WHAT DID GAUSS READ IN THE *APPENDIX*?

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Abstract. In a clear analogy with spherical geometry, Lambert states that in an "imaginary sphere" the sum of the angles of a triangle would be less than π. In this paper we analyze Gauss's reading of Bolyai's *Appendix* in 1832, five years after the publication of *Disquisitiones generales circa superficies curvas*, on the assumption that his investigations into the foundations of geometry were aimed at finding, among the surfaces in $\mathbb{R}^3$, Lambert's hypothetical "imaginary sphere".

We also wish to point out that the close relation between differential geometry and non-Euclidean geometry appears from the very beginning, and not just at the end with Beltrami's model. With this approach, one is able to answer certain natural questions about the history of non-Euclidean geometry; for instance, why Gauss decided not to write anymore after reading the *Appendix.*

In einer deutlichen Analogie mit der Kugelgeometrie behauptet Lambert, dass in einer "imaginären Kugelfläche" die Summe der Winkel eines Dreiecks kleiner als π sein würde. In diesem Artikel analysieren wir die Lesung, die Gauss 1832 – fünf Jahren nach der Veröffentlichung der *Disquisitiones generales circa superficies curvas* – zu Bolyais *Appendix* gemacht hat, unter der Annahme, dass seine Forschung über die Grundlagen der Geometrie darauf gezielt war, Lamberts hypothetische "imaginäre Kugelfläche" unter den Flächen in $\mathbb{R}^3$ zu finden.

Wir möchten auch hiermit hervorheben, dass der enge Zusammenhang zwischen der Differentialgeometrie und der nichteuklidischen Geometrie nicht nur am Ende zusammen mit dem Beltramischen Modell vorkommt, sondern schon am Anfang. In dieser Hinsicht können gewisse naturelle Fragen der Geschichte der nichteuklidischen Geometrie beantwortet werden; zum Beispiel: Warum entschloss sich Gauss nach der Lesung des *Appendix* dazu, nicht mehr über die nichteuklidische Geometrie zu schreiben?

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1. The classical problem

In definition *XXIII* of the *Elements*,[1] Euclid defines "straight parallel lines" as those "straight lines which, being in the same plane and being produced indefinitely in both directions, do not meet one another in either direction."

The *Euclidean Theory of Parallels* is based on the fifth postulate, which states that: "If a straight line falling on two straight lines makes the interior angles on the same side less than two right angles, then the two straight lines, if produced indefinitely, meet on that side on which the angles are less than two right angles."

The 'classical problem' of the Euclidean Theory of Parallels consists of demonstrating that this postulate is a consequence of the other postulates of the *Elements*.

Posidonius had already attempted to solve this problem in the 1st century B.C., when he confused straight parallel lines with equidistant straight lines (see (Bonola, 1955, 2)).

Nevertheless, the problem was resolved negatively two thousand years later at the end of 19th century. The definitive proof of this independence is attributed to Eugenio Beltrami, in 1868,[2] according to the approach adopted by Bernhard Riemann[3].

Beltrami represented the 'new plane' by the points inside a circle, its 'new lines' by chords, and parallel lines by chords meeting at a point on the circumference of the circle. In this way he obtained a geometry that satisfies all of Euclid's postulates except the fifth (see (Beltrami, 1868)). This geometry is called *non-Euclidean geometry*.

In the two thousand years separating Posidonius from Beltrami, many mathematicians believed they had positively resolved the problem of the Euclidean Theory of Parallels.

One of the most important works during this time was Giovanni G. Saccheri's *Euclides ab omni naevo vindicatus*, published in 1733, (Saccheri, 1733). Using Saccheri's quadrilateral — a quadrilateral in which two opposite sides are equal and are perpendicular to the base — Saccheri obtained results from Euclid's postulates, first without using the fifth postulate and later using the negation of it. Beltrami's aim was to find a contradiction and hence prove that the fifth postulate is,

[1]See, for instance, (Euclid, 1956, 154–155).

[2]As Jeremy Gray remarked (Gray, 2004), Beltrami was not aware of this, and it was Roberto Bonola when reading Beltrami who in fact noticed it. See (Bonola, 1955, 177 and 234) and also (Rodríguez, 2011).

[3]The observation that metric relations are independent of coordinates appears clearly in Riemann's Habilitastionsschrift (Riemann, 1867).

in fact, a theorem. He proved, for example, the remarkable fact that this negation implies the existence of asymptotic straight lines. The only error he commited was to consider as ordinary points of the 'new plane' points that are not on this plane. However, the real reason was that some results were much to his dislike, because they went against his Euclidean intuition.

Johann H. Lambert's *Theorie der Parallellinien* (Lambert, 1786) is developed in a similar way, but without arriving at any satisfactory conclusion. In fact he says: "I should almost conclude that the third hypothesis holds on some imaginary sphere."[4] This idea, the analogy with a sphere of imaginary radius, was the most important tool for the discovery of non-Euclidean geometry. We shall use the term 'Analogy' to denote the method consisting of formal substitution of R by the imaginary number Ri in all formulas that appear in the study of the geometry of a sphere of radius R, recalling that $\sin ix = i \sinh x$, and $\cos ix = \cosh x$.[5] The formulas thus obtained will be valid in the new plane.

However, the slow acceptance of complex numbers during the 18th and the early 19th centuries meant that the method of 'The Analogy' was not discussed sufficiently. Carl F. Gauss deserves great recognition in this regard, because in 1831 he was bold enough to defend complex numbers as the numbers that describe the plane.[6] Gauss's argument was that complex numbers constitutes the basic example of a doubly extended quantity, in the same way that real numbers describe the line, the basic example of a simply extended quantity.

In his famous letter of 6 March 1832 on non-Euclidean geometry to Farkas Bolyai, Gauss suggested to Farkas that he should study complex numbers,[7] thus relating non-Euclidean geometry and complex numbers (see Section 7, letter 8).

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[4]"Ich sollte daraus fast den Schlufs machen, die dritte Hypothese komme bey einer imaginären Kugelfläche vor." Although he does not say so explicitly, it is possible that this observation comes from the comparison of the formulas for the area of a triangle (spherical and non-Euclidean)

$$\begin{aligned} A &= R^2(\alpha + \beta + \gamma - \pi), \\ A &= R^2(\pi - \alpha - \beta - \gamma), \end{aligned}$$

the second one deduced synthetically by Lambert, see footnote 18 on page 6.

In 1980 Boris L. Laptev stated that Lambert also arrived at a contradiction. See (Rosenfeld, 1988, 101). See also (Rodríguez, 2006).

[5]We have developed the importance of the 'Analogy' in the discovery of non-Euclidean geometry in (Reventós and Rodríguez, 2005) and (Rodríguez, 2006).

[6]See (Gauss, 1831).

[7]Farkas followed Gauss's advice, see (Kiss, 1999), and Letter 8 of Section 7.

Many articles have been written about the history of non-Euclidean geometry, but we believe that the close relation between classical and differential geometry and the key role played by the imaginary sphere in the discovery of non-Euclidean geometry has not been sufficiently emphasized.[8] These are the main reasons that prompted us to write the present paper.

2. Lambert

Lambert, in section eleven of (Lambert, 1786), says:[9]

> The question is, can it [the fifth axiom] be correctly deduced from the Euclidean postulates together with the other axioms? Or, if these were not sufficient, can other postulates or axioms or both be given such that they have the same evidence as the Euclidean ones and from which the eleventh [fifth] axiom could be proved?
>
> For the first part of this question one can abstract from all that I have previously called *representation of the matter*. And since Euclid's postulates and remaining axioms are already expressed in words, it can and must be required that in the proof one never leans on the matter itself, but carries forward the proof in an absolutely symbolic way. In this respect Euclid's postulates are as so many algebraic equations, that one already has as previously given, and that must be solved for $x, y, z, \ldots$, without looking back to the matter itself.

We shall use the term 'Analytical Program' to refer to this idea by Lambert: the proof of the fifth postulate should not rely on any representation of the matter.

In our opinion, Gauss knew Lambert's work very well: some correspondence between Lambert and Georg S. Klügel[10] exists (see (Engels

[8]In (Gray, 1979), Gray says: "The hyperbolic trigonometry of Lobachevskii and J. Bolyai was not generally taken as a conclusive demonstration of the existence of non-Euclidean geometry until it was given a foundation in the study of intrinsic Riemannian geometry."

[9]A reprint of (Lambert, 1786) can be found in (Engels and Stäckel, 1895, 152–207). The English version included here is due to Albert Dou (Dou, 1970, 401).

[10]Georg S. Klügel (Hamburg 1739 – Halle 1812). In 1760 he entered the University of Göttingen to study theology; but he soon came under the influence of Abraham G. Kastner (see next footnote), who interested him in mathematics and encouraged him to write his thesis on the parallel postulate. Klügel was at the University of Göttingen until 1765 when he moved to Hannover, Helmstedt and finally to Halle. Therefore, at the time of his correspondence with Lambert he was still in Göttingen. See *MacTutor History of Mathematics* and the Note of Klaus Thomas Volkert (Volkert, 2006) presenting the German translation of the Klügel's thesis. The url with the German translation is given in footnote 13.

and Stäckel, 1895, 323)). Klügel was eleven years older than Lambert and his thesis, *Conatuum praecipuorum theoriam parallelarum demonstrandi recensio*, (Klügel, 1763), which he defended on 20 August 1763, was supervised by Abraham G. Kastner.[11] In his thesis, Klügel analyses critically the experiments made so far to prove the parallel postulate. He believed in the independence of the fifth postulate (see also (Kline, 1972, 867)) and stated that Saccheri's results contradicted the experience but not the axioms.[12] For instance, when referring to equidistant lines Klügel in (Klügel, 1763, Section II), says: "It is quite clear that here it is used that a line, which is at the same distance from a straight line, is itself a straight line. This can be concluded by experience and from the judgement of the eyes, not from the nature of the straight line."[13]

Klügel and Johann F. Pfaff (1765–1825) were colleagues at Göttingen, and Pfaff's thesis was also supervised by Kastner. Pfaff was a close friend of Gauss and also his thesis advisor. Gauss even stayed at Pfaff's house for several months (see (Dunnington, 2004, 415)).

Thus, given these circumstances in Göttingen,[14] it seems unlikely that Gauss would be unaware of Lambert's work.[15]

Moreover, this work on theory of parallels was available to Gauss at the Göttingen University Library. The records show that he withdrew Lambert's books *Beiträge zum Gebrauch der Mathematik (3 vols., Berlin, 1765-1772)* in 1795 and *Photometria* in 1797 (see (Gray, 1979, 241) and (Dunnington, 2004, 177)).

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[11]Abraham G. Kastner (Leipzig 1719 – Göttingen 1800). In 1756 he was appointed as professor of mathematics and physics at the University of Göttingen where he taught Gauss and Farkas Bolyai. Another of his students, Johann C. M. Bartels (1769–1836) taught Nikolai I. Lobachevsky (1792–1856). See *MacTutor History of Mathematics*.

[12]Perhaps Saccheri's work came into Gauss's hands via Klügel.

[13]"Satis apparet, sumi hic, lineam, quae a recta aequaliter semper distat, ipsam rectam esse, quod experientia et ex oculorum iudicio, non ex natura lineae rectae colligitur." The English version is our free translation from the German version due to Martin Hellmann accessible at `http://www.uni-koeln.de/math-nat-fak/didaktiken/mathe/volkert/titel.htm`.

[14]Dunnington (Dunnington, 2004, 176) says: "When Gauss went to Göttingen, J. Wildt (1770–1844) gave a trial lecture on the theory of parallels (1795) [...] and in 1801 Seyffer, the professor of astronomy, published two reviews of attempts to prove the parallel axiom [...] Gauss was very close to Seyffer, and their correspondence continued until the latter's death. Their conversations frequently touched on the theory of parallels."

[15]The famous entry [72] in his diary: "Plani possibilitatem demonstravi", is dated 28 July 1797, Göttingen.

In his note of December 1818 on non-Euclidean geometry, sent to Gauss by Christian L. Gerling (1788–1864), Ferdinand K. Schweikart (1780–1857), said: "That this sum [the sum of the three angles in a non-Euclidean triangle] becomes ever smaller, the more content the triangle encloses."[16] In his answer to Gerling, Gauss said: "The defect of the angle sum in the plane triangle from 180° is, for example, not just greater as the area becomes greater, but it is exactly proportional to it."[17] It is possible that Gauss learnt of this result through Lambert.[18]

3. The *Disquisitiones*

The relationship between the *Disquisitiones generales circa superficies curvas*[19] and non-Euclidean geometry can be analyzed according to the two following hypotheses, which enable certain natural questions to be answered.

(1) Gauss was aware that the definitive solution to the problem of the independence of the acute angle hypothesis was not possible with the material representation of points, lines and planes given by pictures. For this reason Gauss adopted Lambert's 'Analytical Program' as the correct method to solve this problem definitively.

(2) Gauss was determined to find a surface that could play the role of the imaginary sphere introduced by Lambert.[20]

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[16] "dass die Summen immer kleiner werden, je mehr Inhalt das Dreieck umfasst." See (Halsted, 1900).

[17] "Der Defect der Winkelsumme im ebenen Dreieck gegen 180° ist z.B. nicht bloss desto grösser, je grösser der Flächeninhalt ist, sondern ihm genau proportional."

[18] Lambert states: "If the third hypothesis is true, [...] that for each triangle the excess of 180° over the sum of its three angles is proportional to the area" ("Wenn es bey der dritten Hypothese möglich wäre, [...] dass bey jedem Triangel der Ueberschuss von 180 Gr. über die Summe seiner drey Winkel dem Flächenraume des Triangles proportional wäre.") Nevertheless Lambert's proof is far from rigorous. In fact, Gauss gave a proof of this in 1832 (Section 7, letter 8) assuming that the area of an ideal triangle is finite.

[19] Henceforth referred to simply as the *Disquisitiones*. It can be found in (Gauss, 1828), or (Dombrowski, 1979).

[20] Among Gauss's manuscripts written between 1823 and 1827 there is the explicit formula for the pseudosphere

$$
\begin{aligned}
y &= R\sin\varphi \\
x &= R\cos\varphi + \log\tan\frac{1}{2}\varphi \\
s &= R\log\frac{1}{\sin\varphi}
\end{aligned}
$$

Lambert's 'Analytical Program' first requires an analytic treatment of the spherical geometry, in such a way that the methods and results obtained therein could be generalized and applied later to the study of any curved surface. The *Disquisitiones* constitutes an attempt by Gauss in this direction. This justifies the occurrence of Theorem *VI* of article 2[21] in *Disquisitiones*.

This theorem, which seems unimportant in the 1827 version, plays an important role in the unpublished version of 1825.[22] In fact, the whole of spherical trigonometry can be analytically deduced from it.

This strengthens even further the idea of looking for a surface analogous to the sphere, but on which the acute angle hypothesis holds: the Lambert imaginary sphere.

However, which surface does this imaginary sphere represent? How are triangles represented on it and what is the sum of their angles?

Perhaps this was one of the reasons that led Gauss to write the *Disquisitiones*. Moreover, at the same time, his discoveries could be applied to geodesy, so that the *Disquisitiones* can also be considered as a first chapter on "advanced geodesy", as Gauss stated in the commented letter to Heinrich C. Schumacher (1780–1850), dated 21 November 1825.

If Gauss had understood, as Riemann himself did, that $\mathbb{R}^2$ can be 'curved' on itself, without embedding it on $\mathbb{R}^3$, he could have developed

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preceded by the words "For the curves whose revolution originates the opposite of the sphere, it is satisfied" ("Für die Curve, durch deren Revolution das Gegenstück der Kugel entsteht, ist:") See (Gauss, 1870–1927, Vol. VIII, p. 265). From this expression it is clear that the curvature is $= -\frac{1}{R^2}$, since the curvature of the surface of revolution obtained by rotating the curve $(x(s), y(s))$ about the x-axis, where s is the arc length, is given by

$$K = -\frac{1}{y}\frac{d^2y}{ds^2}.$$

Observe that this note on the "opposite of the sphere" was written during the preparation period of the *Disquisitiones*.

[21]If L, L', L'', L''' denote four points on the sphere, and A the angle which the arcs L, L', L'', L''' make at their point of intersection, then we shall have

$$\cos LL'' \cdot \cos L'L''' - \cos LL''' \cdot \cos L'L'' = \sin LL' \cdot \sin L''L''' \cdot \cos A$$

[22]Gauss's own words in the 1825 version were: "We shall add here another theorem, which has appeared nowhere else, as far we know, and which can often be used with advantage.", ("Wir fügen noch ein anderes Theorem bei, welches unseres Wissens sonst nirgends vorkommt und öfters mit Nutzen gebraucht werden kann."). See (Gauss, 1870–1927, Vol. VIII, p. 416) or (Gauss, 1902, 88) for the English version.

the geometry corresponding to the hyperbolic length element. By applying the 'Analogy', we obtain this length element directly from the length element of the sphere.

Franz Taurinus (1794–1874) had made much progress in this direction, as can be seen from his writings of 1825 and 1826 on *logarithmic-spherical geometry* (see (Engels and Stäckel, 1895, 255–286) and (Rodríguez, 2006)). 'All he needed to say' was that the triangles he was considering were the geodesic triangles of the geometry of the hyperbolic length element.

Why did Gauss not take this step? We believe that the most natural explanation is that Gauss was looking for this surface, the Lambert imaginary sphere, within $\mathbb{R}^3$, see Section 8.

In an attempt to answer these questions, Gauss found the intrinsic geometry of surfaces. This brilliant discovery was included in the *Disquisitiones*, mainly in the Egregium theorem (Section 12): "If a curved surface is developped upon any other surface whatever, the measure of curvature in each point remains unchanged."[23]

4. Gauss's isolation

In 1794, Adrien-Marie Legendre (1752–1833) published his *Eléments de géomètrie*. In this work, and in later editions, he gave several proofs of the fifth postulate; see (Legendre, 1794), (Legendre, 1833), and (Bonola, 1955, 55–60). Irrespective of whether these proofs were correct or not, it is clear that Legendre was convinced not only of the certainty of this postulate, but also that "he had finally removed the serious difficulties surrounding the foundations of geometry", see (Bonola, 1955, 60). Due to Legendre's great influence, mainly on French mathematicians, we believe that the problem of the Euclidean Theory of Parallels was not sufficiently considered by the great mathematical schools of that time.[24]

[23]"Si superficies curva in quamcunque aliam superficiem explicatur, mensura curvaturae in singulis punctis invariata manet." English version from Peter Dombrowski (Dombrowski, 1979, 38).

[24]Lutzen in (Lützen, 1990), mentioning Karin Reich, says that it was principally due to Joseph Liouville (1809–1882) that Gauss's ideas on differential geometry became known in France: "To be sure, Sophie Germain had read Gauss's *Disquisitiones generales circa superficies curvas* [1828], but during the following 15 years Lame's theories of systems of orthogonal surfaces dominated the French scene, and Gauss's work was forgotten. In 1843, in a paper in Liouville's Journal on this subject, Bertrand admitted that "After having written this memoir, I have learned about a memoir by Mr. Gauss entitled *Disquisitiones generales...*"[Bertrand 1843]. The following year, Bonnet also referred to Gauss. It is not impossible that Liouville himself had called the attention of these two young talents to the *Disquisitones*,

Gauss believed in the importance of this problem, but he felt alone among the great mathematicians of that period.

Nevertheless, he did have a supporting group of friends: Friedrich Bessel (1784–1846), Friedrich L. Wachter[25] (1792–1817), Farkas Bolyai, Gerling, Heinrich W. Olbers (1758–1840), and Schumacher. It was thanks to them that he received news of the important works by Schweikart, Taurinus, and János Bolyai, all of whom were outsiders or amateur mathematicians. Lobachevsky, the other important person in this story,[26] was a professor of mathematics at the peripheral University of Kazan. Although in 1829 he had already published a text on the foundations of the geometry in Russian, it was not until 1840 (perhaps because his ideas on the theory of parallels were ridiculed by his Russian colleagues) that his book *Geometrischen Untersuchungen zur Theorie der Parallellinien*, (Lobachevsky, 1955) appeared, which was read and immediately appreciated by Gauss. Lobachevsky was rector of Kazan University and achieved fame as an educational reformer.

5. The three ds^2 of Bolyai's *Appendix*

In this section we analyze the reading Gauss may have made of Bolyai's *Appendix* (see (Bolyai, 2002)) in 1832, taking into account that this was done five years after the publication of *Disquisitiones* and ten years after Gauss wrote the formula for the curvature of a surface with respect to some conformally euclidean chart. This formula appears in his personal notes with the title "The state of my investigations on the

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and it is certain that when the interest in Gauss's ideas spread in France after 1847 it was due to Liouville." This happened twenty years after the publication of *Disquitiones.*

[25]Wachter was a student of Gauss. In 1816 Wachter suggested to Gauss that a sphere of infinite radius in non-Euclidean space has Euclidean geometry, see (Gauss, 1870–1927, Vol. VIII, pp. 175–176). Nevertheless, Wachter's explanations were quite obscure. For instance, when saying: "Then came the discomfort that on this sphere of infinite radius whose parts are merely symmetric but not congruent and where the radius on one side is infinite but on the other imaginary." ("Es entsteht zwar die eine Unbequemlichkeit daraus, dass die Theile dieser Fläche bloss symmetrisch, nicht, wie bei der Ebene, congruent sind; oder dass der Radius nach der einen Seite hin unendlcih, nach der andern imaginär ist.")

However, as Gray says, in (Dunnington, 2004, 463), "Gauss did not claim to possess knowledge of a new geometry, which surely means that even the ideas he was discussing with Wachter he considered to be hypothetical, and capable of turning out to be false."

[26]The question of priority has been widely studied, see for instance (Gray, 1989, 111).

transformation of surfaces" (Gauss, 1870–1927, Vol. VIII, pp. 374–384); that is, after having acquired a deep knowledge of the role played by the line element ds in geometry.

As is well known, when speaking about the Euclidean Theory of Parallels in 1831, and more particularly about one equivalent formulation of the fifth postulate, Gauss said to Schumacher (letter of 17 May 1831):[27] "In the last few weeks I have begun to put down a few of my own meditations, which are already to some extent nearly 40 years old. These I have never put in writing, so that I have been compelled 3 or 4 times to go over the whole matter afresh in my head. I did not wish it to perish with me."[28]

Nevertheless, some months later, in February 1832, Gauss read Bolyai's *Appendix* and decided to write nothing further on the subject. In a letter to Gerling (14 February 1832) he said: "In addition, I note that in recent days I have received a short work from Hungary on non-Euclidean geometry in which I find all of my ideas and results developed with great elegance, although in a concentrated form that is difficult for one to follow who is not familiar with the subject. The author is a very young Austrian officer, the son of a friend of my youth with whom I had often discussed the subject in 1798, although my ideas at that time were much less developed and mature than those obtained by this young man through his own reflections. I consider this young geometer, v. Bolyai, to be a genius of the first class."[29]

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[27] Gauss's letters on non-Euclidean geometry are commented, using the 'Analogy', in (Reventós and Rodríguez, 2005). See also (Reventós, 2004).

[28] "Von meinen eigenen Meditationen, die zum Theil schon gegen 40 Jahr alt sind, wovon ich aber nie etwas aufgeschrieben habe, und daher manches 3 oder 4 mal von neuem auszusinnen genöthigt gewesen bin, habe ich vor einigen Wochen doch einiges aufzuschreiben angefangen. Ich wünschte doch, dass es nicht mit mir unterginge." (Gauss, 1870–1927, Vol. VIII, p. 220).

[29] "Noch bemerke ich, dass ich dieser Tage eine Schrift aus Ungarn über die Nicht-Euklidische Geometrie erhalten habe, worin ich alle meine eigenen Ideen und Resultate wiederfinde, mit grosser Eleganz entwicklet, obwohl in einer für jemand, dem die Sache fremd ist, wegen der Concentrirung etwas schwer zu folgenden Form. Der Verfasser ist ein sehr junger österreichischer Officier, Sohn eines Jugendfreundes von mir, mit dem ich 1798 mich oft über die Sache unterhalten hatte, wiewohl damals meine Ideen noch viel weiter von der Ausbildung und Reife entfernt waren, die sie durch das eigene Nachdenken dieses jungen Mannes erhalten haben. Ich halte diesen jungen Geometer v. Bolyai für ein Genie erster Grösse." (Gauss, 1870–1927, Vol. VIII, p. 221).

Would Gauss have said this if he had thought that Bolyai's was a mere formal manipulation of concepts, along the lines of Taurinus,[30] without consistency?

Would Gauss have stopped writing his notes if he had not considered that the problem was completely solved?

Moreover, in the above letter to Farkas Bolyai (6 March 1832), he said:

> Now something about the work of your son. If I begin by saying that I must not praise him, surely, you will be startled for a moment; but I cannot do otherwise; praising him would mean praising myself: because all the contents of the work, the way followed by your son, and the results he obtained agree almost from beginning to end with the meditations I had been engaged in partly for 30-35 years already. This extremely surprised me indeed.
>
> It had been my intention to publish nothing of my own work during my life; by the way, I have noted down only a small portion so far. Most people do not even have a right sense of what this matter depends on, and I have met only few to accept with particular interest what I told them. One needs a strong feeling of what in fact is missing and, as to this point, the majority of people lack it. On the other hand, I had planned to write down everything in the course of time so that at least it would not vanish with me some day.
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> Thus I was greatly surprised that now I can save myself this trouble, and I am very glad that it is precisely my old friend's son who so wonderfully outmatched me.[31]

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[30] Taurinus developed non-Euclidean geometry formally using the imaginary sphere. The results were correct, but it was first necessary to prove that this imaginary sphere really existed. This work was commented on extensively by Gauss in his letter to Taurinus (see Section 7, letter 4).

[31] "Jetzt einiges über die Arbeit Deines Sohnes. Wenn ich damit anfange, "dass ich solche nicht loben darf": so wirst Du wohl einen Augenblick stutzen: aber ich kann nicht anders; sie loben hiesse mich selbst loben: denn der ganze Inhalt der Schrift, der Weg, den Dein Sohn eingeschlagen hat, und die Resultate, zu denen geführt ist, kommen fast durchgehends mit meinen eigenen, zum Theile schon seit 30–35 Jahren angestellten Meditationen überein. In der That bin ich dadurch auf das Äusserste überrascht.

"Mein Vorsatz war, von meiner eigenen Arbeit, von der übrigens bis jetzt wenig zu Papier gebracht war, bei meinen Lebzeiten gar nichts bekannt werden zu lassen. Die meisten Menschen haben gar nicht den rechten Sinn für das, worauf es dabei ankommt, und ich habe nur wenige Menschen gefunden, die das, was ich ihnen mittheilte, mit besonderm Interesse aufnahmen. Um das zu können, muss man erst recht lebendig gefühlt haben, was eigentlich fehlt, und darüber sind die meisten Menschen ganz unklar. Dagegen war meine Absicht, mit der Zeit alles so zu Papier zu bringen, dass es wenigstens mit mir dereinst nicht unterginge.

It is in this letter that Gauss suggests the name "parasphere" for the surface called only F by János Bolyai and "horosphere" by Lobachevsky. He says: "For instance, the surface and the line your son calls F and L might be named parasphere and paracycle, respectively: they are, in essence the sphere and circle of infinite radii. One might call hypercycle the collection of all points at equal distance from a straight line with which they lie in the same plane; similarly for hypersphere."[32] Bolyai introduces the surface F, cited in Gauss's letter above, in Section §11 of the *Appendix*.

The first ds^2

In later sections, specifically in §24, Bolyai proves that the relation between the length z of the paracycle (horocycle) cd, the length y of the paracycle ab and the length x of the straight line ac (see Figure 1) is given by

$$z = ye^{-x/R},$$

where R is the constant denoted i by Bolyai (the radius of the imaginary sphere for us).

From this it is easy to see that[33]

$$ds^2 = dx^2 + e^{-2x/R}\, dy^2. \tag{1}$$

This computation, given below, could have been performed by a person with Gauss's knowledge. It is also important to point out that this expression is obtained without trigonometry and without resorting to three dimensions.

Moreover, it hardly seems possible to look at Bolyai's Figure 9, reproduced here in Figure 1, without seeing a system of local coordinates.

"Sehr bin ich also überrascht, dass diese Bemühung mir nun erspart werden kann und höchst erfreulich ist es mir, dass gerade der Sohn meines alten Freundes es ist, der mir auf eine so merkwürdige Art zuvorgekommen ist." (Gauss, 1870–1927, Vol. VIII, pp. 220–221). English version from (Kárteszi, 1987).

[32] "So könnte z. B. die Fläche, die Dein Sohn F nennt, eine Parasphäre, die Linie L ein Paracykel genannt werden: es ist im Grunde Kugelfläche, oder Kreislinie von unendlichem Radius. Hypercykel könnte der Complexus aller Punkte heissen, die von einer Geraden, mit der sie in Einer Ebene liegen, gleiche Distanz haben; eben so Hypersphäre." (Gauss, 1870–1927, Vol. VIII, p. 221). English version from (Kárteszi, 1987, 35).

[33] Observe that this metric, with the change $u = e^{x/R}, v = y/R$, is the metric of the Poincaré half-plane. This could be the "extra work" mentioned by Gray in his comments to the *Appendix*, see page 20 of this paper. Could Gauss have done this "extra work"?

[34] See Figure 4 for all 23 figures in the *Appendix*.

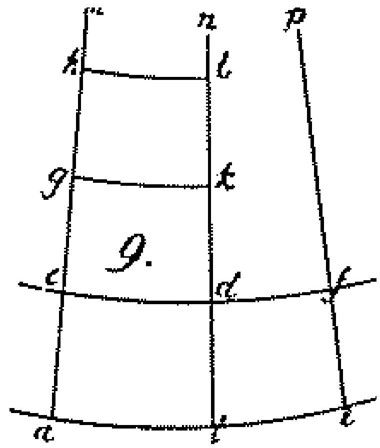


FIGURE 1. Figure 9 in Bolyai's *Appendix*.[34]

In fact, it is clear that the length element, in the sense used by Gauss, can be written in x, y coordinates as

$$ds^2 = dx^2 + f^2(x)dy^2$$

for a certain function $f(x)$, since

- this coordinate system is orthogonal[35] (so the term $dx\,dy$ does not appear),
- the lines $y =$ constant are geodesics parametrized by the arc length (so the coefficient of dx is 1), and
- it is invariant under translation in the y direction (so $f(x, y) = f(x)$).

To find $f(x)$, one takes the curve $\gamma(t) = (x, t)$, for a constant value of x, with $0 \leq t \leq y$ (a portion of a horocycle). The length L of γ is given by

$$L = \int_0^y |\gamma'(t)|dt = \int_0^y f(x)dt = yf(x)$$

However, since $L = ye^{-x/R}$, we have $f(x) = e^{-x/R}$.

[35]The paracycles are orthogonal to the family of parallel straight lines.

The second ds^2

In Section §30, Bolyai gives the length of a circle in terms of its radius r. This relation is[36]

$$L(r) = 2\pi\, R \sinh \frac{r}{R}.$$

Nevertheless, calculations similar to above imply[37] that the metric in cyclic coordinates (r, θ) is given by

$$ds^2 = dr^2 + R^2 \sinh^2 \frac{r}{R}\, d\theta^2. \tag{2}$$

Indeed, it is clear that

$$ds^2 = dr^2 + f^2(r) d\theta^2$$

for a certain function $f(r)$, since this coordinate system is orthogonal[38] (so the term $dr\, d\theta$ does not appear), $\theta =$ constant are geodesics parametrized by the arc length (so the coefficient of dr is 1), and it is invariant under rotation (so $f(r, \theta) = f(r)$).

To find $f(r)$, one takes the curve $\gamma(t) = (r, t)$, for a constant value of r, with $a \leq t \leq b$ (a portion of the circle). The length L of $\gamma(t)$ is given by

$$L = \int_a^b |\gamma'(t)| dt = \int_a^b f(r) dt = (b - a) f(r)$$

However, since $L(r) = 2\pi\, R \sinh \frac{r}{R}$, the length of γ is

$$L = (b - a) R \sinh \frac{r}{R}.$$

[36]This formula is given without proof by Gauss in his letter to Schumacher in 1831, see (Gauss, 1870–1927, Vol. VIII, p. 218). We suggest that the approach adopted by Gauss to prove the formula was the inverse of that taken by Bolyai: Gauss obtained the length of the circumference of the circle from the line element of the imaginary sphere. How Gauss arrived at this formula, so easy to explain according to our hypothesis, has not been sufficiently explained in the literature. Less influence between differential geometry and the discovery of non-Euclidean geometry than we suppose is admitted in much of the literature. For example, Gray says: "there is no evidence that Gauss derived the relevant trigonometric formulae from the profound study of differential geometry that occupied him in the 1820s", see (Gray, 2006). See also the section *Differential geometric foundations of non-Euclidean Geometry* in (Gray, 1987a) or (Gray, 2007, Chapter 20). For a slightly different point of view on this topic see (Scholz, 2004) in German or its Spanish translation by José Ferreirós in (Scholz, 2005). The relation between non-Euclidean and differential geometry it is usually supposed to have first appeared in Beltrami's work (Beltrami, 1868).

[37]This computation does not appear in the *Appendix*; but Gauss would have found it easy to do.

[38]Gauss's lemma, proved in (Gauss, 1828).

Hence, $f(r) = R \sinh \frac{r}{R}$, and the metric of the Bolyai plane in cyclic coordinates is the metric of the imaginary sphere.

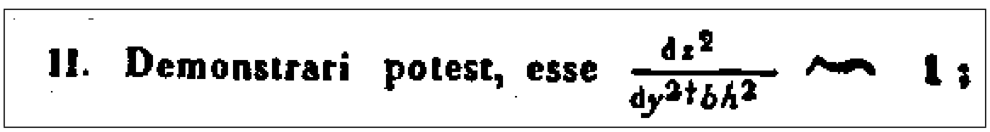
Note that the metric of the sphere in cyclic coordinates is given by $ds^2 = dr^2 + R^2 \sin^2 \frac{r}{R}\, d\theta^2$. Applying here the 'Analogy', we obtain expression (2).

Did Gauss see this in Section §30 of the *Appendix*? Although we are unable to prove it, we are convinced that the answer to this question is affirmative, since Gauss had all the necessary knowledge on differential geometry to perform the above computations, and also because he stopped writing his notes on non-Euclidean geometry after reading Bolyai's work.

We remark that expressions (1) and (2) do not appear explicitly in Bolyai's work.

The third ds^2

In Section §32 of the *Appendix* a metric appears explicitly. Bolyai says:



II. Demonstrari potest, esse $\frac{dz^2}{dy^2+bh^2} \sim 1$;

FIGURE 2. The metric of the *Appendix*.

that is,

$$\textit{It can be proved that} \quad \frac{dz^2}{dy^2 + bh^2} \sim 1;$$

which, using the computation of bh given in Section §27 of the *Appendix*, is equivalent to

$$\frac{ds^2}{dy^2 + \cosh^2 \frac{y}{R}\, dx^2} = 1,$$

that is

$$ds^2 = dy^2 + \cosh^2 \frac{y}{R}\, dx^2, \tag{3}$$

which is the expression of the metric in hypercyclic coordinates.[39]

[39] See Appendix A for details of hypercyclic coordinates.

In fact, expression (3) is apparent to anyone (Gauss, for instance) who knows the local theory of surfaces well.[40]

Specifically, it is clear that

$$ds^2 = dy^2 + f^2(y)dx^2$$

for a certain function $f(y)$, independent of x, since, by Gauss's lemma, this coordinate system is orthogonal (so the term $dx\,dy$ does not appear), the lines x =constant are geodesics (so the coefficient of dy is 1), and it is invariant under translation in the x direction (so $f(x, y) = f(y)$).

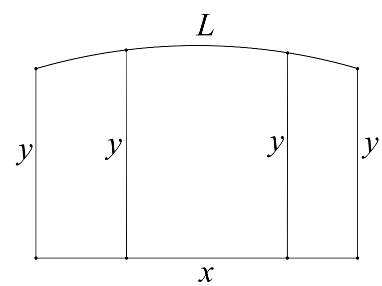


FIGURE 3. Mixed quadrilateral of base x and height y.

To find $f(y)$, one take the curve $\gamma(t) = (t, y)$, for a constant value of y, with $a \leq t \leq b$ (a portion of equidistant). The length of $\gamma(t)$ is

$$L = \int_a^b |\gamma'(t)|dt = \int_a^b f(y)dt = f(y)(b - a)$$

However, in Section §27 of the *Appendix*, Bolyai gives the formula of the length L of the equidistant in terms of the length x of the base and the length y of the height of the mixed quadrilateral (Figure 3). This relation is

$$L = x \cosh \frac{y}{R}.$$

Hence, $f(y) = \cosh \frac{y}{R}$, as we wished to demonstrate.

[40]Unfortunately János Bolyai never knew Gauss's work on the theory of surfaces: Kárteszi in (Kárteszi, 1987, 32), says: "Even of Gauss's results only a small proportion was known to him; for example, he has not heard of the investigations of Gauss in surface theory contained in the work *Disquisitiones generales circa superficies curvas* through his life". This fact may explain how Gauss might have recognized that Bolyai had solved the problem of the theory of parallels, while Bolyai himself did not, and so Gauss gave up writing his notes on the subject. One can only assume that it would have been clear to Gauss that expression (3) is a length element corresponding to a 'surface' of constant negative curvature.

Note that the metric of the sphere in hypercyclic coordinates is given by $ds^2 = dy^2 + \cos^2 \dfrac{y}{R}\, dx^2$. Applying here the 'Analogy', we obtain expression (3).

Bolyai realizes the importance of the 'third ds^2' and finishes Section 32, III, with these words: "The surfaces of bodies may also be determined in S, as well as the *curvatures, the involutes, and evolutes* of any lines, etc."[41]

Curvature

The curvature formula

$$k = -\frac{1}{\sqrt{G}} \frac{\partial^2 \sqrt{G}}{\partial^2 r},$$

known by Gauss since his first version of the *Disquisitiones* in 1825, could be applied to the expressions (1), (2) and (3), with $G = e^{-2x/R}$, $G = R^2 \sinh^2 \frac{y}{R}$ and $G = \cosh^2 \frac{y}{R}$, respectively, to prove that Bolyai's plane is represented by a surface of constant negative curvature $-1/R^2$.

Gauss may have seen that Bolyai's expressions for the metric, equations (2) and (3), could be directly obtained by 'Analogy' from the metric on the sphere written with regard to the cyclic and hypercyclic coordinates, respectively; especially (2), which gives the length of the circle directly, a formula well known to Gauss (letter to Schumacher, 17 May 1831). However, the expression of the metric in paracyclic coordinates, equation (1), cannot appear by 'Analogy', since the concept of paracycle is characteristic of hyperbolic geometry. However, Gauss's manuscripts on the Theory of Parallels of 1831 may be the beginning of a synthetic approach to finding this paracyclic metric, see (Gauss, 1870–1927, Vol. VIII, pp. 202–209).

In Appendix A, we make the change between cyclic, paracyclic and hypercyclic coordinates explicit.

Relation with the consistency

Did Gauss see that the hypercyclic coordinates on the 'new plane' were global, unlike on the sphere, where they are not? In particular, that the imaginary sphere can be covered with only one chart?

Did Gauss see the proof of the consistency in the *Appendix*? The letters to Gerling and Farkas Bolyai referred to above lead us to conjecture that he did, but did he have a clear concept of the problem of the consistency?

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[41] "Superficies quoque corporum in S determinari possunt, nec non *curvaturae, evolutae, evolventesque* linearum qualiumvis etc."

Gauss, who could have done the computations that we perform in this article, did not realize that the problem of the consistency had been solved, because the question 'Which surface of $\mathbb{R}^3$ has one of these metrics?' that he could have been trying to answer was in fact incorrect. Gauss was the founder of the intrinsic geometry of surfaces, but all the length elements (metrics) used by Gauss came from the Euclidean metric of $\mathbb{R}^3$.

This epistemological mistake is very understandable: he was discovering a new world, and like all pioneers he happened to overlook something very important.

It seems that Beltrami also made the same mistake (see footnote 2).

If one assumes that Gauss used the 'Analogy' to find the ds^2 of the imaginary sphere, it is easy to explain all the results of the new geometry that Gauss in his letters showed that he knew. It also explains why he did not include proofs: the use of imaginary numbers was not sufficiently accepted.

However, after reading the *Appendix*, Gauss saw all these results deduced axiomatically and without any reference to imaginary numbers.

6. The drawings in the *Appendix*: Revision of some of Gray's comments

While we agree with Gray's comments on the *Appendix*, see (Gray, 2004, 123–127), we would nevertheless like to make some further remarks, which we trust will contribute to extending the recognition of Bolyai's work, which has already been acknowledged by Gray.

First of all, the coordinates used by Bolyai are the hypercyclic coordinates (the lines x =constant are straight lines, while y =constant are equidistant). It is in this sense that Gray uses the expression "usual system of Cartesian (x, y) coordinates."

Some of Gray's expressions can be considered as a moderate criticism of Bolyai's work; for instance:

- "Without as much as a hint in the direction just outlined, Bolyai supposed that his readers would recognise these arguments."
- "but it requires an interpretation that Bolyai was unwilling to provide."
- "Bolyai escaped the pedagogic problem, not for the first or only time in the *Appendix* by saying: "It can be demonstrated"."

These comments are perfectly understandable if we accept the hypothesis that the *Appendix* was written with the foremost reader in mind:

Gauss himself. In fact, the *Appendix* was sent to Gauss in 1831[42] and the *Tentamen* was published in 1832.

János Bolyai sent a first version of his work to his former professor Herr Johann Walter von Eckwehr in 1825,[43] and "on the prompting of his father" he translated it from German into Latin for publication in *Tentamen*, which was issued in Latin.[44] Given the friendship between Gauss and Farkas, it is logical to assume that Farkas had already decided to send a copy to Gauss.

The writing of the *Appendix* is very concise. We do not know whether for financial difficulties[45] or for mathematical reasons. In his letter to Gerling (see above) Gauss says: [the results of the *Appendix* are developed] "in a concentrated form that is difficult for one to follow who is not familiar with the subject."

The 23 drawings in the *Appendix*, with the caption "Tabula Appendicis" at the top on the right, reproduced here from (Bolyai, 2002, 29) in Figure 4, should not be interpreted as drawings in the Euclidean plane, as might be erroneously inferred from Gray's remark in (Gray, 2004, 124): "He drew a picture of a curve ABC in the familiar Cartesian plane with x- and y- axes and outlined an interpretation of it as a picture of non-Euclidean geometry drawn in a Euclidean plane."

These figures play the same role as the figures that appear in the majority of versions of Euclid's *Elements*: they are guides for the proofs. In fact, Bolyai does not use the Euclidean plane at all. Note that in his few notes on the subject Gauss uses similar drawings.

Nevertheless, a valid objection to Bolyai's drawings is that he represents non-Euclidean segments in the same way that we usually represent Euclidean segments. This problem was skillfully solved by Battaglini,[46] and was the basis for the proof of the consistency given by Beltrami, using a model where non-Euclidean segments were represented by Euclidean ones.

[42]It seems that this copy never arrived, see for instance, (Bonola, 1955, 100), (Gray, 1989, 97) or (Gray, 1987b, 18).

[43]This manuscript has not been found. It seems that it was not returned to János. Perhaps for this reason it was not sent to Gauss until 1831.

[44]See (Bonola, 1955, XXVIII of Halsted's introduction).

[45]Halsted in (Bonola, 1955, XXVIII of Halsted's introduction) remarks that János contributed 104 florins and 50 kreuzers to the printing of the *Appendix*. (The yearly salary of a university professor was about 1300 florins. 60 krazers were equivalent to 1 florin.) In opinion of Barna Szénássy, althought the two Bolyai's had financial difficulties throughout their lives, the economy was not the main reason for the *Appendix* being concise, see (Kárteszi, 1987, 224).

[46]See (Battaglini, 1867), or (Montesinos, 1994) and (Rodríguez, 2011).

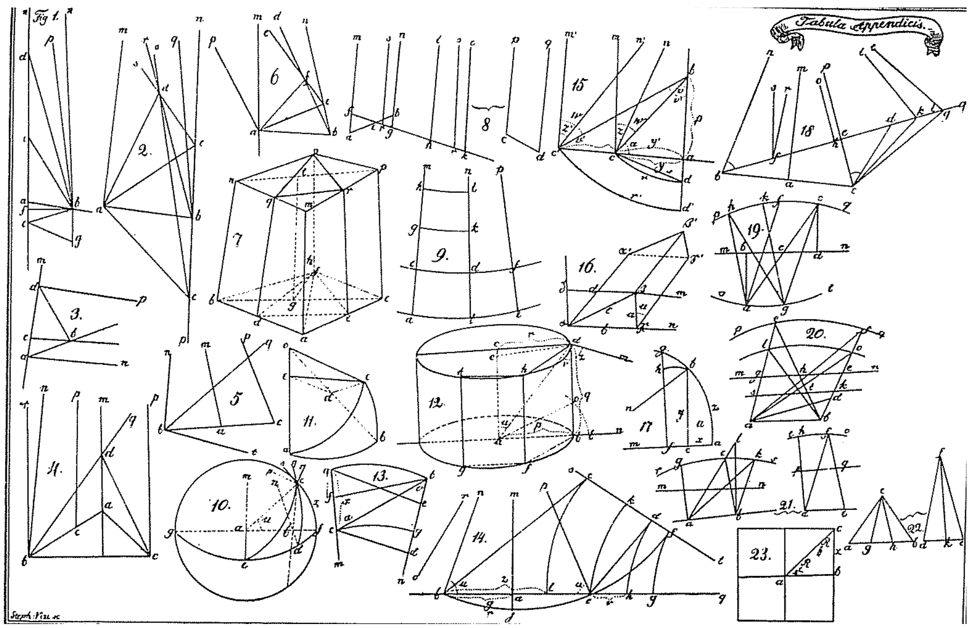

FIGURE 4. All drawings in the *Appendix*.

As Gray says (Gray, 2004, 123), it is a pity that Bolyai did not find the hyperbolic half-plane model: "With a bit of extra work, he could have shown that the entire picture of non-Euclidean two-dimensional geometry could appear in the right half-plane (the region defined by $x > 0$), and that in his new space straight lines were curves of a certain appearance."

However, in order to prove the consistency it is not necessary to have this specific model of hyperbolic geometry. It suffices to have a 'plane' with an appropriate metric, as Bolyai had. But this presupposes the idea of abstract Riemannian manifold, which was Riemann's great contribution many years later.

Finally, we completely agree with Gray (Gray, 2004, 126) when he says: "But the fact that Bolyai got as close as he did to formulating the elements of his new geometry in terms of the calculus is striking testimony to his insight, and seems not to have been appreciated sufficiently in his day or since."

7. The problem of the independence of the fifth postulate

In Gauss's time the consistency of Euclidean geometry was accepted without discussion. But this was not the case with the geometry arising from the negation of the fifth postulate. Perhaps because of their surprising results, a proof of the consistency of this new geometry was demanded.

There are some letters written or received by Gauss in which "astral" geometry or "anti-euclidean" geometry are discussed,[47] and from which we can deduce that Gauss was convinced of the consistency of this new geometry.

We may mention the following:[48]

(1) Gauss to Olbers. Göttingen, 28 April 1817. "Wachter has written a short note on the foundations of the geometry [...] I am coming ever more to the conviction that the necessity of our geometry cannot be proved, at least not by human comprehension nor for human comprehension. Perhaps in another life we will come to other views on the nature of space which are currently unobtainable for us. Until then one must not put Geometry into the same rank as Arithmetic, which stands a priori, but rather in the same rank as, say, Mechanics."[49]

(2) Schweikart's Note to Gauss. Marburg, December 1818. "There is a two-fold geometry, a geometry in the narrow sense, the Euclidean; and an astral study of magnitudes."[50]

(3) Gauss to Gerling. Marburg, 16 March 1819. "The note of Herr Professor Schweikart gave me an incredible amount of pleasure, [...] because although I can really imagine that the Euclidean geometry is not correct,[...]"[51]

(4) Gauss to Taurinus. Göttingen, 8 November 1824. "The assumption that the sum of the three angles is smaller than 180°

[47]See (Gauss, 1870–1927, Vol. VIII, pp. 159–225), Grundlagen der Geometrie, Nachträge zu Band IV, for the complete Gauss's correspondence about the subject.

[48]English translation is quoted from Stanley N. Burris (Burris, 2003) and (Kárteszi, 1987).

[49]"WACHTER hat eine kleine Piece drucken lassen über die ersten Gründe der Geometrie [...] Ich komme immer mehr zu der Überzeugung, dass die Nothwendigkeit unserer Geometrie nicht bewiesen werden kann, wenigstens nicht vom menschlichen Verstande noch für den menschlichen Verstand. Vielleicht kommen wir in einem andern Leben zu andern Einsichten in das Wesen das Raums, die uns jetzt unerreichbar sind. Bis dahin müsste man die Geometrie nicht mit der Arithmetik, die rein a priori steht, sondern etwa mit der Mechanik in gleichen Rang setzen." (Gauss, 1870–1927, Vol. VIII, p. 177).

[50]"Es gibt eine zweifache Geometrie, – eine Geometrie im engern Sinn – die Euklidische; und eine astralische Grössenlehre." (Gauss, 1870–1927, Vol. VIII, p. 180).

[51]"Die Notiz von Hrn. Prof. SCHWEIKART hat mir ungemein viel Vergnügen gemacht, [...] denn obgleich ich mir recht gut die Unrichtigkeit der Euklidischen Geometrie denken kann, [...]" (Gauss, 1870–1927, Vol. VIII, p. 181).

leads to a geometry that is quite different from ours (Euclidean), which is consistent, and which I have developed quite satisfactorily to the point that I can resolve every question in it with the exception of the determination of a constant which does not present itself a priori. [...] All of my efforts to find a contradiction, an inconsistency in this non-Euclidean geometry have been fruitless."[52]

(5) Gauss to Bessel. Göttingen, 27 January 1829. "and my conviction that we cannot completely establish geometry a priori has become stronger."[53]

(6) Bessel to Gauss. Königsberg, 10 February 1829. "From what Lambert has said, and what Schweikart told me, it has become clear that our geometry is incomplete and needs a correction which is hypothetical and which disappears if the sum of the angles of a triangle = 180°."[54]

(7) Gauss to Bessel. Göttingen, 9 April 1830. "My innermost conviction is that the study of space is a priori completely different than the study of magnitudes; our knowledge of the former is missing that complete conviction of necessity (thus of absolute truth) that is characteristic of the latter."[55]

(8) Gauss to Farkas Bolyai.[56] Göttingen, 6 March 1832. "Precisely the impossibility of deciding a priori between Σ and S gives the

[52] "Die Annahme, dass die Summe der 3 Winkel kleiner sei als 180°, führt auf eine eigene, von der unsrigen (Euklidischen) ganz verschiedene Geometrie, die in sich selbst durchaus consequent ist, und die ich für mich selbst ganz befriedigend ausgebildet habe, so dass ich jede Aufgabe in derselben auflösen kann mit Ausnahme der Bestimmung einer Constante, die sich a priori nicht ausmitteln lässt. [...] Alle meine Bemühungen, einen Widerspruch, eine Inconsequenz in dieser Nicht-Euklidischen Geometrie zu finden, sind fruchtlos gewesen." (Gauss, 1870–1927, Vol. VIII, p. 187).

[53] "und meine Überzeugung, dass wir die Geometrie nicht vollständig a priori begründen können, ist, wo möglich, noch fester geworden." (Gauss, 1870–1927, Vol. VIII, p. 200).

[54] "Durch das, was Lambert gesagt hat, und was Schweikart mündlich äusserte, ist mir klar geworden, dass unsere Geometrie unvollständig ist, und eine Correction erhalten sollte, welche hypothetisch ist und, wenn die Summe der Winkel des ebenen Dreiecks = 180° ist, verschwindet." (Gauss, 1870–1927, Vol. VIII, p. 201).

[55] "Nach meiner innigsten Überzeugung hat die Raumlehre in unserm Wissen a priori eine ganz andere Stellung, wie die reine Grössenlehre; es geht unserer Kenntniss von jener durchaus diejenige vollständige Überzeugung von ihrer Nothwendigkeit (also auch von ihrer absoluten Wahrheit) ab, die der letztern eigen ist." (Gauss, 1870–1927, Vol. VIII, p. 201).

[56] This letter is discussed on pages 3 and 11 of this paper.

> clearest proof that Kant was not justified in asserting that space is just the form of our perception. Another equally strong reason is in a brief essay in the Scholarly Notices of Göttingen 1831, article 64, p. 625.[57] Perhaps it will not be a disappointment if you try to procure that volume of the G.G.A. (which may be accomplished through any bookseller in Vienna or Buda[58]), as you also find there, developed in a few pages, the essence of my views concerning imaginary quantities."[59]

The arguments put forward by Gauss in these letters for the belief in the consistency of non-Euclidean geometry were of an inductive and physical type. Inductive: no matter how much he had searched for an inconsistency with the hypothesis of the acute angle, he had been unable to find it. Physical: although Euclidean geometry was a very good candidate for the geometry of the physical space, an "anti-euclidean" geometry with small negative curvature could also provide the answer.

Did Gauss have the concept of a mathematical model? Certainly not, but it is no exaggeration to think that he may have entertained the idea that a surface in the space of three dimensions, with constant negative curvature and without singularities (the "opposite of the sphere"[60] mentioned in footnote 20 had singularities), could be a proof of the possibility of a new plane. We completely agree with (Burago et al., 2001, 158) on this point.[61]

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[57]See (Gauss, 1831). In this paper Gauss gives the geometrical interpretation of complex numbers.

[58]The German name for Buda is Ofen. Budapest became a single city with the unification in 1873 of Buda and Óbuda (Old Buda) together with Pest.

[59]"Gerade in der Unmöglichkeit, zwischen Σ und S a priori zu entscheiden, liegt der klarste Beweis, dass Kant Unrecht hatte zu behaupten, der Raum sei nur Form unserer Anschauung. Einen andern ebenso starken Grund habe ich in einem kleinen Aufsatze angedeutet, der in den Göttingischen Gelehrten Anzeigen 1831 steht Stück 64, pag. 625. Vielleicht wird es Dich nicht gereuen, wenn Du Dich bemühest Dir diesen Band der G.G.A. zu verschaffen (was jeder Buchhändler in Wien oder Ofen leicht bewirken kann), da darin unter andern auch die Quintessenz meiner Ansicht von den imaginären Grössen auf ein Paar Seiten dargelegt ist." (Gauss, 1870–1927, Vol. VIII, p. 224).

[60]"Gegenstück der Kugel."

[61]In his letters, Gauss did express his personal belief that there is no contradiction in the axioms of non-Euclidean geometry. He had an ill-fated, though extremely wise idea of how to construct a model: he wanted to realize hyperbolic geometry as the intrinsic geometry in some surface in $\mathbb{R}^3$ –the same way as spherical geometry is realized by Euclidean spheres. Gauss even found small embedded regions with desired properties (so-called pseudo-spheres), but he was unable to realize the whole plane. This led him to suspect that this might be an indication that a contradiction was still hidden somewhere.

In the above-mentioned letter of 1832 to Farkas Bolyai, Gauss says that he had obtained the same results as his son and in similar ways (see page 11). Nevertheless, in his letter to Schumacher, dated 1846, he says that Lobachevsky had obtained the same results but in a different way:[62] "in the work of Lobachevsky I did not find new results, but the development follows a different approach to the one I took, and indeed Lobachevsky carried out the task in a masterly fashion and in a truly geometric spirit,"[63] (see (Reventós and Rodríguez, 2005, 106)). Perhaps this "different approach" refers to the use of the length element of the imaginary sphere, which he obtained by 'Analogy' (whereas János Bolyai had deduced one of this length elements explicitly, and the other two implicitly). However, as Gauss was unable to show a complete surface[64] in the space of three dimensions with this length element, he did not publish anything. The synthetic rewriting of the Theory of Parallels, which Gauss began in 1831, was far surpassed by the *Appendix*, a complete and masterly synthetic deduction of a hyperbolic arc length.

Perhaps Gauss thought that "anti-euclidean" geometry could emerge by using the geometrical interpretation of complex numbers;[65] that would explain the suggestion made to Farkas Bolyai at the end of the letter. János indeed read Gauss's paper,[66] and developed independently a conception of complex numbers that applied to number theory. As far as we know, J. Bolyai did not relate the new geometrical conception of complex numbers with the problem of consistency of the new geometry.

It is also possible that Gauss made the same suggestion to Riemann; but Riemann was by this time occupied with other mathematical and physical problems that lead him to the discovery of Riemann surfaces (the first example of a topological manifold of dimension two that is not a surface of a three dimensional space: the first example of an abstract manifold!), as well as to a conception of physical space as a perfectly elastic and massless medium formed by an elastic fluid, affected by the energy-momentum of the physical fields within it. Klein compared

[62]Gauss refers to the German version (Lobachevsky, 1955) that does not use differential calculus.

[63]"materiell für mich Neues habe ich also im LOBATSCHEWSKYschen Werke nicht gefunden, aber die Entwickelung ist auf anderm Wege gemacht, als ich selbst eingeschlagen habe, und zwar von LOBATSCHEWSKY auf eine meisterhafte Art in ächt geometrischem Geiste." (Gauss, 1870–1927, Vol. VIII, p. 239).

[64]A surface without singularities, where the straight lines are infinite.

[65]In B we sketch an elementary proof of consistency using complex numbers.

[66]Kiss comments in (Kiss, 1999, 73), that Gauss does not give the correct reference, because the subject was completely developed in another of Gauss's works.

Riemann with Faraday, who had described the electromagnetic field with the idea of "lines of force". With Riemann, geometry became a physical geometry. This idea will be developed in (Rodríguez, 2011).

8. Looking for an imaginary sphere within $\mathbb{R}^3$

Perhaps the most crucial mistake committed by Gauss in this matter was to look for an imaginary sphere within $\mathbb{R}^3$.

In fact, there exists no imaginary sphere in the usual sense of a surface in $\mathbb{R}^3$ of constant negative curvature.[67] Therefore, the search for an imaginary sphere proved to be a futile struggle.[68] It is possible that in 1831 Gauss was aware that he would come to a dead end, and decided to take the deductive point of view. But it was too late: János Bolyai had already followed up this path in the *Appendix*.

The impossibility of finding a complete surface in $\mathbb{R}^3$ of constant negative curvature could have caused Gauss to doubt his belief in the consistency of non-Euclidean Geometry, and may be the main reason why he made no effort to publicize the *Appendix*.

The *Appendix* proves that the problem of the consistency is almost the same in both geometries: the parasphere (a surface of the non-Euclidean space) has Euclidean geometry (see footnote 25). Hence, for symmetry, it seems reasonable to look for an imaginary sphere within $\mathbb{R}^3$.

It would be interesting to answer the following question: Why did Gauss only look at surfaces in three dimensional space?

[67]The interpretation of the Gauss curvature as the product of principal curvatures, and hence equal to $1/R^2$ for a sphere of radius R, appears in the first version of *Disquisitiones* in 1825. In fact, Olinde Rodrigues essentially proved it in 1815 when he proved what today we know as the Gauss-Bonnet theorem, see (Rodrigues, 1815). Indeed, Rodrigues in his study of the integral of the product of the principal curvatures says: "Let us imagine a sphere with radius equal to the unit; and let us move the radius of this sphere in a way such that it will be paralell to all the normals of the piece of the surface which we want to integrate. The area described by the endpoint of this radius will coincide with the value of the desired integral." ("Concevons une sphère d'un rayon égal à l'unité; puis faisons mouvoir le rayon de cette sphère, de manière qu'il soit successivement parallèle à toutes les normales de la portion de surface sur laquelle on veut prendre l'intégrale, l'aire sphérique décrite par l'extrémité de ce rayon, sera la valeur de l'intégrale cherchée.") Free English translation.

[68]In 1910, Hilbert proved that there exists no complete regular surface of constant negative curvature immersed in $\mathbb{R}^3$. In 1955, Kuiper (Kuiper, 1955) proved that such a surface does exist if we change regular for $\mathcal{C}^1$.

A possible answer is that numbers and geometry were on different levels, in the sense that the identification of $\mathbb{R}$, $\mathbb{R}^2$ and $\mathbb{R}^3$ as geometrical objects was still to be clearly made. Gauss, and other contemporary mathematicians, do not identify the set of pairs of real numbers as is done today. It was necessary to wait for Dedekind for the foundation of real numbers; he probably learned from Riemann the importance of thinking about mathematics conceptually, in order to take the definitive step towards the geometrization of $\mathbb{R}^n$.

As Ferreirós has observed, it is precisely with Riemann that the idea of a conceptual mathematics arises; a mathematics that studies manifolds and their mappings (see (Ferreirós, 2000, 93–95) and (Ferreirós, 2007)). Riemann took this giant step because he needed to extend geometric intuition to areas of mathematics different from geometry. However, at the same time, he also found the study of multiply extended quantities useful for thinking about geometry without any spatial intuition (see (Ferreirós, 2000, 94)); Riemann coincides on this point with Lambert and his 'Analytical Program', which was introduced with the hope of solving the classical problem of the Euclidean Theory of Parallels (see (Reventós and Rodríguez, 2005, 16)). This program was completed by Hilbert in his fundamental work on foundations of geometry of 1899 (Hilbert, 1899), using set theory introduced by Cantor. As Hilbert said: "No one shall expel us from the paradise that Cantor has created for us."[69]

9. Non-Euclidean geometry as absolute Euclidean geometry on a reduced scale

A note by Gauss dated about 1840–1846 (Gauss, 1870–1927, Vol. VIII, pp. 255–257) was found in his copy of Lobachevsky's work *Geometrischen Untersuchungen zur Theorie der Parallellinien*, (Lobachevsky, 1955). This note[70] is quite short and Gauss did not give it a title. However, it is referred to as "The spherical and the non-Euclidean geometry"[71] by Stäckel who carefully commented on it.

Although this note was written many years after the Gauss-Bolyai relation discussed in the previous sections, we would like to draw attention to it because it gives a clue as to how Gauss uses differential geometry in order to consider problems of non-Euclidean geometry. Perhaps

[69]"Aus dem Paradies, das Cantor uns geschaffen, soll uns niemand vertreiben können."

[70]In this note, not written for publication, Gauss reveals a part of his method for doing Differential Geometry: he applies classical geometry to a small variation of a diagram.

[71]"Die Sphärische und die Nicht-Euklidische Geometrie."

Gauss was trying to arrive at the same conclusions as Lobatschevsky, but by his own method.

Before continuing, we would like to point out that the formulas[72] which head this note by Gauss (Gauss, 1870–1927, Vol. VIII, p. 255) relating the angles and the sides of a triangle with two unknown functions f, g, and from which he is able to compute f and g, hold for absolute geometry. Indeed they can be deduced solely from absolute geometry – where the side-angle-side criterion holds –, under the hypothesis that in this absolute geometry trigonometric formulas exist that relate the sides and the angles of a triangle, and assuming that this absolute geometry is Euclidean on a reduced scale.

These two hypotheses, together with the relations among the sides of a Saccheri quadrilateral, imply the rectificability of equidistants, and the rectificability of a circle (which are true in absolute geometry and were well-known by Gauss at that time).[73] Using these hypotheses and relations, Gauss's note can be written without any great difficulty. Although Stäckel's explanations in (Gauss, 1870–1927, Vol. VIII, pp. 257–264) are totally clear and can be followed easily, we give here, for the benefit of the reader, the deduction of the first formula adapted to our approach.

Let $\triangle ABC$ be a right-angle triangle with sides a, b, c, and assume that it is changing with time in such a way that the right-angle C remains constant.

Because of the SAS Theorem (side-angle-side), every trigonometric relation between A, B, C, a, b, c can be reduced to a relation between b, c, A. Thus we can assume that the relation $F(b(t), c(t), A(t)) = 0$ holds for each t. Hence, differentiating we obtain

$$F_b \frac{db}{dt} + F_c \frac{dc}{dt} + F_A \frac{dA}{dt} = 0$$

[72]

$$\begin{aligned} ga.\partial b - \sin B.\partial c + fc.\cos B.\partial A &= 0 \\ gb.\partial a - \sin A.\partial c + fc.\cos A.\partial B &= 0 \\ \sin B.\partial a - ga.\cos B.\partial b - fc.\partial A &= 0 \\ gb.\cos A.\partial a - \sin A.\partial b + fc.\partial B &= 0. \end{aligned}$$

[73]The upper side of a Saccheri quadrilateral of equal sides a and base b is equal to $g(a)b$ for some function g, and the length of a circular sector of α radians and radius r is $f(r)\alpha$ for some function f.

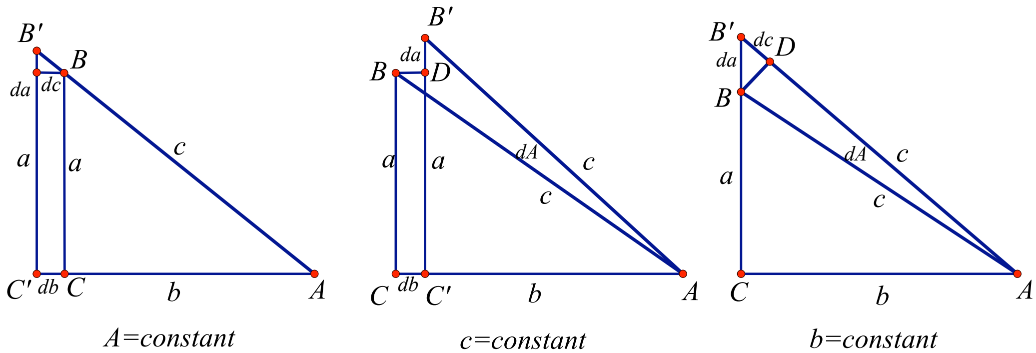

FIGURE 5. Stäckel's diagram redrawn, (Gauss, 1870–1927, Vol. VIII, p. 259).

From the first diagram (Figure 5) we deduce

$$\sin B = \frac{g(a)db}{dc}$$

since the small triangle with hypotenuse dc can be considered as Euclidean, and one of the cathetus is the top side of a Saccheri quadrilateral with base db and height a.

From the second diagram we deduce

$$\cos B = \frac{BD}{BB'} = \frac{-g(a)db}{f(c)dA}$$

since the small triangle $\triangle BB'D$ can be considered as Euclidean and BB' as the arc length of a circular sector of radius c and angle dA. The minus sign comes from the relative position between C and C'.

From the third diagram we deduce

$$\tan B = \frac{BD}{dc} = \frac{f(c)dA}{dc}$$

since the small triangle $\triangle BB'D$ can be considered as Euclidean, and BD as the arc length of a circular sector of radius c and angle dA.

From these three relations we easily compute the partial derivatives F_a, F_b, F_c (up to a constant) and obtain the first Gauss formula in (Gauss, 1870–1927, Vol. VIII, p. 255):

$$g(a)db - \sin B dc + f(c)\cos B dA = 0. \tag{4}$$

In fact, the three above steps followed by Stäckel can be viewed together in the following diagram (Figure 6), since formula (4) says only that

$$FD = FE + ED.$$

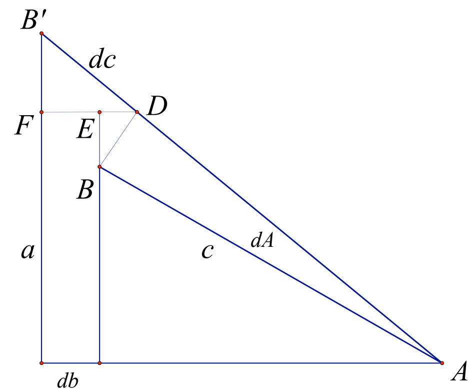

FIGURE 6. Representation of the three differentials.

From (4) and similar expressions, Gauss with his characteristic genius arrives at a second order differential equation, which allows the functions $f(c)$ and $g(a)$ to be computed easily. Gauss assumes that the integration constant is negative ($-kk$ in Gauss's notation), thereby obtaining

$$\begin{aligned} f(x) &= \frac{\alpha}{k}\sin kx \\ g(x) &= \cos kx \end{aligned}$$

and, in particular, the spherical trigonometric formulas for a sphere of radius $1/k$. Nevertheless, if we assume that the integration constant is positive, we obtain, without the introduction of imaginary numbers, the non-Euclidean trigonometric formulas (those corresponding to a sphere of radius i/k).

We also remark that by arriving at these formulas Gauss obtains two of the ds^2 in the *Appendix*: expressions (2) and (3) above.

This note by Gauss is widely commented upon in (Reichardt, 1985, Section 2.3).

In Section 19 of the *Disquisitiones*, Gauss computes $f'(0)$, obtaining $f'(0) = 1$, i.e. the constant α introduced by Gauss in the above computations is 1, if there is a tangent plane in A. In fact $\alpha \neq 1$ only if A is a singular point, such as the vertex of a cone.

In fact Gauss says (Gauss, 1828, Section 19): "Generally speaking, m [$m = \sqrt{G}$][74] will be a function of p, q and mdq the expression for the element of any line whatever of the second system. But in the particular case where all the lines p go out from the same point [...] for an infinitely small value of p the element of a line of the second

[74]Note that, under the hypothesis of radial symmetry, we are assuming, $m = f$.

system (which can be regarded as a circle described with radius p) is equal to pdq, we shall gave for an infinitely small value of p, $m = p$, and consequently, for $p = 0$, $m = 0$ at the same time, and $\frac{\partial m}{\partial p} = 1$."[75,76]

Why was it clear to Gauss in 1827 that the "element of any line" of the second system is pdq, while in 1840 this element is αpdq? The reason could be that in the *Disquisitiones* the argument used is that the metric is defined in the singular point $p = 0$, which is guaranteed because the metrics on the surfaces considered in the *Disquisitiones* come from the ambient metric in $\mathbb{R}^3$.[77]

10. Conclusions

In this paper we analyze a crucial moment in the history of the discovery of non-Euclidean geometry: the reading Gauss made of Bolyai's *Appendix* in 1832. We assume the very plausible hypothesis that Gauss was following Lambert's 'Analytical Program' (described on page 4) and that he was looking, among the surfaces in $\mathbb{R}^3$, for Lambert's hypothetical imaginary sphere (see page 3).

Gauss placed on record this reading in two letters; one to Gerling in February 1832 and another longer letter to Bolyai's father in March 1832.

In the letter to Farkas, Gauss says:

(1) "The way followed by your son, and the results he obtained agree almost from beginning to end with the meditations I had been engaged in partly for 30–35 years already."
(2) "I had planned to write down everything in the course of time [...] now I can save myself this trouble."

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[75]Spivak, in his comments on the *Disquisitiones*, gives a more complete proof of this fact; see (Spivak, 1979, Vol. 2, pp. 84 and 120.).

[76]"Generaliter loquendo m erit functio ipsarum p, q atque $m\,dq$ expressio elementi cuiusvis lineae systematis. In casu speciali autem, ubi omnes lineae p ab eodem puncto proficiscuntur, manifesto pro $p = 0$ esse debet $m = 0$; porro si in hoc casu pro q adoptamus angulum ipsum, quem elementum primum cuiusvis lineae primi systematis facit cum elemento alicuius ex ipsis ad arbitrium electae, quum pro valore infinite parvo ipsius p, elementum lineae secundi systematis (quae considerari potest tamquam circulus radio p descriptus), sit $= p\,dq$, erit pro valore infinite parvo ipsius p, $m = p$, adeoque, pro $p = 0$ simul $m = 0$ et $\frac{dm}{dp} = 1$."

[77]Many years later, the introduction of conical singularities into the study of hyperbolic manifolds led to the introduction of metrics of the type $ds^2 = dp^2 + \alpha^2 R^2 \sin^2 \frac{p}{R} dq^2$, which verify $f'(0) = \alpha$. All these metrics, for different values of α, have the same curvature, but the length of the element of a line $p =$ constant is αpdq.

(3) "Perhaps it will not be a disappointment if you try to procure that volume [...] as you also find there, developed in a few pages, the essence of my views concerning imaginary quantities."

We answer some natural questions arising from these statements by Gauss:

(1) What was the approach adopted by Gauss in his meditations? Was it the same as that adopted by Bolyai?
(2) Why did he feel that there was no longer any need to write anything more about it?
(3) What is the relation between imaginary quantities and the problem of the Euclidean Theory of Parallels?

In Section 5 we see how Bolyai axiomatically deduces a formula for the ds^2 in the hypercyclic coordinate system, which is the system most similar to that of rectangular coordinates in Euclidean geometry. This shows that he wanted to follow the method of the differential geometry of his time: he was looking for an arc length element in the new plane. Gauss says in his letter that the approach adopted by Bolyai is his own. However, Bolyai was not familiar with the *Disquisitiones* and did not recognize the two first ds^2. Nevertheless, it seems clear that they were indeed recognized by Gauss.

In 1831, Gauss gave up searching for Lambert's imaginary sphere in $\mathbb{R}^3$ and opted for the deductive method: after studying the transitivity of parallelism, he described synthetically the paracycle (see (Gauss, 1870–1927, Vol. VIII, pp. 202–209) or (Bonola, 1955, 67–74)). However, in January 1832, after his reading of the *Appendix*, he gave up writing about such a difficult subject: the son of his old friend Farkas had "wonderfully outmatched him".

In the *Appendix*, Bolyai gives the rectification of the paracycle. This should have allowed him to deduce the first ds^2 (see page 12), but he failed to notice it. The new geometry can be deduced from this arc length element with the methods of the *Disquisitiones*; Bolyai had to deduce the third ds^2 to arrive at this conclusion (see page 17).

The problem of consistency still remained to be solved. This was possible if Lambert's imaginary sphere could be found. However, this depended on the consistency of imaginary quantities, a question resolved by Gauss in his note of 1831 (that recommended for reading to F. Bolyai). Gauss was right: in Appendix B we see how the 'Analogy' and the complex plane as conceived by Gauss lead naturally to the Poincaré disc model of the hyperbolic plane. This shows that non-Euclidean geometry is as consistent as Euclidean geometry.

Why did Gauss not publish a review of the *Appendix*? This would have attracted the attention of the mathematical community to this important work.

His fear of the Kantians, led by his colleague Lotze, is not a convincing reason. We provide another possible reason: Gauss hoped to find the imaginary sphere. He knew the pseudosphere (see page 6), but it was not complete. If the singularities were in some sense inevitable, Bolyai's plane would also be inconsistent. Furthermore the *Appendix* fails to satisfy Lambert's 'Analytical Program': it contains 23 drawings (see Section 6), which far from the idea that in the proof "we should never rely on any representation of the matter" (see page 4).

Gauss was isolated (see Section 4) and made the mistake of looking for the imaginary sphere in $\mathbb{R}^3$. In 1910, Hilbert proved that there exists no complete regular surface of constant negative curvature immersed in $\mathbb{R}^3$ (see Section 8). For Gauss, and also for Beltrami (see footnote 2), the length element ds is always the length element of a surface in $\mathbb{R}^3$. The notion of abstract surface had yet to appear.

Section 9 has been added for completeness, in order to see how Gauss uses differential geometry for considering problems of non-Euclidean geometry. We also see how Gauss again addresses singularities and eventually finds the last two ds^2 of the *Appendix*.

Acknowledgements. We are grateful to Professor Jeremy Gray for his valuable comments after a reading of a first draft of this paper and for drawing our attention to the references (Reichardt, 1985) and (Gauss, 1870–1927, Vol. VIII, pp. 175–178 and 250–265).

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Appendix A. Coordinate systems

In the hyperbolic plane, apart from the polar or "cyclic" coordinates and the cartesian or "hypercyclic" coordinates, there are also the "paracyclic" or "horocyclic" coordinates in which one of the distances is measured on paracycles.

Cyclic (r, α). Here, r is the distance between the point P and the origin O; and α is the angle between the straight line PO and a given straight line through O. Observe that $r =$ constant is a hyperbolic circle.

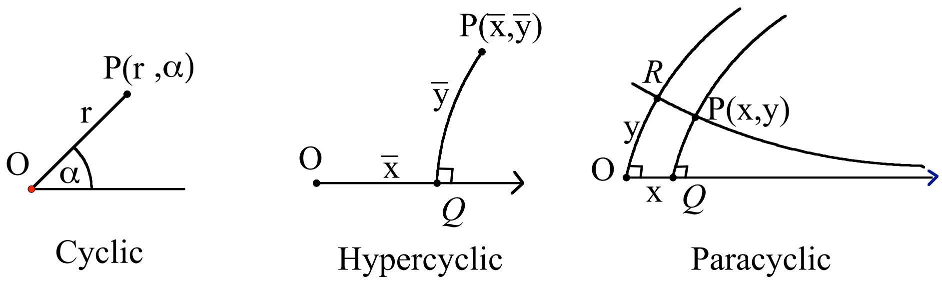

FIGURE 7. Three coordinate systems.

Hypercyclic $(\bar{x}, \bar{y})$. Here, $\bar{x}$ is the distance between the origin O and the point Q, the intersection of the line through P orthogonal to a given line through O; and $\bar{y}$ is the distance between the point P and Q. Observe that $\bar{y} =$ constant is a hypercycle (equidistant).

Both cyclic and hypercyclic coordinates were introduced and widely used by Gauss in the *Disquisitiones*.

Paracyclic (x, y). Here, x is the distance between the origin O and the point Q, the intersection with a given line through O of the horocycle through P and the axis of this line; and y is the length of the horocycle OR, where R is the intersection of the axis through P with the horocycle of this family through O. Observe that $x =$ constant is a paracycle (horocycle).

Recall that three points of the hyperbolic plane determine a straight line, a circle, a hypercycle or a paracycle. The assumption that three points not on a line determine a circle is equivalent to the fifth postulate. In fact, this was the mistake made by Farkas Bolyai in his proof of this postulate.

HYPERCYCLIC-CYCLIC

The change of coordinates cyclic-hypercyclic is immediate applying trigonometry to a right triangle of sides $\bar{x}, \bar{y}$ and hypotenuse r (see (Reventós and Rodríguez, 2005, 120))

$$\begin{aligned}\cosh\frac{r}{R} &= \cosh\frac{\bar{x}}{R}\cosh\frac{\bar{y}}{R}\\ \sinh\frac{\bar{y}}{R} &= \sinh\frac{r}{R}\sin\theta,\end{aligned}$$

From this system we can write: $x = x(r, \theta)$, $y = y(r, \theta)$.

In particular,

$$d\bar{y}^2 + \cosh^2\frac{\bar{y}}{R}\,d\bar{x}^2 = dr^2 + R^2\,\sinh^2\frac{r}{R}\,d\theta^2.$$

Hypercyclic-Paracyclic

Let us assume that the point P has hypercyclic coordinates $(\bar{x}, \bar{y})$, and paracyclic coordinates (x, y).

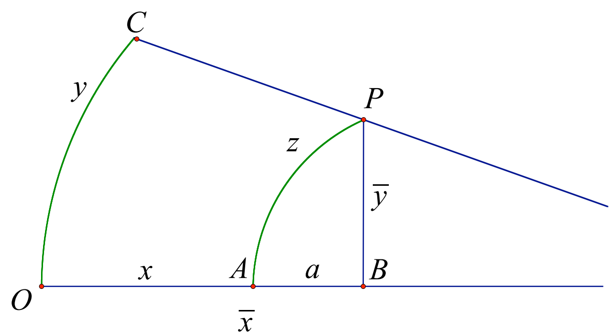

Figure 8. Relation between hypercyclic and paracyclic coordinates.

In the diagram (Figure 8) CO and PA are arcs of horocycles orthogonal to the parallel straight lines CP, OA. The hypercyclic coordinates are given by $\bar{x} = OB$, $\bar{y} = PB$; and the paracyclic coordinates are given by $x = OA$, $y = CO$.

The relation between the length z of the horocycle PA and the length $\bar{y}$ of the straight line PB is

$$z = ye^{-x}. \tag{5}$$

Also

$$z = \sinh \bar{y}. \tag{6}$$

And

$$e^a = \cosh \bar{y}, \tag{7}$$

where $a = AB$. We remark that equations (5), (6) and (7) are given directly in the *Appendix*: Equation (5) in §24 and equations (6) and (7) in §32. Bolyai writes $z = i \cot CBN$, which in our notation is $z = \cot \Pi(\bar{y})$, (we are assuming curvature $= -1$, i.e. $i = 1$), but it is easy to see that $\cot \Pi(\bar{y}) = \sinh \bar{y}$, and thus we have equation (6).

From these equations we can make the change of coordinates explicit:

$$\begin{aligned} \bar{x} &= x + \frac{1}{2}\ln(1 + y^2 e^{-2x}) \\ \bar{y} &= \ln(ye^{-x} + \sqrt{y^2 e^{-2x} + 1}). \end{aligned}$$

In particular,

$$d\bar{y}^2 + \cosh^2 \bar{y}\, d\bar{x}^2 = dx^2 + e^{-2x}\, dy^2.$$

Appendix B. A wasted opportunity

As we commented in Section 2, as a result of Legendre's influence, the French school was not interested in the classical problem of the Euclidean Theory of Parallels. Moreover, Lagrange's analytical point of view spread rapidly throughout the mathematical community and the synthetical approach remained buried until Poincaré unearthed it again.

The 'wasted opportunity' is revealed in the following argument, available to the French school, which allows us to give a construction of the Poincaré disc model of non-Euclidean geometry.

This construction only uses the stereographic projection and the 'Analogy'. So it could easily have been performed by Monge or his school in the École Polytechnique, thirty years before the *Appendix* (which they did not do). This school had as its leitmotif the translation of geometric properties using geometric transformations; in particular, stereographic projections of the quadrics over the plane. For instance, Michel F. Chasles (1793 – 1880) in (Chasles, 1837, 191) says: "From then on, Monge's students successfully cultivated this Geometry of a really new kind, which has often been rightly referred to as the "Monge School", which as we have said consists in introducing into plane Geometry considerations of the three dimensional Geometry."[78]

The stereographic projection between the sphere S_R of radius R and the plane that contains the equator is given by

$$\begin{aligned} p &= \frac{Rx}{R-z} \\ q &= \frac{Ry}{R-z} \end{aligned}$$

with $x^2 + y^2 + z^2 = R^2$.

Equivalently, the image of the point $(x, y, z) \in S_R$ is the complex number $w = p + iq$.

Let us 'translate' the geometry of S_R to the extended complex plane $\mathbb{C}$ via this stereographic projection. First we note that the equator is given by

$$w\bar{w} = R^2.$$

[78]"Depuis, les élèves de Monge cultivèrent avec succès cette Géométrie, d'un genre vraiment nouveau, et à laquelle on a souvent donné, avec raison, le nom d'école de Monge, et qui consiste, comme nous venons de dire, à introduire dans la Géométrie plane des considerations de Géométrie à trois dimensions."

Moreover, if w, w' are the images under the stereographic projection of antipodal points, then

$$w' = -\frac{R^2}{\bar{w}}. \tag{8}$$

Since stereographic projection takes circles to circles, the image of a meridian is a circle in the complex plane. Hence, if $P, Q \in \mathbb{C}$, the 'straight line' PQ is the circle determined by the three points $P, Q, -P^*$, where P^* is the inverse point of P with respect to the circle $w\bar{w} = R^2$.

The 'angles' of this geometry on $\mathbb{C}$ are the angles in S_R. 'Congruent' relations can also be derived in this way. It is the geometry of the sphere considered as $\mathbb{C} \cup \{\infty\}$.

If we now apply the 'Analogy' by changing formally R by Ri in (8), we obtain

$$w' = \frac{R^2}{\bar{w}}.$$

What are the straight lines of this new geometry? If $P, Q \in \mathbb{C}$, the new straight line PQ is the circle determined by the three points P, Q, P^*. Since this circle is orthogonal to the circle $w\bar{w} = R^2$, the new straight lines are circles orthogonal to the boundary of the disc of radius R.

Note that we are obliged to exclude the case $P = P^*$ because the three points must be different. However, the set of points P with $P = P^*$ is the boundary of the disc. Therefore this boundary does not belong to the new geometry.

Thus we have the open disc and its complement, which are 'equal' through inversion. If we consider the open disc with the straight lines defined above, and further consider that 'movements' are generated by inversions, we have the classical Poincaré disc. In other words, we have a model of non-Euclidean geometry and the problem of the consistency is solved. In fact, an inconsistency in non-Euclidean geometry would be translated into an inconsistency in inversion geometry, and hence into an inconsistency in Euclidean geometry. Non-Euclidean geometry is thus as consistent as Euclidean geometry.

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