INTEGRAL GEOMETRY OF PAIRS OF PLANES

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Abstract. We deal with integrals of invariant measures of pairs of planes in euclidean space $\mathbb{E}^3$ as considered by Hug and Schneider. In this paper we express some of these integrals in terms of functions of the visual angle of a convex set. As a consequence of our results we evaluate the deficit in a Crofton-type inequality due to Blashcke.

1. Introducction

The main goal of this paper is to study integrals of invariant measures with respect to euclidean motions in the euclidean space $\mathbb{E}^3$, extended to the set of pairs of planes meeting a compact convex set. To carry out this objective we express these integrals in terms of functions of the dihedral visual angle of the convex set from a line and integrate them with respect to an invariant measure in the space of lines.

The first known formula involving the visual angle of a convex set in the euclidean plane $\mathbb{E}^2$ is Crofton's formula given in [2]. Other results in this direction were obtained by Hurwitz ([8]), Masotti ([10]) and others, in which the use of Fourier series is the main tool. Recently the authors ([3], [4]) have dealt with a general type of integral formulas from the point of view of Integral Geometry.

When trying to generalize these results to higher dimensions the role played by Fourier series in the case of the plane has to be replaced by the use of spherical harmonics. In this sense Theorem 4.1 plays an important role. After stating and proving this result in dimension 3 we were aware of the paper by Hug and Schneider [7] where a more general result in any dimension is proved. In fact the present paper can be considered in some sense as a complement to [7], the novelty been the introduction of the dihedral visual angle.

In Proposition 3.1 we give a characterization of invariant measures in the space of pairs of planes. These will be the kind of measures considered along the paper.

In section 4, using Hug-Schneider's Theorem [7, p. 349], we give an expression for the integral of the sinus of the dihedral visual angle of pairs of planes meeting a given compact convex set K in terms of geometrical properties of K, see formula (4.3); also we characterize the compact convex sets of constant width in terms of invariant measures given by Legendre polynomials in Proposition 4.3.

In section 5 we assign to any invariant measure on the space of pairs of planes an appropriate function of the dihedral visual angle of a given convex set. The

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integral of this function with respect to the measure on the space of lines gives the integral of the above measure extended to those planes meeting the convex set. This result is given in Theorem 5.2. Then we relate this result to Blaschke's work [1]. If K is a convex set of mean curvature M and area of its boundary F, Blaschke proves the formula

$$\int_{G\cap K=\emptyset}(\omega^2-\sin^2\omega)\,dG=2M^2-\frac{\pi^3F}{2}$$

where $\omega=\omega(G)$ is the dihedral visual angle of K from the line G. This equality reveals the significance of the function of the visual angle $\omega^2-\sin^2\omega$. One can ask what role does it play the function $\omega-\sin\omega$; this function, interpreting ω as the visual angle in the plane is significant thanks to Crofton formula. In [1] the inequality

$$\int_{G\cap K=\emptyset}(\omega-\sin\omega)\,dG\geq\frac{\pi}{4}(M^2-2\pi F)$$

is stablished. Here we provide a simple formulation of the deficit in this inequality by means of Theorem 5.4 where it is proved that

$$\int_{G\cap K=\emptyset}(\omega-\sin\omega)dG=\frac{\pi}{4}(M^2-2\pi F)+\pi\sum_{n=1}^{\infty}\frac{\Gamma(n+1/2)^2}{\Gamma(n+1)^2}\|\pi_{2n}(p)\|^2,$$

whith $\pi_{2n}(p)$ the projection of the support function p of K on the vector space of spherical harmonics of degree $2n$.

In section 6 we give a formulation of Theorem 5.2 in terms of Fourier series of the function of the visual angle assigned to an invariant measure. As a consequence one obtains that the integral of any invariant measure in the space of pairs of planes extended to those meeting a compact convex set K is an infinite linear combination of integrals of even powers of the sine of the visual angle of K. From this we exhibit in Proposition 6.3 a simple family of polynomial functions that are in some sense a basis for the integrals considered in Theorem 4.1. In fact every invariant integral can be written as an infinite linear combination of integrals with respect to the invariant measures given by those polynomial functions.

In section 7, motivated by the role played by the integrals of even powers of the sine of the visual angle of a compact set K we compute these integrals in terms of the expansion in spherical harmonics of the support function of K.

2. Preliminaries

Support function. The *support function* of a compact convex set K in the euclidean space $\mathbb{E}^3$ is defined as

$$p_K(u)=\sup\{\langle x,u\rangle : x\in K\}$$

for u belonging to the unit sphere S^2. If the origin O of $\mathbb{E}^3$ is an interior point of K then the number $p_K(u)$ is the distance from the origin to the support plane of K in the direction given by u. The *width* w of K in a direction $u\in S^2$ is $w(u)=p_K(u)+p_K(-u)$.

From now on we will write $p(u)=p_K(u)$ and will assume that $p(u)$ is of class C^2; in this case we shall say that the boundary of K, ∂K, is of class $\mathcal{C}^2$.

Spherical harmonics. Let us recall that a spherical harmonic of order n on the unit sphere S^2 is the restriction to S^2 of an harmonic homogeneous polynomial of degree n. It is known that every continuous function on S^2 can be uniformly approximated by finite sums of spherical harmonics (see for instance [6]).

More precisely, the function $p(u)$ can be written in terms of spherical harmonics as

$$p(u) = \sum_{n=0}^{\infty} \pi_n(p)(u), \tag{2.1}$$

where $\pi_n(p)$ is the projection of the support function p on the vector space of spherical harmonics of degree n. An orthogonal basis of this space is given in terms of the longitude θ and the colatitude φ in S^2 by

$$\{\cos(j\theta)(\sin\varphi)^j \, P_n^{(j)}(\cos\varphi), \quad \sin(j\theta)(\sin\varphi)^j \, P_n^{(j)}(\cos\varphi) : \quad 0 \le j \le n\}$$

where $P_n^{(j)}$ denotes the jth derivative of the nth Legendre polynomial P_n, see [6].

It can be seen that $\pi_0(p) = \mathcal{W}/2 = M/4\pi$ where $\mathcal{W} = 1/4\pi \int_{S^2} w(u)\, du$ is the *mean width* of K, and M is the *mean curvature* of K (cf. [6]). It is clear that $\pi_0(p)$ is invariant under euclidean motions and that $\pi_1(p)$ is not. It is known that $\pi_n(p)$ is also invariant for every $n \neq 1$ (cf. [12]).

As $w(u) = p(u) + p(-u)$, one can easily check that K has constant width if and only if $\pi_n(p) = 0$ for $n \neq 0$ even.

We recall now the Funk-Hecke theorem which gives the value of the integral over the sphere of a given function multiplied by a spherical harmonic. We restrict ourselves to the case of dimension 3.

Theorem 2.1 (Funk-Hecke)**.** *If $F : [-1, 1] \longrightarrow \mathbb{R}$ is a bounded measurable function and Y_n is a spherical harmonic of order n, then*

$$\int_{S^2} F(\langle u, v\rangle) Y_n(v) d\sigma(v) = \lambda_n Y_n(u), \quad u \in S^2$$

with

$$\lambda_n = 2\pi \int_{-1}^{1} F(t) P_n(t)\, dt,$$

where P_n is the Legendre polynomial of degree n.

Measures in the space of planes. The space of planes $\mathcal{A}_{3,2}$ in $\mathbb{E}^3$ is a homogeneous space of the group of isometries of $\mathbb{E}^3$. It can also be considered as a line bundle $\pi : \mathcal{A}_{3,2} \longrightarrow \mathrm{Gr}(3, 2)$ where $\mathrm{Gr}(3, 2)$ is the Grassmannian of planes through the origin in $\mathbb{E}^3$ and $\pi(E)$ is the plane parallel to E through the origin. The fiber on $E_0 \in \mathrm{Gr}(3, 2)$ is identified with $\langle E_0\rangle^\perp$. Each plane $E \in \mathcal{A}_{3,2}$ is then uniquely determined by the pair $(\pi(E), E \cap \langle \pi(E)\rangle^\perp)$. Every pair $(E_0, p) \in \mathrm{Gr}(3, 2) \times \mathbb{R}^3$ determines an element $E_0 + p \in \mathcal{A}_{3,2}$.

We shall also consider the space of affine lines $\mathcal{A}_{3,1}$ in $\mathbb{E}^3$; it is a vector bundle $\pi : \mathcal{A}_{3,1} \longrightarrow \mathrm{Gr}(3, 1)$ where $\mathrm{Gr}(3, 1)$ is the Grassmannian of lines through the origin and every affine line $G \subset \mathbb{E}^3$ can be identified with $(\pi(G), G \cap \langle \pi(G)\rangle^\perp)$ (see for instance [9]).

Both the isometry group of $\mathbb{E}^3$ and the isotropy group of a fixed plane $E \in \mathcal{A}_{3,2}$ are unimodular groups; so the Haar measure of the group of isometries is projected into a isometry-invariant measure m on $\mathcal{A}_{3,2}$.

For a measurable set $B \subset \mathcal{A}_{3,2}$ we consider

$$m(B) = \int_{\mathcal{A}_{3,2}} \chi_B(E) dE := \int_{\mathrm{Gr}(3,2)} \left(\int_{E_0^\perp} \chi_B(E_0 + p) dp \right) d\nu$$

where χ_B is the characteristic function of B, dp denotes the ordinary Lebesgue measure on $E_0^\perp$ and $d\nu$ a normalized isometry-invariant measure on $\mathrm{Gr}(3,2)$ such that $\nu(\mathrm{Gr}(3,2)) = 2\pi$.

More generally, if $f : \mathcal{A}_{3,2} \to \mathbb{R}$ and $\bar{f} : \mathrm{Gr}(3,2) \times \mathbb{R}^3 \to \mathbb{R}$ are related by $\bar{f}(E_0, p) = f(E_0 + p)$ we have

$$\int_{\mathcal{A}_{3,2}} f(E)\, dE := \int_{\mathrm{Gr}(3,2)} \left(\int_{E_0^\perp} \bar{f}(E_0, p) dp \right) d\nu.$$

Notice that the only measures on $\mathcal{A}_{3,2}$ invariant under isometries are those of the form $f(E)dE$ with f a constant function.

In a similar way one can define a normalized isometry-invariant measure on $\mathcal{A}_{3,1}$ that will be denoted by dG.

3. Invariant measures in the space of ordered pairs of planes

We consider measures in the space $\mathcal{A}_{3,2} \times \mathcal{A}_{3,2}$ of pairs of planes in $\mathbb{E}^3$ of the form $m_{\tilde{f}} := \tilde{f}(E_1, E_2) dE_1 dE_2$. We want to study which functions $\tilde{f}$ give an isometry-invariant measure, that is a measure $m_{\tilde{f}}$ satisfying $m_{\tilde{f}}(B) = m_{\tilde{f}}(gB)$ for every euclidean motion g. For instance, it is known that for a given compact convex set K one has $\int_{E \cap K \neq \emptyset} dE = M$. So when $\tilde{f}(E_1, E_2) = 1$ we have

$$\int_{K \cap E_i \neq \emptyset} dE_1 dE_2 = M^2 = 4\pi^2 \mathcal{W}^2, \tag{3.1}$$

where M and $\mathcal{W}$ are the mean curvature and the mean width of K, respectively.

Proposition 3.1. *The measure given by $\tilde{f}(E_1, E_2) dE_1\, dE_2$ in $\mathcal{A}_{3,2} \times \mathcal{A}_{3,2}$ is invariant under isometries of $\mathbb{E}^3$ if and only if $\tilde{f}(E_1, E_2) = f(\langle u_1, u_2 \rangle)$ where $\pi(E_i)^\perp = \langle u_i \rangle$, $i = 1, 2$ and $f : [-1, 1] \to \mathbb{R}$ is an even measurable function.*

Proof. Suppose that $\tilde{f}(E_1, E_2) dE_1\, dE_2$ is invariant. Using the representation of an element $E \in \mathcal{A}_{3,2}$ as a pair $(\pi(E), p)$ where $p = E \cap \langle \pi(E) \rangle^\perp$ we can write

$$\tilde{f}(E_1, E_2) = F(\pi(E_1), p_1; \pi(E_2), p_2)$$

for some $F : (\mathrm{Gr}(3,2) \times \mathbb{E}^3)^2 \to \mathbb{R}$. For any translation τ it is

$$\tilde{f}(E_1 + \tau, E_2 + \tau) = F(\pi(E_1), p_1 + \tau; \pi(E_2), p_2 + \tau).$$

Due to the invariance of $\tilde{f}(E_1, E_2) dE_1\, dE_2$ we have

$$F(\pi(E_1), p_1 + \tau; \pi(E_2), p_2 + \tau) = F(\pi(E_1), p_1; \pi(E_2), p_2)$$

and so F is independent of p_1 and p_2 and we can write $\tilde{f}(E_1, E_2) = H(\pi(E_1), \pi(E_2))$ for some function H on $\mathrm{Gr}(3,2) \times \mathrm{Gr}(3,2)$.

Given $t \in [-1,1]$ consider $(V,W) \in \mathrm{Gr}(3,2)^2$ such that $V = \langle v\rangle^\perp$, $W = \langle w\rangle^\perp$ and $t = \langle v, w\rangle$ with v, w unit vectors. The function $f(t) = H(V,W)$ is well defined since for any rotation θ we have that $H(\theta V, \theta W) = H(V,W)$ and it is even. So it is proved that there exists a measurable and even function $f : [-1,1] \to \mathbb{R}$ such that

$$\tilde{f}(E_1,E_2) = f(\langle u_1, u_2\rangle).$$

If $\tilde{f}$ is as above it is clear that $\tilde{f}(E_1,E_2)dE_1dE_2$ gives rise to an isometry-invariant measure. □

4. Integral of functions of pairs of planes meeting a convex set

Let K be a compact convex set in the euclidean space E_3. According to equality (3.1) it is a natural question to evaluate

$$\int_{E_i\cap K\neq\emptyset} \tilde{f}(E_1,E_2)dE_1\, dE_2,$$

where $\tilde{f}(E_1,E_2)dE_1dE_2$ is an isometry-invariant measure on $\mathcal{A}_{3,2}\times\mathcal{A}_{3,2}$. This can be done in terms of the coefficients of the expansion of the support function of K in spherical harmonics and the coefficients of the Legendre series of the measurable even function $f : [-1,1] \to \mathbb{R}$ such that $\tilde{f}(E_1,E_2) = f(\langle u_1,u_2\rangle)$ (see Proposition 3.1).

The following result is a special case, with a different notation, of Theorem 5 in [7]. However we include a proof of this particular case for reader's convenience.

Theorem 4.1. *Let K be a compact convex set with support function p given in terms of spherical harmonics by (2.1). Let $\tilde{f}(E_1,E_2)dE_1\,dE_2$ be an isometry-invariant measure on $\mathcal{A}_{3,2}\times\mathcal{A}_{3,2}$ and $f : [-1,1]\to\mathbb{R}$ an even measurable function such that $\tilde{f}(E_1,E_2) = f(\langle u_1,u_2\rangle)$ where $\pi(E_i)^\perp = \langle u_i\rangle$, $i = 1,2$. Then*

$$\int_{E_i\cap K\neq\emptyset} \tilde{f}(E_1,E_2)dE_1\, dE_2 = \frac{\lambda_0}{4\pi}M^2 + \sum_{\substack{n=2\\ n\,\text{even}}}^{\infty} \lambda_n\|\pi_n(p)\|^2, \tag{4.1}$$

where $\lambda_n = 2\pi\int_{-1}^1 f(t)P_n(t)\,dt$ with P_n the Legendre polynomial of degree n.

Proof. As $\tilde{f}(E_1,E_2) = f(\langle u_1,u_2\rangle)$ we have that

$$\int_{E_i\cap K\neq\emptyset} \tilde{f}(E_1,E_2)dE_1\, dE_2 = \int_{E_i\cap K\neq\emptyset} f(\langle u_1,u_2\rangle)dE_1\, dE_2.$$

For a fixed plane E_2 in $\mathcal{A}_{3,2}$ and writing $dE_i = dp_i\wedge d\nu$ one has

$$\begin{aligned}\int_{E_1\cap K\neq\emptyset} \tilde{f}(E_1,E_2)dE_1 &= \int_{\mathrm{Gr}(3,2)}\left(\int_{\langle u_1\rangle} f(\langle u_1,u_2\rangle)dp_1\right)d\nu = \\ &= \int_{\mathrm{Gr}(3,2)} f(\langle u_1,u_2\rangle)(p(u_1)+p(-u_1))d\nu = \int_{S^2} f(\langle u,u_2\rangle)p(u)\,du,\end{aligned} \tag{4.2}$$

where the last equality follows from the fact that f is even.

As $p(u) = \sum_{n=0}^{\infty} \pi_n(p)(u)$ (cf. (2.1)) and using Funk-Hecke's theorem we have that

$$\int_{E_1 \cap K \neq \emptyset} \tilde{f}(E_1, E_2) dE_1 = \sum_{n=0}^{\infty} \lambda_n \pi_n(p)(u_2).$$

Notice that f being even one has $\lambda_n = 0$ for n odd. Now performing the integral with respect E_2 we have

$$\begin{aligned}\int_{E_i \cap K \neq \emptyset} \tilde{f}(E_1, E_2) dE_1 \, dE_2 &= \sum_{\substack{n=0 \\ n \text{ even}}}^{\infty} \lambda_n \int_{\mathrm{Gr}(3,2)} \left(\int_{\langle u_2 \rangle} \pi_n(p)(u_2) dp_2 \right) d\nu = \\ &= \sum_{\substack{n=0 \\ n \text{ even}}}^{\infty} \lambda_n \int_{S^2} \pi_n(p)(u) p(u) \, du = \sum_{\substack{n=0 \\ n \text{ even}}}^{\infty} \lambda_n \|\pi_n(p)\|^2,\end{aligned}$$

where we have used the fact that $\pi_n(p)(u) = \pi_n(p)(-u)$ for n even. Taking into account that $\pi_0(p) = M/4\pi$ we have

$$\|\pi_0(p)\|^2 = \frac{M^2}{16\pi^2} \|1\|^2 = \frac{M^2}{4\pi}$$

and then

$$\int_{E_i \cap K \neq \emptyset} \tilde{f}(E_1, E_2) dE_1 \, dE_2 = \frac{\lambda_0}{4\pi} M^2 + \sum_{\substack{n=2 \\ n \text{ even}}}^{\infty} \lambda_n \|\pi_n(p)\|^2.$$

□

Example 1. If $f(t) = \sqrt{1 - t^2}$ then $f(\langle u_1, u_2 \rangle) = \sin(\theta_{12})$ where $0 \leq \theta_{12} \leq \pi$ is the angle between de planes E_1 and E_2 (that is, $\cos \theta_{12} = \pm \langle u_1, u_2 \rangle$ where $\pi(E_i)^{\perp} = \langle u_i \rangle$, $i = 1, 2$). Applying Theorem 4.1 with the corresponding coefficients

$$\lambda_{2n} = 2\pi \int_{-1}^{1} f(t) P_{2n}(t) = -\frac{\Gamma(n + \frac{1}{2}) \Gamma(n - \frac{1}{2})}{n!(n+1)!} \frac{\pi}{2}, \;\; \lambda_0 = \pi^2, \;\; \lambda_{2n+1} = 0$$

(cf. [5], **7.132**), one gets

$$\int_{E_i \cap K \neq \emptyset} \sin(\theta_{12}) dE_1 \, dE_2 = \frac{\pi}{4} M^2 - \frac{\pi}{2} \left(\sum_{n=1}^{\infty} \frac{\Gamma(n + \frac{1}{2}) \Gamma(n - \frac{1}{2})}{n!(n+1)!} \|\pi_{2n}(p)\|^2 \right). \tag{4.3}$$

In the particular case that f is a Legendre polynomial one obtains from Theorem 4.1 the following

Corollary 4.2. *Let K be a compact convex set with support function p given in terms of spherical harmonics by (2.1). Then if P_n is the Legendre polynomial of even degree n, one has*

$$\int_{E_i \cap K \neq \emptyset} P_n(\langle u_1, u_2 \rangle) \, dE_1 \, dE_2 = \frac{4\pi}{2n+1} \|\pi_n(p)\|^2.$$

Proof. In this case $\lambda_m = 0$ for $m \neq n$ and $\lambda_n = \dfrac{4\pi}{2n+1}$. □

Example 2. As the function $f(t) = t^{2n}$ can be written in terms of Legendre polynomials as $t^{2n} = \sum_{k=0}^{n} \mu_{n,k} P_{2k}(t)$ with

$$\mu_{n,k} = \frac{(4k+1)\Gamma(2n+1)\sqrt{\pi}}{2^{2n+1}\Gamma(n-k+1)\Gamma(n+k+3/2)} \tag{4.4}$$

(see [5], **8.922**) we get the following consequence of Corollary 4.2:

$$\int_{E_i \cap K \neq \emptyset} \langle u_1, u_2 \rangle^{2n} \, dE_1 \, dE_2 = \sum_{k=0}^{n} \frac{4\pi}{4k+1} \mu_{n,k} \|\pi_{2k}(p)\|^2$$

where $\mu_{n,k}$ are given by (4.4).

To end this section we analyze equality (4.1) when K is a convex set of constant width. As said this means that $\pi_n(p) = 0$ for $n \neq 0$ even.

Proposition 4.3. *Let K be a compact convex set of constant width $\mathcal{W}$ and let $f : [-1, 1] \longrightarrow \mathbb{R}$ an even bounded measurable function. Then*

$$\int_{E_i \cap K \neq \emptyset} f(\langle u_1, u_2 \rangle) \, dE_1 \, dE_2 = \lambda_0 \pi \mathcal{W}^2, \tag{4.5}$$

with $\lambda_0 = 2\pi \int_{-1}^{1} f(t) dt$. Moreover if the above equality holds when $f(t) = P_{2n}(t)$ where P_{2n} is any Legendre polynomial of even degree $2n$, $n \neq 0$ then K is of constant width.

Proof. Since K is of constant width by (4.1) one gets

$$\int_{E_i \cap K \neq \emptyset} f(\langle u_1, u_2 \rangle) \, dE_1 \, dE_2 = \frac{\lambda_0}{4\pi} M^2$$

and remembering that $M = 2\pi\mathcal{W}$ the equality follows. If equality (4.5) holds for $f(t) = P_n(t)$ with n even, $n \neq 0$, and since the corresponding λ_0 vanishes one has

$$\int_{E_i \cap K \neq \emptyset} P_n(\langle u_1, u_2 \rangle) \, dE_1 \, dE_2 = 0.$$

Therefore by Corollary 4.2 it follows that $\|\pi_n(p)\| = 0$ for every non zero n even and K is of constant width. □

5. Integrals of invariant measures in terms of the visual angle

The aim of this section is to write the integral of a isometry-invariant measure over the pairs of planes meeting a convex set K, given in Theorem 4.1, as an integral of an appropriate function of the visual angle.

Let us precise what we mean by the angle of a plane about a straight line G and the visual angle of a convex set K from a line G not meeting K.

Definition 5.1.

1. Given a straight line G let $(q; e_1, e_2)$ be a fixed affine orthonormal frame in $G^\perp$ with $q \in G$. For each plane E through G let u be the unit normal vector to E pointing from the origin to it. Then the angle α associated to E is the one given by $u = \cos(\alpha)e_1 + \sin(\alpha)e_2$.
2. The visual angle of a convex set K from a line G not meeting K is the angle $\omega = \omega(G)$, $0 \leq \omega \leq \pi$, between the *half-planes* E_1, E_2 through G tangents to K.

If α_i are the angles associated to E_i, $i = 1, 2$ then

$$\cos(\pi - \omega) = \cos(\alpha_2 - \alpha_1) = \langle u_1, u_2 \rangle$$

where u_1, u_2 are the normal unit vectors to E_1, E_2 pointing from the origin, assuming the origin inside K.

The measure $dE_1\, dE_2$ in the space $\mathcal{A}_{3,2} \times \mathcal{A}_{3,2}$ of pairs of planes in $\mathbb{E}^3$ can be written according to Santaló (cf. [11], section II.12.6) as

$$dE_1\, dE_2 = \sin^2(\alpha_2 - \alpha_1)\, d\alpha_1\, d\alpha_2\, dG. \tag{5.1}$$

Then we can prove the following

Theorem 5.2. *Let K be a compact convex set and let $f : [-1, 1] \longrightarrow \mathbb{R}$ be an even continuous function. Let H be the $\mathcal{C}^2$ function on $[-\pi, \pi]$ satisfying*

$$H''(x) = f(\cos(x)) \sin^2(x), \quad -\pi \leq x \leq \pi, \quad H(0) = H'(0) = 0.$$

Then

$$\int_{E_i \cap K \neq \emptyset} f(\langle u_1, u_2 \rangle)\, dE_1\, dE_2 = \pi H(\pi) F + 2 \int_{G \cap K = \emptyset} H(\omega)\, dG, \tag{5.2}$$

where u_i are normal unit vectors to the planes E_i, $\omega = \omega(G)$ is the visual angle from the line G and F is the area of the boundary of K.

Proof. Let $G = q + \langle u \rangle$ with u a unit director vector such that $K \cap G = \emptyset$. Let $E_i, i = 1, 2$ be the supporting planes of K through G. Take now an affine orthonormal frame $\{q; e_1, e_2, u\}$ in $\mathbb{E}^3$ such that $E_1 = q + \langle e_1, u \rangle$. Every plane E through G can be written as $E = q + \langle v_\alpha, u \rangle$ where $v_\alpha = \cos \alpha e_1 + \sin \alpha e_2$ with $\alpha \in [0, \pi)$ and the planes E intersecting K correspond to angles $\alpha \in [0, \omega(G)]$. Then using (5.1) one has

$$\begin{aligned}
&\int_{E_i \cap K \neq \emptyset} f(\langle u_1, u_2 \rangle)\, dE_1\, dE_2 = \\
&\qquad = \int_{G \cap K = \emptyset} \int_0^\omega \int_0^\omega H''(\alpha_2 - \alpha_1)\, d\alpha_1\, d\alpha_2\, dG + \\
&\qquad\qquad + \int_{G \cap K \neq \emptyset} \int_0^\pi \int_0^\pi H''(\alpha_2 - \alpha_1)\, d\alpha_1\, d\alpha_2\, dG.
\end{aligned}$$

Evaluating the inner integrals and taking into account that $\int_{G\cap K\neq\emptyset} dG = \frac{\pi}{2}F$ it follows

$$\int_{E_i\cap K\neq\emptyset} f(\langle u_1,u_2\rangle)\,dE_1\,dE_2 = \\ = \frac{1}{2}\pi(H(\pi)+H(-\pi))F + \int_{G\cap K=\emptyset}(H(\omega)+H(-\omega))\,dG.$$

Since $H(0)=H'(0)=0$ and $H''(x)=H''(-x)$ it is easy to see that $H(x)=H(-x)$ and the result follows. $\square$

Example 3. Consider $f(t)=1$ then $H(x)=(x^2-\sin^2 x)/4$ and

$$M^2 = \int_{E_i\cap K\neq\emptyset} dE_1\,dE_2 = \frac{1}{4}\pi^3 F + \frac{1}{2}\int_{G\cap K=\emptyset}(\omega^2-\sin^2\omega)\,dG, \tag{5.3}$$

which gives the well known Blaschke's formula (cf. [1])

Example 4. Let $f(t)=\sqrt{1-t^2}$ be the function considered in Example 1. In this case the corresponding function H such that

$$H''(x) = f(\cos(x))\sin^2(x) = |\sin^3(x)|$$

is given by

$$H(x) = \frac{2}{3}(|x|-|\sin x|) - \frac{1}{9}|\sin^3 x|.$$

Now, since $\omega\in[0,\pi]$, Theorem 5.2 and equality (4.3) leads to

$$\int_{G\cap K=\emptyset}\left(\omega-\sin\omega-\frac{1}{3!}\sin^3\omega\right)dG = \\ = \pi\left(\frac{3M^2}{16}-\frac{1}{2}\pi F-\frac{3}{8}\sum_{n=1}^{\infty}\frac{\Gamma(n+\frac{1}{2})\Gamma(n-\frac{1}{2})}{n!(n+1)!}\|\pi_{2n}(p)\|^2\right).$$

Example 5. Taking now $f(t)=t^2$, one gets $H(x)=1/16(x^2-\sin^2 x\cos^2 x) = 1/16(x^2-\sin^2 x+\sin^4 x)$. According Exemple 2, Theorem 5.2 and Blaschke formula one gets

$$\frac{1}{3}M^2+\frac{8\pi}{15}\|\pi_2(p)\|^2 = \int_{E_i\cap K\neq\emptyset}\langle u_1,u_2\rangle^2\,dE_1\,dE_2 = \\ = \frac{1}{4}M^2+\frac{1}{8}\int_{G\cap K=\emptyset}\sin^4\omega\,dG, \tag{5.4}$$

and so

$$\int_{G\cap K=\emptyset}\sin^4\omega\,dG = \frac{2}{3}M^2+\frac{64\pi}{15}\|\pi_2(p)\|^2.$$

Notice that in the case that K has constant width one obtains

$$\int_{G\cap K=\emptyset}\sin^4\omega\,dG = \frac{2}{3}M^2.$$

More generally if $f(t)=t^{2n}$ we have the following

Proposition 5.3. *Let K be a compact convex set with mean curvature M. Then one has*

$$\int_{E_i \cap K \neq \emptyset} \langle u_1, u_2 \rangle^{2n}\, dE_1\, dE_2 = -4A_1^{(n)} M^2 + 2 \sum_{r=2}^{n+1} A_r^{(n)} \int_{G \cap K = \emptyset} \sin^{2r} \omega\, dG, \tag{5.5}$$

where

$$A_r^{(n)} = \frac{(-1)^r \binom{n}{r-1}}{4r^2}\, {}_3F_2(1, r+1/2, r-n-1; r, r+1; 1), \quad r = 1, \ldots, n+1.$$

We recall that ${}_3F_2$ is the hypergeometric function given by

$${}_3F_2(a_1, a_2, a_3; b_1, b_2; z) = \sum_{n=0}^{\infty} \frac{(a_1)_n (a_2)_n (a_3)_n}{(b_1)_n (b_2)_n} \frac{z^n}{n!}$$

with $(a)_n = \Gamma(a+n)/\Gamma(a)$ is the Pochhammer symbol.

The integral in the left-hand side of (5.5) was calculated in Exemple 2 in terms of the expansion in spherical harmonics of the support function of K.

Proof. We will apply Theorem 5.2 for $f(t) = t^{2n}$, that is $H''(x) = \cos^{2n} x \sin^2 x$. We search the corresponding $H(x)$. Le us write

$$H''(x) = \sum_{k=0}^{n} \binom{n}{k} (-1)^k \sin^{2(k+1)} x.$$

Integrating twice and using formula **2.511**(2) in [5] we get

$$\begin{aligned} H(x) = &\frac{1}{4}(x^2 - \sin^2 x) + \\ &+ \sum_{k=1}^{n} \binom{n}{k} (-1)^k \left[\alpha_k \frac{x^2}{2} - \frac{1}{2(k+1)} \left(\frac{\sin^{2(k+1)} x}{2(k+1)} + \sum_{r=1}^{k} \beta_r^{(k)} \frac{\sin^{2(k-r+1)} x}{2(k-r+1)} \right) \right], \end{aligned}$$

with

$$\alpha_k = \frac{\Gamma(k+3/2)}{\Gamma(k+2)}, \quad \beta_r^{(k)} = \frac{\Gamma(k-r+1)\Gamma(k+3/2)}{\Gamma(k+1)\Gamma(k-r+3/2)}.$$

This provides for $H(x)$ an expression of the type

$$H(x) = A_0^{(n)} x^2 + A_1^{(n)} \sin^2 x + \cdots + A_{n+1}^{(n)} \sin^{2(n+1)} x.$$

Let us find now the values on $A_r^{(n)}$. Beginning with $A_0^{(n)}$ one gets

$$A_0^{(n)} = \frac{\binom{2n}{n}}{2^{2(n+1)}}.$$

For $r > 0$ one has

$$A_r^{(n)} = \frac{1}{4} \sum_{k=r-1}^{n} \binom{n}{k} \frac{(-1)^{k+1}}{k+1} \frac{\beta_{k-r+1}^{(k)}}{r}, \quad r = 1, \ldots, n+1,$$

or in a more compact form

$$A_r^{(n)} = \frac{(-1)^r \binom{n}{r-1}}{4r^2}\, {}_3F_2(1, r+1/2, r-n-1; r, r+1; 1), \quad r = 1, \ldots, n+1.$$

One easily sees that $A_0^{(n)} = -A_1^{(n)}$ an so

$$H(x) = A_0^{(n)}(x^2 - \sin^2 x) + \sum_{r=2}^{n+1} A_r^{(n)} \sin^{2r} x. \tag{5.6}$$

Finally Theorem 5.2 and Blaschke identity (5.3) give the result. □

5.1. **Crofton's formula in the space.** Blaschke's formula (5.3) reveals the significance of the function of the visual angle $\omega^2 - \sin^2\omega$. One can ask what role the function $\omega - \sin\omega$ plays; this function, interpreting ω as the visual angle in the plane, is significant thanks to Crofton's formula $\int_{P\notin K}(\omega - \sin\omega)dP = L^2/2 - \pi F$, where K is a compact convex set in the plane with area F and length of its boundary L (see [11]).

In [1, p. 85] Blaschke shows that

$$\int_{G\cap K=\emptyset}(\omega - \sin\omega)dG = \frac{1}{4}\int_{u\in S^2} L_u^2 du - \frac{\pi^2}{2}F, \tag{5.7}$$

where L_u is the length of the boundary of the projection of K on $\langle u\rangle^\perp$.

It can be easily seen that $\int_{u\in S^2} L_u du = 2\pi M$ and from this equality, applying Schwarz's inequality, one gets

$$\int_{u\in S^2} L_u^2 du \geq \pi M^2. \tag{5.8}$$

Introducing (5.8) into (5.7) one obtains

$$\int_{G\cap K=\emptyset}(\omega - \sin\omega)dG \geq \frac{\pi}{4}(M^2 - 2\pi F). \tag{5.9}$$

As a consequence of Theorem 5.2 we can now evaluate the deficit in both inequalities (5.8) and (5.9).

Theorem 5.4. *Let K be a compact convex set with support function p, area of its boundary F and mean curvature M. Let L_u be the length of the boundary of the projection of K on $\langle u\rangle^\perp$ and let $\omega = \omega(G)$ be the visual angle of K from the line G. Then*

i) $\displaystyle\int_{u\in S^2} L_u^2 du = \pi M^2 + 4\pi\sum_{n=1}^{\infty}\frac{\Gamma(n+1/2)^2}{\Gamma(n+1)^2}\|\pi_{2n}(p)\|^2,$

ii) $\displaystyle\int_{G\cap K=\emptyset}(\omega - \sin\omega)dG = \frac{\pi}{4}(M^2 - 2\pi F) + \pi\sum_{n=1}^{\infty}\frac{\Gamma(n+1/2)^2}{\Gamma(n+1)^2}\|\pi_{2n}(p)\|^2.$

Moreover equality holds both in (5.8) and (5.9) if and only if K is of constant width.

Proof. We consider $f(t) = 1/\sqrt{1-t^2}$. For this function the corresponding H in Theorem 5.2 is $H(x) = |x| - |\sin x|$. Applying equality (5.2) and Theorem 4.1 with the corresponding λ_{2n}'s given by

$$\lambda_{2n} = 2\pi\int_{-1}^{1} f(t)P_{2n}(t)dt = 2\pi\frac{\Gamma(n+1/2)^2}{\Gamma(n+1)^2},$$

(cf. [5], **7.226**), item ii) follows. Equality i) is a consequence of ii) and (5.7).

The statement about equality in (5.8) and (5.9) is a consequence of the fact that K is of constant width if and only if $\pi_{2n}(p) = 0$ for $n \neq 0$. $\square$

6. A formulation with Fourier series

In this section we give an alternative formulation of Theorem 5.2 in terms of Fourier coefficients of the function $H''(x)$. Since f is even one has that $H''(x) = f(\cos(x))\sin^2(x)$ is an even π-periodic function. Let

$$H''(x) = \frac{1}{2}a_0 + \sum_{n\geq 1} a_{2n}\cos(2nx) \tag{6.1}$$

be the Fourier expansion of $H''(x)$. Integrating twice and taking into account that $H(0) = H'(0) = 0$ one obtains

$$H(x) = \frac{a_0}{4}x^2 + \sum_{n\geq 1} \frac{a_{2n}}{4n^2}(1 - \cos(2nx)). \tag{6.2}$$

Using this expression of the function H, Theorem 5.2 can be written as

Proposition 6.1. *Let K be a compact convex set and let $f : [-1,1] \longrightarrow \mathbb{R}$ be an even continuous function. Let H be the $\mathcal{C}^2$ function on $[-\pi,\pi]$ satisfying*

$$H''(x) = f(\cos(x))\sin^2(x), \quad -\pi \leq x \leq \pi, \quad H(0) = H'(0) = 0. \tag{6.3}$$

If $H(x)$ is given by (6.2) then

$$\begin{aligned} \int_{E_i\cap K\neq\emptyset} f(\langle u_1, u_2\rangle)\, dE_1\, dE_2 = \\ = \frac{a_0}{4}\pi^3 F + \frac{1}{2}\int_{G\cap K=\emptyset}\left(a_0\omega^2 + \sum_{n\geq 1}\frac{a_{2n}}{n^2}(1-\cos(2n\omega))\right) dG, \end{aligned} \tag{6.4}$$

where u_i are normal vectors to the planes E_i, the visual angle from the line G is ω and F denotes the area of the boundary of K.

The right hand side of (6.4) can be written as a linear combination of integrals of even powers of $\sin\omega$. For this purpose we will use Blaschke formula (5.3) and the known equality

$$\cos 2nx = \sum_{m=0}^{n} \alpha_{n,m}\sin^{2m} x \quad \text{with} \quad \alpha_{n,m} = \frac{(-1)^m\, n\, 2^{2m}(n+m-1)!}{(2m)!(n-m)!}, \tag{6.5}$$

which follows easily from the equality $\cos(2nx) = (-1)^n T_{2n}(\sin x)$ where T_{2n} is Chebyshev's polynomial of degree $2n$. We can state

Proposition 6.2. *Let K be a compact convex set and let $f : [-1,1] \longrightarrow \mathbb{R}$ be an even continuous function. Let H be the $\mathcal{C}^2$ function on $[-\pi,\pi]$ satisfying*

$$H''(x) = f(\cos(x))\sin^2(x), \quad -\pi \leq x \leq \pi, \quad H(0) = H'(0) = 0.$$

If $H(x)$ is given by (6.2) then

$$\int_{E_i\cap K\neq\emptyset} f(\langle u_1,u_2\rangle)\,dE_1\,dE_2 = \tag{6.6}$$

$$= a_0M^2 - \frac{1}{2}\sum_{m=2}^{\infty}\left(\sum_{n=m}^{\infty}\frac{a_{2n}}{n^2}\alpha_{n,m}\int_{G\cap K=\emptyset}\sin^{2m}\omega\,dG\right),$$

where u_i are normal vectors to the planes E_i, the visual angle from the line G is ω, F denotes the area of the boundary of K, the coefficients $\alpha_{n,m}$ are given by (6.5) and the coefficients a_{2n} by (6.1).

Proof. Using (6.5) the right hand side of (6.4) is written as

$$\frac{a_0\pi^3}{4}F+\frac{1}{2}\int_{G\cap K=\emptyset}\left(a_0\omega^2-\sum_{n=1}^{\infty}\frac{a_{2n}}{n^2}\alpha_{n,1}\sin^2\omega-\sum_{n=2}^{\infty}\frac{a_{2n}}{n^2}\sum_{m=2}^{n}\alpha_{n,m}\sin^{2m}\omega\right)dG =$$

$$= \frac{a_0\pi^3}{4}F+\frac{1}{2}\int_{G\cap K=\emptyset}\left(a_0(\omega^2-\sin^2\omega)-\sum_{n=2}^{\infty}\frac{a_{2n}}{n^2}\sum_{m=2}^{n}\alpha_{n,m}\sin^{2m}\omega\right)dG$$

where we have used that $\alpha_{n,0}=1, \alpha_{n,1}=-2n^2$ and $a_0=-2\sum_{n=1}^{\infty}a_{2n}$ which is a consequence of the fact that $H''(0)=0$. Using Blaschke formula (5.3) and reordering the double sum the result follows. □

6.1. **A basis for the integrals of invariant measures.** As a consequence of Proposition 6.2 we can exhibit a simple family of polynomial functions that are in some sense a basis for the integrals in Theorem 4.1. Consider the polynomials

$$h_m(t) = m(2mt^2-1)(1-t^2)^{m-2},\ m>1. \tag{6.7}$$

Then for $H''(x)=h_m(\cos(x))\sin^2(x)$ one easily checks that $H(\omega)=\frac{1}{2}\sin^{2m}\omega$ and Theorem 5.2 applied to $h_m(t)$ gives

$$\int_{E_i\cap K\neq\emptyset} h_m(\langle u_1,u_2\rangle)\,dE_1\,dE_2 = \int_{G\cap K=\emptyset}\sin^{2m}\omega\,dG,$$

that together with equation (6.6) leads to the following

Proposition 6.3. *Under the same hypotheses and notation as in Proposition 6.2 one has*

$$\int_{E_i\cap K\neq\emptyset} f(\langle u_1,u_2\rangle)\,dE_1\,dE_2 =$$

$$= a_0M^2-\frac{1}{2}\sum_{m=2}^{\infty}\left(\sum_{n=m}^{\infty}\frac{a_{2n}}{n^2}\alpha_{n,m}\int_{E_i\cap K\neq\emptyset}h_m(\langle u_1,u_2\rangle)\,dE_1\,dE_2\right),$$

where the polynomials h_m are given in (6.7).

So every invariant integral can be written as an infinite linear combination of the integrals of the invariant measures given by the polynomials h_m.

7. Powers of sine function of the visual angle

Equality (6.6) suggests to consider integrals of the form

$$\int_{G\cap K\neq\emptyset} \sin^{2m}\omega\, dG.$$

Integrals of the power of the sine of the visual angle ω for a compact convex set K in the plane were considered in [3] obtaining

$$\int_{P\notin K} \sin^{2m}\omega\ dP = \lambda_0 L^2 + \sum_{\substack{k\geq 2,\\ \text{even}}} \lambda_k c_k^2,$$

for convenient λ_k and where c_k depends on the support function of K. Applying this formula to the projections K_u of a compact convex set K in the euclidean space on the plane $\langle u\rangle^\perp$ and taking into account that $dG = dP \wedge du$ and $\omega_u(P) = \omega(G)$ one gets

$$\int_{G\cap K=\emptyset} \sin^{2m}\omega\, dG = \frac{1}{2}\int_{u\in S^2}\int_{P\notin K_u} \sin^{2m}\omega_u\, dP\, du =$$

$$= \frac{\lambda_0}{2}\int_{u\in S^2} L_u^2 du + \sum_{\substack{k\geq 2,\\ \text{even}}} \frac{\lambda_k}{2}\int_{u\in S^2} c_{k,u}^2 du.$$

The first integral in the right hand side was geometrically interpretated in Theorem 5.4 but the second one has not been handled.

A more direct way to deal with the integral of the power of the sine of the visual angle for a compact convex set K in the euclidean space is by means of our previous results. In fact, we have the following

Proposition 7.1. *Let K be a compact convex set with support function p given in terms of spherical harmonics by (2.1) then*

$$\int_{G\cap K=\emptyset} \sin^{2m}\omega\, dG = \frac{m\sqrt{\pi}(m-2)!}{4\Gamma(m+\frac{1}{2})} M^2 + \sum_{k=1}^{m-1} \beta_{2k}\|\pi_{2k}(p)\|^2, \quad m>1, \tag{7.1}$$

where M is the mean curvature of K and the β_{2k}'s are given by

$$\beta_{2k} = (-1)^{k+1}\frac{m(m-2)!^2\Gamma(k+1/2)\big((2m-1)(2k-1)(k+1)+m\big)}{k!(m-k-1)!\Gamma(m+k+1/2)}\pi. \tag{7.2}$$

Remark 7.2. When K is a convex set of constant width one has $\|\pi_{2k}(p)\|^2 = 0$ and so

$$\int_{G\cap K=\emptyset} \sin^{2m}\omega\, dG = \frac{m\sqrt{\pi}(m-2)!}{4\Gamma(m+\frac{1}{2})} M^2.$$

Proof. Comparing relations (4.1) and (5.2) we have

$$\frac{\beta_0}{4\pi}M^2 + \sum_{k=1}^{\infty}\beta_{2k}\|\pi_{2k}(p)\|^2 = \pi H(\pi)F + 2\int_{G\cap K=\emptyset} H(\omega)\, dG$$

where $H''(x) = f(\cos x)\sin^2 x$ with $H(0) = H'(0) = 0$ and $\beta_{2k} = 2\pi\int_{-1}^{1} f(t)P_{2k}(t)\, dt$. As said above, the function H corresponding to the polynomials h_m given in (6.7) is $H(\omega) = \frac{1}{2}\sin^{2m}\omega$. In this case we have that $\beta_{2k} = 0$ for $k > m-1$ because $h_m(t)$ is a polynomial of degre $2(m-1)$ and $\int_{-1}^{1} h_m(t)P_{2k}(t) = 0$ for $k > m-1$. In order to compute

$$\beta_{2k} = 2\pi\int_{-1}^{1} h_m(x)P_{2k}(x)dx$$

we first observe that, using a Computer algebra system (CAS) we get

$$\int_{-1}^{1} h_m(x)x^{2j}dx = \frac{m(4mj-2j+1)\Gamma(j+1/2)\Gamma(m-1)}{2\Gamma(m+j+1/2)}. \tag{7.3}$$

Now, recalling the expression of the Legendre polynomials

$$P_{2k}(x) = \frac{1}{2^{2k}}\sum_{r=0}^{k}(-1)^r\binom{2k}{r}\binom{4k-2r}{2k}x^{2(k-r)}$$

we have

$$\begin{aligned}\int_{-1}^{1} h_m(x)P_{2k}(x)dx \;&= \\ = \frac{m\Gamma(m-1)}{2^{2k+1}}\sum_{r=0}^{k}(-1)^r&\binom{2k}{r}\binom{4k-2r}{2k}\frac{\Gamma(k-r+1/2)(4m(k-r)-2(k-r)+1)}{\Gamma(m+k-r+1/2)},\end{aligned}$$

but this sum is computable, using again a CAS we have

$$\begin{aligned}&\int_{-1}^{1} h_m(x)P_{2k}(x)dx \;= \\ &\qquad= \frac{1}{2^{2k+1}}m\Gamma(m-1)\Big(\Gamma(m-1)\Gamma(1/2-2k)\Gamma(k-1/2)\cdot \\ &\cdot\left[(m-1)(2k-1)\binom{4k}{2k}((4m-2)k+1)-2(2m-1)k(4k-1)\binom{4k-2}{2k}(2m+2k-1)\right]\cdot \\ &\qquad\qquad\cdot\frac{1}{2\Gamma(1/2-k)\Gamma(m-k)\Gamma(m+k+1/2)}\Big).\end{aligned}$$

Simplifyng this expression and taking into account that $\Gamma(1/2+j)\Gamma(1/2-j) = (-1)^j\pi$ we finally obtain

$$\begin{aligned}\beta_{2k} &= 2\pi\int_{-1}^{1} h_m(x)P_{2k}(x)dx \\ &= (-1)^{k+1}\frac{m(m-2)!^2\Gamma(k+1/2)\big((2m-1)(2k-1)(k+1)+m\big)}{k!(m-k-1)!\Gamma(m+k+1/2)}\pi.\end{aligned}$$

□

References

[1] W. Blaschke. *Vorlesungen uber Integralgeometrie.* VEB Deutscher Verlag der Wissenschaften, Berlin, 3rd edition, 1955.

[2] M.W. Crofton. On the theory of local probability. *Phil.Trans.R.Soc.Lond.*, 158:181–199, 1868.

[3] J. Cufí, E. Gallego, and A. Reventós. Integral geometry about the visual angle of a convex set. *Rendiconti del Circolo Matematico di Palermo*, 2019.

[4] J. Cufí, E. Gallego, and A. Reventós. On the integral formulas of Crofton and Hurwitz relative to the visual angle of a convex set. *Mathematika*, 65(4):874–896, 2019.

[5] I.S. Gradshteyn and I.M. Ryzhiz. *Table of Integrals, Series and Products*. Academic Press, seventh ed edition, 2007.

[6] H. Groemer. *Geometric applications of Fourier series and spherical harmonics*, volume 61 of *Encyclopedia of Mathematics and its Applications*. Cambridge University Press, Cambridge, 1996.

[7] D. Hug and R. Schneider. Integral geometry of pairs of hyperplanes or lines. *Arch. Math. (Basel)*, 115(3):339–351, 2020.

[8] A. Hurwitz. Sur quelques applications geometriques des séries de Fourier. *Annales scientifiques de l'É.N.S., 3ème série*, 19:357–408, dec 1902.

[9] D.A. Klain and G.C. Rota. *Introduction to Geometric Probability*. Lezioni Lincee. Cambridge University Press, 1997.

[10] G. Masotti. La Geometria Integrale. *Rend. Sem. Mat. Fis. Milano*, 25:164–231 (1955), 1953–54.

[11] L.A. Santaló. *Integral geometry and geometric probability*. Cambridge Mathematical Library. Cambridge University Press, Cambridge, second edition, 2004. With a foreword by Mark Kac.

[12] R. Schneider. *Convex bodies: the Brunn-Minkowski theory (second expanded edition)*. Cambridge University Press, 2013.

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