Abel quadratic differential systems of second kind

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Abstract

The Abel differential equations of second kind, named after Niels Henrik Abel, are a class of ordinary differential equation studied by many authors. Here we consider the Abel quadratic polynomial differential equations of second kind denoting this class by $\boldsymbol{QS_{Ab}}$. Firstly we split the whole family of non-degenerate quadratic systems in four subfamilies according to the number of infinite singularities. Secondly for each one of these four subfamilies we determine necessary and sufficient affine invariant conditions for a quadratic system in this subfamily to belong to the class $\boldsymbol{QS_{Ab}}$. Thirdly we classify all the phase portraits in the Poincaré disc of the systems in $\boldsymbol{QS_{Ab}}$ in the case when they have at infinity either one triple singularity (21 phase portraits) or an infinite number of singularities (9 phase portraits). Moreover we determine the affine invariant criteria for the realization of each one of the 30 topologically distinct phase portraits.

Key-words: quadratic differential system, second kind of Abel differential equations, phase portraits.

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1 Introduction and statement of the main results

We consider the class of real quadratic polynomial differential systems

$$\begin{aligned}\dot{x} &= p_0 + p_1(\tilde{a},x,y) + p_2(\tilde{a},x,y) \equiv P(\tilde{a},x,y),\\ \dot{y} &= q_0 + q_1(\tilde{a},x,y) + q_2(\tilde{a},x,y) \equiv Q(\tilde{a},x,y)\end{aligned} \tag{1}$$

where

$$\begin{aligned}&p_0 = a, \quad p_1(\tilde{a},x,y) = cx + dy, \quad p_2(\tilde{a},x,y) = gx^2 + 2hxy + ky^2,\\ &q_0 = b, \quad q_1(\tilde{a},x,y) = ex + fy, \quad q_2(\tilde{a},x,y) = lx^2 + 2mxy + ny^2.\end{aligned}$$

and with $\max(\deg(p), \deg(q)) = 2$. Here the dot denotes derivative with respect to an independent variable t, usually called the time. We denote by $\tilde{a} = (a, c, d, g, h, k, b, e, f, l, m, n)$ the 12-tuple of the coefficients of systems (1), and by **QS** the class of all real quadratic polynomial differential systems, that sometimes we simply will say *quadratic systems*.

There are more than one thousand papers published on **QS**. The main difficulty of studying **QS** comes from the fact that they depend on twelve parameters. So people studied subclasses of **QS** which modulo the affine group action and time rescaling depend on at most three parameters. Without trying to be exhaustive we describe some of these subclasses in the following works: systems in **QS** having a center [55, 62, 64, 78, 88]; systems in **QS** without finite real singularities [36, 76]; systems in **QS** with one anti-saddle and one focus [2]; **QS** with a unique finite singularity [27, 40, 59, 75, 77, 82, 83]; systems in **QS** having the infinity filled of singularities [37, 70]; systems in **QS** having an integrable saddle [18]; systems in **QS** having a weak focus of third order [5, 52]; homogeneous systems in **QS** [84, 85]; Hamiltonian systems in **QS** [3, 4, 42]); bounded systems in **QS** [28, 47]; semilinear systems in **QS** [54]; Darboux integrable systems in **QS** [49, 81]; Lotka-Volterra systems in **QS** [72, 73]; structurally stable systems in **QS** [1, 41]; systems in **QS** having rational first integrals [24, 50, 51]; systems in **QS** having a polynomial inverse integrating factor [25]; systems in **QS** having invariant straight lines of total multiplicity ≥ 4 [65, 67, 68, 69, 71]; systems in **QS** having polynomial first integrals [35]; ... Using modern methods, such as the algebraic and geometric invariants, during the last years better classifications of some subclasses of **QS** where obtained. For example systems in **QS** having a second order weak focus [8], systems in **QS** having one invariant straight line and a weak focus [10], and the complete characterization of the geometric configurations of singularities of systems in **QS** [7, 11, 12, 13, 14, 15].

In this paper we study Abel differential equations of the second kind which are of the form

$$y\frac{dy}{dx} = A(x)y^2 + B(x)y + C(x), \tag{2}$$

with $A(x),\ B(x),\ C(x, y) \in \mathbb{R}(x, y)$. These differential equations can be equivalently written as polynomial differential systems

$$\dot{x} = d(x)y, \quad \dot{y} = a(x)y^2 + b(x)y + c(x),$$

where $A(x) = a(x)/d(x)$, $B(x) = b(x)/d(x)$ and $C(x) = c(x)/d(x)$, with polynomials $a(x), b(x), c(x)$ and $d(x)$ in $\mathbb{R}[x, y]$. In this paper we are interested in studying the *Abel quadratic polynomial differential systems*, i.e., the differential systems of the form

$$\dot{x} = (d_0 + d_1x)y \equiv \widetilde{P}(x, y), \quad \dot{y} = a_0y^2 + (b_0 + b_1x)y + c_0 + c_1x + c_2x^2 \equiv \widetilde{Q}(x, y), \tag{3}$$

coming from the Abel differential equation of second kind (2).

Definition 1. *We say that a non-degenerate quadratic system* (1) *is of Abel type if and only if there exists an affine transformation which brings this system to the form* (3). *We denote the class systems of Abel type by* $\boldsymbol{QS_{Ab}}$.

Some subclasses of $\boldsymbol{QS_{Ab}}$ have already been studied. In the paper [53] is considered the family of systems (3) with $a_0 = 0$ and with Z_2-symmetries. In the paper [33] the family of systems (3) with

$d_1 = 0$ and $a_0 \neq 0$ is analyzed. Finally, in the paper [34] is considered the family of systems (3) with $a_0 \neq 0$ and having a symmetry with respect to an axis or with respect to the origin.

The goal of this paper is firstly to determine necessary and sufficient conditions in terms of affine invariant polynomials for an arbitrary quadratic system to be of Abel type. Secondly to topologically classify all the phase portraits in the Poincaré disc of the systems in $\boldsymbol{QS_{Ab}}$ in the case when they have at infinity either one triple singularity or an infinite number of singularities. Moreover to determine the affine invariant criteria for the realization of each one of the 30 topologically distinct phase portraits.

The affine invariant polynomials which appear in the statement of the next theorem are defined in Section 2. Our main result is the following one.

Main Theorem. *A non-degenerate quadratic system* (1) *(i.e.* $\sum_{i=0}^{4} \mu_i \neq 0$*) belongs to the class* $\boldsymbol{QS_{Ab}}$ *of Abel quadratic systems if and only if* $B_1 = 0$ *and one of the following conditions is satisfied:*

$\mathcal{A}$) *If* $\eta > 0$ *then either*

- $\mathcal{A}_1$) $\theta \neq 0,$ *or*
- $\mathcal{A}_2$) $\theta = 0,\ \widetilde{N} \neq 0,\ H_7 \neq 0,$ *or*
- $\mathcal{A}_3$) $\theta = 0,\ \widetilde{N} \neq 0,\ H_7 = 0,\ B_2 = 0,$ *or*
- $\mathcal{A}_4$) $\theta = 0,\ \widetilde{N} = 0,\ \theta_3 \neq 0,$ *or*
- $\mathcal{A}_5$) $\theta = 0,\ \widetilde{N} = 0,\ \theta_3 = 0,\ B_2 = 0,\ \theta_4 \neq 0,$ *or*
- $\mathcal{A}_6$) $\theta = 0,\ \widetilde{N} = 0,\ \theta_3 = 0,\ B_2 = 0,\ \theta_4 = 0,\ B_3 = 0.$

$\mathcal{B}$) *If* $\eta < 0$ *then either*

- $\mathcal{B}_1$) $\theta \neq 0,\ B_2 \neq 0,$ *or*
- $\mathcal{B}_2$) $\theta \neq 0,\ B_2 = 0,\ \ B_3 = 0,$ *or*
- $\mathcal{B}_3$) $\theta = 0,\ \widetilde{N} \neq 0,\ H_7 \neq 0,\ B_2 \neq 0,$ *or*
- $\mathcal{B}_4$) $\theta = 0,\ \widetilde{N} \neq 0,\ H_7 \neq 0,\ B_2 = 0,\ B_3 = 0,$ *or*
- $\mathcal{B}_5$) $\theta = 0,\ \widetilde{N} = 0,\ B_2 \neq 0,$ *or*
- $\mathcal{B}_6$) $\theta = 0,\ \widetilde{N} = 0,\ B_2 = 0,\ B_3 = 0.$

$\mathcal{C}$) *If* $\eta = 0$ *and* $\widetilde{M} \neq 0$ *then either*

- $\mathcal{C}_1$) $\theta \neq 0,$ *or*
- $\mathcal{C}_2$) $\theta = 0,\ \mu_0 \neq 0,\ \ H_7 \neq 0,$ *or*
- $\mathcal{C}_3$) $\theta = 0,\ \mu_0 \neq 0,\ \ H_7 = 0,\ B_2 = 0,$ *or*
- $\mathcal{C}_4$) $\theta = 0,\ \mu_0 = 0,\ \widetilde{N} \neq 0,\ H_7 \neq 0,$ *or*
- $\mathcal{C}_5$) $\theta = 0,\ \mu_0 = 0,\ \widetilde{N} \neq 0,\ H_7 = 0,\ B_3 = 0,$ *or*
- $\mathcal{C}_6$) $\theta = 0,\ \mu_0 = 0,\ \widetilde{N} = 0,\ \widetilde{K} \neq 0,\ \theta_3 \neq 0,$ *or*
- $\mathcal{C}_7$) $\theta = 0,\ \mu_0 = 0,\ \widetilde{N} = 0,\ \widetilde{K} \neq 0,\ \theta_3 = 0,\ B_3 = 0.$

$\mathcal{D})$ *If* $\eta=0$ *and* $\widetilde{M}=0$ *then either*

- $\mathcal{D}_1)$ $C_2\neq 0,\ \theta\neq 0,$ *or*
- $\mathcal{D}_2)$ $C_2\neq 0,\ \theta=0,\ \widetilde{N}=0,\ B_2\neq 0,$ *or*
- $\mathcal{D}_3)$ $C_2=0,\ H_{10}\neq 0,$ *or*
- $\mathcal{D}_4)$ $C_2=0,\ H_{10}=0,\ H_{12}\neq 0.$

2 The main invariant polynomials associated to the class $\boldsymbol{QS_{Ab}}$

Consider quadratic systems of the form (1). It is known that on the set **QS** acts the group $Aff(2,\mathbb{R})$ of affine transformations on the plane (cf. [66]). For every subgroup $G\subseteq Aff(2,\mathbb{R})$ we have an induced action of G on **QS**. We can identify the set **QS** of systems (1) with a subset of $\mathbb{R}^{12}$ via the map $\mathbf{QS}\longrightarrow\mathbb{R}^{12}$ which associates to each system (1) the 12–tuple $\tilde{a}=(a,c,d,g,h,k,b,e,f,l,m,n)$ of its coefficients. We associate to this group action polynomials in x,y and parameters which behave well with respect to this action, the GL–comitants (GL–invariants), the T–comitants (affine invariants) and the CT–comitants. For their definitions as well as their detailed constructions we refer the reader to the paper [66] (see also [9]).

Next we define the following invariant polynomials associated to the class $\boldsymbol{QS_{Ab}}$:

$$\Big\{\mu_0,\dots,\mu_4,\ \mathbf{D},\ \mathbf{P},\ \mathbf{R},\ \mathbf{S},\ \mathbf{T},\ \mathbf{U},\ \mathcal{T}_1,\dots,\mathcal{T}_4,\ \mathcal{F},\ \mathcal{F}_1,\dots,\mathcal{F}_4,\ \mathcal{H},\ \mathcal{B},\ \mathcal{B}_1,\ \mathcal{B}_2,\ \sigma,\\ \eta,\ \widetilde{M},\ C_2,\ \theta,\ \theta_3,\ \theta_4,\ \widetilde{K},\ \widetilde{N},\ H_7,\ H_9,\ H_{10},\ H_{11},\ H_{12},\ E_1,\ U_1,\ U_2\Big\}. \tag{4}$$

According to [9] (see also [20]) we apply the differential operator $\mathcal{L}=x\cdot\mathbf{L}_2-y\cdot\mathbf{L}_1$ acting on $\mathbb{R}[\tilde{a},x,y]$ with

$$\begin{aligned}\mathbf{L}_1&=2a_{00}\frac{\partial}{\partial a_{10}}+a_{10}\frac{\partial}{\partial a_{20}}+\frac{1}{2}a_{01}\frac{\partial}{\partial a_{11}}+2b_{00}\frac{\partial}{\partial b_{10}}+b_{10}\frac{\partial}{\partial b_{20}}+\frac{1}{2}b_{01}\frac{\partial}{\partial b_{11}},\\ \mathbf{L}_2&=2a_{00}\frac{\partial}{\partial a_{01}}+a_{01}\frac{\partial}{\partial a_{02}}+\frac{1}{2}a_{10}\frac{\partial}{\partial a_{11}}+2b_{00}\frac{\partial}{\partial b_{01}}+b_{01}\frac{\partial}{\partial b_{02}}+\frac{1}{2}b_{10}\frac{\partial}{\partial b_{11}},\end{aligned}$$

to construct several invariant polynomials from the set. More precisely using this operator and the affine invariant $\mu_0=\operatorname{Res}_x\big(p_2(\tilde{a},x,y),q_2(\tilde{a},x,y)\big)/y^4$ we construct the following polynomials

$$\mu_i(\tilde{a},x,y)=\frac{1}{i!}\mathcal{L}^{(i)}(\mu_0),\ i=1,..,4,\quad\text{where}\quad\mathcal{L}^{(i)}(\mu_0)=\mathcal{L}(\mathcal{L}^{(i-1)}(\mu_0)).$$

Using these invariant polynomials we define some new invariants, which according to [9] are responsible for the number and multiplicities of the finite singular points of (1):

$$\begin{aligned}\mathbf{D}&=\Big[3\big((\mu_3,\mu_3)^{(2)},\ \mu_2\big)^{(2)}-\big(6\mu_0\mu_4-3\mu_1\mu_3+\mu_2^2,\ \mu_4\big)^{(4)}\Big]/48,\\ \mathbf{P}&=12\mu_0\mu_4-3\mu_1\mu_3+\mu_2^2,\\ \mathbf{R}&=3\mu_1^2-8\mu_0\mu_2,\\ \mathbf{S}&=\mathbf{R}^2-16\mu_0^2\mathbf{P},\\ \mathbf{T}&=18\mu_0^2(3\mu_3^2-8\mu_2\mu_4)+2\mu_0(2\mu_2^3-9\mu_1\mu_2\mu_3+27\mu_1^2\mu_4)-\mathbf{PR},\\ \mathbf{U}&=\mu_3^2-4\mu_2\mu_4.\end{aligned} \tag{5}$$

In what follows we also need the so-called *transvectant of order k* (see [39], [60]) of two polynomials $f,\, g \in \mathbb{R}[\tilde{a}, x, y]$

$$(f, g)^{(k)} = \sum_{h=0}^{k} (-1)^h \binom{k}{h} \frac{\partial^k f}{\partial x^{k-h} \partial y^h} \; \frac{\partial^k g}{\partial x^h \partial y^{k-h}}.$$

Next we construct the elements $\mathcal{T}_1, \ldots, \mathcal{T}_4$ of the set (4) which are responsible for the number of the vanishing traces corresponding to the finite singularities of systems (1). For this we define a polynomial (which we call *trace polynomial*) as follows.

Following [80] we denote by $\sigma(\tilde{a}, x, y) = \dfrac{\partial p}{\partial x} + \dfrac{\partial q}{\partial y} = \sigma_0(\tilde{a}) + \sigma_1(\tilde{a}, x, y)$ and we observe that the polynomial $\sigma(\tilde{a}, x, y) \in \mathbb{R}[x, y]$ is an affine comitant of systems (1).

Definition 2 ([80])**.** We call *trace polynomial* $\mathfrak{T}(w)$ over the ring $\mathbb{R}[\tilde{a}]$ the polynomial defined as follows

$$\mathfrak{T}(w) = \sum_{i=0}^{4} \frac{1}{(i!)^2} \left(\sigma_1^i,\, \frac{1}{i!} \mathcal{L}^{(i)}(\mu_0) \right)^{(i)} \; w^{4-i} = \sum_{i=0}^{4} \mathcal{G}_i(\tilde{a})\, w^{4-i}, \tag{6}$$

where the coefficients $\mathcal{G}_i(\tilde{a}) = \dfrac{1}{(i!)^2} (\sigma_1^i,\, \mu_i)^{(i)} \in \mathbb{R}[\tilde{a}], \quad i = 0, 1, 2, 3, 4 \; \big(\mathcal{G}_0(\tilde{a}) \equiv \mu_0(\tilde{a})\big)$ are GL–invariants.

Using the polynomial $\mathfrak{T}(w)$ we could construct the above mentioned four affine invariants $\mathcal{T}_4$, $\mathcal{T}_3$, $\mathcal{T}_2$ and $\mathcal{T}_1$:

$$\mathcal{T}_{4-i}(\tilde{a}) = \frac{1}{i!} \frac{d^i \mathfrak{T}}{dw^i}\Big|_{w=\sigma_0} \in \mathbb{R}[\tilde{a}], \quad i = 0, 1, 2, 3 \quad \big(\mathcal{T}_4 \equiv \mathfrak{T}(\sigma_0)\big).$$

In order to construct the remaining invariant polynomials contained in the set (4) we first need to define some elementary bricks which help us to construct these elements of the set.

We remark that the following polynomials in $\mathbb{R}[\tilde{a}, x, y]$ are the simplest invariant polynomials of degree one with respect to the coefficients of the differential systems (1) which are GL-comitants:

$$C_i(x, y) = y p_i(x, y) - x q_i(x, y), \; i = 0, 1, 2; \quad D_i(x, y) \;\; = \frac{\partial}{\partial x} p_i(x, y) + \frac{\partial}{\partial y} q_i(x, y), \; i = 1, 2. \tag{7}$$

Apart from these simple invariant polynomials we shall also make use of the following nine GL-invariant polynomials:

$$\begin{aligned} &T_1 = (C_0, C_1)^{(1)}, \quad T_2 = (C_0, C_2)^{(1)}, \quad T_3 = (C_0, D_2)^{(1)}, \quad T_4 = (C_1, C_1)^{(2)}, \quad T_5 = (C_1, C_2)^{(1)}, \\ &T_6 = (C_1, C_2)^{(2)}, \quad T_7 = (C_1, D_2)^{(1)}, \quad T_8 = (C_2, C_2)^{(2)}, \quad T_9 = (C_2, D_2)^{(1)}. \end{aligned}$$

These are of degree two with respect to the coefficients of systems (1).

We next define a list of T-comitants:

$$
\begin{aligned}
\hat{A}(\tilde{a}) =\ & (C_1, T_8 - 2T_9 + D_2^2)^{(2)}/144,\\
\widehat{B}(\tilde{a},x,y) =\ & \Big\{16D_1(D_2,T_8)^{(1)}(3C_1D_1 - 2C_0D_2 + 4T_2) + 32C_0(D_2,T_9)^{(1)}(3D_1D_2\\
& - 5T_6 + 9T_7) + 2(D_2,T_9)^{(1)}\big(27C_1T_4 - 18C_1D_1^2 - 32D_1T_2 + 32(C_0,T_5)^{(1)}\big)\\
& + 6(D_2,T_7)^{(1)}\big[8C_0(T_8 - 12T_9) - 12C_1(D_1D_2 + T_7) + D_1(26C_2D_1 + 32T_5)\\
& + C_2(9T_4 + 96T_3)\big] + 6(D_2,T_6)^{(1)}\big[32C_0T_9 - C_1(12T_7 + 52D_1D_2)\\
& - 32C_2D_1^2\big] + 48D_2(D_2,T_1)^{(1)}(2D_2^2 - T_8) + 6D_1D_2T_4(T_8 - 7D_2^2 - 42T_9)\\
& - 32D_1T_8(D_2,T_2)^{(1)} + 9D_2^2T_4(T_6 - 2T_7) - 16D_1(C_2,T_8)^{(1)}(D_1^2 + 4T_3)\\
& + 12D_1(C_1,T_8)^{(2)}(C_1D_2 - 2C_2D_1) + 12D_1(C_1,T_8)^{(1)}(T_7 + 2D_1D_2)\\
& + 96D_2^2\big[D_1(C_1,T_6t)^{(1)} + D_2(C_0,T_6)^{(1)}\big] - 4D_1^3D_2(D_2^2 + 3T_8 + 6T_9)\\
& - 16D_1D_2T_3(2D_2^2 + 3T_8) + 6D_1^2D_2^2(7T_6 + 2T_7) - 252D_1D_2T_4T_9\Big\}/(2^8 3^3),
\end{aligned}
$$

$$
\begin{aligned}
\widehat{D}(\tilde{a},x,y) =\ & \big[2C_0(T_8 - 8T_9 - 2D_2^2) + C_1(6T_7 - T_6) - (C_1,T_5)^{(1)} - 9D_1^2C_2\\
& + 6D_1(C_1D_2 - T_5)\big]/36,\\
\widehat{E}(\tilde{a},x,y) =\ & \big[D_1(2T_9 - T_8) - 3(C_1,T_9)^{(1)} - D_2(3T_7 + D_1D_2)\big]/72,\\
\widehat{F}(\tilde{a},x,y) =\ & \big[6D_1^2(D_2^2 - 4T_9) + 4D_1D_2(T_6 + 6T_7) + 48C_0(D_2,T_9)^{(1)} - 9D_2^2T_4\\
& + 288D_1\widehat{E} - 24(C_2,\widehat{D})^{(2)} + 120(D_2,\widehat{D})^{(1)} - 36C_1(D_2,T_7)^{(1)}\\
& + 8D_1(D_2,T_5)^{(1)}\big]/144,\\
\widehat{K}(\tilde{a},x,y) =\ & (T_8 + 4T_9 + 4D_2^2)/72,\\
\widehat{H}(\tilde{a},x,y) =\ & (-T_8 + 8T_9 + 2D_2^2)/72,
\end{aligned}
$$

as well as the needed bricks:

$$
\begin{array}{lll}
A_1(\tilde{a}) = \hat{A}, & A_2(\tilde{a}) = (C_2,\widehat{D})^{(3)}/12, & A_3(\tilde{a}) = \big[\!\big[C_2,D_2)^{(1)},D_2\big)^{(1)},D_2\big)^{(1)}/48,\\
A_4(\tilde{a}) = (\widehat{H},\widehat{H})^{(2)}, & A_5(\tilde{a}) = (\widehat{H},\widehat{K})^{(2)}/2, & A_6(\tilde{a}) = (\widehat{E},\widehat{H})^{(2)}/2,\\
A_7(\tilde{a}) = \big[\!\big[C_2,\widehat{E})^{(2)},D_2\big)^{(1)}/8, & A_8(\tilde{a}) = \big[\!\big[\widehat{D},\widehat{H})^{(2)},D_2\big)^{(1)}/8, & A_9(\tilde{a}) = \big[\!\big[\widehat{D},D_2)^{(1)},D_2\big)^{(1)},D_2\big)^{(1)}/48,\\
A_{10}(\tilde{a}) = \big[\!\big[\widehat{D},\widehat{K})^{(2)},D_2\big)^{(1)}/8, & A_{11}(\tilde{a}) = (\widehat{F},\widehat{K})^{(2)}/4, & A_{12}(\tilde{a}) = (\widehat{F},\widehat{H})^{(2)}/4,\\
A_{14}(\tilde{a}) = (\widehat{B},C_2)^{(3)}/36, & A_{15}(\tilde{a}) = (\widehat{E},\widehat{F})^{(2)}/4, & A_{25}(\tilde{a}) = \big[\!\big[\widehat{D},\widehat{D})^{(2)},\widehat{E}\big)^{(2)}/16,
\end{array}
$$

$$
A_{33}(\tilde{a}) = \big[\!\big[\widehat{D},D_2)^{(1)},\widehat{F}\big)^{(1)},D_2\big)^{(1)},D_2\big)^{(1)}/128, \qquad A_{34}(\tilde{a}) = \big[\!\big[\widehat{D},\widehat{D})^{(2)},D_2\big)^{(1)},\widehat{K}\big)^{(1)},D_2\big)^{(1)}/64.
$$

In the above list the bracket "$[\![$" means a succession of two or up to four parentheses "(" depending on the row where it appears.

Now we can define the remaining invariant polynomials of the set (4):

$$
\begin{aligned}
\mathcal{F}(\tilde{a}) &= A_7,\\
\mathcal{F}_1(\tilde{a}) &= A_2,\\
\mathcal{F}_2(\tilde{a}) &= -2A_1^2A_3 + 2A_5(5A_8 + 3A_9) + A_3(A_8 - 3A_{10} + 3A_{11} + A_{12}) - A_4(10A_8 - 3A_9 + 5A_{10}\\
&\quad + 5A_{11} + 5A_{12}),\\
\mathcal{F}_3(\tilde{a}) &= -10A_1^2A_3 + 2A_5(A_8 - A_9) - A_4(2A_8 + A_9 + A_{10} + A_{11} + A_{12}) + A_3(5A_8 + A_{10} - A_{11} + 5A_{12}),\\
\mathcal{F}_4(\tilde{a}) &= 20A_1^2A_2 - A_2(7A_8 - 4A_9 + A_{10} + A_{11} + 7A_{12}) + A_1(6A_{14} - 22A_{15}) - 4A_{33} + 4A_{34},\\
\mathcal{H}(\tilde{a}) &= -(A_4 + 2A_5),\\
\mathcal{B}(\tilde{a}) &= -(3A_8 + 2A_9 + A_{10} + A_{11} + A_{12}),\\
\mathcal{B}_1(\tilde{a},x,y) &= \Big\{\big(T_7,D_2\big)^{(1)}\big[12D_1T_3 + 2D_1^3 + 9D_1T_4 + 36\big(T_1,D_2\big)^{(1)}\big] - 2D_1\big(T_6,D_2\big)^{(1)}(D_1^2 + 12T_3)\\
&\quad + D_1^2\big[D_1\big(T_8,C_1\big)^{(2)} + 6\big(\big(T_6,C_1\big)^{(1)},D_2\big)^{(1)}\big]\Big\}/144,\\
\mathcal{B}_2(\tilde{a},x,y) &= \Big\{\big(T_7,D_2\big)^{(1)}\big[8T_3\big(T_6,D_2\big)^{(1)} - D_1^2\big(T_8,C_1\big)^{(2)} - 4D_1\big(\big(T_6,C_1\big)^{(1)},D_2\big)^{(1)}\big]\\
&\quad + \Big[\big(T_7,D_2\big)^{(1)}\Big]^2(8T_3 - 3T_4 + 2D_1^2)\Big\}/384,
\end{aligned}
$$

$$
\begin{aligned}
\widetilde{K}(\tilde{a},x,y) &= 4\widehat{K} \equiv \text{Jacob}\,\big(p_2(\tilde{a},x,y), q_2(\tilde{a},x,y)\big),\\
\widetilde{M}(\tilde{a},x,y) &= (C_2,C_2)^{(2)} \equiv 2\text{Hess}\,\big(C_2(\tilde{a},x,y)\big),\\
\widetilde{N}(\tilde{a},x,y) &= \widetilde{K} - 4\widehat{H},\\
\eta(\tilde{a}) &= (\widetilde{M},\widetilde{M})^{(2)}/384 \equiv \text{Discrim}\,\big(C_2(\tilde{a},x,y)\big),\\
\theta(\tilde{a}) &= -(\widetilde{N},\widetilde{N})^{(2)}/2 \equiv \text{Discrim}\,\big(\widetilde{N}(\tilde{a},x,y)\big);\\
\theta_3(\tilde{a}) &= A_8 + A_{11},\\
\theta_4(\tilde{a}) &= A_7,\\
B_1(\tilde{a}) &= \text{Res}_x\left(C_2,\widetilde{D}\right)/y^9 = -2^{-9}3^{-8}\,(B_2,B_3)^{(4)},\\
B_2(\tilde{a},x,y) &= (B_3,B_3)^{(2)} - 6B_3(C_2,\widetilde{D})^{(3)},\\
B_3(\tilde{a},x,y) &= (C_2,\widetilde{D})^{(1)} \equiv \text{Jacob}\left(C_2,\widetilde{D}\right),\\
E_1(\tilde{a}) &= A_{25},\\
\widetilde{U}_1(\tilde{a}) &= A_9 - 54A_1^2,\\
\widetilde{U}_2(\tilde{a}) &= 3A_8 - A_9,\\
H_7(\tilde{a}) &= (\widetilde{N},C_1)^{(2)},\\
H_9(\tilde{a}) &= -\,[\![\widetilde{D},\widetilde{D})^{(2)},\widetilde{D},)^{(1)},\widetilde{D})^{(3)},\\
H_{10}(\tilde{a}) &= [\![\widetilde{N},\widetilde{D})^{(2)},\ D_2)^{(1)},\\
H_{11}(\tilde{a},x,y) &= 8\widehat{H}\big[(C_2,\widetilde{D})^{(2)} + 8(\widetilde{D},D_2)^{(1)}\big] + 3\big[(C_1,\,2\widehat{H} - \widetilde{N})^{(1)} - 2D_1\widetilde{N}\big]^2,\\
H_{12}(\tilde{a},x,y) &= (\widetilde{D},\widetilde{D})^{(2)} \equiv \text{Hessian}(\widetilde{D})
\end{aligned}
$$

We remark that the above invariant polynomials (except $\widetilde{U}_1$ and $\widetilde{U}_2$) were constructed and used in [80], [70] and [17] and only the invariant polynomials $\widetilde{U}_1$ and $\widetilde{U}_2$ are defined here.

3 Preliminary results involving the use of polynomial invariants

We remark that the invariant polynomials $\mu_i(\tilde{a}, x, y)$ $(i = 0, 1, \ldots, 4)$ defined in the previous subsection are responsible for the total multiplicity of the finite singularities of quadratic systems (1). Moreover they detect whether a quadratic system is degenerate or not. More exactly we have the following lemma.

Lemma 1. ([20]) *Consider a quadratic system* (S) *with coefficients* $\boldsymbol{a} \in \mathbb{R}^{12}$. *Then:*

(i) The total multiplicity of the finite singularities of this system is $4 - k$ *if and only if for every* i *such that* $0 \leq i \leq k-1$ *we have* $\mu_i(\boldsymbol{a}, x, y) = 0$ *in* $\mathbb{R}[x, y]$ *and* $\mu_k(\boldsymbol{a}, x, y) \neq 0$.

(ii) The system (S) *is degenerate (i.e.* $\gcd(p, q) \neq$ *constant) if and only if* $\mu_i(\boldsymbol{a}, x, y) = 0$ *in* $\mathbb{R}[x, y]$ *for every* $i = 0, 1, 2, 3, 4$.

On the other hand the invariant polynomials η, $\widetilde{M}$ and C_2 govern the number of real and complex infinite singularities. More precisely, according to [74] (see also [66]) we have the next result.

Lemma 2. *The number of infinite singularities (real and complex) of a quadratic system in* **QS** *is determined by the following conditions:*

(i) 3 *real if* $\eta > 0$;

(ii) 1 *real and* 2 *imaginary if* $\eta < 0$;

(iii) 2 *real if* $\eta = 0$ *and* $\widetilde{M} \neq 0$;

(iv) 1 *real if* $\eta = \widetilde{M} = 0$ *and* $C_2 \neq 0$;

(v) ∞ *if* $\eta = \widetilde{M} = C_2 = 0$.

Moreover, the quadratic systems (1), for each one of these cases, can be brought via a linear transformation to the corresponding case of the following canonical systems $(\mathbf{S}_I) - (\mathbf{S}_V)$:

$$\begin{cases} \dot{x} &= a + cx + dy + gx^2 + (h-1)xy, \\ \dot{y} &= b + ex + fy + (g-1)xy + hy^2; \end{cases} \qquad (\mathbf{S}_I)$$

$$\begin{cases} \dot{x} &= a + cx + dy + gx^2 + (h+1)xy, \\ \dot{y} &= b + ex + fy - x^2 + gxy + hy^2; \end{cases} \qquad (\mathbf{S}_{II})$$

$$\begin{cases} \dot{x} &= a + cx + dy + gx^2 + hxy, \\ \dot{y} &= b + ex + fy + (g-1)xy + hy^2; \end{cases} \qquad (\mathbf{S}_{III})$$

$$\begin{cases} \dot{x} &= a + cx + dy + gx^2 + hxy, \\ \dot{y} &= b + ex + fy - x^2 + gxy + hy^2; \end{cases} \qquad (\mathbf{S}_{IV})$$

$$\begin{cases} \dot{x} &= a + cx + dy + x^2, \\ \dot{y} &= b + ex + fy + xy. \end{cases} \qquad (\mathbf{S}_V)$$

According to [16] (see also [9]) the next proposition is valid.

Proposition 1. *Consider a non-degenerate quadratic differential system. Then:*

(i) this system has one center if and only if one of the following sets of conditions holds

$$
\begin{array}{rl}
(\mathfrak{C}_1) & \mathcal{T}_4=0,\ \mathcal{T}_3\mathcal{F}<0,\ \mathcal{F}_1=\mathcal{F}_2=\mathcal{F}_3\mathcal{F}_4=0;\\
(\mathfrak{C}_2) & \mathcal{T}_4=\mathcal{T}_3=0,\ \mathcal{T}_2>0,\ \mathcal{B}<0,\ \mathcal{F}=\mathcal{F}_1=0;\\
(\mathfrak{C}_3) & \mathcal{T}_4=\mathcal{T}_3=\mathcal{T}_2=\mathcal{T}_1=0,\ \sigma\neq 0,\ \mathcal{F}_1=0, \mathcal{H}<0,\ \mathcal{B}<0,\ \mathcal{F}=0;\\
(\mathfrak{C}_4) & \mathcal{T}_4=\mathcal{T}_3=\mathcal{T}_2=\mathcal{T}_1=0,\ \sigma\neq 0,\ \mathcal{F}_1=0, \mathcal{H}=\mathcal{B}_1=0,\ \mathcal{B}_2<0;\\
(\mathfrak{C}_5) & \sigma=0,\ \mu_0<0,\ \ \mathbf{D}<0,\ \mathbf{R}>0,\ \ \mathbf{S}>0;\\
(\mathfrak{C}_6) & \sigma=0,\ \mu_0=0,\ \ \mathbf{D}<0,\ \mathbf{R}\neq 0;\\
(\mathfrak{C}_7) & \sigma=0,\ \mu_0>0,\ \mathbf{D}>0;\\
(\mathfrak{C}_8) & \sigma=0,\ \mu_0>0,\ \mathbf{D}=0,\ \mathbf{T}<0;\\
(\mathfrak{C}_9) & \sigma=0,\ \mu_0=\mu_1=0,\ \mu_2\neq 0, \mathbf{U}>0,\ \widetilde{K}=0;\\
(\mathfrak{C}_{10}) & \sigma=0,\ \mu_0>0, \mathbf{D}=\mathbf{T}=\mathbf{P}=0,\ \mathbf{R}\neq 0;
\end{array}
\tag{8}
$$

(ii) and it has two centers if and only if one of the following sets of conditions holds

$$
\begin{array}{rl}
(\widehat{\mathfrak{C}}_1) & \mathcal{T}_4=\mathcal{T}_3=0,\ \mathcal{T}_2<0,\ \mathcal{B}<0,\ \mathcal{H}<0,\ \mathcal{F}=\mathcal{F}_1=0;\\
(\widehat{\mathfrak{C}}_2) & \sigma=0,\ \mu_0>0,\ \ \mathbf{D}<0,\ \ \mathbf{R}>0,\ \ \mathbf{S}>0.
\end{array}
\tag{9}
$$

In what follows we also need the next lemma.

Lemma 3. [65] *For the existence of an invariant straight line of a system* (1) *in one (respectively 2; 3 distinct) directions in the affine plane it is necessary that $B_1=0$ (respectively $B_2=0$; $B_3=0$).*

4 The proof of the Main Theorem

We shall consider step by step each one of the subfamilies of quadratic systems defined by the conditions $\mathcal{A}$) - $\mathcal{D}$) which are provided by the Main Theorem.

4.1 The subfamily defined by $\mathcal{A}$): $\eta>0$

According to Lemma 2 we consider systems ($\mathbf{S}$) for which calculations yield:

$$\eta=1,\quad \theta=-8(g-1)(h-1)(g+h). \tag{10}$$

We consider two cases: $\theta\neq 0$ and $\theta=0$.

4.1.1 The case $\theta\neq 0$

Then $(g-1)(h-1)(g+h)\neq 0$ and due to a translation we may assume $d=e=0$, i.e. we get the systems

$$\dot{x}=a+cx+gx^2+(h-1)xy,\quad \dot{y}=b+fy+(g-1)xy+hy^2, \tag{11}$$

for which we calculate

$$B_1 = ab(g-1)^2(h-1)^2\big[(b-a)(g+h)^2 + cf(g-h) + c^2h - f^2g\big] \equiv ab(g-1)^2(h-1)^2\mathcal{H}.$$

So due to $\theta \neq 0$ the condition $B_1 = 0$ is equivalent to $ab\mathcal{H} = 0$ and we consider two subcases: $ab = 0$ and $\mathcal{H} = 0$.

4.1.1.1 The subcase $ab = 0$ We observe that systems (11) keep the form under the change $(x, y, a, b, c, f, g, h) \mapsto (y, x, b, a, f, c, h, g)$ and hence without loss of generality we may consider that the condition $a = 0$ is fulfilled. Then we arrive at the family of systems

$$\dot{x} = x\big[c + gx + (h-1)y\big], \quad \dot{y} = b + fy + (g-1)xy + hy^2, \tag{12}$$

possessing the invariant affine line $x = 0$. It is not too difficult to detect, that after the affine transformation

$$x_1 = x, \quad y_1 = gx + (h-1)y + c$$

we arrive at the systems

$$\dot{x}_1 = x_1y_1, \quad \dot{y} = b' + e'x_1 + l'x_1^2 + (f' + 2m'x_1)y_1 + n'y^2, \tag{13}$$

where b', e', f', l', m' and n' are rational functions of the parameters b, c, f, g, h with the same denominator $(h-1) \neq 0$.

It remains to observe that these systems belong to the family of systems (3).

4.1.1.2 The subcase $\mathcal{H} = 0$ Then the equality

$$(b-a)(g+h)^2 + cf(g-h) + c^2h - f^2g = 0$$

gives us

$$b = a + \frac{(f-c)(fg+ch)}{(g+h)^2} \equiv b_0$$

and this leads to the family of systems

$$\dot{x} = a + cx + gx^2 + (h-1)xy, \quad \dot{y} = b_0 + fy + (g-1)xy + hy^2.$$

Since $g + h \neq 0$ we can apply to these systems the transformation

$$x_1 = (g+h)(x-y) + c - f, \quad y_1 = (g+h)(gx+hy) + fg + ch, \quad t_1 = t/(g+h)$$

with the determinant $(g+h)^3 \neq 0$. As a result we get the family of systems (13) the parameters b', e', f', l', m' and n' of which are rational functions of the parameters a, c, f, g, h with the same denominator $(g+h) \neq 0$. So we again arrive to a subfamily of systems (3).

4.1.2 The case $\theta = 0$

For systems $(\mathbf{S}_I)$ we have

$$\theta = -8(g-1)(h-1)(g+h), \quad \widetilde{N} = (g^2-1)x^2 + 2(g-1)(h-1)xy + (h^2-1)y^2 \tag{14}$$

and therefore the condition $\theta = 0$ yields $(h-1)(g-1)(g+h) = 0$. Without loss of generality we can consider $g = 1$. Indeed, if $h = 1$ (respectively, $g+h = 0$) we can apply the linear transformation which will replace the straight line $x = 0$ with $y = 0$ (respectively, $x = 0$ with $y = x$) reducing this case to $h = 1$.

So we assume $h = 1$ and in this case by (14) for systems $(\mathbf{S}_I)$ we have $\widetilde{N} = (g-1)(1+g)x^2$. We consider two subcases: $N \neq 0$ and $N = 0$.

4.1.2.1 The subcase $N \neq 0$ Then $(g-1)(g+1) \neq 0$ and due to a translation we may assume $e = f = 0$. So we get he family of systems

$$\dot{x} = a + cx + dy + gx^2, \quad \dot{y} = b + (g-1)xy + y^2, \tag{15}$$

for which we calculate

$$\begin{aligned} &B_1 = bd^2g(g-1)^2\big[(b-a)(1+g)^2 + (c+d)(c-dg)\big] \equiv bd^2g(g-1)^2\Phi,\\ &\mu_0 = g^2, \quad H_7 = 4d(g^2-1). \end{aligned}$$

4.1.2.1.1 The possibility $H_7 \neq 0$. This implies $d \neq 0$ and due to $\widetilde{N} \neq 0$ the condition $B_1 = 0$ yields $bg\Phi = 0$. We consider two cases: $\mu_0 \neq 0$ and $\mu_0 = 0$.

a) *The case $\mu_0 \neq 0$.* Then $g \neq 0$ and we get $b\Phi = 0$.

a.1) *The subcase $b = 0$.* Then systems (15) possess the invariant line $y = 0$ and using the transformation

$$x_1 = y, \quad y_1 = (g-1)x + y$$

we arrive at the systems

$$\dot{x}_1 = x_1y_1, \quad \dot{y}_1 = a(g-1) - (c+d-dg)x_1 + cy_1 + \frac{1}{g-1}\big[gx_1^2 - (g-1)x_1y_1 + gy_1^2\big].$$

So we get a subfamily of the family of systems (3).

a.2) *The subcase $\Phi = 0$.* This condition gives

$$b = \frac{a(1+g)^2 - (c+d)(c-dg)}{(1+g)^2} \equiv b_0$$

and systems (15) with $b = b_0$ possess the invariant line $(1+g)(x-y) + c + d = 0$. Then applying the transformation

$$x_1 = (1+g)(x-y) + c + d, \quad y_1 = gx + y \quad (Det = (1+g)^2 \neq 0)$$

we get a subfamily Abel quadratic systems of the form (3):

$$\dot{x}_1 = x_1 y_1, \quad \dot{y}_1 = Q(x, y),$$

where $Q(x, y)$ is a quadratic polynomial the coefficients of which are rational functions of the parameters a, c, d, g, h (denominators are some powers of $(g + 1) \neq 0$).

b) *The case* $\mu_0 = 0$*.* Then we have $g = 0$ and considering systems (15) we obtain the systems

$$\dot{x} = a + cx + dy, \quad \dot{y} = b - xy + y^2.$$

Since $d \neq 0$ we can apply the transformation

$$x_1 = x, \quad y_1 = cx + dy + a$$

which brings the above systems to the form

$$\dot{x}_1 = xy_1, \quad \dot{y}_1 = \frac{1}{d}\big[a^2 + bd^2 + a(2c + d)x_1 + (cd - 2a)y_1 + c(c + d)x_1^2 - (2c + d)x_1 y_1 + y_1^2\big].$$

It is clear that these systems are contained in the family of systems (3) (in the first equation we have $d_1 = 0$ and $d_0 = 1$).

4.1.2.1.2 The possibility $H_7 = 0$.

Then $d = 0$ and we arrive at the family of systems

$$\dot{x} = a + cx + gx^2, \quad \dot{y} = b + (g - 1)xy + y^2, \tag{16}$$

for which we calculate

$$B_1 = 0, \quad B_2 = -648b(-1 + g)^2\big[(b - a)(1 + g)^2 + c^2\big]x^4 \quad \mu_0 = g^2$$

and we consider two cases: $B_2 \neq 0$ and $B_2 = 0$.

a) *The case* $B_2 \neq 0$*.* We claim that for $B_2 \neq 0$ the above systems could not be brought via an affine transformation to the form (3). In order to prove this claim we examine two subcases: $\mu_0 \neq 0$ and $\mu_0 = 0$.

a.1) *The subcase* $\mu_0 \neq 0$*.* Then $g \neq 0$ and systems (16) possess two parallel invariant lines $a + cx + gx^2 = 0$ (which can be real or complex or coinciding).

On the other hand for systems (3) we have $\mu_0 = a_0 c_2 d_1^2$ and the condition $\mu_0 \neq 0$ implies $d_1 \neq 0$. This means that systems (3) possess invariant line $d_0 + d_1 x = 0$ and there does not exist another parallel invariant line in the direction $x = 0$.

It remains to observe that according to Lemma 3 for the existence of invariant lines in two distinct directions for a quadratic system the condition $B_2 = 0$ is necessary. Therefore systems (15) for $B_2 \neq 0$ could not have an invariant affine line in other direction, which could be used for the construction of the needed affine transformation.

a.2) *The subcase* $\mu_0 = 0$*.* Then $g = 0$ and considering (16) we get the systems

$$\dot{x} = a + cx, \quad \dot{y} = b - xy + y^2, \tag{17}$$

for which we have

$$B_1 = \mu_0 = H_7 = 0, \quad \widetilde{N} = -x^2.$$

On the other hand for systems (3) we have $\mu_0 = a_0 c_2 d_1^2$ and the condition $\mu = 0$ gives $a_0 c_2 d_1 = 0$.

If $d_1 = 0$ then for systems (3) we calculate

$$H_7 = -4(b_1^2 - 4a_0 c_2) d_0, \quad \widetilde{N} = -(b_1^2 - 4a_0 c_2) x^2$$

and the conditions $H_7 = 0$ and $\widetilde{N} \neq 0$ implies $d_0 = 0$ which leads to degenerate systems (3).

Assume now $d_1 \neq 0$. This means that systems (3) possess invariant line $d_0 + d_1 x = 0$ in the direction $x = 0$ and therefore $(d_0 + d_1 x)$ is a factor in $\widetilde{P}(x, y)$. Moreover the second factor of $\widetilde{P}(x, y)$ in (3) is y.

On the other hand systems (17) could possess in the direction $x = 0$ either one invariant affine line lines $a + cx = 0$ if $c \neq 0$ or zero lines if $c = 0$. Moreover the right hand side of the first equation does not contain the factor y.

It remains to observe that according to Lemma 3 systems (17) for $B_2 \neq 0$ could not have an invariant affine line in other directions, which could be used for the construction of the needed affine transformation. This completes the prof of our claim.

b) *The case* $B_2 = 0$. Then $b\big[(b - a)(1 + g)^2 + c^2\big] = 0$. We observe that the second factor equals $\Phi\big|_{d=0}$ and we deduce that we can apply the same arguments as previously in the case $H_7 \neq 0$ repeating the steps ***a)*** $b = 0$ and ***b)*** $\Phi = 0$ and considering the condition $d = 0$.

Thus the condition $B_2 = 0$ guarantees the existence of an affine transformation which brings systems (16) to the form (3).

4.1.2.2 The subcase $N = 0$ Considering (14) the condition $\widetilde{N} = 0$ yields $(g - 1)(h - 1) = g^2 - 1 = h^2 - 1 = 0$ and we obtain 3 possibilities: (a) $g = 1 = h$; (b) $g = 1 = -h$; (c) $g = -1 = -h$. The cases (b) and (c) can be brought by linear transformations to the case (a).

So $g = h = 1$ and systems $(\mathbf{S}_I)$ after an additional translation (to make $c = d = 0$ are of the form:

$$\dot{x} = a + dy + x^2, \quad \dot{y} = b + ex + y^2. \tag{18}$$

For these systems we calculate

$$B_1 = -\,d^2 e^2 (4a - 4b + d^2 - e^2), \quad \mu_0 = 1, \quad \theta_3 = -2de, \quad \theta_4 = -(d + e)$$

and we consider two possibilities: $\theta_3 \neq 0$ and $\theta_3 = 0$.

4.1.2.2.1 The possibility $\theta_3 \neq 0$. Then $b = a + (d^2 - e^2)/4$ and we get the systems

$$\dot{x} = a + dy + x^2, \quad \dot{y} = a + (d^2 - e^2)/4 + ex + y^2 \tag{19}$$

possess the invariant line $2x - 2y + d - e = 0$. So by means of the transformation

$$x_1 = 2x - 2y + d - e, \quad y_1 = x + y - (d + e)/2$$

we arrive at the following subfamily of (3):

$$\dot{x}_1 = x_1 y_1, \quad \dot{y}_1 = (4a + 2d^2 + e^2)/2 + (e - d)x_1 + (d + e)y_1 + x_1^2/8 + y_1^2/2. \tag{20}$$

4.1.2.2.2 The possibility $\theta_3 = 0$. Then $de = 0$ and we may consider $d = 0$ due o the change $(x, y, a, b, d, e) \mapsto (x, y, b, a, e, d)$ which conserves the systems. In this case we have

$$B_1 = 0, \quad B_2 = 648e^2(4a - 4b - e^2)x^4, \quad \theta_4 = -e$$

and we consider two cases: $B_2 \neq 0$ and $B_2 = 0$.

1) *The case $B_2 \neq 0$.* We claim that for $B_2 \neq 0$ systems (18) with $d = 0$ could not be brought via an affine transformation to the form (3).

Indeed, for systems (3) we calculate $\mu_0 = a_0 c_2 d_1^2 \neq 0$ (since for (18) we have $\mu = 1$). Hence $d_1 \neq 0$ and these systems possess a single invariant line $d_0 + d_1 x = 0$ in the direction $x = 0$.

On the other hand systems (18) with $d = 0$ possess in the direction $x = 0$ two parallel invariant lines $x^2 + a = 0$, which could be real or complex or coinciding. Taking into account that by Lemma 3 in the case $B_2 \neq 0$ these systems could not invariant lines in other directions we conclude that our claim is proved.

2) *The case $B_2 = 0$.* Then $e(4a - 4b - e^2) = 0$ and we examine two subcases: $\theta_4 \neq 0$ and $\theta_4 = 0$.

***a*)** If $\theta_4 \neq 0$ then we obtain $b = a - e^2/4$ and we get systems (19) with $d = 0$. So applying the transformation

$$x_1 = 2x - 2y - e, \quad y_1 = x + y - e/2$$

we arrive at the family of systems (20) with $d = 0$ which is a subfamily of (3).

***b*)** Assume now $\theta_4 = 0$, i.e. $e = 0$. In this case we get the systems

$$\dot{x} = a + x^2, \quad \dot{y} = b + y^2 \tag{21}$$

for which calculations yield

$$B_1 = B_2 = 0, \quad B_3 = -12(a - b)x^2y^2, \quad \mu_0 = 1.$$

These systems have two couples of parallel lines: $a + x^2 = 0$ (in the direction $x = 0$) and $b + y^2 = 0$ (in the direction $y = 0$) which could be real, or complex, or coinciding.

***b*.1)** If $B_3 \neq 0$ then by Lemma 3 systems (21) could not have other invariant lines.

On the other hand, as it was mentioned earlier, since $\mu_0 \neq 0$ systems (3) have a single line in the direction $x = 0$. So we conclude that systems (21) could not be brought to the form (3) by means of affine transformation.

***b*.2)** Assuming $B_3 = 0$ we obtain $b = a$ and then systems systems (21) possess also the invariant line $y = x$. So applying the transformation $x_1 = x - y$, $y_1 = x + y$ we get the family of systems

$$\dot{x}_1 = x_1 y_1, \quad \dot{y}_1 = 2a + x_1^2/2 + y_1^2/2.$$

Clearly this family is a subfamily of (3).

As all the possibilities in the case $\eta > 0$ are examined we conclude that the statement $\mathcal{A}$) of the Main Theorem is proved.

4.2 The subfamily defined by $\mathcal{B}$): $\eta < 0$

In this case by Lemma 2 we have to consider the systems $(\mathbf{S}_{II})$ for which we calculate:

$$\eta = -1, \quad \theta = 8(1+h)\big[g^2+(h-1)^2\big], \quad \widetilde{N} = (g^2-2h+2)x^2 + 2g(h+1)xy + (h^2-1)y^2. \tag{22}$$

So we examine two cases: $\theta \neq 0$ and $\theta = 0$.

4.2.1 The case $\theta \neq 0$

Then $h+1 \neq 0$ and due to a translation we may assume $c = d = 0$, i.e. we get the systems

$$\dot{x} = a + gx^2 + (h+1)xy, \quad \dot{y} = b + ex + fy - x^2 + gxy + hy^2, \tag{23}$$

for which we calculate

$$\begin{aligned}
B_1 =& -a(h+1)^2(\alpha^2+B\beta^2),\\
B_2 =& -648(\alpha\gamma+\beta\delta)x^4 + 648a(1+h)^2\alpha y^2(6x^2-y^2) - 2592a(1+h)^2\beta xy(x^2-y^2),
\end{aligned}$$

where

$$\begin{aligned}
\alpha =& a(1+g-h)(-1+g+h) - 2bg(-1+h) + f(-e+fg-eh),\\
\beta =& 2ag(-1+h) + b(1+g-h)(-1+g+h) - f^2 - efg + e^2h,\\
\gamma =& -a(g^2-4h-2g^2h) + bg(1+g^2-h^2) + e(egh-f-fg^2-fh),\\
\delta =& -ag(1+g^2-h^2) - b(1+g^2-2h-2g^2h+h^2) - f^2(1+g^2) + e(e+fg)h.
\end{aligned} \tag{24}$$

We observe that he condition $\alpha = \beta = 0$ implies $B_2 = 0$ and so we consider two subcases: $B_2 \neq 0$ and $B_2 = 0$.

4.2.1.1 The subcase $B_2 \neq 0$ Then due to $\theta \neq 0$ the condition $B_1 = 0$ implies $a = 0$ and applying the transformation $x_1 = x$, $y_1 = gx + (1+h)y$ we arrive at the family of systems

$$\dot{x}_1 = x_1y_1, \quad \dot{y}_1 = b(1+h) + (e-fg+eh)x_1 + fy_1 - \frac{1}{1+h}\big[(g^2+(h-1)^2)x_1^2 - 2gx_1y_1 - hy_1^2\big].$$

So we get a subfamily of the family of systems (3).

4.2.1.2 The subcase $B_2 = 0$ Then we obtain $\alpha = \beta = 0$ and considering (24) this condition yields

$$\begin{aligned}
a =& \frac{1}{\big[g^2+(h-1)^2\big]^2}\big[(e+fg-eh)(2egh+fh^2-f-fg^2)\big] \equiv a_0,\\
b =& -\frac{1}{\big[g^2+(h-1)^2\big]^2}\big[efg(h-1)(1+3h) - efg^3 + (h-1)^2(f^2-e^2h) + g^2(f^2+e^2h-2f^2h)\big] \equiv b_0.
\end{aligned}$$

In this case clearly we obtain systems (23) with $a = a_0$ and $b = b_0$ which we denote by (23_0). For these systems calculations yield $B_1 = B_2 = 0$ and

$$B_3 = \frac{3}{\big[g^2 + (h-1)^2\big]^2}(1+h)^2(e+fg-eh)(f+fg^2-2egh-fh^2)(x^2+y^2)^2 \tag{25}$$

We detect that systems (23_0) possess two complex invariant lines:

$$\big[g \pm i(1-h)\big]x + (1-h \mp ig)y - (f \pm ie) = 0$$

in two different directions (intersecting infinite line at complex singularities).

Since for system (3) we have $\theta = 8d_1(b_1^2c_2 - 4a_0c_2^2 + 4a_0c_2d_1 - a_0d_1^2) \neq 0$, we deduce that in order to exist an affine transformation for bringing systems (23_0) to the form (3) we need a real invariant affine line in the third (real) direction.

On the other hand according to Lemma 3 for the existence of invariant affine lines in three distinct directions the condition $B_3 = 0$ is necessary.

So we conclude that in the case $\eta < 0$, $\theta \neq 0$, $B_1 = B_2 = 0$ and $B_3 \neq 0$ a quadratic system could not be brought to an Abel quadratic differential system.

Assume now $B_3 = 0$. Considering the condition $\theta \neq 0$ and (25) we obtain the condition

$$(e+fg-eh)(f+fg^2-2egh-fh^2) = 0.$$

4.2.1.2.1 The possibility $e(1-h)+fg = 0$. If $g = 0$ then due to $\theta \neq 0$ (i.e. $g^2+(h-1)^2 \neq 0$) we obtain $e = 0$ and in this case systems (23_0) have the form

$$\dot{x} = (h+1)xy, \quad \dot{y} = -\frac{f^2}{(h-1)^2} + fy - x^2 + hy^2.$$

Thus we get Abel quadratic systems of the form (3).

Assume now $g \neq 0$. Then we obtain $f = e(h-1)/g$ and systems (23_0) become

$$\dot{x} = gx^2 + (1+h)xy, \quad \dot{y} = -e^2/g^2 + e(h-1)y)/g - x^2 + gxy + hy^2.$$

Applying the transformation $x_1 = x$, $y_1 = gx + (1+h)y$ we arrive at the following subfamily of the family of systems (3):

$$\dot{x}_1 = x_1y_1, \quad \dot{y}_1 = -\frac{e^2(1+h)}{g^2} + 2ex_1 + \frac{e(h-1)}{g}y_1 + \frac{1}{1+h}\big[(g^2+(h+1)^2)x_1^2 + 2gx_1y_1 + hy_1^2\big].$$

4.2.1.2.2 The possibility $f(1+g^2-h^2) - 2egh = 0$. If $g = 0$ then $h^2 - 1 \neq 0$ and we again get $f = 0$ and we arrive at the case considered above.

If $h = 0$ then the condition $f(1+g^2) = 0$ gives $f = 0$ and this leads to the degenerate systems

$$\dot{x} = x(gx+y), \quad \dot{y} = x(e - x + gy).$$

Assume now $gh \neq 0$. Then we calculate $e = f(1+g^2-h^2)/(2gh)$ and after the same transformation applied to systems (23_0) we obtain the systems

$$\begin{aligned}\dot{x}_1 =& x_1y_1,\\ \dot{y}_1 =& \frac{f^2(1+h)\big[g^2+(h+1)^2\big]}{4g^2h} + \frac{-f(h-1)(g^2+(h+1)^2)}{2gh}x_1 + fy_1 - \\ & \frac{(g^2+(h+1)^2)}{1+h}x_1^2 + \frac{2g}{1+h}x_1y_1 + \frac{h}{1+h}y_1^2.\end{aligned}$$

Thus we get Abel quadratic systems of the form (3).

4.2.2 The case $\theta = 0$

According to (22) we have $(h+1)[(h-1)^2+g^2] = 0$ and we consider two subcases: $\widetilde{N} \neq 0$ and $N = 0$.

4.2.2.1 Subcase $N \neq 0$. Then by (22) the condition $\theta = 0$ yields $h = -1$ and in addition we may assume $f = 0$ due to the translation $x \to x$ and $y \to y + f/2$. Hence, we obtain the family of systems

$$\dot{x} = a + cx + dy + gx^2, \quad \dot{y} = b + ex - x^2 + gxy - y^2, \tag{26}$$

for which calculations yield:

$$\begin{aligned}B_1 =& -d^2g(\hat{\alpha}^2+\hat{\beta}^2), \quad H_7 = 4d(4+g^2),\\ B_2 =& -648(\hat{\alpha}\hat{\gamma}+\hat{\beta}\hat{\delta})x^4 + 648a(1+h)^2\hat{\alpha}y^2(6x^2-y^2) - 2592a(1+h)^2\hat{\beta}xy(x^2-y^2),\end{aligned} \tag{27}$$

where

$$\begin{aligned}\hat{\alpha} =& a(g^2-4) + 4bg - 2ce - d(d+e)g,\\ \hat{\beta} =& -4ag + b(g^2-4) + c^2 + d^2 - e^2 + cdg,\\ \hat{\gamma} =& -a(4+3g^2) + bg^3 + c^2g - e(d+e)g + c(dg^2-2e) + d^2g,\\ \hat{\delta} =& -ag^3 - b(4+3g^2) + c^2 + c(d+2e)g + (d+e)(d-e+dg^2).\end{aligned} \tag{28}$$

We observe that he condition $\hat{\alpha} = \hat{\beta} = 0$ implies $B_2 = 0$ and so we examine two possibilities: $B_2 \neq 0$ and $B_2 = 0$.

4.2.2.1.1 The possibility $B_2 \neq 0$. Then the condition $B_1 = 0$ implies $dg = 0$ and we examine two cases: $H_7 \neq 0$ and $H_7 = 0$.

1) *The case $H_7 \neq 0$.* Considering (27) we have $d \neq 0$ and this implies $g = 0$. So we get the family of systems

$$\dot{x} = a + cx + dy, \quad \dot{y} = b + ex - x^2 - y^2,$$

and applying the transformation $x_1 = x$, $y_1 = cx + dy + a$ and $t_1 = t/d$ we arrive at the family of systems

$$\dot{x}_1 = dy_1, \quad \dot{y}_1 = bd^2 - a^2 + (d^2e - 2ac)x_1 + (2a+cd)y_1 - (c^2+d^2)x_1^2 + 2cx_1y_1 - y_1^2.$$

So we get a subfamily of the family of systems (3).

2) *The case* $H_7 = 0$. Then $d = 0$ and we arrive at the systems

$$\dot{x} = a + cx + gx^2, \quad \dot{y} = b + ex - x^2 + gxy - y^2, \tag{29}$$

which could have real straight lines only in the direction $x = 0$. However the right hand side of the first equation does not have as a factor y. Comparing with systems (3) we deduce that there could not exist an affine transformation which brought the above systems to the form (3).

4.2.2.1.2 The possibility $B_2 = 0$. Then we obtain $\hat{\alpha} = \hat{\beta} = 0$ and considering (28) this condition yields

$$\begin{aligned} a =& \frac{1}{(4+g^2)^2}(2c + dg + eg)(-4e + 2cg + dg^2) \equiv a_1, \\ b =& \frac{1}{(4+g^2)^2}\big[cdg^3 + (c^2 - 3d^2 - 4de - e^2)g^2 - 4c(d + 2e)g - 4(c^2 + d^2 - e^2)\big] \equiv b_1. \end{aligned}$$

In this case clearly we obtain systems (26) with $a = a_1$ and $b = b_1$ which we denote by (26_1). For these systems calculations yield $B_1 = B_2 = 0$ and

$$B_3 = -3d^2g(x^2 + y^2)^2, \quad H_7 = 4d(4 + g^2). \tag{30}$$

We detect that systems (26_1) possess two complex invariant lines:

$$(g \pm 2i)x + (2 \mp ig)y + c \mp i(d + e) = 0.$$

We consider two cases: $B_3 \neq 0$ and $B_3 = 0$.

1) *The case* $B_3 \neq 0$. We have two complex invariant lines. But by the same arguments as earlier we deduce that in order to exist an affine transformation for bringing systems (26_1) to the form (3) we need a real invariant affine line in the third (real) direction. However according to Lemma 3 for the existence of invariant affine lines in three distinct directions the condition $B_3 = 0$ is necessary.

So we conclude that in the considered case a quadratic system (26_1) could not be brought to an Abel quadratic system of the form (3).

2) *The case* $B_3 = 0$. Considering (30) the condition $dg = 0$ holds and we consider two subcases: $H_7 \neq 0$ and $H_7 = 0$.

a) *The subcase* $H_7 \neq 0$. Then $d \neq 0$ and the condition $B_3 = 0$ implies $g = 0$. Then systems (26_1) become

$$\dot{x} = -ce/2 + cx + dy, \quad \dot{y} = (c^2 + d^2 - e^2)/4 + ex - x^2 - y^2$$

and applying the transformation $x_1 = x$, $y_1 = cx + dy - ce/2$ and $t_1 = t/d$ we arrive at the following subfamily of systems (3):

$$\dot{x}_1 = dy_1, \quad \dot{y}_1 = (c^2 + d^2)(d^2 - e^2)/4 + (c^2 + d^2)ex_1 + c(d - e)y_1 - (c^2 + d^2)x_1^2 + 2cx_1y_1 - y_1^2.$$

***b*) *The subcase* $H_7 = 0$.** Then $d = 0$ which implies $B_3 = 0$. In this case considering systems (26_1) we arrive at the systems

$$\dot{x} = a_1\big|_{d=0} + cx + gx^2, \quad \dot{y} = b_1\big|_{d=0} + ex - x^2 + gxy - y^2.$$

These systems could possess invariant lines in the unique real direction $x = 0$. However by the same arguments as we present earlier for systems (29) we conclude that there could not exist an affine transformation which brings the above systems to the form (3).

4.2.2.2 Subcase $\widetilde{N} = 0$.

Then from (22) we have $g = h - 1 = 0$ and without loss of generality we may assume $c = d = 0$ via the translation $x \to x - d/2$, $y \to y - c/2$. Hence we obtain the systems

$$\dot{x} = a + 2xy, \quad \dot{y} = b + ex + fy - x^2 + y^2, \tag{31}$$

for which calculations yield:

$$B_1 = -4a(e^2 + f^2)^2, \quad B_2 = -648\big[(e^4 - 8aef + 2e^2f^2 + f^4)x^4 + 16a(e^2 - f^2)x^3y + 48aefx^2y^2 + 16a(f^2 - e^2)xy^3 - 8aefy^4.$$

We observe that the condition $e = f = 0$ implies $B_2 = 0$ and so we consider two possibilities: $B_2 \neq 0$ and $B_2 = 0$.

1) *The possibility* $B_2 \neq 0$. In this case the condition $B_1 = 0$ gives $a = 0$ and evidently systems (31) are of the form (3).

2) *The possibility* $B_2 = 0$. Then considering the condition $B_1 = 0$ we obtain $e = f = 0$ and we get the family of systems

$$\dot{x} = a + 2xy, \quad \dot{y} = b - x^2 + y^2,$$

for which we have $B_3 = -12a(x^2 + y^2)^2$. We detect that these systems possess the following two couples of complex invariant lines:

$$b + ia - (x - iy)^2 = 0, \quad -b + ia - (x + iy)^2 = 0.$$

According to Lemma 3 if $B_3 \neq 0$ then in the real direction $x = 0$ the above systems do not have any invariant line and this means that we could not bring them to the form (3) via an affine transformation.

It remains to observe that for $B_3 = 0$ (i.e. $a = 0$) the above systems are of the form (3).

Thus all the possibilities in the case $\eta < 0$ are examined and we conclude that the statement $\mathcal{B}$) of the Main Theorem is proved.

4.3 The subfamily defined by $\mathcal{C}$): $\eta = 0$, $\widetilde{M} \neq 0$

In this case by Lemma 2 we have to consider the systems $(\mathbf{S}_{III})$ for which calculations yield:

$$\theta = 8h^2(1 - g), \quad \mu_0 = gh^2, \quad \widetilde{N} = (g^2 - 1)x^2 + 2h(g - 1)xy + h^2y^2. \tag{32}$$

We consider two cases: $\theta \neq 0$ and $\theta = 0$.

4.3.1 The case $\theta \neq 0$

Then $(g-1)h \neq 0$ and due to a translation we may assume $d = e = 0$, i.e. we get the systems

$$\dot{x} = a + cx + gx^2 + hxy, \quad \dot{y} = b + fy + (g-1)xy + hy^2, \tag{33}$$

for which we calculate
$$B_1 = -\,a^2 b(g-1)^2 h^4.$$
Therefore due to $\theta \neq 0$ the condition $B_1 = 0$ implies $ab = 0$ and we consider two subcases: $a = 0$ and $b = 0$

4.3.1.1 The subcase $a = 0$ In this case applying the transformation $x_1 = x$, $y_1 = gx + hy + c$ we arrive at the family of systems

$$\dot{x}_1 = x_1 y_1, \quad \dot{y}_1 = c^2 - cf + bh + (c + cg - fg)x_1 + (f - 2c)y_1 + gx_1^2 - x_1 y_1 + y_1^2,$$

which is a subfamily of (3).

4.3.1.2 The subcase $b = 0$ Then systems (33) possess the invariant line $y = 0$ and using the transformation...
$$x_1 = y, \quad y_1 = (g-1)x + hy + f$$
we arrive at the systems

$$\dot{x}_1 = x_1 y_1, \quad \dot{y} = b' + e'x_1 + l'x_1^2 + (f' + 2m'x_1)y_1 + n'y^2,$$

where b', e', f', l', m' and n' are rational functions of the parameters a, c, f, g, h with the same denominator $(g-1) \neq 0$.

It remains to observe that these systems belong to the family of systems (3).

4.3.2 The case $\theta = 0$

By (32) we obtain $h(g-1) = 0$ and we consider two subcases: $\mu_0 \neq 0$ and $\mu_0 = 0$.

4.3.2.1 The subcase $\mu_0 \neq 0$ Considering (32) we obtain $h \neq 0$, $g = 1$ and then we may assume $h = 1$ due to the change $y \to y/h$. Moreover, we may assume $c = d = 0$ via the translation $x \to x - d$ and $y \to y + 2d - c$. So, we obtain the canonical systems

$$\dot{x} = a + x^2 + xy, \quad \dot{y} = b + ex + fy + y^2, \tag{34}$$

for which calculation yields

$$B_1 = -a^2 e^2, \quad B_2 = 648\big[(4a - b)e^2 x^4 + 4ae^2 x^3 y - a^2 y^4\big], \quad H_7 = -4e.$$

The condition $B_1 = 0$ implies $ae = 0$ and we consider two possibilities: $H_7 \neq 0$ and $H_7 = 0$.

4.3.2.1.1 The possibility $H_7 \neq 0$. In this case $e \neq 0$ and we obtain $a = 0$. Then applying the transformation $x_1 = x$, $y_1 = x + y$ we arrive at the family of systems

$$\dot{x}_1 = x_1 y_1, \quad \dot{y}_1 = b + (e - f)x_1 + f y_1 + x_1^2 - x_1 y_1 + y_1^2, \tag{35}$$

which is a subfamily of (3).

4.3.2.1.2 The possibility $H_7 = 0$. Then $e = 0$ and this leads to the systems

$$\dot{x} = a + x^2 + xy, \quad \dot{y} = b + fy + y^2,$$

for which

$$B_1 = 0, \quad B_2 = -648a^2y^4.$$

We observe that these systems possess only two (parallel) invariant lines $b + fy + y^2 = 0$ in the direction $y = 0$ which could be real or complex or could coincide. Moreover by Lemma 3 in the case $B_2 \neq 0$ we do not have any other invariant line in the second direction $x = 0$. Therefore by the same arguments as we presented earlier for systems (29), we conclude that for $B_2 \neq 0$ there cannot exist an affine transformation which brings the above systems to the form (3).

Assuming $B_2 = 0$ we obtain $a = 0$ and using the transformation $x_1 = x$, $y_1 = x + y$ we arrive at the systems (35) with $e = 0$, i.e. we get systems of the form (3).

4.3.2.2 The subcase $\mu_0 = 0$ Since $\theta = 0$ this implies $h = 0$ and for the systems $(\mathbf{S}_{III})$ we have $\widetilde{N} = (g^2 - 1)x^2$ and we examine two possibilities: $\widetilde{N} \neq 0$ and $\widetilde{N} = 0$.

4.3.2.2.1 The possibility $\widetilde{N} \neq 0$. In this case $g - 1 \neq 0$ and we may assume $e = f = 0$ via the translation $x \to x + f/(1-g)$ and $y \to y + e/(1-g)$. This leads to the systems

$$\dot{x} = a + cx + dy + gx^2, \quad \dot{y} = b + (g-1)xy, \tag{36}$$

for which we have

$$B_1 = -bd^4(g-1)^2g^2, \quad \widetilde{N} = (g^2 - 1)x^2, \quad H_7 = 4d(g^2 - 1).$$

So due to $\widetilde{N} \neq 0$ the condition $B_1 = 0$ gives $bdg = 0$ and we consider two cases: $H_7 \neq 0$ and $H_7 = 0$.

1) *The case $H_7 \neq 0$.* Then $d \neq 0$ and we get $bg = 0$.

If $b = 0$ then it is evident that after the interchange $x \leftrightarrow y$ systems (36) become of the form (3).

Assume now $g = 0$. Since $d \neq 0$ we can apply the transformation $x_1 = x$, $y_1 = cx + dy + a$ and this leads to a subfamily of (3):

$$\dot{x}_1 = y_1, \quad \dot{y}_1 = bd + ax_1 + cy_1 + cx_1^2 - x_1 y_1.$$

2) *The case $H_7 = 0$.* Then $d = 0$ and we obtain the systems

$$\dot{x} = a + cx + gx^2, \quad \dot{y} = b + (g-1)xy \tag{37}$$

which possess the invariant lines $a+cx+gx^2=0$ in the real double direction $x=0$ because $C_2=x^2y$.

We calculate

$$B_1=B_2=0,\quad B_3=-3b(g-1)^2x^4$$

and we conclude that an invariant line exists in the direction $y=0$ if and only if $B_3=0$. So by the same arguments as we presented earlier for systems (29), we conclude that for $B_3\neq 0$ there cannot exist an affine transformation which brings the above systems to the form (3).

Assuming $B_3=0$ we obtain $b=0$ (due to $\widetilde{N}\neq 0$) and in the same manner as above by the interchanging $x\leftrightarrow y$ systems (36) become of the form (3).

4.3.2.2.2 The possibility $\widetilde{N}=0$.

In this case $g^2-1=0$, i.e. $g=1$ or $g=-1$.

On the other hand for systems $(\mathbf{S}_{III})$ with $h=0$ we have $\widetilde{K}=g(g-1)x^2$ and we consider two cases: $\widetilde{K}\neq 0$ and $\widetilde{K}=0$.

1) *The case* $\widetilde{K}\neq 0$. Then $g-1\neq 0$ and this implies $g=-1$. In this case we may assume $e=f=0$ via the translation $x\to x+f/2$ and $y\to y+e/2$ and we arrive at the family of systems

$$\dot{x}=a+cx+dy-x^2,\quad \dot{y}=b-2xy, \tag{38}$$

for which calculations yield:

$$B_1=-4bd^4,\quad \theta_3=2d^2.$$

a) If $\theta_3\neq 0$ then the condition $B_1=0$ gives $b=0$ and after interchange $x\leftrightarrow y$ the above systems become of the form (3).

b) Assume now $\theta_3=0$, i.e. $d=0$ and we get the systems (37) with $g=-1$. So we repeat the same steps as before in this particular case and we conclude that the systems (37) could be brought via an affine transformation to the form (3) if and only if either $\theta_3\neq 0$ or $\theta_3=0$ and $B_3=0$.

2) *The case* $\widetilde{K}=0$. Then $g=1$ and we may assume $c=0$ due to the translation $x\to x-c/2$ and $y\to y$. Then we obtain the systems

$$\dot{x}=a+dy+x^2,\quad \dot{y}=b+ex+fy. \tag{39}$$

It is clear that in order to have invariant lines in the direction $x=0$ (respectively $y=0$) the condition $d=0$ (respectively $e=0$) has to be satisfied. However in the case $d=0$ we obtain two parallel complex lines and clearly we could use them for the construction of the transformation which brings these systems to the form (3).

On the other hand if $e=0$ we have the invariant line $fy+b=0$ for $f\neq 0$. However applying the transformation $x_1=fy+b$, $y_1=\gamma x+\delta y+\nu$ with free parameters γ,δ and ν, we arrive at the systems

$$\dot{x}_1=fx_1,\quad \dot{y}_1=\tilde{Q}(x_1,y_1).$$

As it can be observed these systems do not have the form (3).

So we deduce that in the case $\widetilde{N} = 0 = \widetilde{K}$ there cannot exist an affine transformation which brings a system ($\mathbf{S}_{III}$) to an Abel quadratic system of the form (3).

So since all the possibilities in the case $\eta = 0$ and $\widetilde{M} \neq 0$ are examined we deduce that the statement $\mathcal{C}$) of the Main Theorem is proved.

4.4 The subfamily defined by $\mathcal{D}$): $\eta = \widetilde{M} = 0$

According to the conditions provided by Main Theorem we consider two cases: $C_2 \neq 0$ and $C_2 = 0$.

4.4.1 The case $C_2 \neq 0$

Then by Lemma 2 we examine the systems ($\mathbf{S}_{IV}$) for which calculations yield:

$$\eta = \widetilde{M} = 0, \quad C_2 = x^3, \quad \theta = 8h^3. \tag{40}$$

4.4.1.1 The subcase $\theta \neq 0$. Then $h \neq 0$ and due to a translation we may assume $c = d = 0$, i.e. we get the systems

$$\dot{x} = a + gx^2 + hxy, \quad \dot{y} = b + ex + fy - x^2 + gxy + hy^2, \tag{41}$$

for which we calculate $B_1 = -a^3h^6$. So the condition $B_1 = 0$ gives $a = 0$ and then the above systems after the transformation $x_1 = x$, $y_1 = gx + hy$ become

$$\dot{x}_1 = x_1y_1, \quad \dot{y} = bh + (eh - fg)x_1 + fy_1 - hx_1^2 + y^2,$$

i.e. we get a subfamily of (3).

4.4.1.2 The subcase $\theta = 0$. Then $h = 0$ and we calculate

$$B_1 = -d^6g^3, \quad \widetilde{N} = g^2x^2$$

and we consider two possibilities: $\widetilde{N} \neq 0$ and $\widetilde{N} = 0$.

4.4.1.2.1 The possibility $\widetilde{N} \neq 0$. We have $g \neq 0$ and the condition $B_2 = 0$ gives $d = 0$. In this case due to a translation we may assume $e = f = 0$ and this leads to the systems a

$$\dot{x} = a + cx + gx^2, \quad \dot{y} = b - x^2 + gxy. \tag{42}$$

Since for these systems we have $C_2 = x^3$ (i.e. we could have real invariant affine lines only in this direction) we conclude, that besides the parallel invariant lines $a + cx + gx^2 = 0$ the above systems cannot have other invariant lines.

Thus applying the same arguments as we present earlier for systems (29), we deduce that for $\widetilde{N} \neq 0$ there cannot exist an affine transformation which brings systems (42) to the form (3).

4.4.1.2.2 The possibility $\widetilde{N}=0$. Then $g=0$ (this implies $B_1=0$) and we arrive at the systems

$$\dot{x}=a+cx+dy,\quad \dot{y}=b+ex+fy-x^2,$$

for which $B_2=-648d^4x^4$.

1) *The case $\widetilde{B}_2\neq 0$.* We obtain $d\neq 0$ and applying the transformation $x_1=x$, $y_1=cx+dy+a$ we obtain a subfamily of (3):

$$\dot{x}_1=y_1,\quad \dot{y}_1=bd-af+(de-cf)x_1+(c+f)y_1-dx_1^2. \tag{43}$$

2) *The case $\widetilde{B}_2=0$.* Then we get the systems

$$\dot{x}=a+cx,\quad \dot{y}=b+ex+fy-x^2,$$

and since the right hand side of the first equation does not have as a factor y we deduce that there could not exist an affine transformation which brings the above systems to the form (3).

4.4.2 The case $C_2=0$

Then by Lemma 2 we examine the systems $(\mathbf{S}_V)$ which have the infinite line fulfilled with singularities. This family of systems is considered in [70], where are presented a total of 9 canonical forms of this family: $C_2.1-C_2.9$ (see Table 1, page 741).

We observe that the canonical systems $C_2.1-C_2.4$ for $H_{10}\neq 0$ as well as $C_2.5-C_2.7$ for $H_{10}=0$ and $H_{12}\neq 0$ after the additional interchange $x\leftrightarrow y$ have the form

$$\dot{x}=xy,\quad \dot{y}=Q_i(x,y),\quad (i=1,\dots,7)$$

where $Q_i(x,y)$ is the corresponding to $C_2.i$ quadratic polynomial depending of at least one parameter. It is evident that these canonical systems belong to the family (3).

It remains to consider two canonical systems given in Table 1 of [70]:

$$(C_2.8):\ \begin{cases}\dot{x}=x+x^2,\\ \dot{y}=1+xy;\end{cases}\qquad\qquad (C_2.9):\ \begin{cases}\dot{x}=x^2,\\ \dot{y}=1+xy,\end{cases}$$

and we claim that there does not exist an affine transformation bringing any of these two systems to the form (3).

Indeed, for both systems $(C_2.8)$ and $(C_2.9)$ we have: $C_2=0$ and $H_{10}=0=H_{12}$.

On the other hand for systems (3) we calculate

$$C_2=-a_0x^3-b_1x^2y-(c_2-d_1)xy^2$$

and hence the condition $C_2=0$ implies $a_0=b_1=0$ and $d_1=c_2$. Then we get the systems

$$\dot{x}=(d+2hx)y,\quad \dot{y}=b+ex+fy+2hy^2$$

for which calculations yield

$$H_{10} = 36c_1^2c_2^4 = 0, \quad H_{12}\big|_{\{c_1c_2=0\}} = -8c_0^2c_2^4y^2 = 0.$$

If $c_2 \neq 0$ then we obtain $c_1 = 0 = c_0$ and this leads to degenerate systems

$$\dot{x} = (d_0 + d_1x)y, \quad \dot{y} = y(b_0 + d_1y).$$

On the other hand assuming $d_1 = 0$ we get the linear systems

$$\dot{x} = d_0y, \quad \dot{y} = c_0 + c_1x + b_0y.$$

This completes the proof of our claim.

Thus all the cases are examined and we deduce that the the Main Theorem is proved.

4.5 Phase portraits of the quadratic systems from the family $\mathcal{D}$) defined in the Main Theorem

According to Lemma 2 the systems from the family $\mathcal{D}$) defined by the condition $\eta = \widetilde{M} = 0$ could be brought via an affine transformation either to the systems $(\mathbf{S}_{IV})$ (if $C_2 \neq 0$) or to the systems $(\mathbf{S}_V)$ (if $C_2 = 0$). So we examine these two subfamilies separately. We give examples for the realization of each one of the constructed phase portraits of systems (1) belonging to one of the above mentioned two classes in the form (a, c, d, g, h, k), (b, e, f, l, m, n).

4.5.1 Systems $(\mathbf{S}_{IV})$: $\eta = \widetilde{M} = 0$, $C_2 \neq 0$

Theorem 1. *Assume that for a quadratic system the conditions $\eta = \widetilde{M} = 0$, and $C_2 \neq 0$ hold. Then this system belongs to the class $\boldsymbol{QS_{Ab}}$ if and only if either $\theta \neq 0$ or $\theta = \widetilde{N} = 0$ and $B_2 \neq 0$. Moreover its phase portrait is topologically equivalent to one of the pictures given in Figure 1 if and only the following corresponding conditions are verified:*

Picture $S_{IV}.1$	$\Leftrightarrow$	$\theta \neq 0,\ \mathbf{D}<0,\ \mathbf{R}>0,\ \mathbf{S}>0,\ \mu_0<0,\ \neg(\mathfrak{C}_2),\ \widetilde{U}_1\widetilde{U}_2<0;$	$[Conf.(3)]$
Picture $S_{IV}.2$	$\Leftrightarrow$	$\theta \neq 0,\ \mathbf{D}<0,\ \mathbf{R}>0,\ \mathbf{S}>0,\ \mu_0<0,\ \neg(\mathfrak{C}_2),\ \widetilde{U}_1\widetilde{U}_2>0;$	$[Conf.(3)]$
Picture $S_{IV}.3$	$\Leftrightarrow$	$\theta \neq 0,\ \mathbf{D}<0,\ \mathbf{R}>0,\ \mathbf{S}>0,\ \mu_0<0,\ (\mathfrak{C}_2);$	$[Conf.(4)]$
Picture $S_{IV}.4$	$\Leftrightarrow$	$\theta \neq 0,\ \mathbf{D}<0,\ \mathbf{R}>0,\ \mathbf{S}>0,\ \mu_0>0;$	$[Conf.(8)]$
Picture $S_{IV}.5$	$\Leftrightarrow$	$\begin{cases}\theta \neq 0, \mathbf{D}<0, (\mathbf{R}\leq 0)\vee(\mathbf{S}\leq 0) \ \textit{or}\\ \theta=\widetilde{N}=0, B_2\neq 0, \mathbf{U}<0;\end{cases}$	$[Conf.(12)]$
Picture $S_{IV}.6$	$\Leftrightarrow$	$\begin{cases}\theta \neq 0, \mathbf{D}>0, \mu_0<0, \neg(\widehat{\mathfrak{C}}_1), \widetilde{U}_1<0, \ \textit{or}\\ \theta\neq 0, \mathbf{D}=\mathbf{T}=\mathbf{P}=0, \mathbf{R}\neq 0, \mathcal{T}_4\neq 0, \mu_0<0;\end{cases}$	$[Conf.(16)]$
Picture $S_{IV}.7$	$\Leftrightarrow$	$\theta \neq 0,\ \mathbf{D}>0,\ \mu_0<0,\ \neg(\widehat{\mathfrak{C}}_1), \widetilde{U}_1>0;$	$[Conf.(16)]$
Picture $S_{IV}.8$	$\Leftrightarrow$	$\theta \neq 0,\ \mathbf{D}>0,\ \mu_0<0,\ (\widehat{\mathfrak{C}}_1);$	$[Conf.(18)]$
Picture $S_{IV}.9$	$\Leftrightarrow$	$\begin{cases}\theta \neq 0, \mathbf{D}>0, \mu_0>0, \neg(\mathfrak{C}_2), \ \textit{or}\\ \theta\neq 0, \mathbf{D}=\mathbf{T}=\mathbf{P}=0, \mathbf{R}\neq 0, \mathcal{T}_4\neq 0, \mu_0>0, \ \textit{or}\\ \theta=\widetilde{N}=0, B_2\neq 0, \mathbf{U}>0, \neg(\mathfrak{C}_9);\end{cases}$	$[Conf.(23)]$
Picture $S_{IV}.10$	$\Leftrightarrow$	$\begin{cases}\theta \neq 0, \mathbf{D}>0, \mu_0>0, (\mathfrak{C}_2), \ \textit{or}\\ \theta=\widetilde{N}=0, B_2\neq 0, \mathbf{U}>0, (\mathfrak{C}_9);\end{cases}$	$[Conf.(24)]$
Picture $S_{IV}.11$	$\Leftrightarrow$	$\theta \neq 0,\ \mathbf{D}=0,\ \mathbf{T}<0\ \mu_0<0,\ B_2\widetilde{U}_1\neq 0,\ E_1\neq 0;$	$[Conf.(30)]$
Picture $S_{IV}.12$	$\Leftrightarrow$	$\theta \neq 0,\ \mathbf{D}=0,\ \mathbf{T}<0\ \mu_0<0,\ B_2\widetilde{U}_1\neq 0,\ E_1=0;$	$[Conf.(34)]$
Picture $S_{IV}.13$	$\Leftrightarrow$	$\theta \neq 0,\ \mathbf{D}=0,\ \mathbf{T}<0\ \mu_0<0,\ B_2=0;$	$[Conf.(30)]$
Picture $S_{IV}.14$	$\Leftrightarrow$	$\theta \neq 0,\ \mathbf{D}=0,\ \mathbf{T}<0\ \mu_0<0,\ \widetilde{U}_1=0;$	$[Conf.(30)]$
Picture $S_{IV}.15$	$\Leftrightarrow$	$\theta \neq 0,\ \mathbf{D}=0,\ \mathbf{T}<0\ \mu_0>0;$	$[Conf.(37)]$
Picture $S_{IV}.16$	$\Leftrightarrow$	$\begin{cases}\theta \neq 0, \mathbf{D}=0, \mathbf{T}>0, E_1\neq 0 \ \textit{or}\\ \theta\neq 0, \mathbf{D}=\mathbf{T}=\mathbf{P}=\mathbf{R}=0, \mu_0>0;\end{cases}$	$[Conf.(44)]$
Picture $S_{IV}.17$	$\Leftrightarrow$	$\begin{cases}\theta \neq 0, \mathbf{D}=0, \mathbf{T}>0, E_1=0 \ \textit{or}\\ \theta=\widetilde{N}=0, B_2\neq 0, \mathbf{U}=0;\end{cases}$	$[Conf.(47)]$
Picture $S_{IV}.18$	$\Leftrightarrow$	$\theta \neq 0,\ \mathbf{D}=0,\ \mathbf{T}=0\ \mathbf{P}\neq 0;$	$[Conf.(50)]$
Picture $S_{IV}.19$	$\Leftrightarrow$	$\theta \neq 0,\ \mathbf{D}=0,\ \mathbf{T}=0\ \mathbf{P}=0,\ \mathbf{R}\neq 0,\ \mathcal{T}_4=0,\ \mu_0<0;$	$[Conf.(60)]$
Picture $S_{IV}.20$	$\Leftrightarrow$	$\theta \neq 0,\ \mathbf{D}=0,\ \mathbf{T}=0\ \mathbf{P}=0,\ \mathbf{R}\neq 0,\ \mathcal{T}_4=0,\ \mu_0>0;$	$[Conf.(64)]$
Picture $S_{IV}.21$	$\Leftrightarrow$	$\theta \neq 0,\ \mathbf{D}=0,\ \mathbf{T}=0\ \mathbf{P}=0,\ \mathbf{R}=0,\ \mu_0<0;$	$[Conf.(67)]$

The last column in the above table contains the corresponding topological configurations according to the notations given in the set of diagrams provided by the Main Theorem in [16].

Proof: We prove this theorem following the conditions provided by the statement $\mathcal{D}$) of the Main Theorem.

4.5.1.1 The case $\theta \neq 0$. Considering the systems (41) and the corresponding transformed systems we examine the family:

$$\dot{x} = xy, \quad \dot{y} = b + ex + fy - hx^2 + y^2. \tag{44}$$

We shall consider step by step the conditions provided by the the Diagrams 1-6 form [16], taking into account that the conditions $\eta = \widetilde{M} = 0$, and $C_2 \neq 0$ are satisfied.

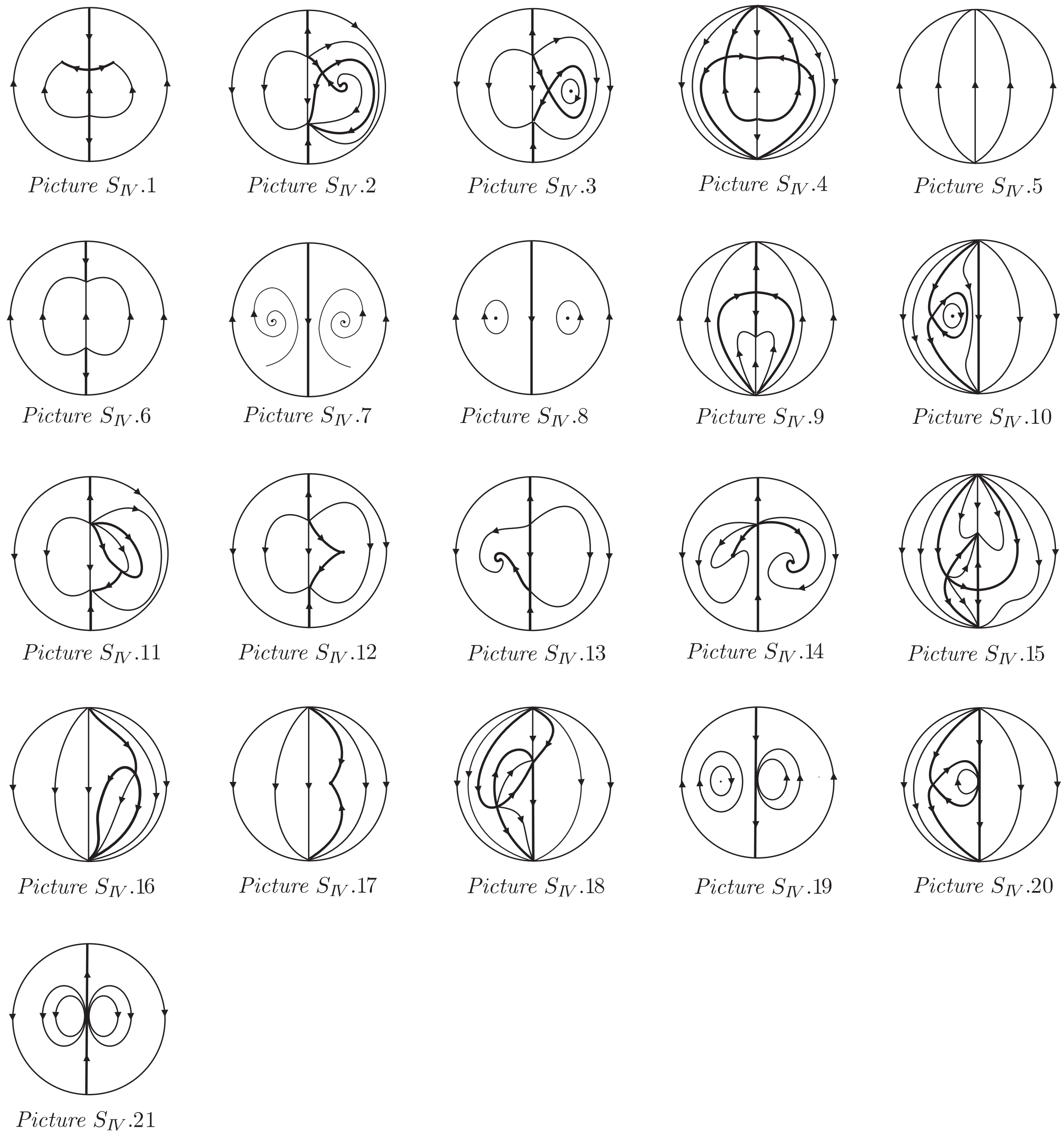

Figure 1: **Global phase portraits of quadratic systems with $\eta = \widetilde{M} = 0$, $C_2 \neq 0$.**

For these systems calculations yield:

$$\begin{gathered}
C_2 = hx^3, \quad \mu_0 = -h, \quad \mathbf{D} = -48b^2(f^2-4b)(e^2+4bh), \quad B_2 = -648b^2h^2x^4, \\
\mathcal{T}_4 = -f^2h(9b-2f^2), \quad \mathcal{T}_3 = -fh(18b-5f^2), \quad \mathcal{T}_2 = -3(3b-f^2)h, \quad \mathcal{F} = 9fh/8, \\
\mathcal{F}_1 = 0, \ \mathcal{F}_2 = -9f^2h^2/2 = -\mathcal{F}_3, \ \mathcal{B} = -9(9e^2+36bh-8f^2h)/8, \ \mathcal{H} = -9h/2, \ \sigma = f+3y.
\end{gathered} \tag{45}$$

We observe that the condition $C_2 \neq 0$ implies $\mu_0 \neq 0$ and according to [9, Table 6.2] the above systems possess finite singularities of total multiplicity four. More exactly we have the singularities $M_{1,2}(0, y_{1,2})$ and $M_{3,4}(x_{3,4}, 0)$, where

$$y_{1,2} = \big(-f \pm \sqrt{f^2 - 4b}\,\big)/2, \quad x_{3,4} = \big(e \pm \sqrt{e^2 + 4bh}\,\big)/(2h). \tag{46}$$

It is clear that on the real invariant line $x = 0$ are located the singularities $M_{1,2}(0, y_{1,2})$ and these singularities are real if $f^2 - 4b > 0$ and they are complex if $f^2 - 4b < 0$.

First we prove the following lemma:

Lemma 4. *For a system* (44) *the conditions* $(\mathfrak{C}_1)$ *as well as the conditions* $(\mathfrak{C}_5)$ - $(\mathfrak{C}_{10})$ *and* $(\widehat{\mathfrak{C}}_2)$ *could not be satisfied.*

Proof: First of all from (45) we obtain that for systems (44) the condition $\sigma = f + 3y \neq 0$ holds. Therefore considering (8) and (9) we deduce that the conditions $(\mathfrak{C}_5)$ - $(\mathfrak{C}_{10})$ and $(\widehat{\mathfrak{C}}_2)$ could not be satisfied for these systems.

It remains to examine the conditions $(\mathfrak{C}_1)$. According to (8) these conditions imply $\mathcal{T}_3 \neq 0$ and $\mathcal{F}_2 = 0$. However considering (45) it is clear that the condition $\mathcal{T}_3 \neq 0$ (i.e. $fh \neq 0$) implies $\mathcal{F}_2 \neq 0$. This completes the proof of the lemma. ∎

According to [9, Table 6.2] all the finite singularities of systems (44) are distinct if $\mathbf{D} \neq 0$ and we have multiple singular points if $\mathbf{D} = 0$. So we examine three subcases: $\mathbf{D} < 0$, $\mathbf{D} > 0$ and $\mathbf{D} = 0$.

4.5.1.1.1 The subcase $\mathbf{D} < 0$. According to [9, Table 6.2, page 124] systems (44) possess either four real distinct finite singularities in the case $\mathbf{R} > 0$, $\mathbf{S} > 0$, or four complex finite singularities if $(\mathbf{R} \leq 0) \vee (\mathbf{S} \leq 0)$.

1) *The possibility* $\mathbf{R} > 0$, $\mathbf{S} > 0$ So systems (44) possess four real distinct finite singularities and following [16, Diagram 1, page 3] we consider two cases: $\mu_0 < 0$ and $\mu_0 > 0$.

a) *The case* $\mu_0 < 0$. According to this diagram we could have either the topological configuration (3) s, a, a, a; S if $\neg(\mathfrak{C}_2)$ or (4) s, a, a, c; S if $(\mathfrak{C}_2)$.

Consider first the configuration (3). It is clear that if the saddle is located on the invariant line $x = 0$ then we have the separatrix connection between the finite saddle and the infinite one. So we need a condition to distinguish this case.

On the other hand denoting by Δ_i $(i = 1, 2, 3, 4)$ the determinant of the linear matrix corresponding to the singular point M_i we calculate

$$\Delta_{1,2} = -2b + (f^2 \pm f\sqrt{f^2 - 4b}\,)/2 \ \Rightarrow\ \Delta_1\Delta_2 = b(4b - f^2).$$

We remark that when two finite singularities coalesce it is important to distinguish if they are located on the invariant line, i.e. if $4b - f^2 = 0$. For systems (44) we have:

$$\widetilde{U}_1 = -27(f^2 - 4b)h/8, \quad \widetilde{U}_2 = 9bh/2. \tag{47}$$

Therefore $\widetilde{U}_1\widetilde{U}_2 = 243b(4b - f^2)h^2/16 = 243\Delta_1\Delta_2h^2/16$ and we conclude that the following remark is valid:

Remark 1. *Assume that two real singularities of a system* (44) *are located on the invariant line* $x = 0$ *and in addition the condition* $\widetilde{U}_1\widetilde{U}_2 \neq 0$ *holds. Then* $\operatorname{sign}(\Delta_1\Delta_2) = \operatorname{sign}(\widetilde{U}_1\widetilde{U}_2)$*, i.e. on the invariant line of this system lies exactly one saddle if and only if* $\widetilde{U}_1\widetilde{U}_2 < 0$.

So, considering the above remark and the fact that we have a single saddle, in the case of the topological configuration (3): $s, a, a, a;\ S$ (see [16]) we obtain *Picture* $S_{IV}.1$ if $\widetilde{U}_1\widetilde{U}_2 < 0$ and *Picture* $S_{IV}.2$ if $\widetilde{U}_1\widetilde{U}_2 > 0$. The corresponding example are:

Picture $S_{IV}.1$ if $\neg(\mathfrak{C}_2)$ and $\widetilde{U}_1\widetilde{U}_2 < 0$ [Ex: $(0, 0, 0, 0, 1/2, 0)$, $(1/8, 0, 1, -1, 0, 1)$];

Picture $S_{IV}.2$ if $\neg(\mathfrak{C}_2)$ and $\widetilde{U}_1\widetilde{U}_2 > 0$ [Ex: $(0, 0, 0, 0, 1/2, 0)$, $(-3/4, 2, 1, -1, 0, 1)$];

Consider now the configuration (4): $s, a, a, c;\ S$. Since we have a center (i.e. the conditions $(\mathfrak{C}_2)$ hold), considering [78] (see also [79]) we get the unique phase portrait given by *Picture* $S_{IV}.3$ [Ex: $(0, 0, 0, 0, 1/2, 0)$, $(-1, \sqrt{5}, 0, -1, 0, 1)$].

b) *The case* $\mu_0 > 0$. In this case by [16, Diagram 1, page 3] we could have either the topological configuration (8) $s, s, a, a;\ N$ if $\neg((\widehat{\mathfrak{C}}_1) \vee (\widehat{\mathfrak{C}}_2))$ or (9) $s, s, c, c;\ N$ if $(\widehat{\mathfrak{C}}_1) \vee (\widehat{\mathfrak{C}}_2)$.

However the configuration (9) with two centers is not realizable for systems (44) because by Lemma 4 the conditions $(\widehat{\mathfrak{C}}_2)$ are incompatible.

Consider now the conditions $(\widehat{\mathfrak{C}}_1)$. According to (45) and (9) the condition $\mathcal{H} = -9h/2 < 0$ is necessary but this implies $h > 0$ which contradicts $\mu_0 = -h > 0$. This completes the proof of our claim.

It remains to examine the configuration (8) $s, s, a, a;\ N$. It is not too difficult to convince ourself that both saddles could not be located on the invariant line $x = 0$ (since $\Delta_1 + \Delta_2 = f^2 - 4b > 0$). If both singularities on $x = 0$ are nodes, then we get *Picture* $S_{IV}.4$ [Ex: $(0, 0, 0, 0, 1/2, 0)$, $(-3/4, 1, 1, 1, 0, 1)$]. We claim that if we have a saddle and a node on $x = 0$ then the phase portrait is topologically equivalent to *Picture* $S_{IV}.4$. This results from the following lemma:

Lemma 5. *The only possible phase portrait of systems* (44) *with configuration* $s, s, a, a;\ N$ *and a saddle and a node on* $x = 0$*, is topologically equivalent to Picture* $S_{IV}.4$.

Proof. We will prove first that *Picture* $S_{IV}.4$ is the only generic phase portrait that we may have in family (44) and later we will prove that no other non-generic phase portrait may exist. So assume we are looking first for a generic phase portrait.

The ordinates of both singularities on $x = 0$ must have the same sign, otherwise, a) if both singularities on $y = 0$ had the same sign of abscissa, three of them would form a triangle inside of which is the remaining singularity, and this forces three points of index $+1$ (respectively -1) and one of index -1 (respectively $+1$) and this is incompatible with $\mu_0 > 0$; or b) singularities on $y = 0$ have different signs and all form a quadrilateral but again this is incompatible with the result of Berlinski (see [21]) since we get that a saddle and an anti-saddle occupy opposite vertices.

So, the ordinates of both singularities on $x = 0$ have the same sign, and due to a symmetry we may assume both positive. Again, the abscissa of both singularities on $y = 0$ must have the same sign, otherwise this contradicts the result of Berlinski [21]. By means of another symmetry we may consider them also to be positive. And since we do not have yet applied a time change, we may assume that the node at $[0:1:0]$ is attractor.

The isocline on which $\dot{y}=0$ is a real ellipse since the homogeneous quadratic part of this ellipse (see systems (44)) is $-hx^2+y^2$ and $h<0$ by $\mu_0>0$. The other possibility would be a complex conic or a single point, but since we have 4 real singularities, we could only have a real ellipse.

A generic phase portrait of these systems must be topologically equivalent to a structurally stable quadratic system of family 3 (see [1]) which we portray in Figure 2. We have already mentioned that the triple node at infinity behaves as a simple node.

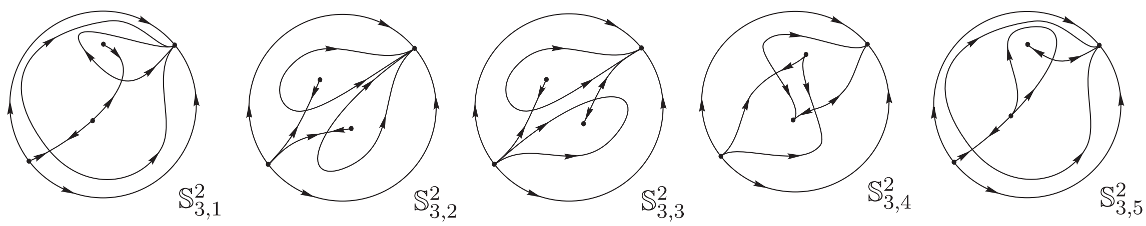

Figure 2: **Structurally stable quadratic systems of Family 3 of [1].**

Four of these phase portraits contain what is called in some papers (see [1, 6]) a *basin*, that is, a saddle sending two of its separatrices to a same singularity and enclosing at least one finite anti-saddle in the region formed by the two separatrices. The only case which does not contain any basin is $\mathbb{S}^2_{3,4}$, is topologically equivalent to *Picture* $S_{IV}.4$. We claim then that a system (44) cannot contain any basin.

By (46) we have that $y_{1,2}=\big(-f\pm\sqrt{f^2-4b}\big)/2$ and we are assuming that both ordinates are positives and we have that $f^2-4b>0$. By means of a change of parameters $b=(f^2-u^2)/4$ we may write them simply as $y_{1,2}=(-f\pm u)/2$ and their determinants are $\Delta_{1,2}=u(u\mp f)/2$. Due to the change $u\to -u$ we get $y_1\leftrightarrow y_2$ and we may assume $0<y_1<y_2$. Then the singularity on $(0,y_1)$ must be a saddle and correspondingly $(0,y_2)$ is a node, otherwise we will have $\Delta_1>0>\Delta_2$ and the conditions $\{0<-f+u<-f-u, u(-f+u)>0, u(f+u)<0\}$ are clearly incompatible.

So we assume we have a basin from the finite saddle at $(0,y_1)$ and that the finite node is at $(0,y_2)$ with $0<y_1<y_2$, so the basin formed by this saddle must end at the infinite node $[0:1:0]$. Then we put the other two singularities on $(x_1,0)$ and $(x_2,0)$ with $0<x_1<x_2$. The saddle must be on $(x_2,0)$ by Berlinski. We plot also with doted lines the isocline on which $\dot{y}=0$. This is an ellipse and the remaining component $y=0$ of the isocline $\dot{x}=xy=0$ (see Figure 3). The eigenvectors of the saddle $(0,y_1)$ are $(1,0)$ and $(0,1)$, then the right separatrix of the saddle $(0,y_1)$ in the $(1,0)$ direction must enter inside the ellipse. But after entering, it must leave it again if we want it to arrive at $[0:1:0]$, and in order to leave the ellipse it must cross it again (at a point we may call p) with slope zero. Then a straight line passing through the point p and the saddle $(0,y_1)$ will have three contact points. So, the left separatrix of $(0,y_1)$ must go to the infinite singularity $[0:1:0]$ and the other to the anti-saddle at $(x_1,0)$ (remember that we are looking for a generic phase portrait). Then, since both anti-saddles already receive a separatrix from the saddle $(0,y_1)$, the saddle at $(x_2,0)$ cannot form a basin by itself since there is no anti-saddle inside the basin. So we have proved that if the

phase portrait is generic it must be $\mathbb{S}^2_{3,4}$ which is topologically equivalent to *Picture* $S_{IV}.4$ in Figure 1.

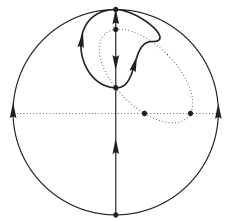

Figure 3: **Impossibility of a portrait with a basin.**

Since $\mathbb{S}^2_{3,4}$ is the only structurally stable possible phase portrait with these conditions, we cannot have a higher codimension phase portrait (with connections of separatrices) since a small perturbation that keeps the infinity untouched would produce a structurally stable phase portrait which we have proved it is not possible to be. So the lemma is proved.

□

2) *The possibility* $(\mathbf{R} \le 0) \vee (\mathbf{S} \le 0)$. It was mentioned earlier that in this case we have four complex singularities. According to [16, Diagram 1, page 3] we could have a single topological configuration (12) N, which leads to the *Picture* $S_{IV}.5$: [Ex: $(0,0,0,0,1/2,0)$, $(1,0,0,1,0,1)$]

4.5.1.1.2 The subcase $\mathbf{D} > 0$. According to [9, Table 6.2, page 124] systems (44) possess two real and two complex finite singularities. Considering the coordinates (46) of the singularities $M_{1,2}(0, y_{1,2})$ we observe that they are real if $f^2 - 4b > 0$ and they are complex if $f^2 - 4b < 0$.

On the other hand for systems (44) we have $\widetilde{U}_1 = -27(f^2 - 4b)h/8$ and $\mu_0 = -h$ and hence, $\operatorname{sign}(f^2 - 4b) = \operatorname{sign}(\mu_0 \widetilde{U}_1)$.

Remark 2. *Assume that for a quadratic system* (44) *the condition* $\mathbf{D}\mu_0 \neq 0$ *holds. Then the finite singularities located on the invariant line* $x = 0$ *of this system are real if* $\mu_0 \widetilde{U}_1 > 0$ *and they are complex if* $\mu_0 \widetilde{U}_1 < 0$.

1) *The possibility* $\mu_0 < 0$. Since $\eta = \widetilde{M} = 0$, by [16, Diagram 1, page 4] systems (44) could have either the topological configuration (16) a, a; S if $\neg((\mathfrak{C}_1) \vee (\widehat{\mathfrak{C}}_1))$, or (17) a, c; S if $(\mathfrak{C}_1)$, or (18) c, c; S if $(\widehat{\mathfrak{C}}_1)$.

We observe that by Lemma 4 the conditions $(\mathfrak{C}_1)$ from (8) are incompatible for systems (44). This means that the topological configuration (17) could not be realizable for these systems.

Since $\mu_0 < 0$, considering Remark 2 it is not difficult to detect that in the case of the configuration (16) a, a; S we get the *Picture* $S_{IV}.6$ if $\widetilde{U}_1 < 0$ and the *Picture* $S_{IV}.7$ if $\widetilde{U}_1 > 0$.

On the other hand the configuration (18) c, c; S leads to the *Picture* $S_{IV}.8$. exhibit three examples of realization of the pictures:

Picture $S_{IV}.6$: [Ex: $(0,0,0,0,1/2,0)$, $(-1,2,1,-2,0,1)$];

Picture $S_{IV}.7$: [Ex: $(0,0,0,0,1/2,0)$, $(2,0,-1,-1,0,1)$];

Picture $S_{IV}.8$: [Ex: $(0,0,0,0,1/2,0)$, $(1,0,0,-1,0,1)$].

2) *The possibility* $\mu_0 > 0$. Considering the condition $\eta = \widetilde{M} = 0$, by [16, Diagram 1, page 3] systems (44) could either have the topological configuration (23) $s, a;\ N$ if $\neg((\mathfrak{C}_2) \vee (\mathfrak{C}_7))$, or (24) $s, c;\ N$ $(\mathfrak{C}_2) \vee (\mathfrak{C}_7)$. However by Lemma 4 the conditions $(\mathfrak{C}_7)$ could not be satisfied.

Thus considering Remark 2 and the condition $\mu_0 > 0$ we deduce that the configuration (23) $s, a;\ N$ with the condition $\neg(\mathfrak{C}_2)$ leads to the *Picture* $S_{IV}.9$ if $\widetilde{U}_1 > 0$ and to the *Picture* (a) (see Figure 4) if $\widetilde{U}_1 < 0$. We observe that the last phase portrait is topologically equivalent to the *Picture* $S_{IV}.9$.

On the other hand the configuration (24) $c, c;\ S$ (with the conditions $(\mathfrak{C}_2)$) leads to the *Picture* $S_{IV}.10$. The realization of these phase portraits is proved by the next examples:

Picture $S_{IV}.9$: [Ex: $(0,0,0,0,1/2,0)$, $(1/8,0,1,1,0,1]$;

Picture (a), Fig.4 : [Ex: $(0,0,0,0,1/2,0)$, $(2,3,-1,1,0,1]$;

Picture $S_{IV}.10$: [Ex: $(0,0,0,0,1/2,0)$, $(1/2,2,0,1,0,1)$].

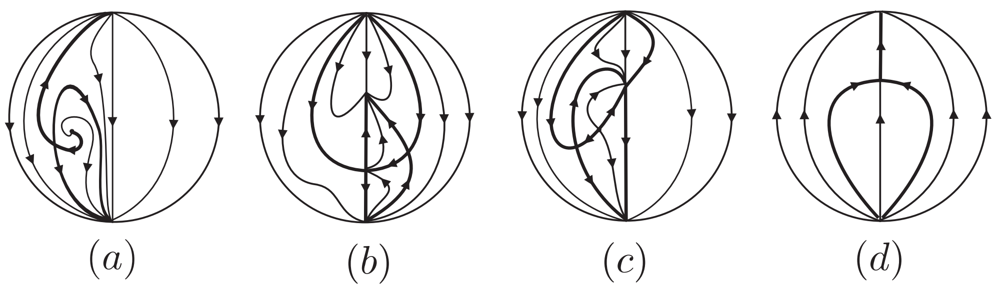

Figure 4: **Some phase portraits of quadratic systems with** $\eta = \widetilde{M} = 0$**,** $C_2 \neq 0$**.**

4.5.1.1.3 The subcase $\mathbf{D} = 0$. If $\mathbf{T} \neq 0$ then according to [9, Table 6.2, page 125] systems (44) possess one double real singular point and two distinct finite singularities. Moreover these two singular points are real if $\mathbf{T} < 0$ and complex $\mathbf{T} > 0$. In the case $\mathbf{T} = 0$ and $\mu_0 \neq 0$ by [9, Table 6.2] these systems possess at most two finite singularities of total multiplicity four.

Considering (45) we detect that the condition $\mathbf{D} = 0$ gives three possibilities: 1) $b = 0$; 2) $f^2 - 4b = 0$ and 3) $(e^2 + 4bh) = 0$. Taking into account the values of the invariant polynomials $\mathbf{D}$, B_2 and $\widetilde{U}_1$ from (45) and (47) it is easy to determine, that due to $\mu_0 \neq 0$ the three mentioned possibilities could be distinguished by means of these invariant polynomials. More precisely, considering also the coordinates (46) of the finite singularities $M_{1,2}(0, y_{1,2})$ and $M_{3,4}(x_{3,4}, 0)$ we have the next remark.

Remark 3. (i) *The following conditions are equivalent:*

1) $b = 0 \quad \Leftrightarrow \quad B_2 = 0$;
2) $f^2 - 4b = 0 \quad \Leftrightarrow \quad \widetilde{U}_1 = 0$;
3) $(e^2 + 4bh) = 0 \quad \Leftrightarrow \quad \mathbf{D} = 0,\ \ B_2\widetilde{U}_1 \neq 0$.

(ii) *In the case* $B_2 = 0$ *(respectively* $\widetilde{U}_1 = 0$*;* $\mathbf{D} = 0$, $B_2\widetilde{U}_1 \neq 0$*) the singular point* M_4 *coalesces with* M_1 *(respectively* M_2 *with* M_1*;* M_4 *with* M_3*).*

(*iii*) *the condition* $B_2 = 0$ *(i.e.* $b = 0$*) implies* $\mathbf{T} = -3e^2 f^2 x^2 y^2 (fhx - ey)^2 \leq 0$.

In what follows we consider three cases: $\mathbf{T} < 0$, $\mathbf{T} > 0$ and $\mathbf{T} = 0$.

1) *The possibility* $\mathbf{T} < 0$. Then all three finite singularities (one of them is double) are real and following [16, Diagram 1, page 4] we consider two cases: $\mu_0 < 0$ and $\mu_0 > 0$.

a) *The case* $\mu_0 < 0$. Since $\eta = \widetilde{M} = 0$, according to this diagram we could have either the topological configuration (30) a, a, sn; S if $E_1 \neq 0$ or (34) a, a, cp; S if $E_1 = 0$.

Considering Remark 3 (i) we examine three subcases: $B_2 \widetilde{U}_1 \neq 0$; $B_2 = 0$ and $\widetilde{U}_1 = 0$.

***a.*1)** *The subcase* $B_2 \widetilde{U}_1 \neq 0$. Then by Remark 3 ($i$) the condition $\mathbf{D} = 0$ yields $e^2 + 4bh = 0$ and we obtain $b = -e^2/(4h)$. In this case for systems (44) we calculate:

$$\mathbf{T} = -3e^2(e^2 + f^2 h) x^2 (ehx^2 + 2fhxy - ey^2)^2/(16h), \quad E_1 = -e^2 f(e^2 + f^2 h)/(8h) \tag{48}$$

and since $\mathbf{T} < 0$, the condition $E_1 = 0$ is equivalent to $f = 0$.

By Remark 3 (ii) we deduce that in this case the singularities located outside the invariant line coalesced. So in the case $E_1 \neq 0$ the configuration (30) a, a, sn; S leads to the phase portrait given by *Picture* S_{IV}.11 (see Figure 1).

If $E_1 = 0$ we have the topological configuration (34) a, a, cp; S which leads to the *Picture* S_{IV}.12.

The corresponding examples are:

Picture S_{IV}.11: [Ex: $(0, 0, 0, 0, 1/2, 0)$, $(-1, 2, -1, -1, 0, 1)$];

Picture S_{IV}.12: [Ex: $(0, 0, 0, 0, 1/2, 0)$, $(-1, -2, 0, -1, 0, 1)$].

***a.*2)** *The subcase* $B_2 = 0$. Then by Remark 3 we have $b = 0$ and in this case we obtain:

$$\mathbf{T} = -3e^2 f^2 x^2 y^2 (fhx - ey)^2, \quad E_1 = -e^2 f^3/2 \tag{49}$$

and evidently the condition $\mathbf{T} \neq 0$ implies $E_1 \neq 0$. So in this case we could have only the configuration (30) a, a, sn; S. Taking into account Remark 3 (ii) we arrive at the *Picture* S_{IV}.13: [Ex: $(0, 0, 0, 0, 1/2, 0)$, $(0, -1, -1, -1, 0, 1)$].

***a.*3)** *The subcase* $\widetilde{U}_1 = 0$. By Remark 3 we have $b = f^2/4$ and in this case we obtain:

$$\mathbf{T} = -3f^2(e^2 + f^2 h) y^2 (-fhx^2 + 2exy + fy^2)^2/16, \quad E_1 = f^3(e^2 + f^2 h)/16. \tag{50}$$

Clearly that the condition $\mathbf{T} \neq 0$ implies $E_1 \neq 0$ and again we could have only the configuration (30) a, a, sn; S. In this case according to Remark 3 (ii) in this case the singularities located on the invariant line coalesced. Therefore we arrive at the *Picture* S_{IV}.14: [Ex: $(0, 0, 0, 0, 1/2, 0)$, $(1/4, 1, -1, -1, 0, 1)$].

b) *The case* $\mu_0 > 0$. In this case by [16, Diagram 1, page 4] we could have either the topological configuration (37) s, a, sn; N if $E_1 \neq 0$, or (40) s, a, cp; N if $E_1 = 0$ and $\neg(\mathfrak{C}_8)$, or (41) s, c, cp; N if $E_1 = 0$ and $(\mathfrak{C}_8)$. However by Lemma 4 the conditions $(\mathfrak{C}_8)$ are incompatible for systems (44). So it remains to examine the phase portraits given by the topological configurations (37) and (40).

***b*.1)** *The subcase* $B_2\widetilde{U}_1 \neq 0$. Then by Remark 3 (i) we have $e^2 + 4bh = 0$, i.e. $b = -e^2/(4h)$ and we calculate:

$$\mathbf{T} = -3e^2(e^2 + f^2h)x^2(ehx^2 + 2fhxy - ey^2)^2/(16h), \quad E_1 = -e^2 f(e^2 + f^2h)/(8h)$$

and since $\mathbf{T} < 0$, the condition $E_1 = 0$ is equivalent to $f = 0$. However for $f = 0$ we get $\mathbf{T} = -3e^6x^2(hx^2 - y^2)^2/(16h)$ and therefore the condition $\mathbf{T} < 0$ implies $h > 0$ and this contradicts $\mu_0 = -h > 0$.

Thus in the case $B_2\widetilde{U}_1 \neq 0$ we could only have the configuration (37) s, a, sn; N and considering Remark 3 (ii) there are two singularities (saddle and node) on the invariant line $x = 0$ and a saddle-node outside. Since the triple infinite node behaves as a simple one, the possible generic phase portraits that may appear in these systems under the current conditions must be topologically equivalent to one of the 9 codimension 1 structurally unstable phase portraits ranging from $\mathbb{U}^1_{A,2}$ to $\mathbb{U}^1_{A,10}$ from [6]. We do not plot them to save space. Note that they are simply the five phase portraits given in Figure 2 where one anti-saddle had coalesced with a saddle. However if any one of them cpold be then by a small perturbation that leaves the infinity untouched, we could produce the respective structurally stable phase portrait. Since we have proved that only $\mathbb{S}^2_{3,4}$ is realizable, then the only codimension 1 realizable phase portrait for these systems is $\mathbb{U}^1_{A,7}$ which is topologically equivalent to *Picture* S_{IV}.15.

For the same reason, as we proved that $\mathbb{S}^2_{3,4}$ was unique, here we cannot have other phase portraits with separatrix connections and hence $\mathbb{U}^1_{A,7}$ is also unique. As an example of *Picture* S_{IV}.15 we may take [Ex: $(0,0,0,0,1/2,0)$, $(1/8,1,-1,2,0,1)$]. Moreover, the next two cases must also be topologically equivalent to *Picture* S_{IV}.15 with the difference of the relative position of the saddle-node with respect to the the invariant straight line.

***b*.2)** *The subcase* $B_2 = 0$. According to (49) in this case the condition $\mathbf{T} \neq 0$ implies $E_1 \neq 0$ and we could only have the configuration (37) s, a, sn; N. Taking into account Remark 3 (ii) we arrive at the *Picture* (b), Fig.4: [Ex: $(0,0,0,0,1/2,0)$, $(0,-1,-1,1,0,1)$].

***b*.3)** *The subcase* $\widetilde{U}_1 = 0$. Considering (50) we conclude that the condition $\mathbf{T} \neq 0$ implies $E_1 \neq 0$ and again systems (44) could have only the configuration (37) s, a, sn; N. In this case taking into account Remark 3 (ii) we get the *Picture* (c), Figure 4: [Ex: $(0,0,0,0,1/2,0)$, $(1/4,1,-1,1/2,0,1)$].

We remark that the phase portraits *Picture* (b) and *Picture* (c) from Figure 4 are topologically equivalent to the *Picture* S_{IV}.15.

2) *The possibility* $\mathbf{T} > 0$. According to [16, Diagram 1, page 4] systems (44) possess one real (double) and two complex singularities. Moreover in this case we could have either the topological configuration (44) sn; N if $E_1 \neq 0$ or (47) cp; N if $E_1 = 0$.

According to Remark 3 (iii) the condition $B_2 = 0$ implies $\mathbf{T} < 0$ and therefore we examine two cases: $\widetilde{U}_1 \neq 0$ and $\widetilde{U}_1 = 0$.

***a*)** *The case* $\widetilde{U}_1 \neq 0$. In this case the condition $\mathbf{D} = 0$ gives $b = -e^2/(4h)$ and we obtain the values of $\mathbf{T}$ and E_1 given in (48). Clearly the condition $\mathbf{T} > 0$ implies $e^2h(e^2 + f^2h) < 0$ and then the condition $E_1 = 0$ is equivalent to $f = 0$.

So in the case $E_1 \neq 0$ we get the *Picture* $S_{IV}.16$: [Ex: $(0,0,0,0,1/2,0)$, $(1,-2,-1,1,0,1)$].

If $E_1 = 0$ we have a cusp and this leads to the phase portrait given by *Picture* $S_{IV}.17$: [Ex: $(0,0,0,0,1/2,0)$, $(1,-2,0,1,0,1)$].

b) *The case* $\widetilde{U}_1 = 0$. Then $b = f^2/4$ and in this case we obtain the values of $\mathbf{T}$ and E_1 given in (50). Evidently the condition $\mathbf{T} > 0$ implies $E_1 \neq 0$ and we could have only the configuration (44) sn; N. Then we obtain a phase portrait topologically equivalent to the *Picture* $S_{IV}.16$: [Ex: $(0,0,0,0,1/2,0)$, $(1/4,0,-1,1,0,1)$].

2) *The possibility* $\mathbf{T} = 0$. Since $\mathbf{D} = 0$, according to [16, Diagram 1, page 5] we consider two cases: $\mathbf{P} \neq 0$ and $\mathbf{P} = 0$.

a) *The case* $\mathbf{P} \neq 0$. Then by [9, Table 6.2] systems (44) possess two double finite singularities, which are real if $\mathbf{PR} > 0$ and complex if $\mathbf{PR} < 0$. However we have the next lemma.

Lemma 6. *The conditions* $\mathbf{D} = 0 = \mathbf{T}$ *and* $\mathbf{P} \neq 0$ *imply for a system* (44) $\mathbf{PR} > 0$ *and* $B_2 \neq 0$.

Proof: Suppose first that the condition $B_2 = 0$. Then $b = 0$ and for systems (44) we have:

$$\mathbf{D} = 0, \quad \mathbf{T} = -3e^2 f^2 x^2 y^2 (fhx - ey)^2, \quad \mathbf{P} = e^2 f^2 x^2 y^2,$$

and clearly the condition $\mathbf{P} \neq 0$ implies $\mathbf{T} \neq 0$, i.e. we get a contradiction.

So $B_2 \neq 0$ and then the condition $\mathbf{D} = -48b^2(f^2 - 4b)(e^2 + 4bh) = 0$ gives $(f^2 - 4b)(e^2 + 4bh) = 0$. We claim that the condition $\mathbf{D} = 0 = \mathbf{T}$ implies $f^2 - 4b = e^2 + 4bh = 0$.

Indeed, assuming $b = f^2/4$ we obtain:

$$\mathbf{D} = 0, \quad \mathbf{T} = -3(e^2 + f^2 h) y^2 \mathbf{P}, \quad \mathbf{P} = f^2 (fhx^2 - 2exy - fy^2)^2/16$$

and therefore the conditions $\mathbf{T} = 0$ and $\mathbf{P} \neq 0$ imply $(e^2 + f^2 h) = 0$ and $f \neq 0$. So we have $h = -e^2/f^2$ and we get $e^2 + 4bh = 0$ and this proves our claim.

On the other hand for $b = f^2/4$ and $h = -e^2/f^2$ calculations yield

$$\mathbf{D} = \mathbf{T} = 0, \quad \mathbf{P} = (ex + fy)^4/16, \quad \mathbf{R} = e^2(ex + fy)^2/f^2, \quad \mathcal{T}_4 = e^2 f^2/4 \neq 0$$

and we observe that $\mathbf{PR} > 0$ and this completes the proof of the lemma. ∎

Considering the conditions $\mathbf{D} = \mathbf{T} = 0$, $\mathbf{PR} > 0$ and $\mathcal{T}_4 \neq 0$, according to [16, Diagram 1, page 5] we arrive at the unique topological configuration (50) sn, sn; N. According to Remark 3 (*ii*) we have one saddle-none on the invariant line and another outside. As a result we arrive at the *Picture* $S_{IV}.18$: [Ex: $(0,0,0,0,1/2,0)$, $(1/4,0,-1,1,0,1)$]. There are other topologically different phase portraits with two finite saddle-nodes and one infinite node as it is pointed out in [19] but in this case, the existence of the invariant straight line, or simply continuity arguments from the cases already studied s, a, sn; N and s, s, a, a; N, give that there is only one possible phase portrait in these conditions.

b) *The case* $\mathbf{P} = 0$. We prove the following lemma:

Lemma 7. *Assume that for a system* (44) *the condition* $\mathbf{D} = \mathbf{T} = \mathbf{P} = 0$ *holds. Then the following conditions are equivalent:*

$$\begin{array}{lcl} \mathbf{R} \neq 0,\ \mathcal{T}_4 \neq 0 & \Leftrightarrow & b = e = 0,\ f \neq 0; \\ \mathbf{R} \neq 0,\ \mathcal{T}_4 = 0 & \Leftrightarrow & b = f = 0,\ e \neq 0; \\ \mathbf{R} = 0 & \Leftrightarrow & b = e = f = 0. \end{array}$$

Proof: The condition $\mathbf{D} = 0$ yields $b(f^2 - 4b)(e^2 + 4bh) = 0$ and we consider all three cases given by this relation.

(i) If $b = 0$ then for systems (44) we have:

$$\mathbf{T} = -3e^2 f^2 x^2 y^2 (fhx - ey)^2, \quad \mathbf{P} = e^2 f^2 x^2 y^2, \quad \mathcal{T}_4 = 2f^4 h.$$

It is clear that the condition $\mathbf{T} = \mathbf{P} = 0$ implies $ef = 0$ and therefore $\mathbf{R} = 3f^2h^2x^2 + 3e^2y^2$. We observe that the condition $f = 0$ is equivalent to $\mathcal{T}_4 = 0$.

Thus in the case $\mathbf{R} \neq 0$ we have either $b = e = 0$ and $f \neq 0$ if $\mathcal{T}_4 \neq 0$, or $b = f = 0$ and $e \neq 0$ if $\mathcal{T}_4 = 0$.

(ii) Assuming $b = f^2/4$ we obtain:

$$\mathbf{T} = -3(e^2 + f^2 h) y^2 \mathbf{P}, \quad \mathbf{P} = f^2 (fhx^2 - 2exy - fy^2)^2/16, \quad \mathcal{T}_4 = -f^4 h/4$$

and clearly the condition $\mathbf{T} = \mathbf{P} = 0$ yields $f = 0$ and we have $\mathcal{T}_4 = 0$ and $\mathbf{R} = 3e^2y^2$. So if $\mathbf{R} \neq 0$ we get the conditions $b = f = 0$ and $e \neq 0$.

(iii) Suppose now that the condition $b = -e^2/(4h)$ holds. Then we calculate:

$$\mathbf{T} = -3h(e^2 + f^2 h) x^2 \mathbf{P}, \quad \mathbf{P} = e^2 (ehx^2 + 2fhxy - ey^2)^2/(16h^2), \quad \mathcal{T}_4 = f^2(9e^2 + 8f^2 h)/4$$

and evidently the condition $\mathbf{T} = \mathbf{P} = 0$ gives $e = 0$ and in this case we obtain $\mathbf{R} = 3f^2h^2x^2$ and $\mathcal{T}_4 = 2f^4 h$. Therefore the condition $\mathbf{R} \neq 0$ implies $\mathcal{T}_4 \neq 0$ and in this case we have $b = e = 0$ and $f \neq 0$.

It remains to observe that in all three cases **(i)**, **(ii)** and **(iii)** the condition $\mathbf{T} = \mathbf{P} = \mathbf{R} = 0$ gives $b = e = f = 0$ and this completes the proof of the lemma. ∎

In what follows we consider each one of the subcases provided by Lemma 7.

***b*.1)** *The subcase* $\mathbf{R} \neq 0$, $\mathcal{T}_4 \neq 0$. By Lemma 7 we have $b = e = 0$ and considering [16, Diagram 1, page 6] we calculate:

$$E_3 = -f^2 h/4, \quad \mathcal{T}_4 = 2f^4 h, \quad \mu_0 = -h.$$

If $\mu_0 < 0$ then $h > 0$ and this implies $E_3 < 0$. Then by [16, Diagram 1, page 6] we arrive at the configurations (16) a, a; S. This leads to the *Picture* $S_{IV}.6$: [Ex: $(0,0,0,0,1/2,0)$, $(0,0,1,-1,0,1]$.

Assuming $\mu_0 > 0$ we obtain $E_3 > 0$ and by the same Diagram 1 from paper [16] we obtain either the configuration (23) s, a; N if $\neg(\mathfrak{C}_{10})$ or (24) s, c; N if $(\mathfrak{C}_{10})$. However by Lemma 4 the conditions $(\mathfrak{C}_{10})$ are not compatible for systems (44).

On the other hand the configuration (23) s, a; N leads to the phase portrait which is equivalent to *Picture* $S_{IV}.9$: [Ex: $(0,0,0,0,1/2,0)$, $(0,0,1,1,0,1]$.

b.2) *The subcase* $\mathbf{R} \neq 0$, $\mathcal{T}_4 = 0$. By Lemma 7 we have $b = f = 0$ (this implies $\mathcal{T}_3 = 0$) and $e \neq 0$. Therefore we obtain $E_3 = -e^2/4 < 0$ and following [16, Diagram 1, page 6] we need to distinguish two cases: $\mu_0 < 0$ and $\mu_0 > 0$.

If $\mu_0 < 0$ then by [16, Diagram 1, page 5] we get either the configuration (59) a, es; N if $\neg(\mathfrak{C}_3)$ or (60) c, es; N if $(\mathfrak{C}_3)$. Considering the conditions $(\mathfrak{C}_3)$ from (8) in the case $b = e = 0$ we obtain:

$$\mathcal{T}_4 = \mathcal{T}_3 = \mathcal{T}_2 = \mathcal{T}_1 = 0,\ \sigma = 3y \neq 0,\ \mathcal{F} = \mathcal{F}_1 = 0,\ \mathcal{H} = -9h/2,\ \mathcal{B} = -81e^2/8 < 0.$$

Since $\mu_0 < 0$ (i.e. $h > 0$) we have $\mathcal{H} < 0$ and we deduce that the conditions $(\mathfrak{C}_3)$ are satisfied in the considered case. So we could have only the configuration (60) c, es; N which leads to the *Picture* $S_{IV}.19$: [Ex: $(0,0,0,0,1/2,0)$, $(0,1,0,-1,0,1]$.

Assume now $\mu_0 > 0$. Since $E_3 < 0$ and $\mathcal{T}_4 = \mathcal{T}_3 = 0$ by [16, Diagram 1, page 5] we have the unique configuration (64) s, es; N which leads to the *Picture* $S_{IV}.20$: [Ex: $(0,0,0,0,1/2,0)$, $(0,1,0,1,0,1]$

b.3) *The subcase* $\mathbf{R} = 0$. By Lemma 7 we have $b = e = f =$ and this leads to the homogeneous quadratic systems (44)

$$\dot{x} = xy, \quad \dot{y} = -hx^2 + y^2.$$

In this case by [16, Diagram 1, page 6] we could have either the topological configuration (67) ee; S if $\mu_0 < 0$, or (44) sn; N if $\mu_0 > 0$.

In the first case we arrive at the *Picture* $S_{IV}.21$: [Ex: $(0,0,0,0,1/2,0)$, $(1/4,1,-1,1,0,1)$].

The configuration (44) sn; N leads to a phase portrait topologically equivalent with *Picture* $S_{IV}.16$: [Ex: $(0,0,0,0,1/2,0)$, $(1/4,1,-1,1,0,1)$].

4.5.1.2 The case $\theta = \widetilde{N} = 0$, $B_2 \neq 0$.

Considering the systems (43) we shall examine the family:

$$\dot{x} = y, \quad \dot{y} = b + ex + fy + hx^2. \tag{51}$$

For these systems calculations yield:

$$\begin{gathered} C_2 = -hx^3,\ \eta = \widetilde{M} = 0,\ \mu_0 = \mu_1 = 0,\ \mu_2 = h^2x^2, \\ \mathbf{U} = h^2(e^2 - 4bh)x^4y^2,\ \kappa = \widetilde{K} = \tilde{L} = 0,\ \mathcal{T}_4 = \mathcal{B}_1 = 0,\ \sigma = f. \end{gathered} \tag{52}$$

Since $\mu_0 = \mu_1 = 0$ and the condition $C_2 \neq 0$ implies $\mu_2 \neq 0$, according to [9, Table 6.2] the above systems possess finite singularities of total multiplicity two. More exactly we have the singularities $M_{1,2}(x_{1,2}, 0)$, where

$$\big(-e \pm \sqrt{e^2 - 4bh}\big)/(2h), \quad \text{and} \quad \operatorname{sign}(e^2 - 4bh) = \operatorname{sign}(\mathbf{U}). \tag{53}$$

So we examine three subcases: $\mathbf{U} > 0$, $\mathbf{U} < 0$ and $\mathbf{U} = 0$.

4.5.1.2.1 The subcase $\mathbf{U} > 0$. Then considering (52) by [16, Diagram 3, page 9] we obtain either the configuration (23) $s, a;\ \binom{2}{3}\ P - P$ if $\neg(\mathfrak{C}_9)$ or (24) $s, c;\ \binom{2}{3}\ P - P$ if $(\mathfrak{C}_9)$.

On the other hand comparing the conditions $(\mathfrak{C}_9)$ from (8) with (52) we deduce that all the conditions are satisfied except $\sigma = 0$, because for systems (51) we have $\sigma = f$. So we deduce that in the considered case the conditions $(\mathfrak{C}_9)$ are satisfied if and only if $f = 0$.

Thus we obtain that the configuration (23) $s, a;\ \binom{2}{3}\ P - P$ leads to a phase portrait topologically equivalent with *Picture* $S_{IV}.9$: [Ex: $(0, 0, 1, 0, 0, 0)$, $(0, 1, -1, -1, 0, 0)$].

In a similar way we detect that the configuration (24) $s, c;\ \binom{2}{3}\ P - P$ leads to a phase portrait topologically equivalent with *Picture* $S_{IV}.10$: [Ex: $(0, 0, 1, 0, 0, 0)$, $(0, 1, 0, -1, 0, 0)$].

4.5.1.2.2 The subcase $\mathbf{U} < 0$. By [16, Diagram 3, page 9] we get the topological configuration (12) $\binom{2}{3}\ P - P$. This configuration leads to a phase portrait topologically equivalent with *Picture* $S_{IV}.5$: [Ex: $(0, 0, 1, 0, 0, 0)$, $(-1, 0, -1, -1, 0, 0)$].

4.5.1.2.3 The subcase $\mathbf{U} = 0$. In this case systems (51) possess a double singular point which could be a saddle-node or a cusp. But since the conditions $\kappa = \widetilde{K} = \tilde{L} = 0$, $\mathcal{T}_4 = \mathcal{B}_1 = 0$ hold, according to [16, Diagram 3, page 11] we arrive at the topological configuration (47) $cp;\ \binom{2}{3}\ P - P$. This configuration leads to a phase portrait topologically equivalent with *Picture* $S_{IV}.17$: [Ex: $(0, 0, 1, 0, 0, 0)$, $(-1, 2, 0, -1, 0, 0)$].

Since all the cases are examined Theorem 1 is proved. ∎

4.5.2 Systems $(\mathbf{S}_V)$: $C_2 = 0$

These systems have the infinite line filled with singularities and this family is considered in [70], where a total of 9 canonical forms of this family are presented: $C_2.1 - C_2.9$ (see Table 1, page 741).

Following [70] and considering our Main Theorem we arrive at the next result.

Theorem 2. *Assume that for a quadratic system the condition $C_2 = 0$ holds. Then this system belongs to the class $\boldsymbol{QS_{Abel}}$ if and only if the condition $H_{10}^2 + H_{12}^2 \neq 0$ is satisfied. Moreover its phase portrait is topologically equivalent to one of the pictures given in Figure 5 if and only the following corresponding conditions are verified:*

Picture $C_2.1 \Leftrightarrow H_{10} \neq 0,\ H_9 < 0;$
Picture $C_2.2(a) \Leftrightarrow H_{10} \neq 0,\ H_9 > 0,\ H_7 \neq 0;$
Picture $C_2.2(b) \Leftrightarrow H_{10} \neq 0,\ H_9 > 0,\ H_7 = 0;$
Picture $C_2.3 \Leftrightarrow H_{10} \neq 0,\ H_9 = 0,\ H_{12} \neq 0;$
Picture $C_2.4 \Leftrightarrow H_{10} \neq 0,\ H_9 = 0,\ H_{12} = 0;$
Picture $C_2.5(a) \Leftrightarrow H_{10} = 0,\ H_{12} \neq 0,\ H_{11} > 0,\ \mu_2 < 0;$
Picture $C_2.5(b) \Leftrightarrow H_{10} = 0,\ H_{12} \neq 0,\ H_{11} > 0,\ \mu_2 > 0;$
Picture $C_2.6 \Leftrightarrow H_{10} = 0,\ H_{12} \neq 0,\ H_{11} < 0;$
Picture $C_2.7 \Leftrightarrow H_{10} = 0,\ H_{12} \neq 0,\ H_{11} = 0.$

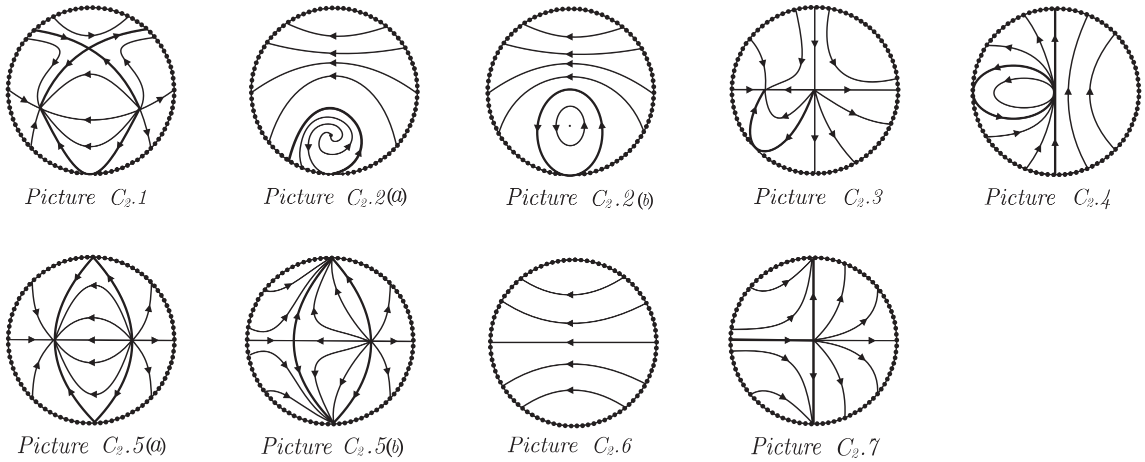

Figure 5: **Global phase portraits of quadratic systems with $C_2 = 0$.**

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