distinct phase portraits without limit cycle. From these phase portraits 18 of them appear also with a limit cycle and all of them are presented in the list below. The phase portraits corresponding to these labels will appear along the paper.*

- $QS77_1^{(2)}$, $QS77_2^{(2)}$, $QS77_{2(1)LC}^{(2)}$, $QS77_3^{(2)}$,$QS77_{3(1)LC}^{(2)}$, $QS77_1^{(3)}$, $QS77_2^{(3)}$, $QS77_3^{(3)}$, $QS77_4^{(3)}$, $QS77_5^{(3)}$;
- $QS78_1(Vul_6)$, $QS78_5(Vul_5)$;

2020 *Mathematics Subject Classification.* Primary 58K45, 34C05, 34A34.

Key words and phrases. quadratic vector fields, infinite and finite singularities, affine invariant polynomials, Poincaré compactification, configuration of singularities, topological equivalence relation.

- $QS79_1^{(2)}$, $QS79_{1(1)LC}^{(2)}$, $QS79_2^{(2)}$, $QS79_3^{(2)}$, $QS79_1^{(3)}$, $QS79_2^{(3)}$, $QS79_3^{(3)}$;
- $QS80_1^{(2)}$, $QS80_{1(1)LC}^{(2)}$, $QS80_2^{(2)}$, $QS80_3^{(2)}$, $QS80_4^{(2)}$, $QS80_1^{(3)}$, $QS80_2^{(3)}$;
- $QS81_1^{(2)}$, $QS81_{1(1)LC}^{(2)}$, $QS81_2^{(2)}$, $QS81_3^{(2)}$, $QS81_1^{(3)}$, $QS81_2^{(3)}$;
- $QS82_1^{(3)}$, $QS82_2^{(3)}$, $QS82_3^{(3)}$, $QS82_{3(1)LC}^{(3)}$, $QS82_1^{(4)}$, $QS82_2^{(4)}$;
- $QS88_1^{(2)}$;
- $QS89_1^{(2)}$, $QS89_{1(1)LC}^{(2)}$, $QS89_1^{(3)}$;
- $QS90_1^{(4)}$;
- $QS91_1^{(2)}$, $QS91_{1(1)LC}^{(2)}$, $QS91_2^{(2)}$, $QS91_{2(1)LC}^{(2)}$, $QS91_3^{(2)}$, $QS91_1^{(3)}$, $QS91_2^{(3)}$, $QS91_{2(1)LC}^{(3)}$, $QS91_3^{(3)}$, $QS91_1^{(4)}$;
- $QS92_1(Vul_{14})$;
- $QS93_1^{(2)}$, $QS93_{1(1)LC}^{(2)}$, $QS93_1^{(3)}$;
- $QS94_1(Vul_{13})$;
- $QS95_1^{(3)}$, $QS95_{1(1)LC}^{(3)}$, $QS95_2^{(3)}$, $QS95_1^{(4)}$;
- $QS106_1^{(3)}$, $QS106_2^{(3)}$, $QS106_3^{(3)}$, $QS106_4^{(3)}$, $QS106_1^{(4)}$, $QS106_2^{(4)}$, $QS106_3^{(4)}$, $QS106_4^{(4)}$, $QS106_5^{(4)}$;
- $QS107_1^{(4)}$;
- $QS108_1^{(3)}$, $QS108_2^{(3)}$, $QS108_3^{(3)}$, $QS108_4^{(3)}$, $QS108_5^{(3)}$, $QS108_{5(1)LC}^{(3)}$;
- $QS109_1^{(3)}$, $QS109_2^{(3)}$, $QS109_3^{(3)}$, $QS109_4^{(3)}$, $QS109_5^{(3)}$, $QS109_6^{(3)}$, $QS109_7^{(3)}$, $QS109_{7(1)LC}^{(3)}$, $QS109_1^{(4)}$, $QS109_2^{(4)}$, $QS109_3^{(4)}$, $QS109_4^{(4)}$; $QS109_5^{(4)}$;
- $QS110_1^{(3)}$, $QS110_2^{(3)}$, $QS110_3^{(3)}$, $QS110_4^{(3)}$, $QS110_5^{(3)}$, $QS110_{5(1)LC}^{(3)}$, $QS110_1^{(4)}$, $QS110_2^{(4)}$, $QS110_3^{(4)}$;
- $QS111_1^{(4)}$, $QS111_{1(1)LC}^{(4)}$, $QS111_2^{(4)}$, $QS111_3^{(4)}$, $QS111_4^{(4)}$, $QS111_5^{(4)}$, $QS111_6^{(4)}$, $QS111_1^{(5)}$, $QS111_2^{(5)}$, $QS111_3^{(5)}$;
- $QS113_1^{(4)}$;
- $QS114_1^{(4)}$;
- $QS115_1^{(4)}$;
- $QS116_1^{(5)}$;
- $QS119_1^{(5)}$, $QS119_1^{(6)}$;
- $QS120_1^{(5)}$;
- $QS121_1^{(5)}$;
- $QS123_1^{(3)}$, $QS123_1^{(4)}$;
- $QS124_1^{(3)}$;
- $QS125_1^{(3)}$;
- $QS126_1^{(3)}$;
- $QS127_1^{(4)}$;
- $QS129_1^{(3)}$, $QS129_1^{(4)}$;
- $QS130_1^{(4)}$;
- $QS135_1^{(3)}$, $QS135_{1(1)LC}^{(3)}$, $QS135_2^{(3)}$, $QS135_3^{(3)}$, $QS135_1^{(4)}$, $QS135_2^{(4)}$;
- $QS136_1^{(3)}$, $QS136_2^{(3)}$, $QS136_3^{(3)}$, $QS136_{3(1)LC}^{(3)}$, $QS136_1^{(4)}$, $QS136_2^{(4)}$;
- $QS137_1^{(3)}$;
- $QS138_1^{(3)}$, $QS138_1^{(4)}$;
- $QS142_1^{(3)}$;
- $QS143_1^{(3)}$;
- $QS144_1^{(4)}$;
- $QS145_1^{(4)}$, $QS145_1^{(5)}$;
- $QS146_1^{(5)}$;
- $QS152_1^{(4)}$, $QS152_2^{(4)}$, $QS152_3^{(4)}$, $QS152_1^{(5)}$;
- $QS153_1^{(5)}$;
- $QS154_1^{(4)}$, $QS154_2^{(4)}$, $QS154_3^{(4)}$, $QS154_1^{(5)}$;

- $QS155_1^{(5)}$;
- $QS156_1^{(4)}$;
- $QS157_1^{(4)}$;
- $QS158_1^{(5)}$;
- $QS160_1^{(4)}$, $QS160_1^{(5)}$;
- $QS162_1^{(5)}$;
- $QS163_1^{(5)}$;
- $QS164_1^{(5)}$;
- $QS165_1^{(5)}$;
- $QS166_1^{(5)}$;
- $QS167_1^{(6)}$;
- $QS168_1^{(6)}$;
- $QS169_1^{(6)}$;
- $QS197_1^{(6)}$.

2. Preliminary

In book [2] the authors already started the study of phase portraits having a single connection of separatrices (without considering limit cycles) and stated that there are five types:

(a) Heteroclinic between two finite singularities;
(b) Homoclinic (involves two separatrices of the same singularity);
(c) Heteroclinic between a finite and an infinite singularities;
(d) Heteroclinic between an infinite singularity and its opposite;
(e) Heteroclinic between an infinite singularity and another infinite singularity different from its opposite, in fact an adjacent one since if there is another singularity between them, the stabilities of the finite separatrices of the infinite singularities would be the same.

Even this description was done for polynomial differential systems of codimension equal to one, it can also be extended with some modifications to polynomial systems with codimension greater than one. The main change is that now an infinite singularity could make a connection with itself. Since this is homoclinic we will call it $(b)_\infty$ connection.

We start from the topological distinct local configurations of infinite singularities given in [3] (see also [14]) altogether with topological classification of configurations of singularities given in [4].

In the monograph [3] there are 46 different configurations at infinity, among which 6 have the infinite filled up with singularities and hence at infinity they do not have isolated singularities. From the other 40 it easily could be detected that 27 of them have one nilpotent or intricate infinite singularity which is topologically non-equivalent to a node or a saddle or a semi-elemental saddle-node. Moreover in [3] (see Figure 6.2) there are presented the 31 topologically distinct local configurations of infinite singular points for degenerate quadratic systems. Among them there exist only one possessing an isolate nilpotent singular point presented there by QD_{18}^{∞}. We will describe the singularities at infinity using the notation given in [4, Section 2].

Some of these configurations of infinite singularities force exactly the number of finite singularities that a system may have, whereas other configurations have several possibilities. In order to determine the exact number of finite singularities we will use [4]. All this information is summarized in Table 1.

In this table we indicate in the first column the codes of these 28 configurations of the infinite singular points from [3]. In the second column we give the topological description of the corresponding infinite singularities. In the remaining eight columns we give the code (numeration given in the diagrams from the article [4]) of the global topological configurations of singularities (finite and infinite) which corresponds to each infinite

Table 1

#	Finite multiplicity / Config. of infin. SPs \ # of real FSPs	$m_f = 3$			$m_f = 2$			$m_f=1$	$m_f=0$
		3	1	2	2	0	1	1	0
8	$H-H,\ N$					(137)			
9	$E-H,\ N$	(77)	(88)	(106) (107)					
10	$E-E,\ N$				(123)				
11	$E-H, \binom{1}{1}SN$				(136)	(143)	(154) (155)		
12	$E-E,\ \binom{1}{1}SN$							(160)	
13	$PH-H, N$				(135)	(142)	(152) (153)		
14	$E-PH,\ N$							(156)	
15	$E-PH,\ \binom{1}{1}SN$								(166)
16	$E-H,\ S$	(79)	(89) (119)	(108) (113)					
17	$E-E,\ S$				(129)	(138)	(145)		
19	$PH-PH,\ N$				(124)		(144)		
21	$PH–PH, \binom{1}{1}SN$							(161)	
23	$HHH-H,\ N$	(81)	(93) (121)	(110) (115)					
24	$HHH-E,\ N$								(165)
25	$HHH-HP,\ N$							(157)	
26	$PHP-E,\ S$	(80)	(91) (120)	(109) (114)					
27	$HHH–HHH, N$				(126)				
28	$PHP-PHP,\ N$								(164)
29	$PHP-PHP,\ S$				(125)				
32	$EH-P$				(130)		(146)		(169)
33	$EE-P$							(162)	
35	$EE-HH$								(168)
36	$EH-HE$								(167)
37	$HHP-P$	(82)	(95)	(111) (116)					
38	$HPH-P$								
39	$HE-PHH$							(158)	
40	$HHP-PHH$				(127)				
QD^{∞}_{18}	$\binom{1}{2}E-H, (\ominus\ [\vert];\emptyset)$								(197)

configuration. This information is split in several columns according to the total multiplicity m_f of finite singularities (from zero to 3) and also the number of real finite singularities. We have omitted by purpose the cases with a finite center, even if some configurations could have infinite nilpotent or intricate singularities. The reason is that systems with centers are already completely studied.

In paper [5] the authors consolidate the notion of codimension related to polynomial differential systems. Even this concept has been widely used before, in paper [5], the concept gathers a bigger relevance when it may expand beyond the low codimension degrees to bigger ones, and whichever the equivalence relation that it is used. The relation between multiplicity of singularities and their codimension is also presented.

We must make now a recall about notation. Along more than 100 years, mathematicians have classified quadratic phase portraits and have given them all kind of names. Most times, a same phase portrait appears in different papers having different names. It is maybe a good moment now to establish a systematic way to name all of them. And we propose the next:

We have 208 different topological configurations of singularities. This implies that two phase portraits having different configurations of singularities, cannot be topologically equivalent. So the 208 configurations of singularities provides a nice skeleton on which we can classify the phase portraits. Many of the topological configurations will have just one phase portrait, some may have several realizable phase portraits, and a few configurations may have some dozens of phase portraits. We already have the configuration (39): $s, a, sn; \binom{0}{2} SN, N$ with 99 confirmed distinct phase portraits.

Notation 1. *We propose to call each phase portrait as $QSr_a^{(b)}$ where QS stands for "quadratic differential system", 'r' is the number of the configuration of singularities from [4], 'b' is the topological codimension of the phase portrait and 'a' is simply a cardinal to enumerate the different phase portraits which have the same configuration and codimension. In case the phase portrait has limit cycles we will add to the sub-index a vector like $QS3^{(0)}_{1(i,j,k)LC}$, where i, j and k denote the number of limit cycles that a phase portrait may have around anti-saddles. We are aware that a quadratic system can have at most two foci (i.e. at most two nests of limit cycles). But configuration like $QS3$ with three finite anti-saddles may have limit cycle around any of them or even two of them. So we need a vector with three digits to describe all the possibilities.*

The use of the codimension for the notation allows us to reduce the maximum value that may appear as cardinal, but more important, it helps us to describe the different phase portraits and to detect which ones can (or cannot) bifurcate from others. That is, it helps us to locate the "neighbors" of the phase portraits.

This notation was already introduced for the first time in [5] but there only few examples of its use were given. In this paper we will use it widely.

In paper [5] the authors also assigned the topological codimension to each one of the 208 topological configurations of singularities from [4]. Thus, the topological codimension of the phase portraits that we can obtain from each configuration of singularities will have the same codimension or greater if the phase portrait shows one or more separatrix connections, which are not already forced by the configuration of singularities.

In Section 3 we use some canonical forms of the quadratic systems, provided by the following lemma (we keep the notations from [15] and from Table 6.1 in [3]):

Lemma 1 ([15], [3]). *Assume that a quadratic system possesses finite singular points of total multiplicity $m_f \leq 2$. Then via an affine transformation this system could be brought to one of the canonical forms, correspondingly with the number of finite singularities.*

1) In the case of two real distinct finite singularities ($m_f = 2$):

$$14a) \quad \begin{cases} \dot{x} = & cx + dy - cx^2 + 2duxy, \\ \dot{y} = & ex + fy - ex^2 + 2fuxy, \quad (cf - de)(2u + 1)u \neq 0; \end{cases}$$

$$14b) \quad \begin{cases} \dot{x} = & -(g + ku^2)x - 2hy + gx^2 + 2hxy + ky^2, \\ \dot{y} = & ux + y, \quad g - 2hu + ku^2 \neq 0. \end{cases}$$

2) In the case of two complex distinct finite singularities ($m_f = 2$):

$$15a)\quad \begin{cases} \dot{x} = & a + hux + 2hxy + ay^2, \\ \dot{y} = & b + mux + 2mxy + by^2, \quad am - bh \neq 0; \end{cases}$$

$$15b)\quad \begin{cases} \dot{x} = & a + cx + gx^2 + 2hxy + ay^2, \\ \dot{y} = & x, \quad a \neq 0. \end{cases}$$

3) In the case of one double real singularity ($m_f = 2$):

$$16a)\quad \begin{cases} \dot{x} = & dy + gx^2 + 2dxy, \\ \dot{y} = & fy + lx^2 + 2fxy, \quad fg - dl \neq 0; \end{cases}$$

$$16b)\quad \begin{cases} \dot{x} = & cx + dy, \quad c^2n - 2cdm + dl^2 \neq 0, \\ \dot{y} = & lx^2 + 2mxy + ny^2. \end{cases}$$

3) In the case of one real singularity ($m_f = 1$):

$$17a)\quad \begin{cases} \dot{x} = & cx + dy + (2c + d)x^2 + 2dxy, \\ \dot{y} = & ex + fy + (2e + f)x^2 + 2fxy, \quad cf - de \neq 0; \end{cases}$$

$$17b)\quad \begin{cases} \dot{x} = & x + dy, \quad (de - f)(l^2 + m^2) \neq 0, \\ \dot{y} = & ex + fy + lx^2 + 2mxy - d(ld - 2m)y^2. \end{cases}$$

4) In the case of the non-existence of finite singularities ($m_f = 0$):

$$18a)\quad \begin{cases} \dot{x} = & h + gx^2 + 2hxy, \\ \dot{y} = & m + lx^2 + 2mxy, \quad hl - gm \neq 0; \end{cases}$$

$$18b)\quad \begin{cases} \dot{x} = & y, \quad l^2 + m^2 \neq 0 \\ \dot{y} = & 1 + fy + 2mxy + ny^2; \end{cases}$$

$$18c)\quad \begin{cases} \dot{x} = & x, \quad l^2 + m^2 \neq 0, \\ \dot{y} = & 1 + ex + lx^2 + 2mxy; \end{cases}$$

$$18d)\quad \begin{cases} \dot{x} = & 1, \quad l^2 + m^2 + n^2 \neq 0, \\ \dot{y} = & ex + fy + lx^2 + 2mxy + ny^2. \end{cases}$$

Moreover the invariant polynomial $\widetilde{K}$ defined in [3] is not zero for the systems 14a), 15a), 16a), 17a) *and* 18a) *and it vanishes for other normal forms.*

Following Proposition 5.10 of [1] we present a similar one adapted to the type of singularities for all quadratic systems.

Proposition 1. *In a quadratic system with at least two pairs of infinite singular points, one of them having parabolic sectors at both sides at infinity, a finite singularity having at least three separatrices cannot send two of them with the same stability to one infinite singularity and the third separatrix to the opposite infinite singularity.*

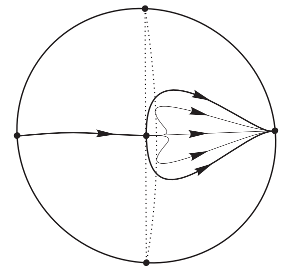
Proof: The proof Proposition 5.10 of [1] was done providing that a finite singularity was an elemental saddle and now we enlarge the possibility to have a semi-elemental or nilpotent saddle or saddle-node.

A finite nilpotent saddle or saddle-node in a quadratic system implies the existence of an invariant straight line passing through the point. It is not too hard to check that in these case there appear also to many contact points with a straight line passing through the second infinite singularity.

In the case of a semi-elemental saddle the proof given in [1] is still valid. Assume that we have a semi-elemental saddle-node which sends two separatrices with the same stability to the same infinite singularities. Then the straight line $\mathcal{L}$ that connects the second infinite singularity with the finite saddle node may either cross one of the mentioned separatrices or not. In the first case the proof follows the same pattern as in [1].

In the second case the separatrices of the saddle-node that go to the same infinite singularity must arrive to the saddle-node must be tangent to the line $\mathcal{L}$ (because they could not cross the line).

On the other hand the orbits of the parabolic sector of the saddle-node must arrive transversal to the line $\mathcal{L}$ and this produces contact points with a parallel line to $\mathcal{L}$ closed to the saddle-node (see the presented picture).



Remark 1. *We have detected a minor error in the paper [5]. More precisely in **Diagram 3** on page 11 in the branch*

$$\mu_0 = \mu_1 = 0,\ \mu_2 \neq 0,\ \mathbf{U} > 0,\ \kappa = 0,\ \widetilde{K} = 0,\ \eta = 0, \widetilde{M} \neq 0,\ \tilde{L} \neq 0$$

the given configuration (135): $s, a; \binom{2}{2} PH - H, N$ occurs only when $\kappa_1 \neq 0$. In the case $\kappa_1 = 0$ the corresponding configuration must be $s, a; \binom{2}{2} PH - PH, N$ but this is not a new topological configuration because it coincides with configuration (124).

3. Results

We now examine one by one each one of the 28 configurations of infinite singularities given in Table 1. Due to the technique used for the study of different configurations of singularities we will order them according to their complexity. The configuration QD_{18}^{∞} will appear as a border case inside infinite configuration 11 (more exactly in topological configuration (154)).

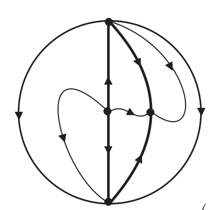
3.1. **The configuration 40:** $HHP - PHH$**.** According to [4] this configuration of infinite singularities leads to the unique global topological configuration of singularities: (127) $a, a;\ HHP - PHH$. We recall that from [5], configuration (127) has topological codimension 4. Moreover a quadratic system possesses the configuration (127) if and only if the following conditions hold:

$$\mu_0 = \mu_1 = 0,\ \mu_2 < 0,\ \mathbf{U} > 0,\ \kappa = 0,\ \widetilde{K} > 0, \widetilde{M} = 0,\ C_2 \neq 0. \tag{2}$$

Since $\widetilde{K} \neq 0$, according to Lemma 1 a system with two distinct finite singularities belongs to the canonical form 14a). For these systems the above conditions imply $d = 0$, $c = -2fu \neq 0$, $2u + 1 < 0$ and $e \neq 0$. Moreover we may assume $f = 1 = e$ due to the rescaling $(x, y, t) \mapsto (x, ey/f, t/f)$. Therefore we get the family of systems

$$\dot{x} = 2u(x-1)x, \quad \dot{y} = x + y - x^2 + 2uxy, \quad u < -1/2. \tag{3}$$

The systems (3) have the invariant straight lines $x = 0$ and $x = 1$ and the nodes $M_1(0,0)$ and $M_2(1,0)$. This leads to the unique phase portrait given by $QS127_1^{(4)}$.



$QS127_1^{(4)}$

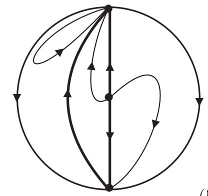
3.2. **The configuration 39:** $HE - PHH$**.** According to [4] this configuration of infinite singularities leads to the unique global topological configuration of singularities: (158) $a;\ HE - PHH$. We recall that from [5], configuration (158) has topological codimension 5. Moreover a quadratic system possesses the configuration (158) if and only if the following conditions hold:

$$\mu_0 = \mu_1 = \mu_2 = 0,\ \mu_3 \neq 0,\ \kappa = 0,\ \widetilde{K} > 0, \tilde{L} = 0,\ C_2 \neq 0,\ K_3 < 0. \tag{4}$$

Since $\widetilde{K} \neq 0$, according to Lemma 1 a system with a unique finite singularity belongs to the canonical form 17a). For these systems the above conditions imply $d = 0$, $cf > 0$, $f - c = 0$ and $c(c + 2e) > 0$. Moreover we may assume $c = 1$ due to a time rescaling and we get the family of systems

$$\dot{x} = x(1 + 2x), \quad \dot{y} = ex + y + (2e+1)x^2 + 2xy, \quad 1 + 2e > 0. \tag{5}$$

The systems (5) have the invariant straight lines $x = 0$ and $x = -1/2$ and the node $M_1(0,0)$. This leads to the unique phase portrait given by $QS158_1^{(5)}$.



$QS158_1^{(5)}$

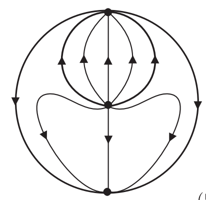
3.3. **The configuration 38:** $HPH - P$**.** According to [4] this configuration of infinite singularities leads to the unique global topological configuration of singularities: (163) a; $HPH - P$. We recall that from [5], configuration (163) has topological codimension 5. Moreover a quadratic system possesses the configuration (163) if and only if the following conditions hold:

$$\mu_0 = \mu_1 = \mu_2 = 0,\ \mu_3 \neq 0,\ \kappa = \widetilde{K} = \eta = \widetilde{M} = 0,\ \mu_3 K_1 > 0,\ K_3 < 0.$$

Since $\widetilde{K} = 0$, according to Lemma 1 a system with a unique finite singularity belongs to the canonical form 17b). For these systems the above conditions imply $d = m = 0$, $l \neq 0$, $f > 0$ and $f(f-2) > 0$. Moreover we may assume $l = 1$ due to the rescaling $y \to ly$ and we get the family of systems

$$\dot{x} = x, \quad \dot{y} = ex + fy + x^2, \quad f > 2.$$

These systems possess one invariant straight line $x = 0$ and the node $M_1(0,0)$. This leads to the unique phase portrait given by $QS163_1^{(5)}$.



$QS163_1^{(5)}$

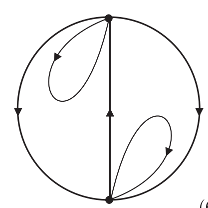
3.4. **The configuration 36:** $EH - HE$**.** According to [4] this configuration of infinite singularities leads to the unique global topological configuration of singularities: (167) $EH - HE$. We recall that from [5], configuration (167) has topological codimension 6. Moreover a quadratic system possesses the configuration (167) if and only if the following conditions hold:

$$\mu_0 = \mu_1 = \mu_2 = \mu_3 = \eta = \widetilde{M} = 0,\ \mu_4 \neq 0,\ C_2 \neq 0,\ \widetilde{K} \neq 0,\ K_3 < 0.$$

Since $\widetilde{K} \neq 0$, according to Lemma 1 a system with no finite singularities belongs to the canonical form 18a). For these systems the above conditions imply $h = 0$, $g = 2m$ and $lm < 0$. Moreover we may assume $l = 1$ due to the rescaling $y \to ly$ and we get the family of systems

$$\dot{x} = 2mx^2, \quad \dot{y} = m + x^2 + 2mxy, \quad m < 0.$$

These systems possess one invariant straight line $x = 0$ which is of multiplicity at least two and one infinite sinularity of total multiplicity 7. This leads to the unique phase portrait given by $QS167_1^{(6)}$.



$QS167_1^{(6)}$

3.5. **The configuration 35:** $EE - HH$**.** According to [4] this configuration of infinite singularities leads to the unique global topological configuration of singularities: (168) $EE - HH$. We recall that from [5], configuration (168) has topological codimension 6. Moreover a quadratic system possesses the configuration (168) if and only if the following conditions hold:

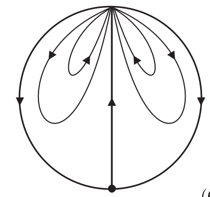
$$(6) \qquad \mu_0 = \mu_1 = \mu_2 = \mu_3 = \eta = \widetilde{M} = 0,\ C_2 \neq 0,\ \widetilde{K} = 0,\ K_1 \neq 0,\ \mu_4 < 0.$$

Since $\widetilde{K} = 0$, according to Lemma 1 a system with no finite singularities belongs to the canonical forms 18b) - 18d). However in this case the conditions (6) could be satisfied only for the canonical systems 18c) and these conditions imply $m = 0$ and $l < 0$.

So we consider the normal form

$$\dot{x} = x, \quad \dot{y} = 1 + ex + lx^2, \quad l < 0.$$

which corresponds exactly to the global configuration (168).
These systems possess one invariant straight line $x = 0$ and this leads to the unique phase portrait given by $QS168_1^{(6)}$.



$QS168_1^{(6)}$

3.6. **The configuration 33:** $EE-P$**.** According to [4] this configuration of infinite singularities leads to the unique global topological configuration of singularities: (162) s; $EE-P$. We recall that from [5], configuration (162) has topological codimension 5. Moreover a quadratic system possesses the configuration (162) if and only if the following conditions hold:

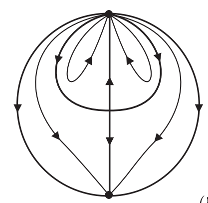
$$\mu_0 = \mu_1 = \mu_2 = \kappa = \widetilde{K} = \eta = \widetilde{M} = 0, \ \mu_3 K_1 < 0.$$

We observe that these conditions lead to systems with $m_f = 1$ and the condition $\widetilde{K} = 0$ holds. Therefore according to Lemma 1 we have to consider the canonical forms 17b). The above conditions imply $d = m = 0$, $l \neq 0$ and $f < 0$.

As a result we arrive at the the normal form

$$\dot{x} = x, \quad \dot{y} = ex + fy + lx^2, \quad f < 0.$$

which corresponds exactly to the global configuration (162). These systems possess one invariant straight line $x = 0$ and the saddle $M_1(0, 0)$. As a result we get the unique phase portrait given by $QS162_1^{(5)}$.



$QS162_1^{(5)}$

3.7. **The configuration 29:** $PHP - PHP, S$**.** According to [4] this configuration of infinite singularities leads to the unique global topological configuration of singularities: (125) a, a; $PHP - PHP, S$. We recall that from [5], configuration (125) has topological codimension 3. Moreover a quadratic system possesses the configuration (125) if and only if the following conditions hold:

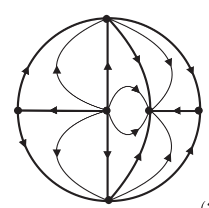
$$\mu_0 = \mu_1 = \kappa = 0, \ \mathbf{U} > 0, \ \widetilde{K} > 0, \ \tilde{L} < 0, \ \widetilde{M} \neq 0, \ \mu_2 < 0. \tag{7}$$

Then according to [3] the finite anti-saddles must be nodes. Since the configuration (125) has two real distinct singularities and $\widetilde{K} \neq 0$, by Lemma 1 we have to consider the systems 14a). So the above conditions imply $d = 0$, $cfu < 0$, $1 + 2u < 0$ and $c(c + 2fu) < 0$. Moreover we may assume $f = 1$ due to a time rescaling and we get the family of systems

$$\dot{x} = cx(1 - x), \quad \dot{y} = ex + y - ex^2 + 2uxy, \tag{8}$$

with $u < -1/2$ and $0 < c < -2u$. We observe that the above systems have the invariant straight lines $x = 0$ and $x = 1$ and the nodes $M_1(0, 0)$ and $M_2(1, 0)$.

We observe that the above systems have the invariant straight lines $x = 0$ and $x = 1$ and the nodes $M_1(0, 0)$ and $M_2(1, 0)$. It is not too difficult to detect, that we could not have neither an (e) connection nor a (d) connection. And since the nodes are located on the invariant lines this leads to the unique phase portrait given by $QS125_1^{(3)}$.



$QS125_1^{(3)}$

3.8. **The configuration 28:** $PHP - PHP, N$**.** According to [4] this configuration of infinite singularities leads to the unique global topological configuration of singularities (164): $PHP - PHP, N$, which belongs to the class $m_f = 0$, i.e. we have no finite singular points. We recall that from [5], configuration (164) has topological codimension 5.

According to [4] a quadratic system possesses the configuration (164) if and only if the following conditions hold:

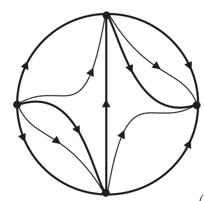
$$\mu_0 = \mu_1 = \mu_2 = \mu_3 = 0,\ \mu_4 \neq 0,\ \eta = 0,\ \widetilde{M} \neq 0,\ \kappa = 0,\ \widetilde{K} < 0,\ \widetilde{R} < 0. \tag{9}$$

Since $m_f = 0$ and $\widetilde{K} \neq 0$, by Lemma 1 we have to consider the systems 18a). We determine that the conditions (9) imply $h = 0$, $gm < 0$, and $g(g + 2m) < 0$. Moreover we may assume $g = 1$ due to a time rescaling and we get the 2-parameter family of systems

$$\dot{x} = x^2, \quad \dot{y} = m + lx^2 + 2mxy, \quad m < -1/2.$$

We observe that these systems possess the invariant line $x = 0$ (which is at least double).

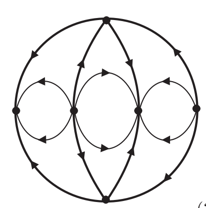
It is not too hard to detect that this invariant line forms a (d) connection of separatrices, i.e. the conditions (9) imply the existence of a separatrix connection for a quadratic system. Therefore this connection will not increase the codimension of the global phase portrait. Thus phase portrait $QS164_1^{(5)}$ has the codimension 5 given directly by its configuration of singularities. This is the first case we meet here in which the configuration of singularities enforces the existence of a separatrix connection. That is, if by means of a perturbation we try to break the separatrix connection, we would change also the intricate singularity.



$QS164_1^{(5)}$

3.9. **The configuration 27:** $HHH - HHH, N$**.** According to [4] this configuration of infinite singularities leads to the unique global topological configuration of singularities: (126) a, a; $HHH - HHH$, N. We recall that from [5], configuration (126) has topological codimension 3. Moreover for this configuration, the same conditions (7) are satisfied except the condition $\tilde{L} < 0$. Instead of it, in order to have at infinity exactly the configuration of singularities (126) the condition $\tilde{L} > 0$ is necessary and then we arrive at the canonical form (8) for which the conditions $u < -1/2$ and $c > -2u$ are fulfilled.

The systems (8) have the invariant straight lines $x = 0$ and $x = 1$ and the nodes $M_1(0, 0)$ and $M_2(1, 0)$, which are located on these lines. Since at infinity we have an intricate singular point possessing only hyperbolic sectors it is clear that the invariant lines passing through this intricate point must be the separatrices of the corresponding sectors. This leads to the unique phase portrait given by $QS126_1^{(3)}$.



$QS126_1^{(3)}$

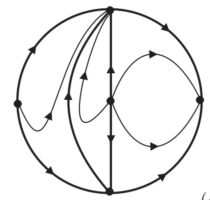
3.10. **The configuration 25:** $HHH - HP, N$**.** According to [4] this configuration of infinite singularities leads to the unique global topological configuration of singularities: (157) a; $HHH - HP$, N. We recall that from [5], configuration (157) has topological codimension 4. Moreover a quadratic system possesses the configuration (157) if and only if the following conditions hold:

$$\mu_0 = \mu_1 = \mu_2 = \kappa = 0,\ \widetilde{K} > 0,\ \tilde{L} > 0,\ \mu_3 \neq 0. \tag{10}$$

Since the above conditions lead to systems with $m_f = 1$ and $\widetilde{K} \neq 0$, according to Lemma 1 we have to consider the canonical form 17a). The conditions (10) imply for these systems $d = 0$, $cf > 0$ and $c(c - f) > 0$. We may assume $f = 1$ (due to a time rescaling) and we arrive at the family of systems:

$$\dot{x} = cx(1 + 2x), \quad \dot{y} = ex + y + (2e + 1)x^2 + 2xy, \quad c > 1. \tag{11}$$

We observe that these systems possess two invariant lines $x = 0$ and $x = -1/2$ and the finite anti-saddle $M_1(0,0)$ is located on the invariant line $x = 0$ (and clearly it is a node). Since the invariant lines pass through the infinite intricate singularity which on one part possess only three hyperbolic sectors we conclude that these invariant lines must serve as separatrices for these sectors. So it is not so difficult to detect that we arrive at the unique phase portrait given by $QS157_1^{(4)}$.



$QS157_1^{(4)}$

3.11. **The configuration 24:** $HHH-E,\, N$**.** According to [4] this configuration of infinite singularities leads to the unique global topological configuration of singularities (165), which belongs to the class $m_f = 0$, i.e. we have not finite singular points. We recall that from [5], configuration (165) has topological codimension 5.

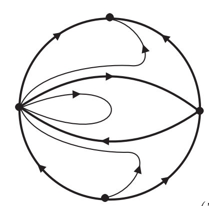
According to [4] a quadratic system possesses the configuration (165) if and only if the following conditions hold:

$$\mu_0 = \mu_1 = \mu_2 = \mu_3 = 0,\ \eta = 0,\ \widetilde{M} \neq 0,\ \kappa = 0,\ \widetilde{K} = 0,\ \tilde{L} \neq 0,\ \kappa_1 = 0,\ K_2 > 0,\ \mu_4 < 0. \tag{12}$$

Since $m_f = 0$ and $\widetilde{K} = 0$ by Lemma 1 we have to consider one of the systems 18b) - 18d). It could be checked directly that the conditions (12) could be satisfied only for the canonical systems 18b) and these conditions imply $m = 0$, $n < 0$ and $f^2 - 4n > 0$. So we get the family of systems

$$\dot{x} = y, \quad \dot{y} = 1 + fy + ny^2, \quad n < 0.$$

Since $f^2 - 4n > 0$ we deduce that the invariant lines $1 + fy + ny^2 = 0$ are real and these lines pass through the infinite intricate singularity possessing on one side of the line $Z = 0$ three hyperbolic sectors. Consequently these lines must be the separatrices for the hyperbolic sectors and, since we have not finite singularities, it is clear that in the region delimited by the invariant lines we must have at infinity an elliptic sector. As a result we arrive at the unique phase portrait given by $QS165_1^{(5)}$.



$QS165_1^{(5)}$

3.12. **The configuration 21:** $PH - PH,\, \binom{1}{1}SN$**.** According to [4] this configuration of infinite singularities leads to the unique global topological configuration of singularities: (161) a; $PH-PH,\ \binom{1}{1}SN$. We recall that from [5], configuration (161) has topological codimension 4, and a quadratic system possesses the configuration (161) if and only if the following conditions hold:

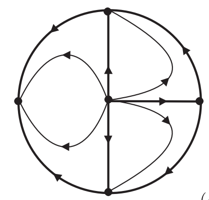
$$\mu_0 = \mu_1 = \mu_2 = 0,\ \eta = \kappa = 0,\ \widetilde{M} \neq 0,\ \widetilde{K} = \tilde{L} = \kappa_1 = 0,\ \mu_3 K_1 > 0. \tag{13}$$

Since the above conditions lead to systems with $m_f = 1$ and $\widetilde{K} = 0$, according to Lemma 1 we have to consider the canonical form 17b). Then the conditions (13) imply $d = 0$, $f > 0$ and $m \neq 0$. We may assume $m = 1$ due to the rescaling $x \to x/m$ and we arrive at the family of systems:

$$\dot{x} = x, \quad \dot{y} = ex + fy + lx^2 + 2xy, \quad f > 0. \tag{14}$$

Since the anti-saddle $M_1(0,0)$ is located on the invariant line $x = 0$ of these systems it is clear that M_1 is a node. It is not too difficult to detect that we could not have a (d) connection formed by the separatrices of the infinite intricate point because both separatrices of the intricate singularity have the same stability. We can neither have a (e) connection with the separatrix of the infinite saddle-node, because this produces a graphic which must contain a focus (or a center) inside.

Moreover the separatrices of the intricate singularity can not have the opposite stability with respect to the separatrix of the saddle-node otherwise we obtain a contradiction with the existence of the invariant line $x = 0$. Then all 3 separatrices have the same stability and therefore they must have the finite singularity as a limit point. It is easy to prove that in fact the separatrices of the intricate point are parts of the invariant line. As a result we arrive at the unique phase portrait given by $QS161_1^{(4)}$.



$QS161_1^{(4)}$

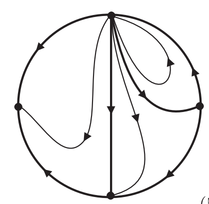
3.13. **The configuration 15:** $E - PH$, $\binom{1}{1}SN$**.** According to [4] this configuration of infinite singularities leads to the unique global topological configuration of singularities: (166) which belongs to the class $m_f = 0$, i.e. we have no finite singular points. We recall that from [5], configuration (166) has topological codimension 5. The affine invariant conditions which define this topological configuration are:

$$\mu_0 = \mu_1 = \mu_2 = \mu_3 = 0,\ \mu_4 \neq 0,\ \eta = \kappa = 0,\ \widetilde{M} \neq 0,\ \widetilde{K} = \tilde{L} = \kappa_1 = 0,\ K_1 \neq 0. \tag{15}$$

Since $m_f = 0$ and $\widetilde{K} = 0$, by Lemma 1 we have to consider one of the systems 18b) - 18d). It could be checked directly that the conditions (15) could be satisfied only for the canonical systems 18c) and these conditions imply $m \neq 0$. So we may assume $m = 1$ due to the rescaling $x \to x/m$ and we arrive at the family of systems:

$$\dot{x} = x, \quad \dot{y} = 1 + ex + lx^2 + 2xy.$$

We observe that these systems possess an invariant line $x = 0$ which connects two sides of the intricate infinite singular point $N_1[0:1:0]$. Therefore the affine separatrix of the infinite saddle-node $N_2[1 : -l/2 : 0]$ must come from (go to) the infinite singularity $N_1[0:1:0]$ creating an elliptic sector. As a result we obtain the phase portrait $QS166_1^{(5)}$.



$QS166_1^{(5)}$

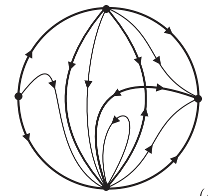
3.14. **The configuration 14:** $E - PH, N$**.** According to [4] this configuration of infinite singularities leads to the unique global topological configuration of singularities: (156) s; $E - PH, N$. We recall that from [5], configuration (156) has topological codimension 4. By [4] a quadratic system possesses the configuration (156) if and only if the following conditions hold:

$$\mu_0 = \mu_1 = \mu_2 = 0,\ \mu_3 \neq 0,\ \kappa = 0,\ \widetilde{K} < 0. \tag{16}$$

Since the configuration (156) contains a single finite singular point (which is a saddle) and $\widetilde{K} \neq 0$, according to Lemma 1 we have to consider the canonical form 17a). Then the conditions (16) imply $d = 0$ and $cf < 0$. We may assume $f = 1$ (due to a time rescaling) and we arrive at the family of systems (11) but with the condition $c < 0$ and in this case the 2-parameter family of systems (11) possesses exactly the configuration (156).

We observe that systems (11) have two invariant lines $x = 0$ and $x = -1/2$ as well as the saddle $M_1(0, 0)$ located on the invariant line $x = 0$. Moreover, we observe that these invariant lines pass through the infinite intricate singularity and hence, two separatrices of the saddle (the semi-lines) go to the infinite intricate singularity. Since one of the separatrices of the finite elemental saddle lies on the domain bordered by the invariant lines $x = 0$ and $x = -1/2$, we deduce that this implies the existence of the elliptic sector between these two invariant lines, bordered by the two separatrices of the finite saddle. The last separatrix of the finite saddle is located at the semi-plane $x > 0$ and since we could not have two elliptic sectors, this separatrix must go to the infinite elemental node.

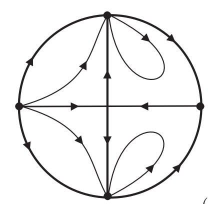
Thus we deduce that the hyperbolic sector of the intricate point must be located on the left of the invariant line $x = -1/2$ and we observe, that whether this invariant line serves as the separatrix bordering the hyperbolic sector or not, we arrive at the same phase portrait. However for accuracy we could examine the local behavior of the trajectories in the vicinity of the intricate singularity at infinity and convince ourselves that the invariant line $x = -1/2$ indeed serves as a separatrix for this singularity. In such a way we get the phase portrait given by $QS156_1^{(4)}$.


$QS156_1^{(4)}$

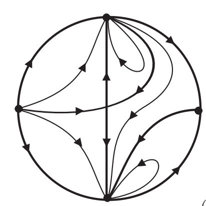
3.15. **The configuration 12:** $E-E$, $\binom{1}{1}SN$**.** According to [4] this configuration of infinite singularities leads to the unique global topological configuration of singularities: (160) s; $E-E$, $\binom{1}{1}SN$. We recall that from [5], configuration (160) has topological codimension 4. By [4] a quadratic system possesses the configuration (160) if and only if the conditions (13) hold, except for the last one: instead of $\mu_3 K_1 > 0$ must be $\mu_3 K_1 < 0$. As a result we consider systems (14), for which instead of $f > 0$ the condition $f < 0$ is satisfied.

We observe that these systems have the invariant line $x = 0$ on which the finite saddle is located. Moreover this line passes through the infinite intricate singularity and this means that the two separatrices of the finite saddle go to intricate singularity (each one in its direction). Without loss of generality we may assume that the finite separatrix of the infinite semi-elemental saddle-node is located on the right hand of the invariant line $x = 0$. So on this Poincaré semi-disc (defined by $x > 0$) there are two separatrices and we have two logically distinct possibilities: there exists a (c) connection and there does not exist any separatrix connection.

The first possibility is realizable, for example, if $e = l(f-1)/2$, because in this case systems (14) possess the invariant line $y = -lx/2$ which consists of two separatrices of finite saddle going to the infinite saddle-node and produces a (c) connection. As a result we get the phase portrait given by $QS160_1^{(5)}$. The codimension is 5 due to the codimension 4 of the configuration of singularities plus the added codimension given by the (c) connection.


$QS160_1^{(5)}$

Assume now that there does not exist a separatrix connection. Then both separatrices on the Poincaré semi-disc defined by $x > 0$ must go to the intricate infinite singularity. And clearly they must go in different directions, otherwise we get two elliptic sectors on the same part of the line $Z = 0$. Evidently this produces two elliptic sectors on different sides with respect to the line $Z = 0$.


$QS160_1^{(4)}$

Now we examine the behavior of the separatrix of the finite saddle located on the semi-plane $x < 0$. Evidently this separatrix can not go/arrive to/from the infinite intricate singularity because in this case we get two elliptic sectors on the same part of the line $Z = 0$. So the unique possibility for this separatrix is to go to the parabolic sector of the infinite saddle-node. In such a way we get the phase portrait given by $QS160_1^{(4)}$ (here we have not a separatrix connection).

The coherence of codimension between these two phase portraits is clear. By means of a perturbation, one may break the separatrix connection in $QS160_1^{(5)}$ and obtain $QS160_1^{(4)}$.

3.16. **The configuration 10:** $E-E, N$**.** According to [4] this configuration of infinite singularities leads to the unique global topological configuration of singularities: (123) $s, s; E-E, N$. We recall that from [5], configuration (123) has topological codimension 3. The affine invariant conditions which define this topological configuration are:

$$\mu_0 = \mu_1 = 0,\ \mathbf{U} > 0,\ \kappa = 0,\ \widetilde{K} < 0,\ \mu_2 < 0. \tag{17}$$

Since the configuration (123) contains two real distinct singularities (which are saddles) and $\widetilde{K} \neq 0$, by Lemma 1 we have to consider the systems 14a) for which conditions (17) imply $d = 0$, $-cfu < 0$ and $1 + 2u < 0$.

Moreover we may assume $f = 1$ due to a time rescaling and we get the family of systems (8) for which the conditions $u < -1/2$ and $c < 0$ hold.

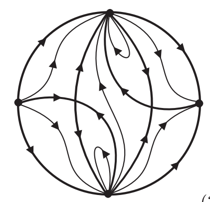
We observe that the systems (8) have the invariant straight lines $x = 0$ and $x = 1$ and on these lines the saddles $M_1(0, 0)$ and $M_2(1, 0)$ are located. So the invariant lines consist of four separatrices of the finite saddles and hence they connect finite saddles with the infinite intricate singularity (in each part two separatrices). Since on the lines $x = 0$ and $x = 1$ we have

$$\frac{dy}{dt}\Big|_{x=0} = y, \quad \frac{dy}{dt}\Big|_{x=1} = y(1 + 2u)$$

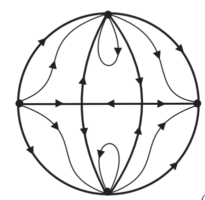
and due to the condition $u < -1/2$ we deduce that the flow on the half-line $x = 0$ (respectively $x = 1$) and $y > 0$ goes upwards (respectively downwards). Inside the domain bordered by the invariant lines $x = 0$ and $x = 1$ there are located two separatrices: one of the saddle M_1 and another of the saddle M_2. Clearly these separatrices have also different stabilities.

So we could have three logically distinct situations: (i) we have an (a) connection between the saddles; (ii) both separatrices go to the intricate singularity in different sides and (iii) both separatrices go to the intricate singularity to the same side.

However the third possibility leads to the existence of at least two elliptic sectors on the same side of the line $Z = 0$, which is impossible. The first two possibilities lead to the existence of two elliptic sectors on different sides of the line $Z = 0$. This means that the remaining two separatrices outside the mentioned domain could go only to the elementary node at infinity (each one to its side), otherwise we get again two elliptic sectors on the same part of the line $Z = 0$, which is impossible.



$QS123_1^{(3)}$

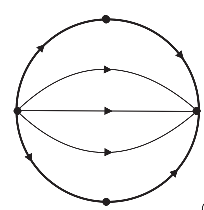


$QS123_1^{(4)}$

We observe that an (a) connection exists when $e = 0$, because in this case systems (20) possess the invariant line $y = 0$ passing through the saddles $M_1(0, 0)$ and $M_2(1, 0)$. As a result we arrive to the phase portrait $QS123_1^{(4)}$.

In the case when we have not a separatrix connection we get the phase portrait $QS123_1^{(3)}$.

3.17. **The configuration 8:** $H - H, N$**.** According to [4] this configuration of infinite singularities leads to the unique global topological configuration of singularities: (137) $H - H, N$. We recall that from [5], configuration (137) has topological codimension 3. In this case we have two finite complex singularities and it is not difficult to detect that this leads to the unique phase portrait given by $QS137_1^{(3)}$.



$QS137_1^{(3)}$

Up to here every configuration at infinity has forced the finite configuration of singularities. Now we start with configurations at infinity which allows several options for finite singularities.

3.18. **The configuration 32:** $EH - P$**.** According to [4] this configuration of infinite singularities leads to the following three global topological configurations of singularities:

$$(169)\ \ EH - P; \quad (146)\ \ sn;\ EH - P; \quad (130)\ \ s, a;\ EH - P.$$

3.18.1. **The topological configuration** (169)**.** We recall that from [5], configuration (169) has topological codimension 6. According to [4] the affine invariant conditions which define this topological configuration are:

$$\mu_0 = \mu_1 = \mu_2 = \mu_3 = 0,\ \mu_4 \neq 0,\ \widetilde{M} = 0,\ C_2 \neq 0,\ \widetilde{K} = K_1 = 0,\ K_3 \neq 0. \tag{18}$$

So the configuration (169) is from the class $m_f = 0$ and since $\widetilde{K} = 0$, according to Lemma 1 a system in this class must belong to one of the canonical forms $18b$) - $18d$). It could be checked directly that the conditions (18) could be satisfied only for the canonical systems $18d$), i.e. we consider the following family of systems:

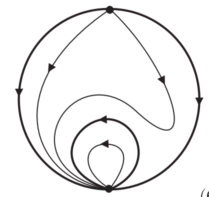
$$\dot{x} = 1, \quad \dot{y} = ex + fy + lx^2 + 2mxy + ny^2.$$

For these systems the conditions $\widetilde{M} = 0$ and $C_2K_3 \neq 0$ imply $m = n = 0$ and $fl \neq 0$. Then due to the rescaling $(x, y, t) \mapsto (x/f, ey/f^2, t/f)$ we may assume $f = 1 = e$ in the case $e \neq 0$. If $e = 0$ then we apply the rescaling $(x, y, t) \mapsto (x/f, y, t/f)$ obtaining $f = 1$. Thus we arrive at the family of systems

$$\dot{x} = 1, \quad \dot{y} = ex + y + lx^2, \quad l \neq 0, \ e \in \{0, 1\}.$$

We observe that these systems possess at infinity one intricate singularity of multiplicity 7 having one elliptic and one hyperbolic sector on one side of the infinite line $Z = 0$ and one parabolic sector at the other side.

Since according to [3] (see Diagram 8.1) the geometrical configuration of the intricate singularity is $\binom{4}{3}\ \widehat{P}_\curlywedge EH_\curlywedge - \widehat{P}$ there is no parabolic sector beside the hyperbolic sector. Therefore the hyperbolic sector is formed by an affine separatrix plus a separatrix at infinity. As a result we arrive at the unique phase portrait given by $QS169_1^{(6)}$.



$QS169_1^{(6)}$

3.18.2. **The topological configuration** (146)**.** We recall that from [5], configuration (146) has topological codimension 5. By [4] the affine invariant conditions which define this topological configuration are:

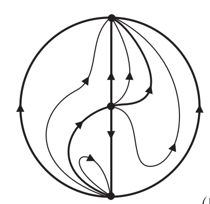
$$\mu_0 = \mu_1 = 0, \ \mu_2 \neq 0, \ \mathbf{U} = \kappa = \tilde{L} = 0, \ \widetilde{K} > 0, \ C_2 \neq 0.$$

So the configuration (146) is from the class $m_f = 2$ and moreover there exists a double finite singularity. Since $\widetilde{K} \neq 0$, according to Lemma 1 such a system belongs to the canonical forms 16a). For these systems the conditions $\kappa = \tilde{L} = 0$ and $\widetilde{K} \neq 0$ imply $d = 0$ and $g = 2f \neq 0$ and the conditin $C_2 \neq 0$ imolies $l \neq 0$. Moreover we may assume $f = 1$ due to a time rescaling. Therefore we get the family of systems

$$\dot{x} = 2x^2, \quad \dot{y} = y + lx^2 + 2xy, \quad l \neq 0.$$

We observe that the above systems possess the invariant line $x = 0$ (of multiplicity at least two) and the saddle-node (0,0) is located on this line. Moreover both semi-lines of this invariant line are the separatrices of the saddle-node and the separatrices are both repelors.

We have another separatrix in the semi-plane $x < 0$ and without loss of generality we may consider that the parabolic sector of infinite singularity is located in the semi-plane $y > 0$. Therefore the finite separatrix can only come from the opposite region formed by parabolic sector. As a result we obtain the elliptic sector and this leads to the unique phase portrait given by $QS146_1^{(5)}$.



$QS146_1^{(5)}$

Observation 1. *It is worth to notice that this phase portrait has codimension 5 while the previous one has codimension 6. If configuration (169) could be obtained with two complex finite singularities it would have codimension 4. But this is not possible by [3, Diagram 9.4]. In other words one can not perturb phase portrait* $QS146_1^{(5)}$ *to produce two complex finite singularities (or perturb* $QS169_1^{(6)}$ *so to extract two finite complex singularities from infinity). It is possible to perturb it to produce two real finite singularities as we will see in the next case, but perturbing it in two complex affects infinite singularities.*

3.18.3. **The topological configuration** (130)**.** We recall that from [5], configuration (130) has topological codimension 4. According to [4] the affine invariant conditions which define this topological configuration are:

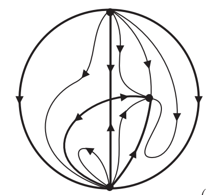
$$\mu_0 = \mu_1 = 0, \ \mu_2 > 0, \ \mathbf{U} > 0, \ \kappa = \widetilde{M} = 0, \ \widetilde{K} > 0, \ C_2 \neq 0. \tag{19}$$

So the configuration (130) is from the class $m_f = 2$ and moreover there exist two distinct finite singularities which are a saddle and an anti-saddle. Since $\widetilde{K} \neq 0$, according to Lemma 1 such a system belongs to the canonical forms 14a). for which the conditions (19) imply $d = 0$, $c = -2fu \neq 0$, $2u + 1 > 0$ and $e \neq 0$. Moreover we may assume $f = 1 = e$ due to the rescaling $(x, y, t) \mapsto (x, ey/f, t/f)$. Therefore we get the family of systems

$$\dot{x} = 2u(x - 1)x, \quad \dot{y} = x + y - x^2 + 2uxy, \quad 0 \neq u > -1/2. \tag{20}$$

We observe that the above systems possess two invariant lines $x = 0$ and $x = 1$ and on each one of them there is located a singular point. As one of the finite singularities is a saddle (we may assume to be located on $x = 0$) we conclude that one separatrix of this saddle is located between the two parallel invariant lines.

Without loss of generality we may assume that the parabolic sector of the infinite intricate singularity is located on the semi-plane $y > 0$ and that it is an repelor. This forces the direction of the flow of every separatrix. Therefore the separatrix between the invariant lines must go to the finite node and the other separatrix must border the infinite elliptic sector. Thus we obtain the unique phase portrait given by $QS130_1^{(4)}$.



$QS130_1^{(4)}$

3.19. **The configuration 19:** $PH-PH,\ N$**.** According to [4] this configuration of infinite singularities leads to the following two global topological configuration of singularities:

$$(144)\ \ sn;\ PH-PH,\ N; \qquad (124)\ \ s,a;\ PH-PH,\ N.$$

Similarly as in the configuration **32** one cannot perturb the saddle-node into complex singularities and maintain the infinite intricate singularity.

3.19.1. **The topological configuration** (144)**.** We recall that from [5], configuration (144) has topological codimension 4. Taking into consideration [4] we determine that the configuration (144) could occur if and only if one the following three sets of the conditions are satisfied:

$$\begin{aligned} &(a)\ \ \mu_0=\mu_1=0,\ \mu_2\neq 0,\ \mathbf{U}=\kappa=0,\ \widetilde{K}<0;\\ &(b)\ \ \mu_0=\mu_1=0,\ \mu_2\neq 0,\ \mathbf{U}=\kappa=0, \widetilde{K}>0,\ \tilde{L}>0;\\ &(c)\ \ \mu_0=\mu_1=0,\ \mu_2\neq 0,\ \mathbf{U}=\kappa=0,\ \widetilde{K}=0,\ \tilde{L}\neq 0,\ \mathcal{T}_4=0,\ \eta=\kappa_1=0. \end{aligned} \tag{21}$$

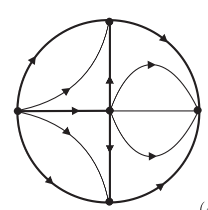
So the configuration (144) is from the class $m_f = 2$ and moreover there exists a double finite singularity. According to Lemma 1 we arrive at the family of systems $16a$) in the cases (a) and (b) (since $\widetilde{K}\neq 0$) and at systems $16b$) in the case (c) (when $\widetilde{K}=0$).

For systems $16a$) we obtain the following dependencies: $d = 0$ and either $fg < 0$ or $fg > 0$ and $g(g-2f) > 0$. We may assume $f = 1$ due to a time rescaling and we arrive at the family of systems

$$\dot{x}=gx^2,\quad \dot{y}=y+lx^2+2xy,\quad g(g-2)>0. \tag{22}$$

We observe that the above systems possess the invariant line $x = 0$ (of multiplicity at least two) and the saddle-node (0,0) is located on this line. Moreover both semi-lines of this invariant line are the separatrices of the saddle-node and the separatrices are both repelors.

We have another separatrix in the semi-plane $x < 0$ if $g > 0$ and we detect that the node at infinity is located a located at $N[l/2:0:0]$ and it is an attractor on the side $x > 0$. This implies the existence of the unique phase portrait given by $QS144_1^{(4)}$. We point out that even this phase portrait contains two separatrix connections, its codimension does not grow above the value 4 already provided by the configuration of singularities because the connections are already forced.



$QS144_1^{(4)}$

In the case $g < 0$ we obtain a phase portrait which is symmetrical to $QS144_1^{(4)}$ with respect to invariant line $x = 0$.

For systems $16b$) the conditions (c) from (21) imply

$$n\neq 0,\quad l=m^2/n,\quad m(dm-cn)=0, dm-cn\neq 0$$

and therefore we obtain $m = l = 0$. So we arrive at the systems

$$\dot{x}=cx+dy,\quad \dot{y}=ny^2,\quad cn\neq 0$$

and due to the rescaling $(x, y, t) \mapsto (x, cy/n, t/c)$ we may assume $c = n = 1$.

Evidently the above system has the invariant line $y = 0$ on which is located a saddle-node. At infinity we have a simple node $N_1[0:1:0]$ and the intricate one $N_2[1:0:0]$. This leads to the phase portrait topologically equivalent to $QS144_1^{(4)}$.

3.19.2. **The topological configuration** (124)**.** We recall that from [5], configuration (124) has topological codimension 3. By [4] the affine invariant conditions which define this topological configuration are:

$$(23) \qquad \mu_0 = \mu_1 = 0,\ \mu_2 > 0,\ \mathbf{U} > 0,\ \kappa = 0 \ \text{ and either } \ \widetilde{K} < 0, \ \text{ or } \ \widetilde{K} > 0,\ \widetilde{M} \neq 0,\ \tilde{L} > 0.$$

Moreover according to Remark 1 there exist the following third possibility:

$$(24) \qquad \mu_0 = \mu_1 = 0,\ \mu_2 \neq 0,\ \mathbf{U} > 0,\ \kappa = \widetilde{K} = \eta = 0, \widetilde{M} \neq 0,\ \tilde{L} \neq 0,\ \kappa_1 = 0.$$

According to Lemma 1 a system with two distinct finite singularities belongs to one of the canonical forms 14*a*) or 14*b*). More exactly in the case (23) we have $\widetilde{K} \neq 0$ and hence we have to consider systems 14*a*), whereas in the case (24) ($\widetilde{K} = 0$) and we have to examine the canonical systems 14*b*).

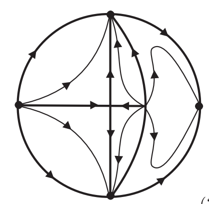
We examine first the conditions (23). The condition $\kappa = 0$ implies $d = 0$. Then we calculate

$$\widetilde{K} = -4cfux^2,\ \mathbf{U} = c^4 f^4 (1+2u)^2 x^4 y^2,\ \widetilde{M} = -8(c+2fu)^2 x^2,\ \tilde{L} = 8c(c+2fu)^2 x^2,\ \mu_2 = c^2 f^2 (1+2u) x^2.$$

Since $c \neq 0$ we may assume $c = 1$ due to a time rescaling and we consider the family of systems

$$\dot{x} = x(1-x), \quad 1+2u > 0 \text{ and either} fu > 0 \text{ or } fu < 0, 1+2fu > 0,$$
$$\dot{y} = ex + fy - ex^2 + 2fuxy,$$

By means of a symmetry with respect to the line $x = 1/2$ we may assume that the finite saddle is at the origin of the coordinates. As a result we arrive at the unique phase portrait given by $QS124_1^{(3)}$.


$QS124_1^{(3)}$

Now we consider the conditions (24) which must be satisfied for systems 14*b*). We calculate:

$$\eta = 4g^2(h^2 - gk), \quad \tilde{L} = 8g(gx^2 + 2hxy + ky^2)$$

and therefore the condition $\tilde{L} \neq 0$ implies $g \neq 0$. Then we may assume $g = 1$ due to a time rescaling and hence the condition $\eta = 0$ yields $k = h^2$. Then we calculate

$$\mu_2 = (hu-1)^2(x+hy)^2, \quad \kappa_1 = -32h^2u(hu-1)$$

and clearly the conditions $\kappa_1 = 0$ and $\mu_2 \neq 0$ imply $hu = 0$. This leads to the canonical form

$$\dot{x} = -x - 2hy + (x+hy)^2, \quad \dot{y} = ux + y, \quad hu = 0.$$

Assume first $u = 0$. Then the above systems possess three invariant lines: $y = 0$, $x = -hy$ and $x = 1 - hy$. The last two are parallel and one of them passes through the finite node and another one through the finite saddle. So for the same arguments as above the existence of these two parallel lines leads to the phase portrait $QS124_1^{(3)}$.

Suppose now $h = 0$. Then we get the invariant line $x = 0$ and $x = 1$ which again pass through the saddle and node respectively. Clearly we get the same phase portrait $QS124_1^{(3)}$.

3.20. **The configuration 17:** $E - E$, S**.** According to [4] this configuration of infinite singularities leads to the following three global topological configuration of singularities:

$$(138) \ \ E - E,\ S; \quad (145) \ \ sn;\ E - E,\ S; \quad (129) \ \ s, a;\ E - E,\ S.$$

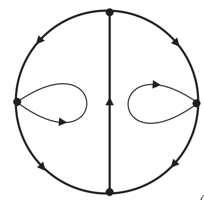
3.20.1. **The topological configuration** (138)**.** We recall that from [5], configuration (138) has topological codimension 3. Taking into consideration [4] we determine that the configuration (144) could occur if and only if one the following three sets of the conditions are satisfied:

$$(25)\qquad \begin{array}{c} (a)\ \ \mu_0=\mu_1=0,\ \mu_2\neq 0,\ \mathbf{U}<0,\ \kappa=0,\ \widetilde{K}>0,\ \tilde{L}<0;\\ (b)\ \ \mu_0=\mu_1=\mu_2=\mu_3=0,\ \mu_4\neq 0,\ \eta=0,\ \widetilde{M}\neq 0,\ \kappa=0,\ \widetilde{K}>0,\ \tilde{L}<0;\\ (c)\ \ \mu_0=\mu_1=\mu_2=\mu_3=0,\ \mu_4\neq 0,\ \eta=0,\ \widetilde{M}\neq 0,\ \kappa=0,\ \widetilde{K}=\tilde{L}=\kappa_1=K_1=0,\ L_1<0. \end{array}$$

According to Lemma 1 we arrive at the family of systems 15a) in the case (a), at systems 18a) in the case (b) and at systems 18d) in the case (c).

The conditions provided by the set (a) from (25) implies $m=0$, $bh>0$ and $b(b-2h)<0$. Moreover we may assume $b=1$ due to a time rescaling. Therefore we get the family of systems

$$(26)\qquad \dot{x}=a+hux+2hxy+ay^2,\quad \dot{y}=1+y^2,\quad h>1/2.$$

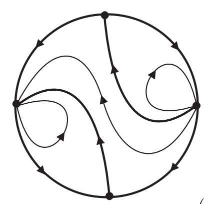


$QS138_1^{(4)}$

The intricate infinite singularity of these systems is located at the point $N_1[1:0:0]$, whereas the infinite saddle is at the point $N_2[a/(1-2h):1:0]$.

Since there are no real finite singularities and there are two separatrices of the infinite saddle we have two logically distinct possibilities: there exists a (d) connection and there does not exist any separatrix connection.

The first possibility is realizable, for example, if $a=0$, because in this case systems (26) possess the invariant line $x=0$ which consists of two separatrices of the infinite saddle and produces a (d) connection. As a result we get the phase portrait given by $QS138_1^{(4)}$.



$QS138_1^{(3)}$

If there does not exist a (d) connection, we arrive at the unique first portrait defined by $QS138_1^{(3)}$ of codimension 3.

Next we consider the conditions (b) from (25). For systems 18a) we calculate $\kappa=128h^2(hl-gm)$ and $\mu_4=(gm-hl)^2x^4$. So clearly the conditions $\kappa=0$ and $\mu_4\neq 0$ imply $h=0$ and $gm\neq 0$. Then we may assume $g=1$ due to a time rescaling and considering the remaining conditions from (b) we arrive at the following family of systems

$$\dot{x}=x^2,\quad \dot{y}=m+lx^2+2mxy,\quad m>1/2.$$

Since the above systems possess the invariant line $x=0$ we conclude that they have the phase portrait $QS138_1^{(4)}$.

In the case of conditions (c) from (25) we have to consider the systems 18d) for which we calculate

$$\mu_4=(lx^2+2mxy+ny^2)^2,\quad \tilde{L}=8n(lx^2+2mxy+ny^2).$$

Therefore the conditions $\tilde{L}=0$ and $\mu_4\neq 0$ imply $n=0$. Then the condition $L_1=-64m^3x^2<0$ implies $m>0$ and we arrive at the family of systems

$$(27)\qquad \dot{x}=1,\quad \dot{y}=ex+fy+lx^2+2mxy,\quad m>0.$$

Since there are no real finite singularities and there are two separatrices of the infinite saddle we have two logically distinct possibilities: there exists a (d) connection and there does not exist any separatrix connection.

The first possibility is realizable, for example, if $f=l=0$, because in this case systems (27) possess the invariant line $e+2my=0$ which consists of two separatrices of the infinite saddle and produces a (d) connection.

In such a way we obtain two phase portraits (possessing or not possessing a (d) connection) which are topologically equivalent to $QS138_1^{(4)}$ and $QS138_1^{(3)}$, respectively.

3.20.2. **The topological configuration** (145)**.** We recall that from [5], configuration (145) has topological codimension 4. According to [4] the affine invariant conditions which define this topological configuration are:

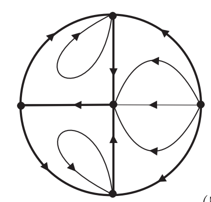
$$\mu_0 = \mu_1 = 0,\ \mu_2 \neq 0,\ \mathbf{U} = 0,\ \kappa = 0,\ \widetilde{K} > 0,\ \tilde{L} < 0.$$

So the configuration (145) is from the class $m_f = 2$ and moreover there exists a double finite singularity. According to Lemma 1 we arrive again at the family of systems $16a$). The above conditions imply the following dependences for the coefficients of these systems : $d = 0$, $fg > 0$ and $g(2f - g) > 0$. We may assume $f = 1$ due to a time rescaling and we arrive at the family of systems (22) but with the condition $0 < g < 2$.

We observe that the systems (22) possess the invariant line $x = 0$ (of multiplicity at least two) and the saddle-node (0,0) is located on this line. Moreover both semi-lines of this invariant line are the separatrices of the saddle-node and the separatrices are both repelors. We have another separatrix in the semi-plane $x < 0$ and we detect that the saddle at infinity is located at $N[1 : l/(g - 2) : 0]$. Due to the condition $0 < g < 2$ the saddle has a unstable affine separatrix on the semi-plane $x < 0$.

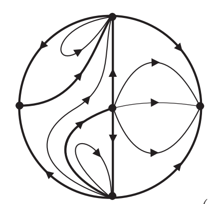
So on this Poincaré semi-disc (defined by $x < 0$) there are two separatrices and we have two logically distinct possibilities: there exists a (c) connection and there does not exist any separatrix connection.

The first possibility is realizable, for example, if $l = 0$, because in this case systems (22) possess the invariant line $y = 0$ which for $x < 0$ consists of two separatrices of the infinite saddle going to the finite saddle-node and produces a (c) connection. As a result we get the phase portrait given by $QS145_1^{(5)}$.



$QS145_1^{(5)}$

Assume now that there does not exist a separatrix connection. Then both separatrices on the Poincaré semi-disc defined by $x < 0$ must go to the intricate infinite singularity. And clearly they must go in different directions, otherwise we get two elliptic sectors on the same part of the line $Z = 0$. Evidently this produces two elliptic sectors on different sides with respect to the line $Z = 0$. This implies the existence of the unique phase portrait given by $QS145_1^{(4)}$.



$QS145_1^{(4)}$

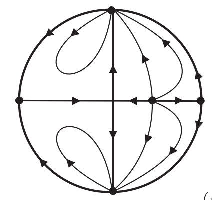
3.20.3. **The topological configuration** (129)**.** We recall that from [5], configuration (129) has topological codimension 3. By [4] the affine invariant conditions which define this topological configuration are:

$$\mu_0 = \mu_1 = 0,\ \mu_2 > 0,\ \mathbf{U} > 0,\ \kappa = 0,\ \widetilde{K} > 0,\ \tilde{L} < 0,\ \widetilde{M} \neq 0.$$

So the configuration (129) is from the class $m_f = 2$ and moreover there exists only two real finite singularities which are a saddle and an anti-saddle. So since $\widetilde{K} \neq 0$, by Lemma 1 such a system belongs to the canonical forms $14a$) for which these conditions imply $d = 0$, $cfu < 0$, $1 + 2u > 0$ and $c(c + 2uf) < 0$. We may assume $c = 1$ via a time rescaling and we get the systems

$$\dot{x} = x(1 - x), \quad \dot{y} = ex - ex^2 + fy + 2fuxy, \quad u < -1/2, \quad uf < -1/2. \tag{28}$$

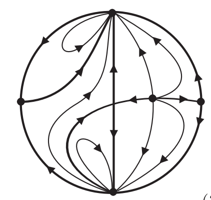
We observe that by means of a symmetry with respect to the line $x = 1/2$ we may assume that the finite saddle is at the origin of the coordinates. Therefore on the semi-plane $x < 0$ we have two separatrices: one of the infinite saddle and one of the finite saddle. So we have two logically distinct possibilities: there exists a (c) connection and there does not exist any separatrix connection.



$QS129_1^{(4)}$

The first possibility is realizable, for example, if $e = 0$, because in this case systems (28) possess the invariant line $y = 0$ whose semi-line $x < 0$ is the (c) connection under discussion. As a result we get the phase portrait given by $QS129_1^{(4)}$.

If there does not exist a (c) connection we arrive at the unique first portrait defined by $QS129_1^{(3)}$.

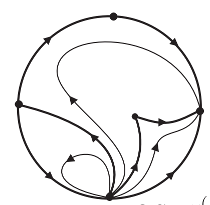


$QS129_1^{(3)}$

3.21. **The configuration 11:** $E-H, \binom{1}{1}SN$**.** According to [4] this configuration of infinite singularities leads to the following five global topologically configurations of singularities, two of which being equivalent:

$$(143)_a\ E-H, \binom{1}{1}SN;\ \ (143)_b\ E-H, \binom{3}{1}SN.;\ \ (154)\ sn;\ E-H, \binom{1}{1}SN;$$
$$(155)\ cp;\ E-H, \binom{1}{1}SN;\ \ (136)\ s, a;\ E-H, \binom{1}{1}SN.$$

We note that quadratic systems with a finite nilpotent singular point are already classified (see for instance, [8]). According to [8] a quadratic system with a finite cusp and infinite singularities with the configuration $E-H, \binom{1}{1}SN$ possesses a unique phase portrait given by picture 22 in Figure 20 (see [8, page 200]). We denote this picture which corresponds to the configuration (155) by $QS155_1^{(5)}$.

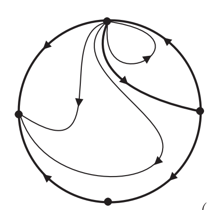


$QS155_1^{(5)}$

In what follows we consider the configurations (143), (154) and (136).

According to [3] the infinite singularity $\binom{1}{2}E-H$ is nilpotent, whereas the infinite saddle-nodes $\binom{1}{1}SN$ and $\binom{3}{1}SN$ are semi-elemental. Thus the systems which possess the topological configurations (136), (154) and $(143)_a$ belong to the family of systems with $m_f = 2$, whereas in the case of the configuration $(143)_b$ the systems are in the class with $m_f = 0$.

3.21.1. **The topological configurations** $(143)_a$ **and** $(143)_b$**.** We recall that from [5], configuration (143) has topological codimension 3. We should have to study two different normal forms, one for $(143)_a$ and another for $(143)_b$. But given the reduced number of singularities of these configurations, we can do simply a topological argument to find the phase portrait.



$QS143_1^{(3)}$

Since we have not real finite singularities and only one finite separatrix belongs to the infinite saddle-node, it is clear that this separatrix must go to the nilpotent infinite singularity forming an elliptic sector. This leads to the unique phase portrait given by $QS143_1^{(3)}$.

3.21.2. **The topological configuration (154).** We recall that from [5], configuration (154) has topological codimension 4. According to [4] the affine invariant conditions which define this topological configuration are:

$$\mu_0 = \mu_1 = 0,\ \mu_2 \neq 0,\ \mathbf{U} = 0,\ \kappa = \widetilde{K} = \tilde{L} = 0,\ \widetilde{M} \neq 0,\ \mathcal{T}_4 = 0,\ \mathcal{B}_1 \neq 0.$$

So the configuration (154) is from the class $m_f = 2$ and moreover there exists a double finite singularity. Since $\widetilde{K} = 0$, according to Lemma 1 we have to consider the canonical form 16b) and these conditions imply $n = 0$ and $cdm(ld - 2cm) \neq 0$. We may assume $c = d = m = 1$ due to the rescaling $(x, y, t) \mapsto \left(\dfrac{cx}{m}, \dfrac{c^2y}{dm}, \dfrac{t}{c}\right)$, and we arrive at the family of systems:

$$\dot{x} = x + y, \quad \dot{y} = lx^2 + 2xy, \quad l \neq 2. \tag{29}$$

We have 1-parameter family for which one of the bifurcation points is $l = 2$ because in this case we get a degenerate system (which by [4] possesses the global topological configuration (197)). By the way, we recall that from [5], configuration (197) has topological codimension 6.

We observe that the value $l = 0$ is also a bifurcation value because in this case we get an invariant line which gives a (c) connection.

Since for $l = 0$ (respectively for $l = 2$) we get a special system without parameters we determine that the corresponding phase portrait is given in Figure 1 by $QS154_1^{(5)}$ (respectively, $QS197_1^{(6)}$ which is the phase portrait generated by the infinite configuration QD_{18}^{∞}).

One could find that there is a lack of coherence here since phase portrait $QS154_1^{(5)}$ has topological codimension 5 and phase portrait $QS197_1^{(6)}$ has topological codimension 6, while both may be perturbated inside this family into systems of codimension 4. However, the configuration of singularities (197) has been proved that it may be perturbed into other configurations of singularities of codimension 5, and thus any phase portrait obtained from configuration (197) will have at least codimension 6, even if it appears in a normal form which suggests a lower codimension.

We claim that there are not more bifurcation values for the parameter l and in each one of the three intervals defined by $l = 0$ and $l = 2$ we get a unique phase portrait.

Indeed, it could be easily proved the following statements: *i*) the elliptic sector of the nilpotent infinite singularity $N_1[0 : 1 : 0]$ is always on the semi-plane $y > 0$; *ii*) on the semi-plane $x > 0$ the semi-elemental infinite saddle-node $N_2[1 : -l/2 : 0]$ has the hyperbolic sectors if $l < 2$ and it has parabolic sectors if $l > 2$; *iii*) the finite saddle-node has the center manifold (the separatrix with the eigenvalue zero) on the semi-plane $y > 0$ (respectively $y < 0$) if $l < 2$ (respectively $l > 2$); *iv*) the flow on the line $y = 0$ goes down if $l < 0$ and up if $l > 0$. According to these conditions we arrive in unique mode to the phase portraits given by $QS154_1^{(4)}$ if $l < 0$; by $QS154_2^{(4)}$ if $0 < l < 2$ and by $QS154_3^{(4)}$ if $l > 2$ (see Figure 1).

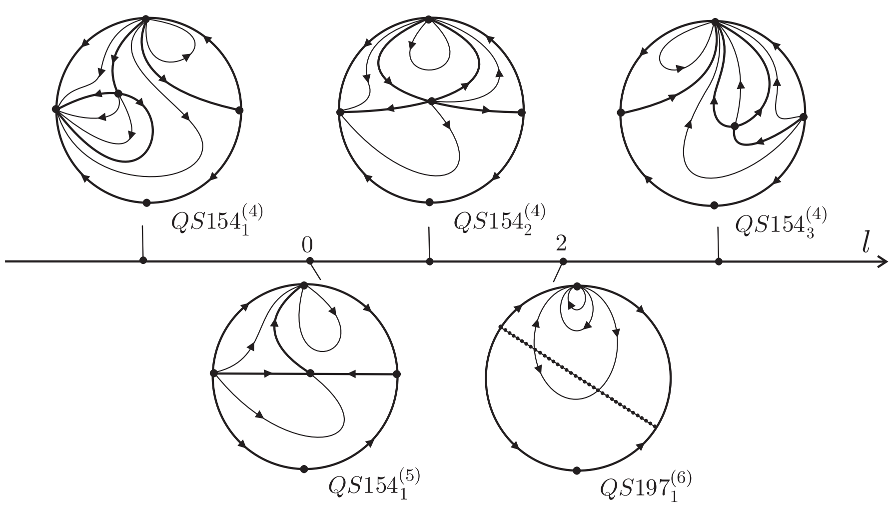


Figure 1. Bifurcation diagram for the family of systems (29).

3.21.3. **The topological configuration** (136)**.** We recall that from [5], configuration (136) has topological codimension 3. According to [4] the affine invariant conditions which define this topological configuration are:

$$\mu_0 = \mu_1 = 0, \ \mu_2 \neq 0, \ \mathbf{U} > 0, \ \eta = \kappa = \widetilde{K} = \tilde{L} = 0, \ \widetilde{M} \neq 0.$$

For the systems with this configuration it is more convenient to examine canonical form associated to infinite singularities. Since $\eta = 0$ and $\widetilde{M} \neq 0$ we have to consider the canonical form $(\mathbf{S}_{III})$ (see [3], Section 6.4), which possess two real distinct infinite singularities, i.e. we consider the following systems:

$$\dot{x} = a + cx + dy + gx^2 + hxy,$$
$$\dot{y} = b + ex + fy + (g-1)xy + hy^2.$$

For these systems we calculate $\kappa = -16h^2 = 0$, i.e. $h = 0$ and then we have $\tilde{L} = 8gx^2 = 0$ which yields $g = 0$. Then we obtain

$$\mu_0 = \mu_1 = 0, \ \mu_2 = -cdxy, \ \widetilde{K} = 0.$$

So due to the condition $\mu_2 \neq 0$ (i.e. $cd \neq 0$) we may assume $c = d = 1$ due to the rescaling $(x, y, t) \mapsto (cx, c^2y/d, t/c)$ and moreover since we have two real finite singularities, due to a translation we may assume that one of the singularities is located at the origin of coordinate, i.e. $a = b = 0$. As a result we arrive at the 2-parameter family of systems

$$\dot{x} = x + y, \quad \dot{y} = ex + fy - xy, \tag{30}$$

with the condition $e - f \neq 0$ due to $U = (e-f)^2x^2y^2(x+y)^2 > 0$. These systems possess the finite singularities $M_1(0,0)$ and $M_2(f-e, e-f)$ (one saddle and one anti-saddle). More exactly the singularity $M_1(0,0)$ is a saddle if $f - e < 0$ and it is an anti-saddle if $f - e > 0$. We observe that without loss of generality we may assume $f - e < 0$ due to the change $(x, y, e, f) \mapsto (x + f - e, \ y - f + e, \ f, \ e)$.

The family of systems (30) depends on two parameters e and f and we are going to examine the bifurcation diagram in these parameters. Using the tools described in the monograph [3] we obtain that the relevant bifurcation curve is $e = -1$ and namely, on the semi-plane $f - e < 0$ any system from the family defined by the points located on the line $e = -1$ possesses a weak focus. The invariant polynomials also detect that the straight line $f = -1$ which has the same geometrical meaning as the line $e = -1$ but in this case the corresponding systems possess a weak saddle. However this bifurcation will not have any topological effect as we will see below.

On the other hand studying the possible existence of invariant straight lines, we detect then they do exist when $ef = 0$. However meanwhile the case $e = 0$ implies he existence of a separatrix (c)-connection the other case never produces a separatrix connection.

Finally, since the bifurcation $e = -1$ implies the existence of a weak focus, there must be a limit cycle on one of the sides of this bifurcation and it is expected also to exist another bifurcation (possible non-algebraic) which implies the existence of a separatrix connection forming a graphic where the limit cycle must disappear. We detect that this bifurcation exists inside the band $-1 < e < 0$, it ends at the point $(-1, -1)$. This bifurcation curve clearly exists for $f < -1$ and it can not exists for $f > -1$ because it would cross then the line of weak saddle. The coexistence of a loop with a weak saddle produces a known bifurcation which emits a limit cycle without breaking the loop. So we would have the possibility of a phase portrait with two limit cycles and this region would end also at the point $(-1, -1)$. On the other hand at this point we have the phase portrait $QS155_1^{(5)}$ which has a cusp. However according to [9] polynomials systems with the cusp of multiplicity n cannot bifurcate more than $n - 1$ limit cycles. Therefore a cusp of multiplicity two as we have in $QS155_1^{(5)}$ can only bifurcate at most one limit cycle.

Summarising all the reasons presented above we arrive at the bifurcation diagram given in FIGURE 2.

This bifurcation diagram presents four bidimensional regions and three one dimensional bifurcation curves. Also on the line $f = e$ we see the cases with a double finite singularity which correspond to the topological configurations (154) and (155) investigated earlier. The bifurcation diagram includes the codes of the new phase portraits presented in FIGURE 3. The phase portrait $QS136_{3wf}^{(3)}$ having a weak focus is topologically equivalent to $QS136_3^{(3)}$. The phase portrait $QS136_{3LC}^{(3)}$ having a limit cycle is topologically equivalent modulo

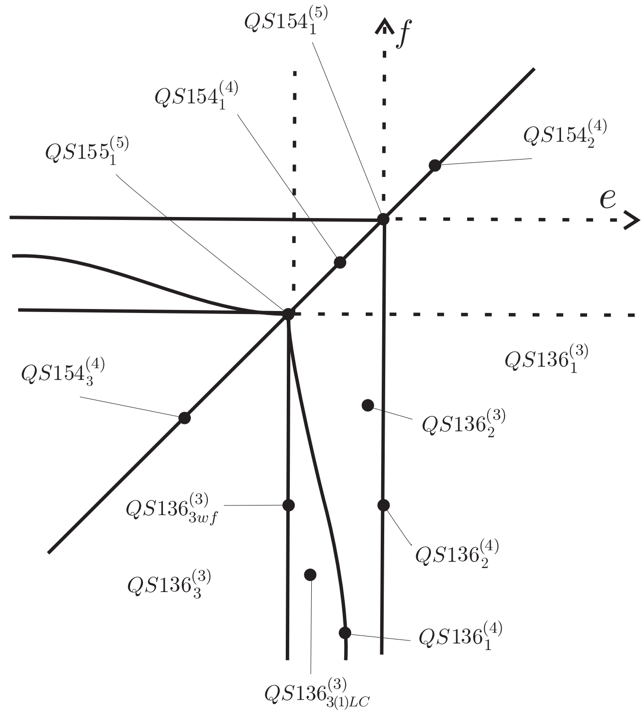


FIGURE 2. Bifurcation diagram for the family of systems (30).

limit cycles to $QS136_3^{(3)}$. The reader may observe from the diagram the coherence of the codimensions assigned to each phase portrait.

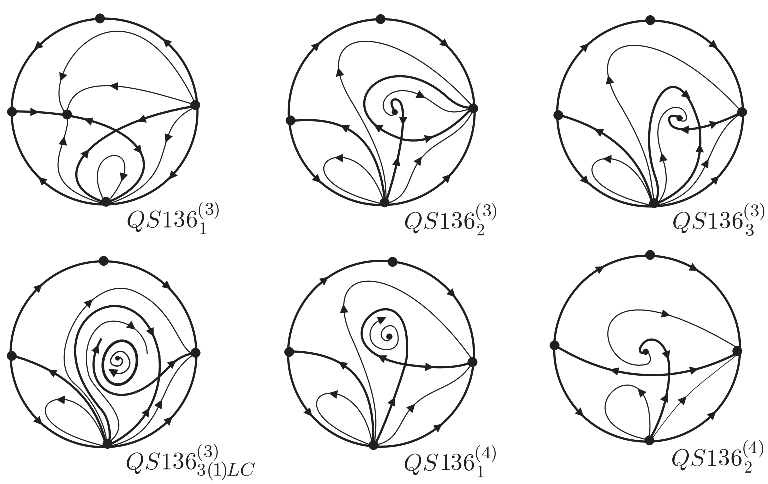
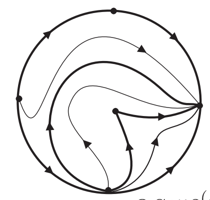


FIGURE 3. The phase portraits for the family of systems (30).

3.22. **The configuration 13:** $PH-H,N$**.** According to [4] this configuration of infinite singularities leads to the following four global topologically distinct configurations of singularities:

(142) $PH-H,N$; (152) sn; $PH-H,N$; (153) cp; $PH-H,N$; (135) s,a; $PH-H,N$.

3.22.1. **The topological configuration** (153)**.** As we mentioned earlier the class of quadratic systems with a finite nilpotent singular point is already classified (see for instance, [8]). However this classification is not complete because there exists a quadratic system with a finite cusp and infinite singularities with the configuration $PH-H,N$ which was omitted in [8]. Indeed it is not too difficult to determine that the system $\dot{x}=y$, $\dot{y}=(x-y)^2$ possesses the phase portrait $QS153_1^{(5)}$ which corresponds to the configuration (153).



$QS153_1^{(5)}$

Next we prove that $QS153_1^{(5)}$ is the unique possible phase portrait which corresponds to configuration of singularities (153). According to [4] the affine invariant conditions which define this topological configuration are:

$$\mu_0=\mu_1=0,\ \mu_2\neq 0,\ \mathbf{U}=0,\ \kappa=\widetilde{K}=0,\ \tilde{L}\neq 0,\ \eta=\mathcal{T}_4=0,\ \kappa_1\neq 0, \mathcal{B}_1=0.$$

So the configuration (153) is from the class $m_f=2$ and moreover there exists a double finite singularity. Since $\widetilde{K}=0$, according to Lemma 1 we have to consider the form $16b$). For these systems we have $\tilde{L}=8n(lx^2+2mxy+ny^2)\neq 0$ which implies $n\neq 0$. Then we may assume $n=1$ due to a time rescaling. Therefore the condition $\eta=4(m^2-l)=0$ gives us $l=m^2$. In this case we calculate

$$\mu_2=(c-dm)^2(mx+y)^2,\ \varkappa_1=32m(c-dm),\ \mathcal{B}_1=2c^2m(c-dm)^2$$

and due to $\kappa_1\neq 0$ the condition $\mathcal{B}_1=0$ gives us $c=0$. Moreover since $dm\neq 0$ we may assume $d=m=1$ due to the rescaling $(x,y,t)\mapsto(dx,dmy,t/(dm))$ and this leads to the system $\dot{x}=y$, $\dot{y}=(x-y)^2$ possessing the phase portrait $QS153_1^{(5)}$.

In what follows we consider the configurations (142), (152) and (135). We point out that the systems which possess such configurations belong to the family of systems with $m_f\leq 2$.

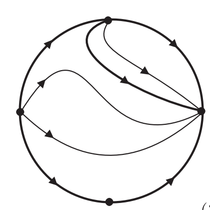
According to [3] the infinite singularity $PH-H$ is nilpotent of multiplicity 4, whereas the infinite node is elemental.

3.22.2. **The topological configuration** (142)**.** We recall that from [5], configuration (142) has topological codimension 3. Taking into consideration [4] we determine that the configuration (142) could occur if and only if one the following two sets of the conditions are satisfied:

$$(a)\ \ \mu_0=\mu_1=0,\ \mu_2\neq 0,\ \mathbf{U}<0,\ \kappa=\eta=0,\ \widetilde{K}=0,\ \kappa_1\neq 0,\ \tilde{L}\neq 0;$$

$$(b)\ \ \mu_0=\mu_1=\mu_2=\mu_3=0,\ \mu_4\neq 0,\ \eta=0,\ \widetilde{M}\neq 0,\ \kappa=0,\ \widetilde{K}=0,\ \tilde{L}\neq 0,\ \kappa_1\neq 0.$$

So the configuration (142) in the case (a) is from the class $m_f=2$ and in the case (b) is from the class $m_f=0$. Moreover there exists only two complex finite singularities in the case (a) and no finite singularities in the case (b). Since we have not real finite singularities and only one finite separatrix belonging to infinite nilpotent singularity, we conclude that this separatrix must go to the infinite node, which is adjacent to the parabolic part of the nilpotent singularity. This leads to the unique phase portrait given by $QS142_1^{(3)}$.



$QS142_1^{(3)}$

3.22.3. **The topological configuration (152).** We recall that from [5], configuration (152) has topological codimension 4. By [4] the affine invariant conditions which define this topological configuration are:

$$\mu_0=\mu_1=0,\ \mu_2\neq 0,\ \mathbf{U}=0,\ \kappa=\eta=\mathcal{T}_4=0,\ \widetilde{K}=0,\ \tilde{L}\neq 0,\ \kappa_1\neq 0,\ \mathcal{B}_1\neq 0. \tag{31}$$

So the configuration (152) is from the class $m_f=2$ and moreover there exists a double finite singularity. Since $\widetilde{K}=0$, according to Lemma 1 such a system belongs to the canonical systems $16b$) for which the condition $\tilde{L}\neq 0$ implies $n\neq 0$. So we may assume $n=1$ due to a time rescaling and then the condition

$\eta = 4(m^2 - l) = 0$ gives $l = m^2$ and then the conditions (31) yield $cm(c - dm) \neq 0$. We may assume $c = m = 1$ due to the rescaling $(x, y, t) \mapsto (cx/m, cy, t/c)$ and we arrive at the family of systems:

$$\dot{x} = x + dy, \quad \dot{y} = (x + y)^2, \quad d \neq 1. \tag{32}$$

We determine that for $d = -1$ the above system possess the invariant parabola $\Phi(x, y) = x^2 - 2y + 2xy + y^2$ which passes through the finite saddle-node $(0, 0)$ and the infinite nilpotent point producing a (c) connection. Thus we have two bifurcation values of the parameter d: $d = -1$ and $d = 1$. So these two bifurcation points split the parameter line d in five parts, including the bifurcation points. We will provide the phase portraits in each one of these regions and then we will prove that they are the only realizable for systems (32).

Thus for the values of the parameter d we obtain the corresponding phase portraits given in Figure 4.

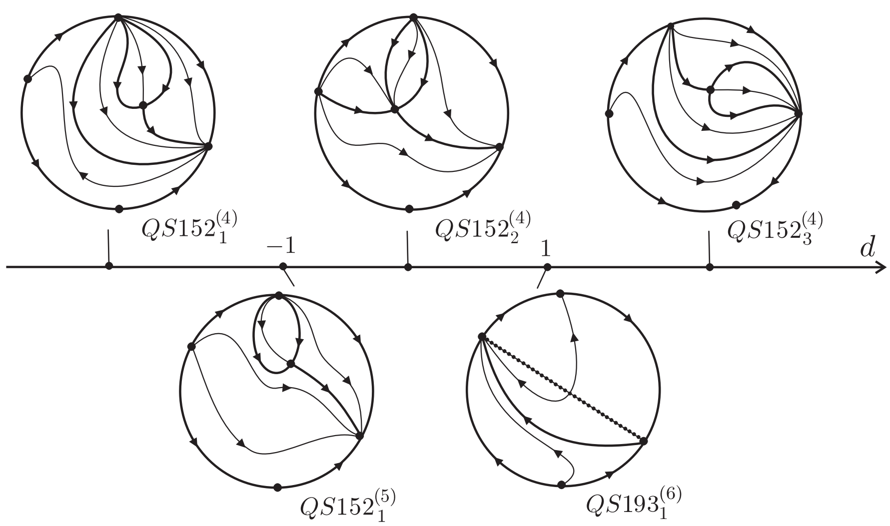


Figure 4. Bifurcation diagram for the family of systems (32).

In order to prove that these are the only realizable phase portraits for systems (32) we have to prove that any other potential phase portraits compatible with the singularities of these systems cannot be realizable. The technique used to prove this is similar to the one used in papers like [1] and [2]. In these papers the authors start by producing all potential phase portraits compatible with certain properties, prove the realizable ones by means of the corresponding examples and the impossibilities of the others by some geometrical arguments.

In order to produce all the potential phase portraits for systems (32) we start from the phase portrait $QS142_1^{(3)}$ which has the same infinite singularities and no real finite singularities.

The potential phase portraits of systems (32) will be obtained by adding a finite saddle-node to the phase portrait $QS142_1^{(3)}$. We observe that in this phase portrait there exits a unique separatrix which splits the Poincaré disc in two canonical regions. We call the *minor region* the one whose border has only two infinite singularities and the *major region* possessing on hos border four infinite singularities.

If the finite saddle-node is located in interior of minor region we get two realizable phase portraits: $QS152_1^{(4)}$ and $QS152_3^{(4)}$. We claim that the saddle-node cannot be in the major region. Indeed supposing the contrary we arrive at the phase portraits $QS152_1^I$ or $QS152_2^I$ given in Figure 5. As we can see the saddle-node is sending two separatrices to the same infinite singularity and the other to the opposite one. Then by Proposition 1 we deduce that these phase portraits are impossible.

Assume now that the finite saddle-node appear in the phase portrait $QS142_1^{(3)}$ as the ω-limit of the infinite separatrix. Then this separatrix either forms a (c) connection with one of the three separatrices of the finite saddle-node or it goes to the parabolic part of this saddle-node. These four possibilities are given in Figure 5 (three of them with a (c) connection) and they generate four potential phase portraits, only two of them are realizable. The phase portraits $QS152_3^I$ and $QS152_4^I$ are not realizable because a perturbation of $QS152_3^I$ would produce $QS152_1^I$ and a perturbation of $QS152_4^I$ would produce $QS152_2^I$.

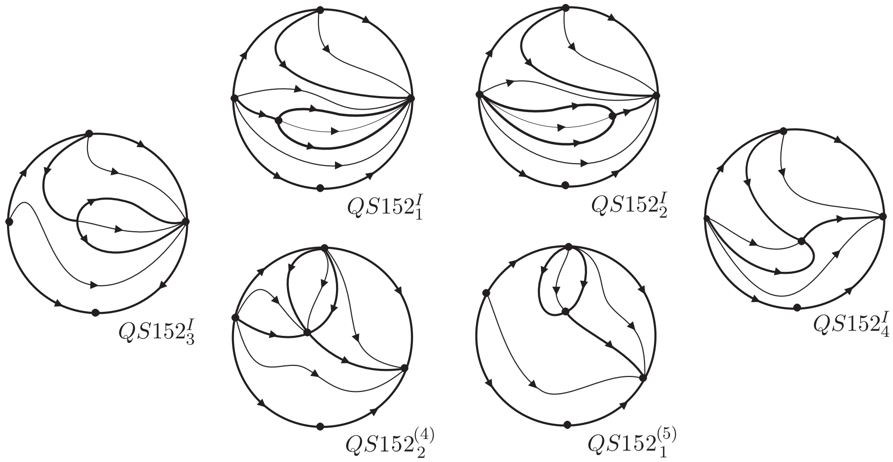


FIGURE 5. Potential phase portraits for the family of systems (32).

3.22.4. **The topological configuration (135).** We recall that from [5], configuration (135) has topological codimension 3. By [4] the affine invariant conditions which define this topological configuration are:

$$\mu_0 = \mu_1 = 0,\ \mu_2 \neq 0,\ \mathbf{U} > 0,\ \kappa = \eta = 0,\ \widetilde{K} = 0,\ \widetilde{M} \neq 0,\ \tilde{L} \neq 0,\ \kappa_1 \neq 0.$$

Since $\eta = 0$ and $\widetilde{M} \neq 0$ we have to consider the canonical form $(\mathbf{S}_{III})$ (see [3], Section 6.4), which possess two real distinct infinite singularities, i.e. we consider the following systems:

$$\dot{x} = a + cx + dy + gx^2 + hxy,$$
$$\dot{y} = b + ex + fy + (g-1)xy + hy^2.$$

For these systems we calculate $\kappa = -16h^2 = 0$, i.e. $h = 0$ and we obtain

$$\mu_0 = 0,\ \mu_1 = dg(g-1)^2x,\ \widetilde{K} = 2g(g-1)x^2,\ \tilde{L} = 8gx^2,$$

So the conditions $\widetilde{K} = 0$ and $\tilde{L} \neq 0$ imply $g = 1$ and then the condition $\mu_2 \neq 0$ gives $f \neq 0$. Then we have $\kappa_1 = -32d \neq 0$. Due to a translation we may also assume $a = b = 0$ and moreover we may consider $f = 1 = d$ due to the rescaling $(x, y, t) \mapsto (fx, f^2y/d, t/f)$. Now we calculate $\mathbf{U} = (e-c)^2x^4(ex+y)^2 >$ and this implies $e - c \neq 0$. As a result we arrive at the 2-parameter family of systems

$$\dot{x} = cx + y + x^2, \quad \dot{y} = ex + y, \quad e - c \neq 0. \tag{33}$$

Even we have one normal form with just two parameters constructed above and could produce a bifurcation diagram for it obtaining the realizable phase portraits, we will also need some other arguments to discard non realizable ones. So we chose to start with the second line.

We start again with the phase portrait $QS142_1^{(3)}$ where the only separatrix splits the Poincaré disc in two canonical regions, mentioned above: a minor and a major region. Now we must introduce a finite saddle and a finite anti-saddle in this phase portrait. It is not possible to put one singularity in each of the canonical regions because each region is only compatible with global index zero in its interior (see [1, Corollary 4.9]). Therefore both finite singularities must be inside the same canonical region ar at least one singularities must be in the common border of the regions.

We point out that both singularities cannot be in the major region due to the same argument as the saddle node of the configuration (152) cannot be in the major region (see phase portraits $QS152_1^I$ or $QS152_2^I$ given in FIGURE 5).

Both singularities can be in the minor region and then we obtain two phase portraits $QS135_1^{(3)}$ and $QS135_2^{(3)}$ without separatrix connection and one phase portrait $QS135_1^{(4)}$ with separatrix connection (see FIGURE 6). The phase portraits without separatrix connection could have limit cycles. However this is irrelevant for the topological classification of phase portraits modulo limit cycles but anyway since we have detected one limit cycle in the considered configuration and we present it here as $QS135_{1LC}^{(3)}$.

However the reader may observe that there are two other potential phase portraits. One of them is the same as $QS135_1^{(4)}$ but changing the orientation of the finite anti-saddle and we denote it by $QS135_2^{(4)?}$. The other is the one that we obtain by adding the limit cycle to $QS135_2^{(3)}$ and we denote it by $QS135_{2LC}^{(3)?}$.

We can prove the existence of $QS135_1^{(4)}$ and $QS135_{1LC}^{(3)}$ by means of a bifurcation diagram and numerical examples. The other two do not appear in the bifurcation diagram but this does not discard that they could exist in a small island in the parameter space of the corresponding systems. This phenomena was first pointed out in [2] where the authors studied structurally unstable phase portraits of quadratic systems with codimension one. In this paper 7 skeletons of separatrices showing the same feature that happens with $QS135_1^{(4)}$ (a graphic with an internal anti-saddle with two possible stabilities of which only one appears to be realizable), were conjectured to produce one realizable and one non-realizable phase portrait. If case $QS135_2^{(4)?}$ were realizable, by means of a perturbation, $QS135_{2LC}^{(3)?}$ would also be realizable. Even the opposite is not compulsory, the existence of $QS135_{1LC}^{(3)?}$ would be a good starting point to look for an example of $QS135_2^{(4)?}$. Using the same arguments as in [2], we will also conjecture that $QS135_2^{(4)?}$ and $QS135_{2LC}^{(3)?}$ are non realizable, and so we add the "?" to their code.

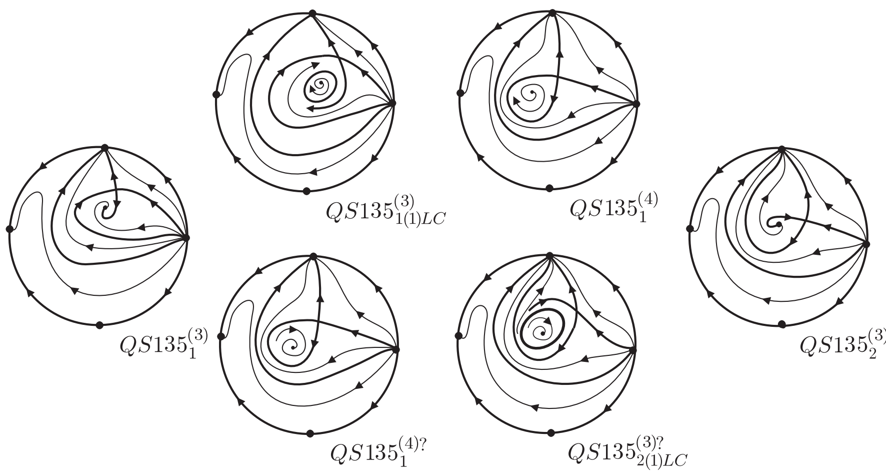


FIGURE 6. Phase portraits for the topological configuration (135).

Next we consider the case when at least one of the finite singularities appears on the separatrix on the phase portrait $QS142_1^{(3)}$. First we assume that this point is a saddle and we obtain the scheme (a) given in FIGURE 7. Notice that all the phase portraits generated by this scheme will have a (c) separatrix connection.

Suppose that the finite anti-saddle is unstable (i.e. a repelor). There is only one attractor point which is at infinity. Then we have one generic possibility and two non-generic. In the generic possibility the attractor at infinity must receive the two unstable separatrices of the saddle. This leads to the phase portrait $QS135_1^I$ which is unrealizable because a perturbation of it would be in contradiction with Proposition 1.

The two non-generic possibilities could be obtained when two finite separatrices form a loop obtaining two phase portraits: $QS135_2^I$ and $QS135_3^I$. We point out that both phase portraits are non-realizable because the phase portrait $QS135_1^I$ can be obtained by a perturbation of any of them.

Admit now that the finite anti-saddle is stable (i.e. an attractor). Then we have two non-generic possibilities for the α-limit of the unstable separatrix of the saddle ($QS135_4^I$ and $QS135_2^{(4)}$) and two non-generic possibilities: $QS135_5^I$ and $QS135_6^I$ (all of them given in FIGURE 7).

We determine that $QS135_i^I$, $i = 4, 5, 6$ are also non-realizable because the phase portrait $QS135_1^I$ (or a version of the same portrait with limit cycle) can be obtained by a perturbation of any of them.

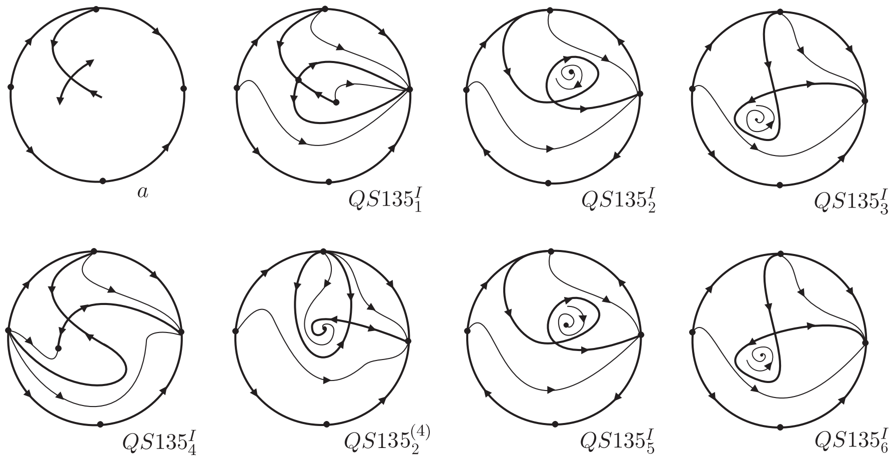
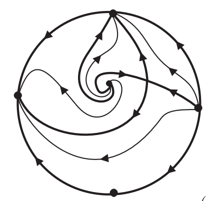


FIGURE 7. Phase portraits for the topological configuration (135) (cont.)

Assume finally that the finite anti-saddle appears on the separatrix. Then it must be an attractor. On the phase plane there exists a saddle located anywhere. We point out that on the Poincaré disc there exist two sources and two sinks and every source already sends some orbits to each sink close to the infinity as well as along the separatrix of the nilpotent singularity. Therefore each source and sink must received on separatrix from the finite saddle, producing the phase portrait $QS135_3^{(3)}$.



$QS135_3^{(3)}$

Next we present for each one of the realizable phase portraits the corresponding numeric example using the canonical form (33) with $e = -2$ and one free parameter c. In the case of the existence of a loop we will give an approximate set of parameters.

$QS135_1^{(3)}$ $c = -11/10$; $QS135_{1LC}^{(3)}$ $c = -9/10$; $QS135_2^{(3)}$ $c = -6/10$;
$QS135_3^{(3)}$ $c = 0$; $QS135_1^{(4)}$ $c \approx -0.72\ldots$; $QS135_2^{(4)}$ $c = -1/2$.

3.23. **The configuration 37:** $HHP - P$**.** According to [4] this configuration of infinite singularities leads to the following three global topological configuration of singularities:

$$(95)_a\ a;\ \binom{3}{3}HHP-P;\quad (95)_b\ a;\ \binom{1}{3}HHP-P;\quad (95)_c\ n_{(3)};\ \binom{1}{3}HHP-P;$$
$$(116)\ \ a, cp;\ HHP-P;\quad (111)\ \ a, sn;\ HHP-P;\quad (82)\ \ s, a, a;\ HHP-P.$$

According to [3] if a system possesses finite singularities of total multiplicity three, then this system cannot have intricate infinite singularities. So in the cases of the configurations $(95)_b$, $(95)_c$, (116), (111) and (82) the infinite singularity is a nilpotent singular point, whereas in the case $(95)_a$ at infinity we have an intricate singularity. We recall that by $n_{(3)}$ we denote a finite node of multiplicity 3.

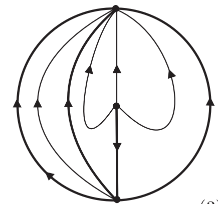
3.23.1. **The topological configuration** $(95)_a$**.** We recall that according to [5] the configuration (95) has topological codimension 3, but the configuration $(95)_a$ has a greater geometrical codimension and by [4] the affine invariant conditions which define this topological configuration are:

$$\mu_0 = \mu_1 = \mu_2 = 0,\ \mu_3 \neq 0,\ \kappa = 0,\ \widetilde{K} > 0,\ \tilde{L} = 0,\ C_2 \neq 0,\ K_3 > 0. \tag{34}$$

Since the configuration $(95)_a$ is from the class $m_f = 1$ and $\widetilde{K} \neq 0$, by Lemma 1 such a system belongs to the canonical systems $17a$). For these systems the conditions (34) imply $d = 0$, $f = c \neq 0$ and $c(c + 2e) > 0$.

Moreover we may assume $c = 1$ due to a time rescaling and we get the family of systems (5) but for these systems the condition $1 + 2e > 0$ must be satisfird.

According to [3] the finite anti-saddle must be a node. It is clear that systems (5) have the invariant straight lines $x = 0$ and $x = -1/2$ on which is located the node $M_1(0, 0)$. Due to the above conditions this leads to the unique phase portrait given by $QS95_1^{(3)}$.



$QS95_1^{(3)}$

3.23.2. **The topological configuration** $(95)_b$**.** According to [4] the affine invariant conditions which define this topological configuration are:

$$\mu_0 = 0,\ \mu_1 \neq 0,\ \mathbf{D} > 0,\ \kappa = 0,\ \widetilde{K} > 0,\ C_2 \neq 0,\ \tilde{L} = 0. \tag{35}$$

So the configuration $(95)_b$ is from the class $m_f = 3$ and according to [3] (see Table 6.1) such a system must belong to the canonical forms 11). So we consider the family of systems

$$\begin{aligned} \dot{x} &= 2(h - gu)x + g(u^2 + 1)y + gx^2 - 2hxy, \\ \dot{y} &= 2(m - lu)x + l(u^2 + 1)y + lx^2 - 2mxy, \quad gm - hl \neq 0, \end{aligned}$$

for which the conditions (35) imply: $h = 0$, $gm < 0$, $g + 2m = 0$ and $l \neq 0$. Moreover due to a time rescaling we may assume $m = 1$ and we arrive at the 2-parameter family of systems

$$\begin{aligned} \dot{x} &= 4ux - 2x^2 - 2(1 + u^2)y, \quad l \neq 0 \\ \dot{y} &= 2(1 - lu)x + l(u^2 + 1)y + lx^2 - 2xy. \end{aligned} \tag{36}$$

According to [3] the above systems possess either a node or a focus. Moreover the focus can be either strong or weak of order one, and in order to have a weak focus the condition $l = -4u/(u^2 + 1)$ is necessary and sufficient. It clear that the phase portraits of the systems (36) must have two separatrices, both at infinity and on the same part of the equator. Note also that due the change $(x, y, t, l, u) \mapsto (-x, y, -t, -l, -u)$ (which conserves systems (36)) there exists a symmetry in the parameter space with respect to the line $l = u$. So we will consider $u < 0$.

So we can have a separatrix connection ($(b)_\infty$ connection) or not, and both possibilities are realizable. Moreover in the case $l = -2u/(1+u^2)$ the loop can be realizable by the parabola $2(1+u^2)y = -x^2+2ux+1+u^2$. However we cannot know if the loop may exist in anothe r form including non-algebraic one. If the condition $l = -2u/(1 + u^2)$ holds (i.e. there exists an algebraic loop), the phase portrait is given by $QS95_1^{(4)}$ having a stable focus inside the loop (see FIGURE 8).

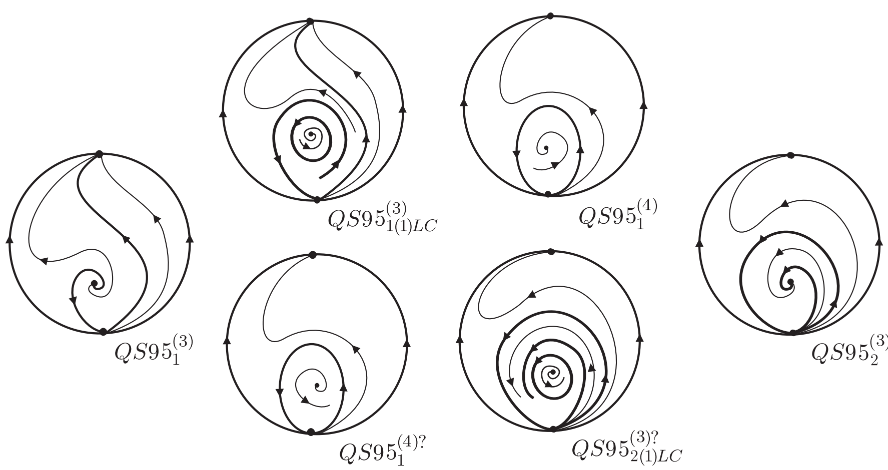


FIGURE 8. Phase portraits for the topological configuration (95)

If $l > 0$ and $l < -2u/(1+u^2)$ then the loop connection is broken and we get the phase portrait $QS95_2^{(3)}$. If $-2u/(1+u^2) < l < -4u/(1+u^2)$ then the loop connection is broken in other direction producing a limit cycle and we denote the obtained phase portrait as $QS95_{1LC}^{(3)}$. For $l = -4u/(1+u^2)$ the limit cycle disappears in a Hopf bifurcation and we get a phase portrait which is topological equivalent to the phase portrait which is obtained for $l > -4u/(1+u^2)$. We denote this phase portrait by $QS95_1^{(3)}$ and it is topological equivalent to the one obtained earlier for configuration $(95)_a$ but now without invariant lines.

Assuming $l < 0$ we obtain again $QS95_1^{(3)}$ due to the symmetry mentioned above.

We would like to point out that there are two other potential phase portraits that we denote by $QS95_1^{(4)?}$ and $QS95_{1LC}^{(3)?}$ given also in FIGURE 8. In case they exist they must live in an island of the parameter space as it was explained earlier discussing the topological configuration (135). So we conjecture them to be unrealizable.

3.23.3. **The topological configuration** $(95)_c$**.** According to [4] and [3] the affine invariant conditions which define this topological configuration are:

$$\mu_0 = 0,\ \mu_1 \neq 0,\ \mathbf{D} = \mathbf{P} = 0,\ \kappa = 0,\ \widetilde{K} > 0,\ \mathcal{F}_1 \neq 0,\ \tilde{L} = 0. \tag{37}$$

So the configuration $(95)_c$ is from the class $m_f = 3$ containing a triple finite anti-saddle which is a node. According to [3] (see Table 6.1) such a system must belong to the canonical forms 13). So we consider the family of systems

$$\dot{x} = gy + gx^2 + 2hxy, \quad \dot{y} = ly + lx^2 + 2mxy, \quad gm - hl \neq 0,$$

for which the conditions (37) imply: $h = 0$, $gm > 0$, $g - 2m = 0$ and $l \neq 0$. Moreover due to a time rescaling we may assume $m = 1$ and we arrive at the following 1-parameter family of systems

$$\dot{x} = 2(x^2 + y) \quad \dot{y} = ly + lx^2 + 2xy, \quad l \neq 0.$$

Due to the rescaling $(x, y, t) \mapsto (-x, y, -t)$ we may assume $l > 0$. Then the finite triple node is a repelor and we arrive in unique mode to a phase portrait which is topologically equivalent to $QS95_1^{(3)}$.

3.23.4. **The topological configuration** (111)**.** We recall that from [5], configuration (111) has topological codimension 4. According to [4] and [3] the affine invariant conditions which define this topological configuration are:

$$\mu_0 = 0,\ \mu_1 \neq 0,\ \mathbf{D} = \kappa = 0,\ \mathbf{P} \neq 0,\ \widetilde{K} > 0,\ E_1 \neq 0,\ C_2 \neq 0,\ \tilde{L} = 0. \tag{38}$$

So the configuration (111) is from the class $m_f = 3$ containing on the phase plane one semi-elemental saddle-node and one elemental anti-saddle. According to [3] (see Table 6.1) such a system must belong to the canonical forms 12). So we consider the family of systems

$$\dot{x} = cx + cy - cx^2 + 2hxy, \quad \dot{y} = ex + ey - ex^2 + 2mxy, \quad cm - eh \neq 0, \tag{39}$$

for which the conditions (38) imply: $h = 0$, $cm < 0$, $c + 2m = 0$ and $e(e - 2m) \neq 0$. Moreover due to a time rescaling we may assume $m = 1$ and we arrive at the following 1-parameter family of systems

$$\dot{x} = 2(-x - y + x^2) \quad \dot{y} = ex - ex^2 + ey + 2xy, \quad e(e - 2) \neq 0. \tag{40}$$

For $e = -2$ we have the invariant parabola $-2x + x^2 - 2y = 0$. Moreover for $e = 1$ we have the second invariant parabola $-x + x^2 - 2y = 0$.

We need also the invariant polynomial $\mathcal{B}_1$ in order to distinguish if a focus is weak or not. For systems (40) we calculate $\mathcal{B}_1 = -8(e-2)^2(e+4)$. As a result we obtain the following algebraic bifurcation values of the parameter e: $e \in \{-4, -2, 0, 1, 2\}$. Moreover we detect that there exists one non-algebraic bifurcation value $-3.168 < e^* < -3.167$, where limit cycle disappear in a loop.

Following the parameter line we have the phase portraits given in FIGURE 9.

We point out that among the phase portraits corresponding to the topological configuration (111), on the bifurcation diagram we also present two phase portraits which are on the border of systems (40): one with

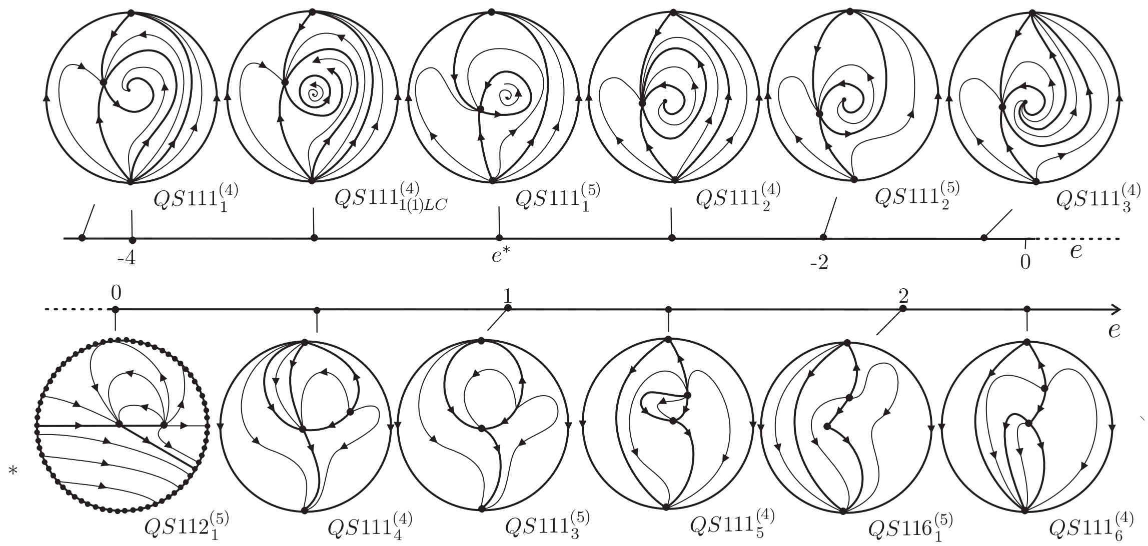


FIGURE 9. Bifurcation diagram for the systems (40)

the infinite line filled up with singularities ($QS112_1^{(5)}$) and another with the cusp ($QS116_1^{(5)}$). Both phase portraits have codimension 5 according to [5].

We mention that there are other potential phase portraits for the configuration (111), as for example, a similar to the phase portrait $QS111_1^{(5)}$, but with the internal anti-saddle having opposite stability. Following the argumentation applied up to now, either this phase portrait lives in an island in the corresponding parameter space or must be conjectured impossible.

Remark 2. *We remark that up to this moment we have detected all realizable phase portraits, proved the non-existence of some potential ones as well as conjectured some of them as impossible. But starting from this configuration and the ones remaining to study, the number of potential phase portraits increases a lot. And the technique applied here is not so effective as the technique used in papers like [1] and [2] for a complete clarification of the situation in discussion. So we let for another paper using those techniques, the study of the rest of potential phase portraits of this configuration and the ones that may remain from the remaining ones.*

3.23.5. **The topological configuration** (116)**.** . From [5], configuration (116) has topological codimension 5. According to [4] and [3] the affine invariant conditions which define this topological configuration are:

$$\mu_0 = 0,\ \mu_1 \neq 0,\ \mathbf{D} = \kappa = 0,\ \mathbf{P} \neq 0,\ \widetilde{K} > 0,\ E_1 = 0,\ \tilde{L} = 0. \tag{41}$$

So the configuration (116) is from the class $m_f = 3$ containing on the phase plane one nilpotent cusp and one elemental anti-saddle. According to [3] (see Table 6.1) such a system must belong to the canonical forms 12). So we consider the family of systems (39) for which the conditions (41) imply: $h = 0$, $cm < 0$, $c + 2m = 0$ and $e - 2m = 0$. Moreover due to a time rescaling we may assume $m = 1$ and we arrive at the system which belong to the family (40) for $e = 2$. According to bifurcation diagram presented in FIGURE 9 we have the phase portrait $QS116_1^{(5)}$

3.23.6. **The topological configuration** (82)**.** We recall that from [5], configuration (82) has topological codimension 3. According to [4] and [3] the affine invariant conditions which define this topological configuration are:

$$\mu_0 = 0,\ \mu_1 \neq 0,\ \mathbf{D} < 0,\ \kappa = 0,\ \widetilde{K} > 0,\ C_2 \neq 0,\ \tilde{L} = 0. \tag{42}$$

So the configuration (82) is from the class $m_f = 3$ containing on the phase plane three real elemental singularities: and saddle and two anti-saddles. According to [3] (see Table 6.1) such a system must belong to the canonical forms 10). So we consider the family of systems

$$\dot{x} = cx + dy - cx^2 + 2hxy, \quad \dot{y} = ex - ex^2 + fy + 2mxy, \quad (eh - cm)(fh - dm) \neq 0,$$

for which the conditions (42) imply: $h = 0$, $cm < 0$, $c + 2m = 0$ and $de \neq 0$. Moreover we may assume $m = d = 1$ due to the rescaling $(x, y, t) \mapsto (x, my/d, t/m)$. Therefore we arrive at the following 2-parameter family of systems

$$\dot{x} = -2x + y + 2x^2, \quad \dot{y} = ex + fy - ex^2 + 2xy, \quad e(e + 2f)(e + 2f + 4) \neq 0.$$

In order to obtain a symmetry in the bifurcation diagram of the above systems we replace f by $f - 1$ obtaining the systems

$$\dot{x} = -2x + y + 2x^2, \quad \dot{y} = ex + (f - 1)y - ex^2 + 2xy, \quad e(e + 2f - 2)(e + 2f + 2) \neq 0. \tag{43}$$

Moreover we detect that the transformation $(x, y, t, e, f) \mapsto (-x+1, y, -t, -e, -f)$ conserves the above systems.

Thus we conclude that in order to detect all the possible phase portraits of the systems (43) it is sufficient to consider the condition $e > 0$.

We determine that systems (43) possess the following three finite singularities (one saddle and two anti-saddles):

$$M_1(0, 0), \quad M_2(1, 0), \quad M_3\Big(\frac{1}{4}(2 - e - 2f),\ \frac{1}{8}\big(4 - (e + 2f)^2\big)\Big).$$

We also detect that these systems could have one of the following tree invariant parabolas:

$$\begin{aligned}\Phi_1(x, y) =& -(4 + f - 1)x + x^2 + y = 0 \quad \text{if } \ e = 2(f + 3);\\ \Phi_2(x, y) =& f - 2 + (1 - f)x + x^2 + y = 0 \quad \text{if } \ e = 2(f - 3);\\ \Phi_3(x, y) =& -x + x^2 + y = 0 \quad \text{if } \ e = -f.\end{aligned}$$

We have to study the two parameter family of quadratic systems (43) and we will apply the same technique as used in papers like [1], [2] and [6].

We begin by detecting all the relevant bifurcations related to singularities plus those related to invariant parabolas and later we add the bifurcations related to non-algebraic separatrix connections. For these family of systems the relevant algebraic bifurcations are:

- $\mathbf{D} = -192(e + 2f - 2)^2(e + 2f + 2)^2 = 0$, where tho finite singularities coalesce;
- $\mathcal{B}_1 = 2(f - 3)(f + 3)(3e + 4f) = 0$, where we have a finite weak singularity;
- $\zeta_{22} = -288(e - 6 - 2f)(e + 6 - 2f)(e + f) = 0$, where there exist at least one invariant parabola;
- $C_2 = ex^3 = 0 \Rightarrow e = 0$, where we have the infinite line filled up with singularities.

These bifurcations split the parameter space into a set of regions of dimensions 2, 1 and 0. We will denote the two-dimensional regions by S_i with $i \in \mathbb{N}$.

The one-dimension regions are denoted by kL_i with $i \in \mathbb{N}$ and $k = 2$ (respectively 3; 4; 5) if $\mathbf{D} = 0$ (respectively $\mathcal{B}_1 = 0$; $\zeta_{22} = 0$; $C_2 = 0$) and $k = 7$ if the region corresponds to a non-algebraic bifurcation[1].

The zero-dimension regions (just points) are denoted by $m.kP_i$ with $i \in \mathbb{N}$ where m and k correspond to the two geometrically most relevant bifurcations curves that intersect at these points (the values of m and k are defined above).

Moreover in the bifurcation diagram we have used dashed lines if that part of the bifurcation diagram is not relevant, i.e. does not implies a topological change.

In a more generic system we usually find the bifurcation $\mathbf{D} = 0$ so we obtain different sign of $\mathbf{D}$ on different sides of the bifurcation. By [3, Diagram 6.1] changing the sign of $\mathbf{D}$ two of the real singularities after coalescence pass to two complex singularities. However for systems (43) we have $\mathbf{D} \leq 0$ (see its value above)

[1]The number used to denote different bifurcations are not in correlative order because we use the same codes applied in many previous papers for similar bifurcations

and then the two finite singularities that coalesce due to this bifurcations split also in two real. So for example $S_4 \equiv S_9$, $S_6 \equiv S_7$ and $7L1 \equiv 7L2$ and even $S_1 \equiv S_{19}$ because we make a path from one to the other passing through non-topological relevant bifurcations. On this bifurcation curve (defined by $\mathbf{D} = 0$) we will find the same phase portraits that we obtained before when study the topological configuration (111) and (116).

Next we prove the existence of non-algebraic bifurcation and describe its position. On the region $3L3$ systems (43) possess a weak focus and the invariant $\mathcal{B}_1$ changes the sign passing through the bifurcation implying the change of stability of the focus (by Hopf bifurcation). This implies the existence of a limit cycle in one of two regions having as a common border $3L_3$. Since the region producing limit cycle does not extend to the all region S_5 there must be a bifurcation curve $7L_1$ splitting S_5 from S_6 and the existence of this curve is numerically confirmed.

We point out that the non-algebraic bifurcation curve must cut the bifurcation curve $\mathbf{D} = 0$ at a point $2.7P_1$ separating $2L_2$ from $2L_3$ and enters the next region as $7L_2$ splitting S_7 from S_8.

We claim that this non-algebraic curve must end at $2.3P_1$ without crossing any other algebraic bifurcation. Indeed it can not cross $3L_4$ because $\mathcal{B}_1$ changes the sign when crossing $3L_4$ and then the point of intersection would have to imply a weak focus of higher degree or a center which are not possible in this family of systems by [3, Theorem 6.2]. We point out that at $2.3P_1$ we have a phase portrait having a cusp which by Bogdanov-Takens bifurcation may produce phase portraits with limit cycles or loops as we have in the Diagram given in FIGURE 10. In case the non-algebraic bifurcation would cross $3L_5$ we would have a weak saddle having a loop which would implies the existence of two limit cycle closed to this point. Once inside region S_{16} the non-algebraic bifurcation would have to arrive at the point $2.3P_1$ because it can not cross the other borders of S_{16}. Even it is possible that a cusp may produce more that one limit cycle this is not possible when the conditions for the weak focus are are the conditions given by statement (e_2) of Theorem 6.2 from [3]. This completes the proof of our claim.

The list of phase portraits that the family of systems (43) can have is given in FIGURE 12. Here are listed only topologically distinct phase portraits of these systems. Moreover for each one of these phase portraits we will indicate all the regions in which it exists as well the definitive name in our codification. First we enumerate the phase portraits of the topological configuration (82).

(1) $QS82_1^{(3)}$ is the name of the phase portrait S_1 which is equivalent to the phase portraits corresponding to the regions: S_2, S_4, S_9, S_{13}, S_{14}, S_{15}, S_{19}, S_{20}, $3L_2$, $3L_6$, $3L_9$, $4L_1$, $4L_6$ and $4L_7$.
(2) $QS82_2^{(3)}$ is the name for $S_3 \equiv S_5 \equiv S_8 \equiv S_{16} \equiv S_{18} \equiv S_{21} \equiv 3L_1 \equiv 3L_5 \equiv 3L_8$.
(3) $QS82_3^{(3)}$ is the name for $S_{10} \equiv S_{11} \equiv S_{12} \equiv 3L_3 \equiv 3L_4 \equiv 3L_7$.
(4) $QS82_{3(1)LC}^{(3)}$ is the name for $S_6 \equiv S_7 \equiv S_{17}$.
(5) $QS82_1^{(4)}$ is the name for $4L_2 \equiv 4L_3 \equiv 4L_4 \equiv 4L_5 \equiv 4L_8 \equiv 4L_9 \equiv 3.4P_1 \equiv 3.4P_2 \equiv 3.4P_31$.
(6) $QS82_2^{(4)}$ is the name for $7L_1 \equiv 7L_2 \equiv 7L_3$.

Next we describe the phase portraits which appear on the border of normal form (43).

(1) $QS111_1^{(4)}$ is the phase portrait corresponding to the regions $2L_1 \equiv 2.3P_1$.
(2) $QS111_{1(1)LC}^{(4)}$ for $2L_2$.
(3) $QS111_1^{(5)}$ for $2.7P_1$.
(4) $QS111_2^{(4)}$ for $2L_3$.
(5) $QS111_2^{(5)}$ for $2.4P_1$.
(6) $QS111_3^{(4)}$ for $2L_4$.
(7) $QS112_1^{(5)}$ for $2.5P_1 \equiv 2.5P_2$.
(8) $QS111_4^{(4)}$ for $2L_5$.
(9) $QS111_3^{(5)}$ for $2.4P_2$.
(10) $QS111_5^{(4)}$ for $2L_6$.
(11) $QS116_1^{(5)}$ for $2.3P_1$.

(12) $QS111_6^{(4)}$ for $2L_7$.

In the border of systems (43) there also appears the phase portrait corresponding to region $5L_1$ whose topological configuration of singularity is (83). By [4] this configuration has codimension 4 and we denote the corresponding phase portrait by $QS83_1^{(4)}$. This phase portrait also occurs in the regions $5L_2$, $5L_3$, $5L_4$, $5L_5$, $5L_6$, $4.5P_l$, $4.5P_2$ and $4.5P_3$.

We point out the bifurcation diagram given in Figure 10 may contains islands with different phase portrait inside as it was mentioned earlier (see Remark 2). So we could not consider this classification being complete until a complete topological study of all potential phase portraits will be done.

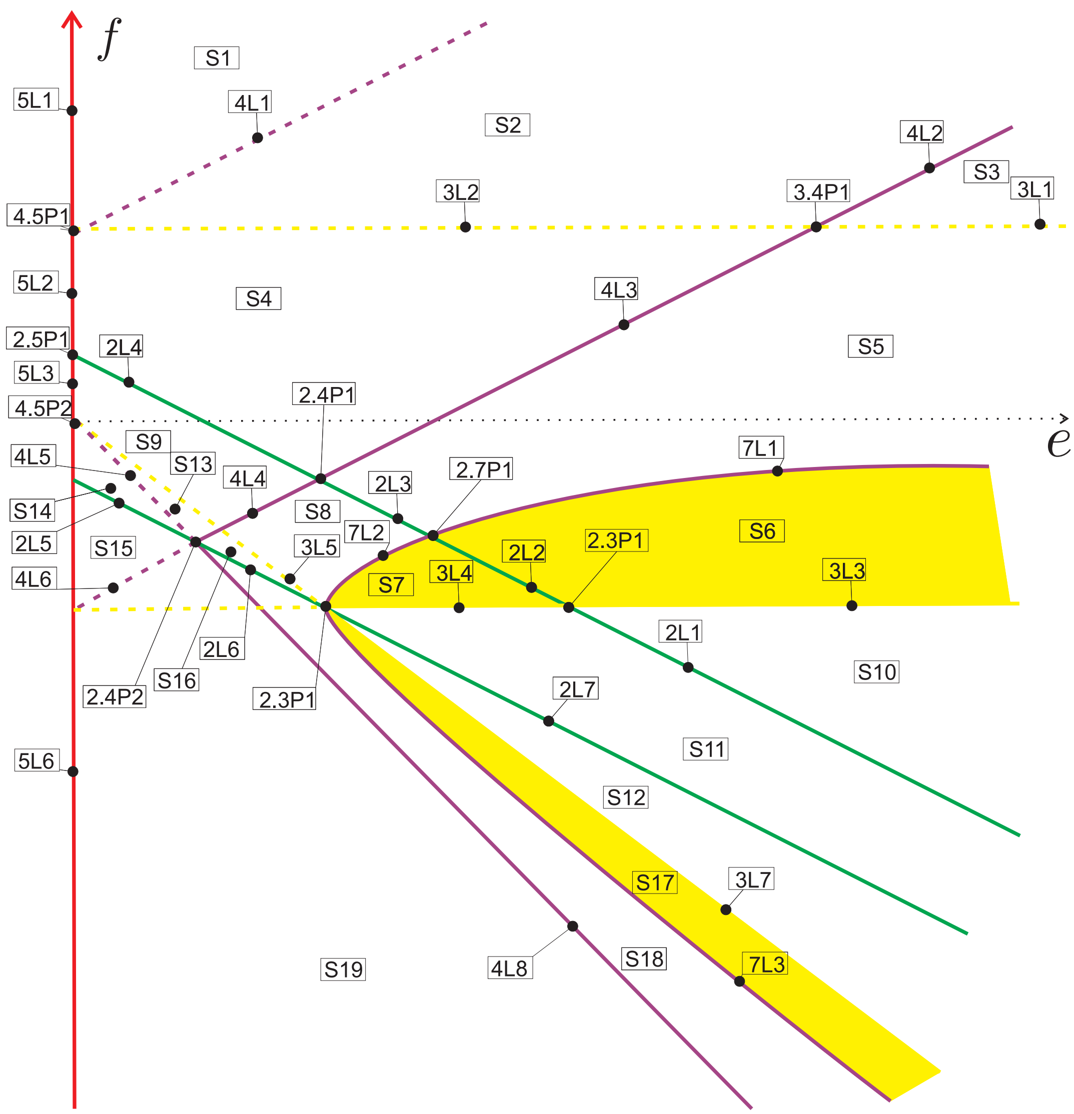


Figure 10. Bifurcation diagram for the systems (43)

3.24. **The configurations 9: $E-H,N$; 16: $E-H,S$, 23: $HHH-H,N$ and 26: $PHP-E,S$.**

We will study all four configurations together since all the information concerning their classification is contained in a set of bifurcation diagrams given in [6].

According to [4] each one the the mentioned configurations of infinite singularities leads to the following global topological configurations of singularities, correspondingly:

- 9: $E-H,N$

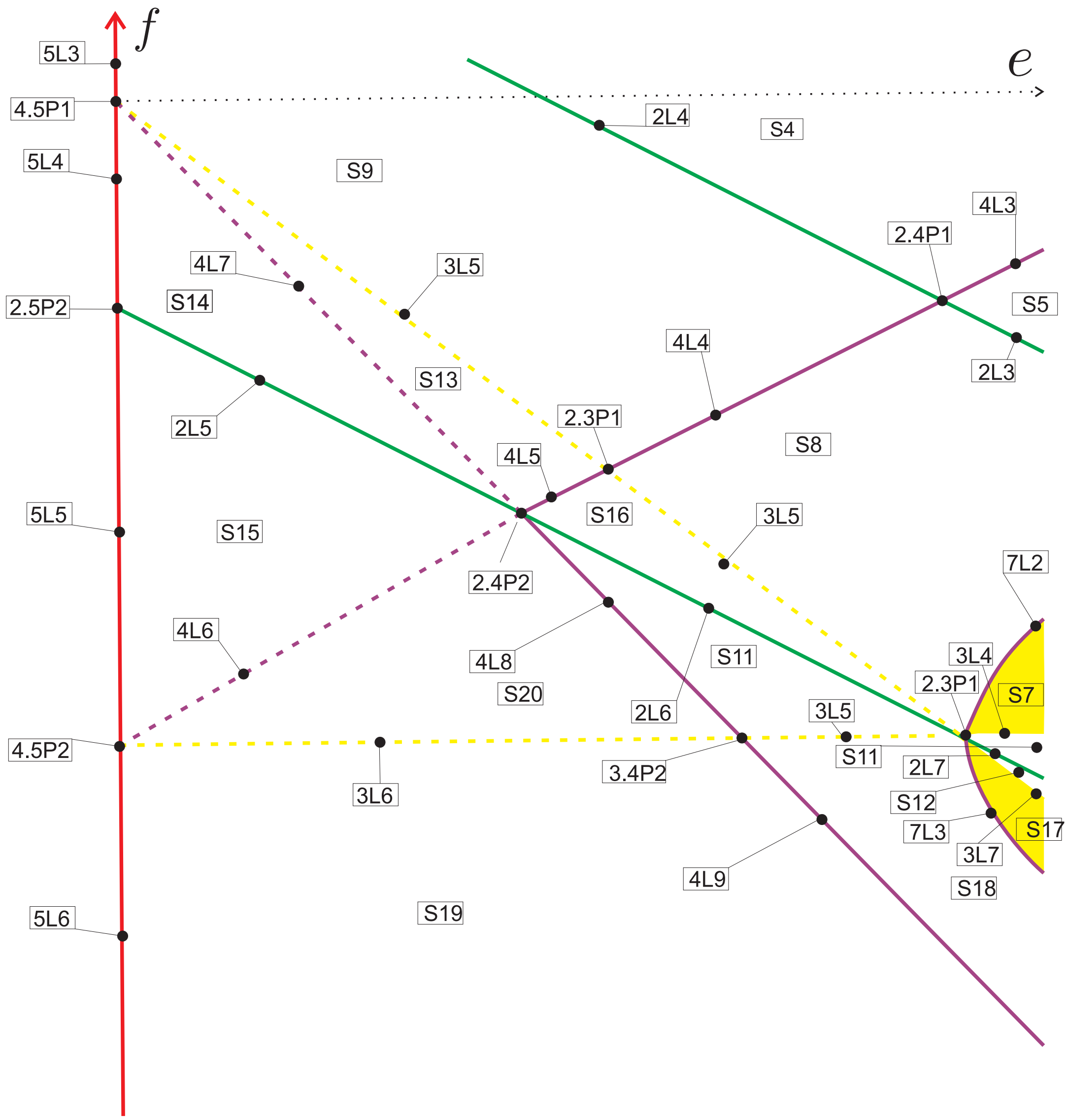


FIGURE 11. Magnification of part of FIGURE 10

(77): $s, s, a;\ \binom{1}{2}E - H, N$
(88): $s;\ \binom{1}{2}E - H, N$
(106): $s, sn;\ \binom{1}{2}E - H, N$
(107): $s, cp;\ \binom{1}{2}E - H, N$

- 16: $E - H, S$

(79): $s, a, a;\ \binom{1}{2}E - H, S$
(89): $a;\ \binom{1}{2}E - H, S$
(119): $es;\ \binom{1}{2}E - H, S$
(108): $a, sn;\ \binom{1}{2}E - H, S$
(113): $a, cp;\ \binom{1}{2}E - H, S$

- 23: $HHH - H, N$

(81): $s, a, a;\ \binom{1}{2}HHH - H, N$
(93): $a;\ \binom{1}{2}HHH - H, N$
(121): $es;\ \binom{1}{2}HHH - H, N$
(110): $a, sn;\ \binom{1}{2}HHH - H, N$

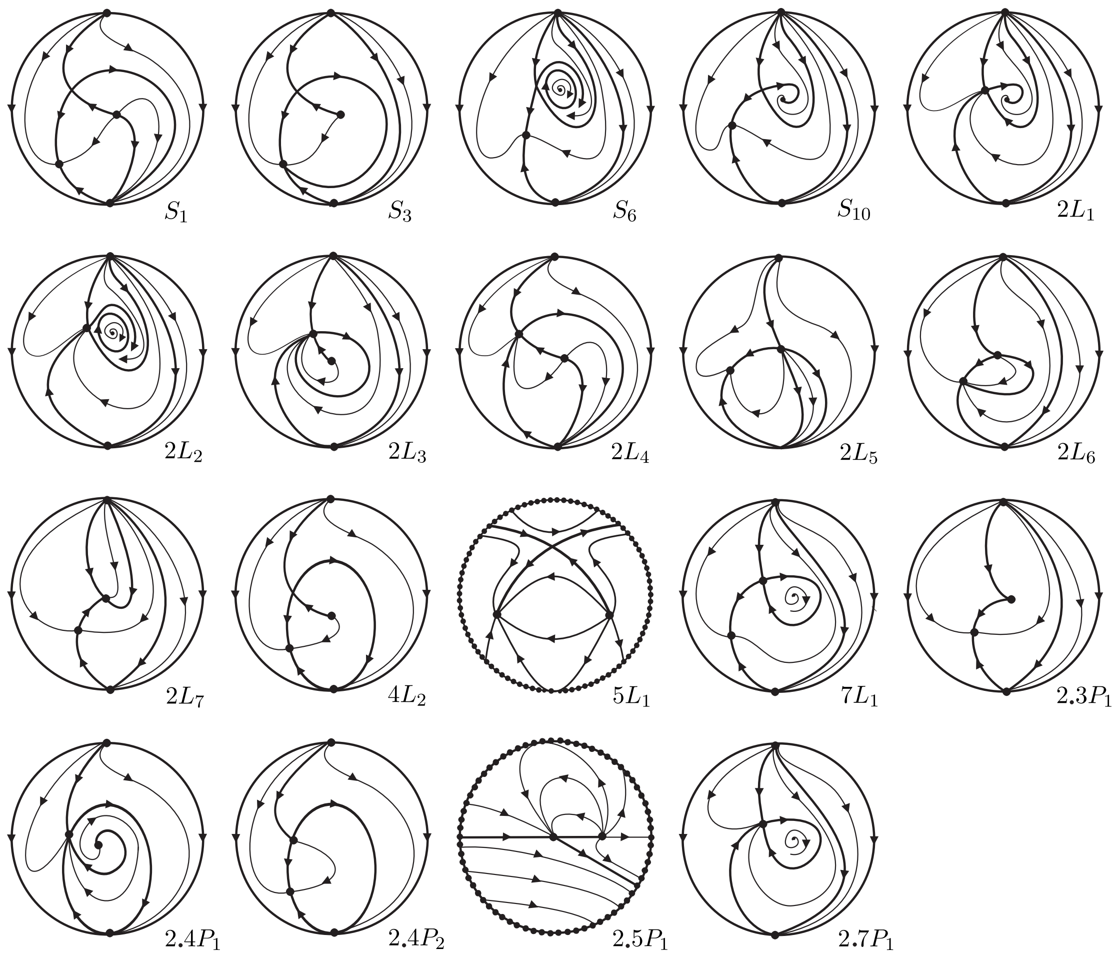


FIGURE 12. Phase portraits of systems (43)

(115): a, cp; $\binom{1}{2}HHH - H, N$

- 26: $PHP - E, S$

(80): s, a, a; $\binom{1}{2}PHP - E, S$
(91): a; $\binom{1}{2}PHP - E, S$
(120): es; $\binom{1}{2}PHP - E, S$
(109): a, sn; $\binom{1}{2}PHP - E, S$
(114): a, cp; $\binom{1}{2}PHP - E, S$

In paper [6] the authors have classified all the phase portraits of the family of quadratic systems having an infinite elliptic saddle or a nilpotent saddle obtaining 124 distinct phase portraits. Most of them belong to the 19 topological configurations presented above, whereas a few others are in the borders of the studied normal forms. We point out that the phase portrait corresponding to the borders may belong to some more degenerate configurations (possessing a nilpotent or intricate singularity) studied above and having already a name. However there could be some phase portraits obtained earlier in other papers and following our strategy (see Notation 1) we attached to each one of them the corresponding name.

Next we extract from paper [6] all the phase portraits and split them in 4 sets corresponding to different topological configurations of infinite singularities (i.e. 9, 16, 23 and 26) given above. In addition we present also a fifth set containing the phase portraits corresponding to the borders of the normal forms considered in [6].

For each phase portrait we attach two labels: one is the definitive name which we give according to Notation 1 and another one corresponds to the labels given in paper [6]. In this paper three normal forms **A, B** and **C**. The phase portrait **A** (respectively **B**; **C**) are given in Fig.1,2 and 3 (respectively Fig.4; Fig.5). We have insert the corresponding letter A, B or C in front of the label given by authors in [6] in order to recognize the normal form possessing the corresponding portraits.

Considering [6], [4] and Notation 1 we arrive at the following set of four lemmas.

Lemma 2. *The topological configuration of infinite singularities 9 has the 20 realizable phase portraits given in* FIGURE *13.*

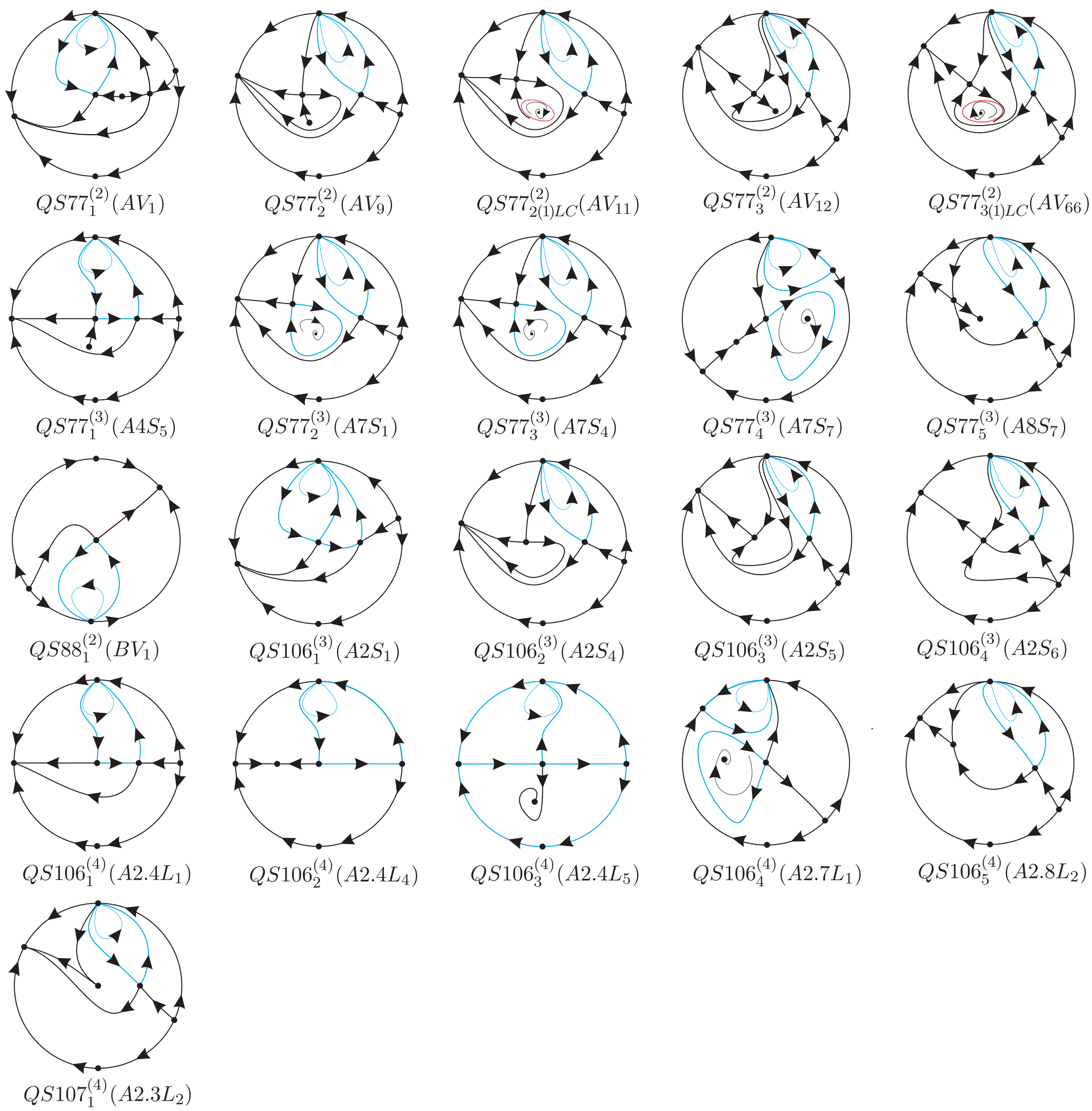


FIGURE 13. Phase portraits with the configuration of infinite singularities 9

Lemma 3. *The topological configuration of infinite singularities 16 has the 19 realizable phase portraits given in* FIGURE *14.*

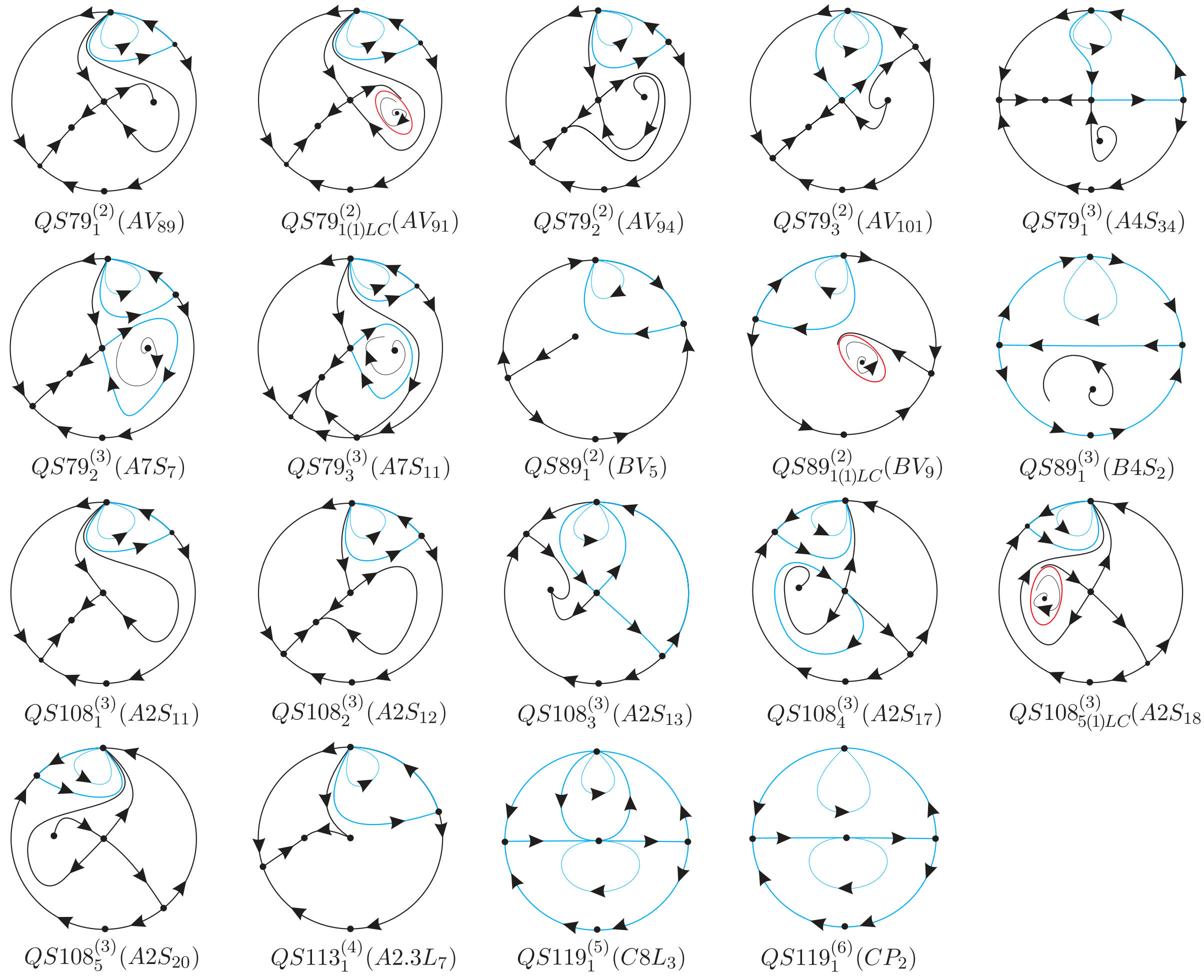


FIGURE 14. Phase portraits with the configuration of infinite singularities 16

Lemma 4. *The topological configuration of infinite singularities 23 has the 20 realizable phase portraits given in* FIGURE *15.*

Lemma 5. *The topological configuration of infinite singularities 26 has the 31 realizable phase portraits given in* FIGURE *16.*

Lemma 6. *The borders of the normal forms* **A, B** *and* **C** *of [6] produce the 33 realizable phase portraits given in* FIGURE *17.*

4. Miscelanea

When the authors of [3] split the set of 1764 geometric configurations of singularities into 208 topologically classes in [4], the goal was to concentrate the research of topologically different phase portraits into a more reduced set of classes with the assurance that if a phase portrait belonged to one class, could not belong to another one. In this sense it seemed clear that a phase portrait having the geometric configuration of singularities $s, n; \overline{\binom{0}{3}}N$, or $s, n; \overline{\binom{2}{1}}N$, Ⓒ, Ⓒ or $\overline{s}_{(3)}, n;$ N, Ⓒ, Ⓒ or $s, \overline{n}_{(3)};$ N, Ⓒ, Ⓒ would not have a phase portrait topologically different from the ones that may be obtained for the most generic configuration $s, a; N$. Of course, il will be normal that the most generic configuration may have phase portraits that cannot be obtained in more degenerated configurations.

The phase portraits with a finite intricate singularity are already known, and those with a finite nilpotent singularity appeared in [8] (with a couple of mistakes that we have pointed out here). The phase portraits

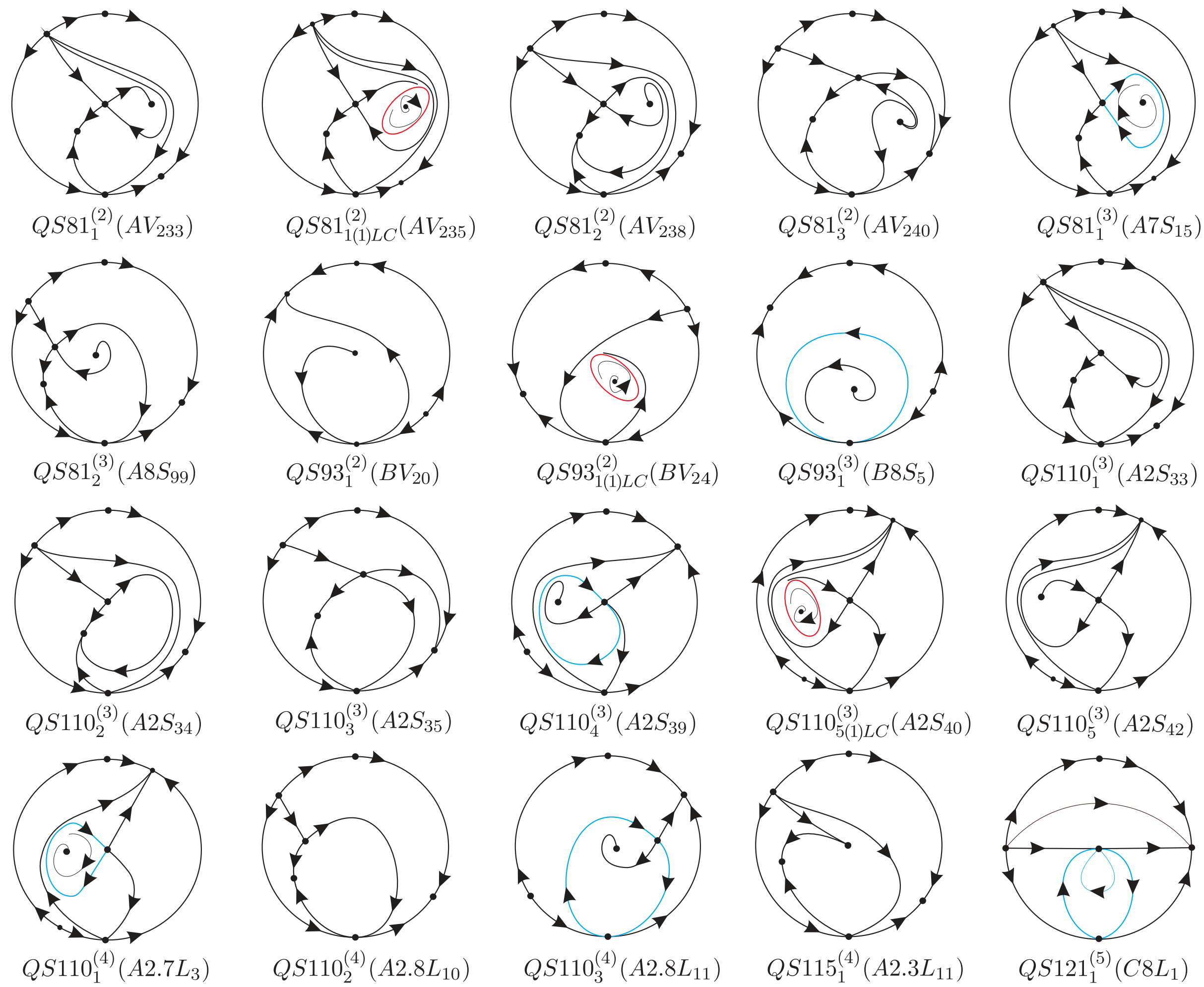


FIGURE 15. Phase portraits with the configuration of infinite singularities 23

of quadratic systems with the infinite line filled with singularities are also known, and the phase portraits of degenerate quadratic systems are easy to find and will be done in a future paper so to assign a name to each one of them.

Thus, the goal of this current paper was to determine all the topologically different phase portraits of quadratic systems having a nilpotent or intricate singularity at infinity, which cannot be obtained with elemental and semi-elemental singularities. In this way we complete the less generic phase portraits of quadratic systems. We want to be sure that after these steps are done, all other phase portraits remaining to be classified have at most semi-elemental singularities (finite or infinite).

However since we have not proved that any phase portrait obtained from a geometric very degenerate configuration of singularities must always be realizable for a less degenerate configuration, such possibilities must be checked carefully. Indeed the next subsection deals with one of these possibilities.

4.1. **The phase portrait $QS91_1^{(4)}$.** After having produced the first draft of phase portraits which correspond to topological configuration (91): a; $\binom{1}{2}PHP-E, S$ we thought that it was complete. However comparing our results with some papers with which this paper has intersections, we realized that at least one phase portrait was missing in our work. The fact is that the reduction of geometric configurations of singularities into topological ones, produces that some configurations with intricate or nilpotent singularities become identified with configurations with more generic singularities. So it is not guaranteed that certain phase portraits may be realizable only when the singularity is the intricate one and it is not realizable with a topologically equivalent

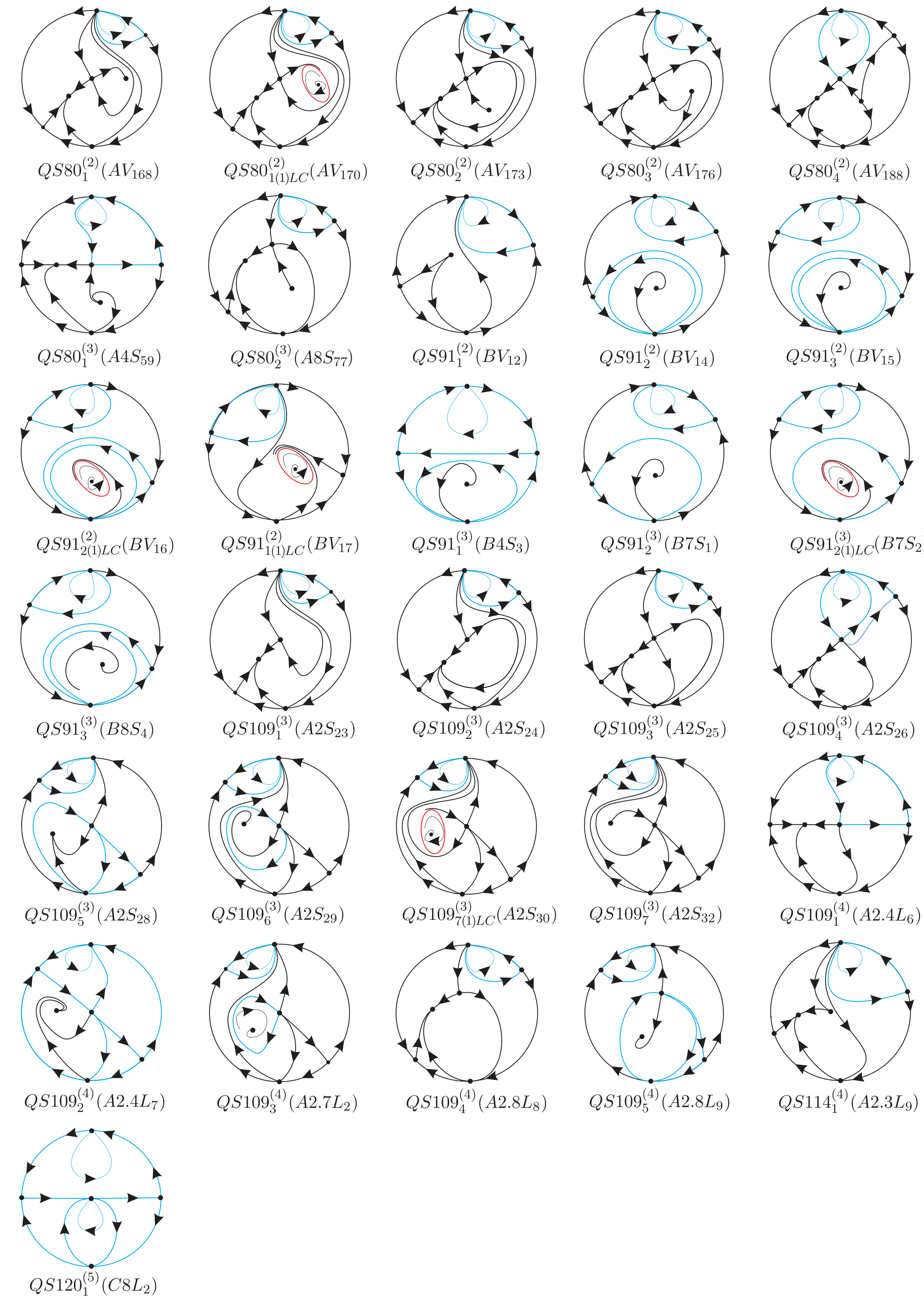


FIGURE 16. Phase portraits in the border with the configuration of infinite singularities 26

nilpotent, semi-elemental or elemental, even if it is locally topologically equivalent. This has forced us to

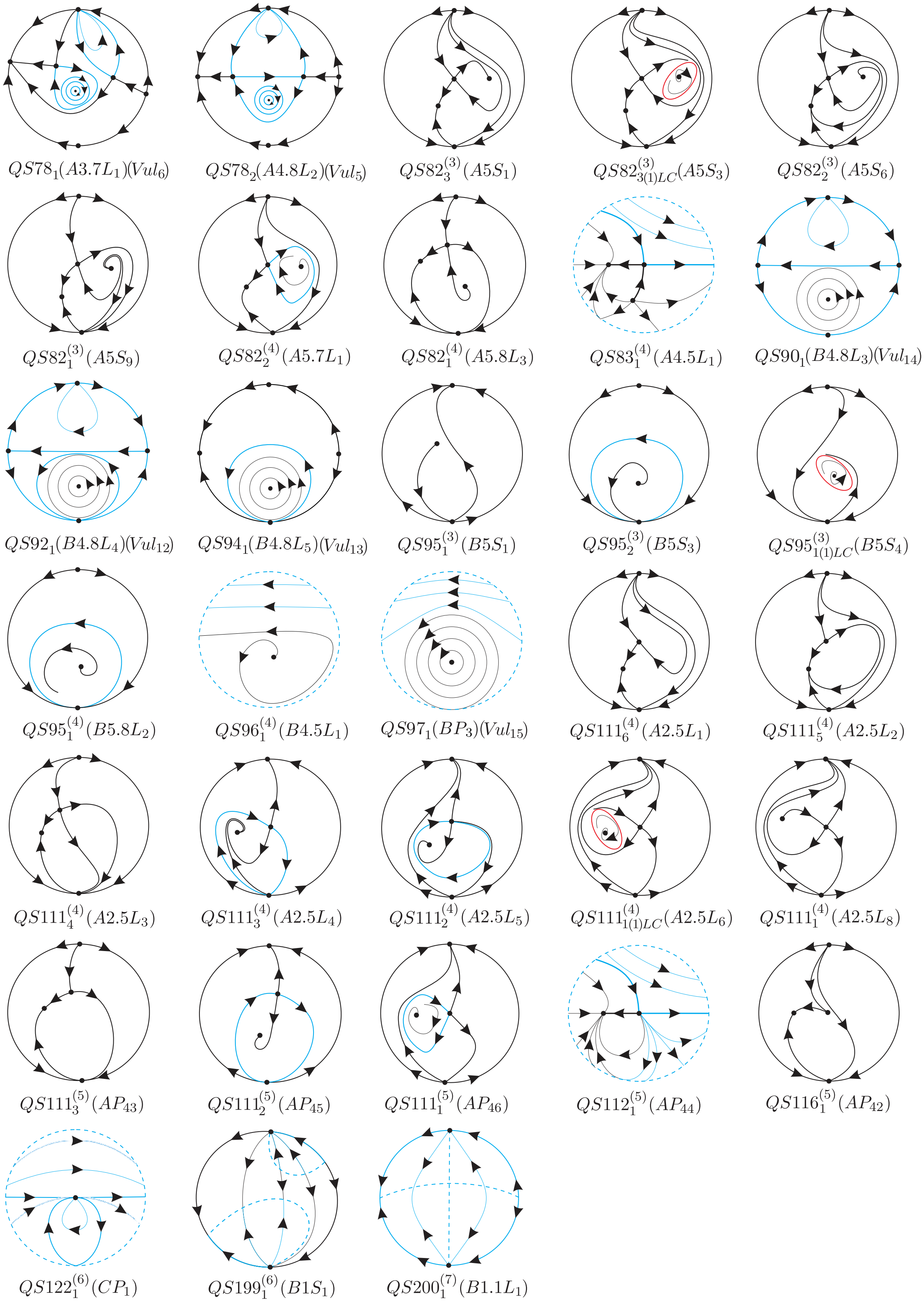
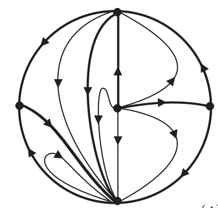


FIGURE 17. Phase portraits corresponding to the borders of normal forms **A, B** and **C** of [6]

investigate here all the possible cases in which this phenomenon may happen, even though at the end, it has appeared a single case which it is the one we first discovered. This has forced us to investigate here all the possible cases in which this phenomenon may happen, even though at the end, it has appeared a single case which it is the one we first discovered. More precisely, the phase portrait that has helped us to complete the study is the phase portrait E24 from [7] that we rename here as $QS91_1^{(4)}$. This phase portrait has the geometric configuration of singularities $n\,;\ \binom{3}{2}\ \widehat{P}H\widehat{P}-\widehat{P}\,\widehat{P}E,\ S$ which is topologically equivalent to $a\,;\ \binom{1}{2}\ PHP\!-\!E,\ S$, that is the configuration (91) inside the infinite configuration 23.



$QS91_1^{(4)}$

Notice that infinite configurations 9, 16, 23 and 26 have not been studied in the same way as all previous configurations, that is, we have not studied all the possible geometric conditions that lead to every topological configuration of singularities, thus confirming that every case is considered. For the cases 9, 16, 23 and 26, we have relied in paper [6] but in this paper only the case $m_f = 3$ is studied, i.e. the non elemental singularity at infinity must be always a nilpotent singularity of multiplicity 3 formed by the coalescence of a finite singularity with two infinite singularities. So, if a phase portrait with geometric configuration $n\,;\ \binom{3}{2}\ \widehat{P}H\widehat{P}-\widehat{P}\,\widehat{P}E,\ S$ is not realizable with a topologically equivalent singularity, then we are missing it.

We will start by proving that the mentioned geometric configuration has only the phase portrait $QS91_1^{(4)}$ and we confirm that the codimension must be 4 due to the multiplicity of the infinite intricate singularity.

Indeed, according to [4] and [3] the affine invariant conditions which define this geometric configuration are:

$$\mu_0 = \mu_1 = \mu_2 = 0,\ \mu_3 \neq 0,\ \kappa = 0,\ \widetilde{K} > 0,\ \tilde{L} > 0. \tag{44}$$

So this configuration is from the class $m_f = 1$ containing on the phase plane one elemental anti-saddle. In this case, instead of using a normal form from Lemma 1 which is based on the finite singularities, we better use a normal form based on infinite singularities. So we consider the family of systems

$$\begin{aligned} \dot{x} =& \quad a + cx + dy + gx^2 + hxy, \\ \dot{y} =& \quad b + ex + fy + (g-1)xy + hy^2. \end{aligned}$$

Since there must exist one finite singularity, we may assume that $a = b = 0$. For this system we obtain that $\kappa = -16h^2$, thus we set $h = 0$. Then $\tilde{K} = 2(g-1)gx^2 > 0$ and $\mu_1 = d(g-1)^2gx+0$. So $d = 0$ and $(g-1)g > 0$. Moreover, $\mu_2 = fg(c - cg + fg)x^2$ and $\mu_3 = cfx^2\big[-egx + (cg - c - fg)y\big]$. Since $\mu_3 \neq 0$ then $f \neq 0$ and $\mu_2 = 0$ implies $c = -fg/(1-g)$. Then we obtain the family system

$$\begin{aligned} \dot{x} =& \quad \tfrac{fg}{1-g}x + gx^2, \\ \dot{y} =& \quad ex + fy + (g-1)xy. \end{aligned}$$

This system has clearly two invariant vertical lines $x(x - f/(1-g)) = 0$ and this is incompatible with the presence of an infinite nilpotent singularity at $[0:1:0]$. The finite singular point is on $x = 0$ and the infinite saddles have only one candidate to send their separatrices. So the phase portrait is unique and corresponds to $QS91_1^{(4)}$.

4.2. **Other possible cases.** After the case studied in the previous subsection, it is compulsory to detect all other possible situations investigated in this paper where this phenomenon may occur. And it is also important to detect which geometric configurations with nilpotent and intricate singularities have been reduced to topological configurations with semi-elemental or elemental singularities. In case there appeared some new phase portraits, they must be included in this paper so to let for a future work the most generic cases with the assurance that we do not have missed phase portraits of this class.

These cases will not be hard to study since all of them imply that at least two finite singularities have escaped to infinity. In the next list we describe the generic configuration of singularities and add those geometrical configurations (with infinite nilpotent or intricate singularities) which are reduced to the most generic topological one. We will name them adding a letter to the corresponding code. We have not displayed all the geometric features as N^f or N^∞ and so on so not to make the list too long, but we have reduced them to a representative which covers the changes in the multiple singularities that could produce a new phase portrait.

(1) The configuration (93) $a\,;\ \binom{1}{2}\,HHH{-}H,\ N$ may also be present as $(93)_b\ a\,;\ \widehat{\binom{3}{2}}\,HHH{-}H,\ N$;
(2) The configuration (12) N may also be present as $(12)_b$ ©, © ; $\binom{2}{3}\,P{-}P$ or $(12)_c\ \emptyset\,;\ \binom{4}{3}\,P{-}P$;
(3) The configuration (14) $\binom{0}{2}\,SN,\ N$ may also be present as $(14)_b\ \emptyset\,;\ \binom{4}{2}\,PH{-}HP,\ N$;
(4) The configuration (23) $s, a\,; N$ may also be present as $(23)_b\ s, a\,;\ \binom{2}{3}\,P{-}P$;
(5) The configuration (44) $sn\,; N$ may also be present as $(44)_b\ \overline{sn}\,;\ \binom{2}{3}\,P{-}P$;
(6) The configuration (47) $cp\,; N$ may also be present as $(47)_b\ \widehat{cp}\,;\ \binom{2}{3}\,P{-}P$;
(7) The configuration (84) $a\,;\ \binom{1}{1}\,SN$ may also be present as $(84)_b\ a\,;\ \binom{3}{3}\,HH{-}PP$.

The study of all the cases is quite simple since there are very few separatrices to control.

1: The case $(93)_b$ is the only one which implies a reduction of multiplicity passing from a nilpotent singularity (of multiplicity 5) to another nilpotent singularity (of multiplicity 3). But the system has only two separatrices and the pase portraits that can be obtained from $(93)_b$ are the same that could be obtained from (93).

2: The configuration $(12)_b$ has no separatrices, so the only possible phase portrait is the same that one can obtain from (12).

3: Consider the configuration $(14)_b$. According to [4] and [3] the affine invariant conditions which define this geometric configuration are:

$$\mu_0 = \mu_1 = \mu_2 = \mu_3 = 0,\ \mu_4 \neq 0,\ \eta = \kappa = 0,\ \widetilde{M} \neq 0, \widetilde{K} > 0,\ \tilde{L} > 0. \tag{45}$$

So this configuration is from the class $m_f = 0$. Since $\widetilde{K} \neq 0$, according to Lemma 1 such a system belongs to the canonical form $18a$) for which we have

$$\kappa = 128h^2(hl - gm) = 0,\quad \widetilde{K} = -4(hl - gm)x^2 \neq 0.$$

Therefore we get $h = 0$ and then we obtain $\mu_4 = g^2m^2x^4 \neq 0$. Due to a time rescaling we can assume $g = 1$ and we get the family of systems

$$\dot{x} = x^2,\quad \dot{y} = m + lx^2 + 2mxy,\quad 0 < m < 1/2.$$

These systems have the invariant straight line $x = 0$ which split the plane in two semi-planes. But we have to determine if this straight line is formed by separatrices of the intricate infinite singularity or not. This can be verified by computing the systems in the local chart U_2 and doing the required sequence of blow-ups to the intricate singularity $N[0:1:0]$ we confirmed that the vertical invariant straight line $x = 0$ is formed by both separatrices of the intricate infinite singularity.

Therefore it is easy to determine that in each one of semi-planes there exist a unique source of orbits and one sink of orbits. So we obtain the unique phase portrait which is topologically equivalent to the phase portrait of the system:

$$\dot{x} = x^2 + 2xy,\quad \dot{y} = -1 - 9x^2/8 - 2xy.$$

It remains to point out that the above system has the geometrical configuration ©, © ; $\overline{\binom{0}{2}}SN\,\overline{\binom{2}{1}}N$ which is topological equivalent to configuration $(14)_b$ and to (14) but only has semi-elemental singularities.

4: The configurations $(23)_b$, $(44)_b$ and $(47)_b$ have an intricate singular point at infinity which is topologically equivalent with an elemental node (the same as configurations $(12)_b$ and $(12)_c$). So we conclude that we cannot obtain any phase portrait which is topologically different from the ones that can be obtained with an elemental infinite node.

Remark 3. *During the investigation of the configurations* $(44)_b$ *and* $(47)_b$ *we detected a misprint in the paper [4, Diagram 3, page 13]: in the last two branches corresponding to configurations* $\{44\}$ *and* $\{47\}$ *instead of* $\mathcal{B}_1$ *must be* $\mathcal{B}_4$ *(see the last two branches in the Diagram 9.2 on page 284 in [3]).*

5: The configuration $(84)_b$ has an infinite intricate singularity which is topologically equivalent to $\binom{1}{1}SN$. The systems have a unique separatrix which must go to the finite anti-saddle (or to a limit cycle surrounds it) .

Thus all the configurations from the above list have been investigated and we have proved that no other phase portrait of a quadratic system implying the existence of an infinite nilpotent or intricate singularity could exist beyond the set of the phase portraits provided by the statement of Theorem 1.

Acknowledgments

The first two authors are partially supported by the Ministerio de Economia, Industria y Competitividad, Agencia Estatal de Investigación grants MTM2016-77278-P (FEDER) and MDM-2014-0445, the Agència de Gestio d'Ajuts Universitaris i de Recerca grant 2017SGR1617, and the H2020 European Research Council grant MSCA-RISE-2017-777911. The third author is supported by NSERC. The fourth author is supported by the Program SATGED 011303, Moldova State University and by the grant number 21.70105.31 ŞD.

The third and the fourth authors are very thankful for the hospitality provided by the Departament de Matemàtiques during their visits of Universitat Autònoma de Barcelona.

References

[1] J. C. Artés, R. Kooij and J. Llibre, *Structurally stable quadratic vector fields*, Memoires Amer. Math. Soc., 134(639), 1998. 108 pp.

[2] J. C. Artés, J. Llibre and A.C. Rezende, *Structurally unstable quadratic vector fields of codimension one*. Universitext, Birkhäuser, New York–Berlin, 2018, 283 pp.

[3] J.C. Artés, J. Llibre, D. Schlomiuk and N. Vulpe, *Geometric configurations of singularities of planar polynomial differential systems* [*A global classification in the quadratic case*]. Birkhäuser/Springer, Cham, [2021], ©2021. xii+699 pp. ISBN: 978-3-030-50569-1; 978-3-030-50570-7

[4] J. C. Artés, J. Llibre, D. Schlomiuk and N. I. Vulpe, *Global topological configurations of singularities for the whole family of quadratic differential systems*, Qual. Theory Dyn. Syst. 19 (2020), no. 1, Paper No. 51, 32 pp.

[5] J. C. Artés, J. Llibre, D. Schlomiuk and N. I. Vulpe, *Codimension in planar polynomial differential systems*, Prepublications (Preprint). Núm. 01/2024, Universitat Autònoma de Barcelona, 47 pp.

[6] J.C. Artés, M.C. Mota and A.C. Rezende, *Quadratic systems possessing an infinite elliptic–saddle or an infinite nilpotent saddle*, J. Bifur. Chaos Appl. Sci. Engrg. 45 pp. (submitted January 2024)

[7] B. Coll, A. Gasull, J. Llibre *Quadratic systems with a unique finite rest point.* Publ. Mat. **32** (1988), no. 2, 199–259.

[8] P. de Jager, Phase portraits for quadratic systems with a higher order singularity with two zero eigenvalues, *J. Differential Equations*, **87** (1990), 169–204.

[9] Han Maoan, *Bifurcation of limit cycles and the cusp of order* n. A Chinese summary appears in Acta Math. Sinica 40 (1997), no. 2, 319. Acta Math. Sinica (N.S.) 13 (1997), no. 1, 64?75

[10] J.W. Reyn, *Phase portraits of quadratic systems with finite multiplicity one.* Nonlinear Anal. **28** (1997), no. 4, 755–778.

[11] J. Reyn, *Phase portraits of quadratic systems without finite critical points.* Nonlinear Anal. **27** (1996), no. 2, 207–222.

[12] J. Reyn, Phase portraits of planar quadratic systems. Mathematics and Its Applications (Springer), 583. Springer, New York, 2007. xvi+334 pp.

[13] J.W. Reyn and X. Huang, X. *Phase portraits of quadratic systems with finite multiplicity three and a degenerate critical point at infinity.* Rocky Mountain J. Math. **27** (1997), no. 3, 929–978.

[14] D. Schlomiuk and N. I. Vulpe, Geometry of quadratic differential systems in the neighborhood of infinity, *J. Differential Equations*, **215** (2005), no. 2, 357–400.

[15] N. Vulpe, *Characterization of the finite weak singularities of quadratic systems via invariant theory.* Nonlinear Analysis, **74** (2011), No. 4, 6553–6582.

[1] Departament de Matemàtiques, Universitat Autònoma de Barcelona, 08193 Bellaterra, Barcelona, Spain

Email address: joancarles.artes@uab.cat, jaume.llibre@uab.cat

[2] Département de Mathématiques et de Statistiques, Université de Montréal

Email address: dasch@dms.umontreal.ca

[3] State university of Moldova. Vladimir Andrunachievici Institute of Mathematics and Computer Science, 5 Academiei str, Chişinău, MD-2028, Moldova

Email address: nvulpe@gmail.com