COVERING DIMENSION OF CUNTZ SEMIGROUPS II

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Abstract. We show that the dimension of the Cuntz semigroup of a C^*-algebra is determined by the dimensions of the Cuntz semigroups of its separable sub-C^*-algebras. This allows us to remove separability assumptions from previous results on the dimension of Cuntz semigroups.

To obtain these results, we introduce a notion of approximation for abstract Cuntz semigroups that is compatible with the approximation of a C^*-algebra by sub-C^*-algebras. We show that many properties for Cuntz semigroups are preserved by approximation and satisfy a Löwenheim-Skolem condition.

1. Introduction

The Cuntz semigroup of a C^*-algebra A encodes the comparison theory of positive elements in A and its stabilization in a partially ordered, abelian monoid $\mathrm{Cu}(A)$. This invariant was introduced by Cuntz [Cun78] in his pioneering work on simple C^*-algebras, and it continues to play an important role to this day. For example, it was used by Toms to distinguish his groundbreaking examples of non-isomorphic simple, nuclear C^*-algebras with the same K-theoretic data [Tom08], to classify algebras and morphisms in [Rob12], and it was a key feature in some recent breakthroughs in the structure theory of C^*-algebras [Thi20b, APRT18].

In [TV21], we introduced a notion of covering dimension (see Definition 2.8) for Cuntz semigroups and their abstract counterparts, the Cu-semigroups as introduced in [CEI08] and extensively studied in [APT18, APT20]. Among other results, we proved the expected permanence properties (recalled in Proposition 2.9), studied the relation between the dimension of $\mathrm{Cu}(A)$ and the nuclear dimension of A, and computed the dimension of Cuntz semigroups of simple, $\mathcal{Z}$-stable C^*-algebras.

The goal of this paper is to further develop the results from [TV21] and provide additional tools to compute the dimension of Cuntz semigroups and Cu-semigroups.

Our first main result is a new permanence property: the dimension of Cuntz semigroups behaves well with respect to approximation by sub-C^*-algebras. Here, we say that a C^*-algebra A is *approximated* by a collection of sub-C^*-algebras $A_\lambda \subseteq A$ if for every $a_1, \ldots, a_n \in A$ and $\varepsilon > 0$ there exist $\lambda \in \Lambda$ and $b_1, \ldots, b_n \in A_\lambda$ such that $\|b_j - a_j\| < \varepsilon$ for $j = 1, \ldots, n$. (This is stronger than requiring that $\bigcup_\lambda A_\lambda$ is dense in A. On the other hand, the subalgebras are not required to be nested.) For example, a C^*-algebra is *locally finite-dimensional* (sometimes called *locally AF*) if and only if it is approximated by a family of finite-dimensional sub-C^*-algebras.

Theorem A (3.8)**.** *Let A be a C^*-algebra that is approximated by a family of sub-C^*-algebras $A_\lambda \subseteq A$. Then* $\dim(\mathrm{Cu}(A)) \leq \sup_\lambda \dim(\mathrm{Cu}(A_\lambda))$.

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To prove this result, we introduce a notion of approximation for Cu-semigroups (see Definition 3.1) and we show that if a C^*-algebra A is approximated by a family of sub-C^*-algebras $A_\lambda \subseteq A$, then $\mathrm{Cu}(A)$ is approximated by the corresponding Cuntz semigroups $\mathrm{Cu}(A_\lambda)$; see Proposition 3.7. For every fixed n, we prove that the property of having dimension at most n is preserved by approximations (see Proposition 3.4), which then gives Theorem A above.

Our second main result shows that the dimension of Cuntz semigroups satisfies the Löwenheim-Skolem condition.

Theorem B (6.2)**.** *Let A be a C^*-algebra. Then, for every separable sub-C^*-algebra $B_0 \subseteq A$ there exists a separable sub-C^*-algebra $B \subseteq A$ such that $B_0 \subseteq B$ and $\dim(\mathrm{Cu}(B)) \leq \dim(\mathrm{Cu}(A))$.*

By combining Theorems A and B, we obtain the following characterization of the dimension of the Cuntz semigroup of a C^*-algebra in terms of its separable sub-C^*-algebras:

Corollary C (6.3)**.** *Let A be a C^*-algebra, and let $n \in \mathbb{N}$. Then $\dim(\mathrm{Cu}(A)) \leq n$ if and only if every finite (or countable) subset of A is contained in a separable sub-C^*-algebra $B \subseteq A$ satisfying $\dim(\mathrm{Cu}(B)) \leq n$.*

The permanence properties from [TV21] together with Theorems A and B show that associating to each C^*-algebra the dimension of its Cuntz semigroup is a well-behaved invariant that satisfies all but one property of a noncommutative dimension theory; see Paragraph 6.4. The failing property is the compatibility with minimal unitizations; see Example 6.5. It remains open if associating to each C^*-algebra the dimension of the Cuntz semigroup of its minimal unitization is well-behaved; see Question 6.6.

To prove Theorem B, we show that every C^*-algebra A admits a large collection of separable sub-C^*-algebras $B \subseteq A$ such that the induced map $\mathrm{Cu}(B) \to \mathrm{Cu}(A)$ is an order-embedding (see Proposition 6.1) and we prove the Löwenheim-Skolem condition for the dimension of Cu-semigroups: Given a Cu-semigroup S and a countably based sub-Cu-semigroup $T_0 \subseteq S$, there exists a countably based sub-Cu-semigroup $T \subseteq S$ such that $T_0 \subseteq T$ and $\dim(T) \leq \dim(S)$; see Lemma 5.5.

In Section 4 we investigate when a submonoid T of a Cu-semigroup S is a sub-Cu-semigroup. In analogy to topological derived sets and Cantor-Bendixson derivatives, we introduce the associated Cu-semigroups T' and $\delta(T)$ (see Definitions 4.8 and 4.13). In particular, we show that $T \subseteq S$ is a sub-Cu-semigroup if and only if $T = T'$; see Proposition 4.11. Further, using the sub-Cu-semigroup $\delta(T)$, we prove that the sub-Cu-semigroups of a Cu-semigroup form a complete lattice when ordered by inclusion; see Theorem 4.15.

Our results also provide a characterization of the dimension of a Cu-semigroup through its countably based sub-Cu-semigroups:

Theorem D (5.7)**.** *Let S be a* Cu*-semigroup, and let $n \in \mathbb{N}$. Then $\dim(S) \leq n$ if and only if every finite (or countable) subset of S is contained in a countably based sub*-Cu-*semigroup $T \subseteq S$ satisfying $\dim(T) \leq n$.*

As an application of Theorem D, we generalize some results from [TV21] by removing the assumption of countable basedness; see Propositions 5.8 and 5.9.

2. Preliminaries

In the next paragraphs, we briefly recall the definition of (abstract) Cuntz semigroups. We refer to [APT11] and [APT18] for details.

2.1. *The Cuntz semigroup.* Given a C^*-algebra A, we use A_+ to denote the set of its positive elements. For $a, b \in A_+$, one says that a is *Cuntz subequivalent* to b, in symbols $a \precsim b$, if $a = \lim_n r_n b r_n^*$ for some sequence $(r_n)_n$ in A. One also writes $a \sim b$, and says that a is *Cuntz equivalent* to b, if $a \precsim b$ and $b \precsim a$.

The *Cuntz semigroup* of A is the set of equivalence classes $\mathrm{Cu}(A) := (A \otimes \mathcal{K})_+ / \sim$, where $A \otimes \mathcal{K}$ denotes the stabilization of A. One endows $\mathrm{Cu}(A)$ with the partial order induced by $\precsim$. Further, addition of orthogonal elements in $(A \otimes \mathcal{K})_+$ induces an abelian monoid structure on $\mathrm{Cu}(A)$. This turns $\mathrm{Cu}(A)$ into a positively ordered monoid, that is, every element $x \in \mathrm{Cu}(A)$ satisfies $0 \leq x$, and if $x, y, z \in \mathrm{Cu}(A)$ satisfy $x \leq y$, then $x + z \leq y + z$.

Given $a \in A \otimes \mathcal{K}$, we denote its class in $\mathrm{Cu}(A)$ by $[a]$.

2.2. *Abstract Cuntz semigroups.* In [CEI08] it was shown that, beyond being a positively ordered monoid, the Cuntz semigroup of a C^*-algebra always satisfies four additional properties. To formulate them, we need to recall the *way-below relation*: An element x in a partially ordered set is said to be *way-below* (or *compactly contained* in) y, denoted by $x \ll y$, if for every increasing sequence $(z_n)_n$ that has a supremum z satisfying $y \leq z$ there exists $n \in \mathbb{N}$ such that $x \leq z_n$.

The properties introduced in [CEI08], and that the Cuntz semigroup of a C^*-algebra always satisfies, are:

- (O1) Every increasing sequence has a supremum.
- (O2) Every element is the supremum of a $\ll$-increasing sequence.
- (O3) Given $x' \ll x$ and $y' \ll y$, we have $x' + y' \ll x + y$.
- (O4) Given increasing sequences $(x_n)_n$ and $(y_n)_n$, we have $\sup_n x_n + \sup_n y_n = \sup_n (x_n + y_n)$.

Moreover, it was also proved in [CEI08] that every *-homomorphism $\varphi\colon A \to B$ between two C^*-algebras A and B induces an order-preserving monoid morphism $\mathrm{Cu}(A) \to \mathrm{Cu}(B)$ that also preserves suprema of increasing sequences and the way-below relation.

It follows that the Cuntz semigroup defines a functor from the category of C^*-algebras and *-homomorphisms to the category Cu of Cu-semigroups and Cu-morphisms, which are defined as follows: A Cu-*semigroup* (also called *abstract Cuntz semigroup*) is a positively ordered monoid satisfying (O1)-(O4). A Cu-*morphism* between Cu-semigroups S and T is an order-preserving monoid morphism $S \to T$ that preserves suprema of increasing sequences and the way-below relation.

2.3. *Additional properties.* In addition to (O1)-(O4), the following properties are known to be satisfied by the Cuntz semigroup of every C^*-algebra (see [APT18, Proposition 4.6], [Rob13] and [APRT19, Proposition 2.2] respectively):

- (O5) Given $x+y \leq z$, $x' \ll x$ and $y' \ll y$, there exists c such that $x'+c \leq z \leq x+c$ and $y' \ll c$.
- (O6) Given $x' \ll x \leq y+z$ there exist $v \leq x, y$ and $w \leq x, z$ such that $x' \leq v+w$.
- (O7) Given $x_1' \ll x_1 \leq w$ and $x_2' \ll x_2 \leq w$ there exists x such that $x_1', x_2' \ll x \leq w, x_1 + x_2$.

It is common to use (O5) when $y = 0$, that is, for $x' \ll x \leq z$. In this case, (O5) implies that there exists an element c such that $x' + c \leq z \leq x + c$.

A Cuntz semigroup is said to be *weakly cancellative* if, whenever $x+z \ll y+z$, we have $x \ll y$. It was shown in [RW10, Theorem 4.3] that stable rank one C^*-algebras have weakly cancellative Cuntz semigroups.

The following result contains a characterization of (O5) that will be used in Proposition 3.3 to show that (O5) is preserved by approximation of Cu-semigroups, and in Proposition 5.3 to show that it satisfies the Löwenheim-Skolem condition.

Analogous characterizations of (O6) and (O7) are shown in Propositions 2.5 and 2.6 below.

Recall that a subset B of a Cu-semigroup S is said to be a *basis* if for every $x', x \in S$ satisfying $x' \ll x$ there exists $y \in B$ such that $x' \ll y \ll x$. A Cu-semigroup is said to be *countably based* if it contains a countable basis.

Cuntz semigroups of separable C^*-algebras are countably based (see, for example, [APS11, Lemma 1.3]).

Proposition 2.4. *Let S be a* Cu*-semigroup. Then S satisfies (O5) if and only if there exists a basis $B \subseteq S$ with (equivalently, every basis $B \subseteq S$ has) the following property: for all $x', x, y', y, z', z \in B$ satisfying*

$$x + y \ll z', \quad x' \ll x, \quad y' \ll y, \quad z' \ll z,$$

there exists $c \in B$ such that

$$x' + c \ll z, \quad z' \ll x + c, \quad \textit{and} \quad y' \ll c.$$

Proof. First, assume that S satisfies (O5), and let $B \subseteq S$ be a basis. To verify that B has the stated property, let $x', x, y', y, z', z \in B$ satisfy $x + y \ll z'$, $x' \ll x$, $y' \ll y$ and $z' \ll z$. Choose $z'' \in S$ satisfying $z' \ll z'' \ll z$. Applying (O5) for $x + y \leq z''$, $x' \ll x$ and $y' \ll y$, we obtain $a \in S$ such that

$$x' + a \leq z'' \leq x + a, \quad \text{and} \quad y' \ll a.$$

Using that $z' \ll z''$ and $y' \ll a$, choose $a' \in S$ such that

$$z' \ll x + a', \quad \text{and} \quad y' \ll a' \ll a.$$

Since B is a basis, we obtain $c \in B$ with $a' \ll c \ll a$. Then c has the desired properties.

Next, assume that $B \subseteq S$ is a basis with the stated property. The proof is similar to that of [APT18, Theorem 4.4(1)]. To verify that S satisfies (O5), let x', x, y', y, z be elements in S such that

$$x + y \leq z, \quad x' \ll x, \quad \text{and} \quad y' \ll y.$$

Since B is a basis, we can choose a $\ll$-decreasing sequence $(x_n)_n$ in B such that

$$x' \ll \ldots \ll x_{n+1} \ll x_n \ll \ldots \ll x_1 \ll x_0 \ll x.$$

Further, we can choose $c_0', c_0 \in B$ with

$$y' \ll c_0' \ll c_0 \ll y.$$

Using that $x_0 + c_0 \ll z$, we can take a $\ll$-increasing sequence $(z_n)_n$ in B with supremum z, and such that $x_0 + c_0 \ll z_0$.

We have

$$x_0 + c_0 \ll z_0, \quad x_1 \ll x_0, \quad c_0' \ll c_0, \quad \text{and} \quad z_0 \ll z_1.$$

By assumption, we obtain $c_1 \in B$ such that

$$x_1 + c_1 \ll z_1, \quad z_0 \ll x_0 + c_1, \quad \text{and} \quad c_0' \ll c_1.$$

Choose $c_1' \in B$ such that $z_0 \ll x_0 + c_1'$ and $c_0' \ll c_1' \ll c_1$.

Then

$$x_1 + c_1 \ll z_1, \quad x_2 \ll x_1, \quad c_1' \ll c_1, \quad \text{and} \quad z_1 \ll z_2.$$

By assumption, we obtain $c_2 \in B$ such that

$$x_2 + c_2 \ll z_2, \quad z_1 \ll x_1 + c_2, \quad \text{and} \quad c_1' \ll c_2.$$

Choose $c_2' \in B$ such that $z_1 \ll x_1 + c_2'$ and $c_1' \ll c_2' \ll c_2$.

Proceeding in this manner inductively, we obtain a $\ll$-increasing sequence $(c'_n)_n$ such that

$$x' + c'_n \leq x_n + c_n \ll z_n \leq z, \quad \text{and} \quad z_n \ll x_n + c'_{n+1} \leq x + c'_{n+1}$$

for each n. Therefore, the supremum $c := \sup_n c'_n$ satisfies $x' + c \leq z \leq x + c$, as desired. $\square$

Proposition 2.5. *Let S be a* Cu*-semigroup. Then S satisfies (O6) if and only if there exists a basis $B \subseteq S$ with (equivalently, every basis $B \subseteq S$ has) the following property: for all $x', x, y', y, z', z \in B$ satisfying*

$$x \ll y' + z', \quad x' \ll x, \quad y' \ll y, \quad z' \ll z,$$

there exist $v, w \in B$ such that

$$x' \ll v + w, \quad v \ll x, y, \quad \textit{and} \quad w \ll x, z.$$

Proof. Assuming that S satisfies (O6), one can use the same methods as in the proof of Proposition 2.4 to see that every basis $B \subseteq S$ satisfies the desired condition.

Next, assume that $B \subseteq S$ is a basis with the stated property. To verify that S satisfies (O6), let $x', x, y, z \in S$ satisfy

$$x' \ll x \leq y + z.$$

Using that B is a basis, choose $a', a \in B$ such that $x' \ll a' \ll a \ll x$. Thus, one has $a \ll y + z$, and we can choose $b', b, c', c \in B$ satisfying

$$a \ll b' + c', \quad b' \ll b \ll y, \quad \text{and} \quad c' \ll c \ll z.$$

By assumption, we obtain $v, w \in B$ such that

$$x' \ll a' \ll v + w, \quad v \ll a, b, \quad \text{and} \quad w \ll a, c.$$

Since $a \ll x$, $b \ll y$ and $c \ll z$, the elements v and w have the desired properties. $\square$

The next results are proved with the same methods as Proposition 2.5. We omit the proofs.

Proposition 2.6. *Let S be a* Cu*-semigroup. Then S satisfies (O7) if and only if there exists a basis $B \subseteq S$ with (equivalently, every basis $B \subseteq S$ has) the following property: for all $x'_1, x_1, x'_2, x_2, w', w \in B$ satisfying*

$$x'_1 \ll x_1 \ll w', \quad x'_2 \ll x_2 \ll w', \quad \textit{and} \quad w' \ll w,$$

there exists $x \in B$ such that

$$x'_1, x'_2 \ll x \ll w, x_1 + x_2.$$

Proposition 2.7. *A* Cu*-semigroup S is weakly cancellative if and only if there exists a basis $B \subseteq S$ with (equivalently, every basis $B \subseteq S$ has) the following property: for all $x', x, y', y, z', z \in B$ satisfying $x' \ll x$, $y' \ll y$ and $z' \ll z$ with $x + z \ll y' + z'$, we have $x' \ll y$.*

We recall the definition of (covering) dimension for Cu-semigroups from [TV21, Defintion 3.1]:

Definition 2.8. Let S be a Cu-semigroup. Given $n \in \mathbb{N}$, we write $\dim(S) \leq n$ if, whenever $x' \ll x \ll y_1 + \ldots + y_r$ in S, then there exist $z_{j,k} \in S$ for $j = 1, \ldots, r$ and $k = 0, \ldots, n$ such that:

(i) $z_{j,k} \ll y_j$ for each j and k;
(ii) $x' \ll \sum_{j,k} z_{j,k}$;
(iii) $\sum_{j=1}^{r} z_{j,k} \ll x$ for each $k = 0, \ldots, n$.

We set $\dim(S) = \infty$ if there exists no $n \in \mathbb{N}$ with $\dim(S) \leq n$. Otherwise, we let $\dim(S)$ be the smallest $n \in \mathbb{N}$ such that $\dim(S) \leq n$. We call $\dim(S)$ the *(covering) dimension* of S.

The following result summarizes the permanence properties shown in [TV21].

Proposition 2.9 ([TV21, Propositions 3.5, 3.7, 3.9])**.** *Given a* Cu*-semigroup S and an ideal $I \subseteq S$, we have:*

$$\dim(I) \leq \dim(S), \quad \textit{and} \quad \dim(S/I) \leq \dim(S).$$

Given Cu*-semigroups S and T, we have:*

$$\dim(S \oplus T) = \max\{\dim(S), \dim(T)\}.$$

Given an inductive limit of Cu*-semigroups $S = \varinjlim_{\lambda \in \Lambda} S_\lambda$, we have*

$$\dim(S) \leq \liminf_{\lambda} \dim(S_\lambda).$$

3. Approximation of Cu-semigroups and C^*-algebras

In this section, we introduce a notion of *approximation* for a Cu-semigroup S by a family of Cu-morphisms $S_\lambda \to S$; see Definition 3.1. The definition ensures that any 'reasonable' property passes to the approximated Cu-semigroup, and we show this specifically for the property of having dimension at most n; see Proposition 3.4.

If S is an inductive limit of a system of Cu-semigroups S_λ, then the canonical maps $S_\lambda \to S$ approximate S; see Proposition 3.5. Another natural source of approximation comes from C^*-algebras: If a C^*-algebra A is approximated by a family of sub-C^*-algebras $A_\lambda \subseteq A$ (we recall the definition before Proposition 3.7), then $\mathrm{Cu}(A)$ is approximated by the Cu-morphisms $\mathrm{Cu}(A_\lambda) \to \mathrm{Cu}(A)$ induced by the inclusions $A_\lambda \to A$; see Proposition 3.7.

Note that in Definition 3.1 below we do not insist that the Cu-morphisms $S_\lambda \to S$ are order-embeddings. The reason is that this would exclude the abovementioned sources of approximations. Indeed, the natural maps $S_\lambda \to S$ to an inductive limit are not necessarily order-embeddings. Further, if $A_\lambda \subseteq A$ is a sub-C^*-algebra, then the induced Cu-morphism $\mathrm{Cu}(A_\lambda) \to \mathrm{Cu}(A)$ need not be an order-embedding (consider, for example, $\mathbb{C} \subseteq \mathcal{O}_2$).

Definition 3.1. Let S be a Cu-semigroup and let $(S_\lambda, \varphi_\lambda)_{\lambda \in \Lambda}$ be a family of Cu-semigroups S_λ and Cu-morphisms $\varphi_\lambda \colon S_\lambda \to S$.

We say that the family $(S_\lambda, \varphi_\lambda)_{\lambda \in \Lambda}$ *approximates* S if the following holds: Given finite sets J and K, given elements $x_j', x_j \in S$ for $j \in J$, and given functions $m_k, n_k \colon J \to \mathbb{N}$ for $k \in K$, such that $x_j' \ll x_j$ for all $j \in J$ and such that

$$\sum_{j \in J} m_k(j) x_j \ll \sum_{j \in J} n_k(j) x_j'$$

for all $k \in K$, there exist $\lambda \in \Lambda$ and $y_j \in S_\lambda$ for $j \in J$ such that $x_j' \ll \varphi_\lambda(y_j) \ll x_j$ for each $j \in J$, and such that

$$\sum_{j \in J} m_k(j) y_j \ll \sum_{j \in J} n_k(j) y_j$$

for all $k \in K$.

Remark 3.2. In Definition 3.1, we think of J as the index set for a collection of variables, and for each $k \in K$ we think of the pair (m_k, n_k) as the encoding of a 'formula'. We say that S is approximated by the S_λ if every finite collection of elements in S that satisfy certain formulas can be approximated by a collection of elements in some S_λ that satisfy the same formulas.

Assume that the Cu-semigroup S is approximated by the family $(S_\lambda, \varphi_\lambda)_{\lambda\in\Lambda}$. Definition 3.1 ensures that every 'reasonable' property of Cu-semigroups passes from the approximating family to S. In Proposition 3.3 we show this for weak cancellation, (O5), (O6) and (O7), and in Proposition 3.4 we prove it for the property '$\dim(\_) \leq n$'.

We do not formalize the notion of 'formula' or 'reasonable property' for Cu-semigroups since this would go into the direction of developing a model theory for Cu-semigroups, which is an elaborate task that will be taken up elsewhere.

Proposition 3.3. *Let S be a* Cu*-semigroup that is approximated by $(S_\lambda, \varphi_\lambda)_{\lambda\in\Lambda}$. If each S_λ is weakly cancellative, then so is S. Similarly, if each S_λ satisfies (O5) (respectively, (O6) or (O7)), then so does S.*

Proof. First, assume that each S_λ is weakly cancellative. To see that S is also weakly cancellative, we will use Proposition 2.7. Thus, let $x', x, y', y, z', z \in S$ satisfy $x' \ll x$, $y' \ll y$, $z' \ll z$ and $x + z \ll y' + z'$.

Since S is approximated by $(S_\lambda, \varphi_\lambda)_{\lambda\in\Lambda}$, there exist $\lambda \in \Lambda$ and elements $u, v, w \in S_\lambda$ such that

$$x' \ll \varphi_\lambda(u) \ll x, \quad y' \ll \varphi_\lambda(v) \ll y, \quad z' \ll \varphi_\lambda(w) \ll z, \quad \text{and} \quad u + w \ll v + w.$$

Since S_λ is weakly cancellative, one gets $u \ll v$ and, consequently,

$$x' \ll \varphi_\lambda(u) \ll \varphi_\lambda(v) \ll y,$$

as required.

Next, assume that each S_λ satisfies (O5). We show that S satisfies the property of Proposition 2.4. Let $x', x, y', y, z', z \in S$ satisfy

$$x + y \ll z', \quad x' \ll x, \quad y' \ll y, \quad z' \ll z.$$

We need to find $c \in S$ such that

$$x' + c \ll z, \quad z' \ll x + c, \quad \text{and} \quad y' \ll c.$$

By assumption, there exist $\lambda \in \Lambda$ and $u, v, w \in S_\lambda$ such that

$$x' \ll \varphi_\lambda(u) \ll x, \quad y' \ll \varphi_\lambda(v) \ll y, \quad z' \ll \varphi_\lambda(w) \ll z, \quad \text{and} \quad u + v \ll w.$$

Since φ_λ is a Cu-morphism and $u + v \ll w$, we can choose $u', v' \in S_\lambda$ such that

$$x' \ll \varphi_\lambda(u'), \quad u' \ll u, \quad y' \ll \varphi_\lambda(v'), \quad \text{and} \quad v' \ll v.$$

Using that S_λ satisfies (O5), we obtain $a \in S_\lambda$ such that

$$u' + a \leq w \leq u + a, \quad \text{and} \quad v' \ll a.$$

Then $c := \varphi_\lambda(a)$ has the desired properties.

The statements for (O6) and (O7) are proved with similar methods, using Propositions 2.5 and 2.6 respectively. □

Proposition 3.4. *Let S be a* Cu*-semigroup that is approximated by a family $(S_\lambda, \varphi_\lambda)_{\lambda\in\Lambda}$. Then $\dim(S) \leq \sup_{\lambda\in\Lambda} \dim(S_\lambda)$.*

Proof. Set $n := \sup_{\lambda\in\Lambda} \dim(S_\lambda)$, which we may assume to be finite. To verify $\dim(S) \leq n$, let $x' \ll x \ll y_1 + \ldots + y_r$ in S. Choose $y_1', \ldots, y_r' \in S$ such that

$$x' \ll x \ll y_1' + \ldots + y_r', \quad y_1' \ll y_1, \quad \ldots, \quad \text{and} \quad y_r' \ll y_r.$$

Using that S is approximated by $(S_\lambda, \varphi_\lambda)_{\lambda\in\Lambda}$, we obtain $\lambda \in \Lambda$ and elements $v, w_1, \ldots, w_r \in S_\lambda$ such that

$$x' \ll \varphi_\lambda(v) \ll x, \quad y_1' \ll \varphi_\lambda(w_1) \ll y_1, \quad \ldots, \quad \text{and} \quad y_r' \ll \varphi_\lambda(w_r) \ll y_r,$$

and such that

$$v \ll w_1 + \ldots + w_r.$$

Since φ_λ is a Cu-morphism and $x' \ll \varphi_\lambda(v)$, there exists $v' \in S_\lambda$ such that

$$x' \ll \varphi_\lambda(v'), \quad \text{and} \quad v' \ll v.$$

We have $v' \ll v \ll w_1 + \cdots + w_r$ in S_λ. Using that $\dim(S_\lambda) \leq n$, we obtain elements $z_{j,k} \in S_\lambda$ for $j = 1, \ldots, r$ and $k = 0, \ldots, n$ satisfying conditions (i)-(iii) in Definition 2.8. It is now easy to check that the elements $\varphi_\lambda(z_{j,k}) \in S$ satisfy conditions (i)-(iii) in Definition 2.8 for $x' \ll x \ll y_1 + \ldots + y_r$, as desired. □

Proposition 3.5. *Let $S = \varinjlim_{\lambda\in\Lambda} S_\lambda$ be an inductive limit of* Cu*-semigroups, and let $\varphi_\lambda \colon S_\lambda \to S$ be the* Cu*-morphisms into the limit. Then the family $(S_\lambda, \varphi_\lambda)_{\lambda\in\Lambda}$ approximates S.*

Proof. For $\lambda \leq \mu$ in Λ, let $\varphi_{\mu,\lambda} \colon S_\lambda \to S_\mu$ denote the connecting Cu-morphism of the inductive system. We will use the following conditions, which were shown in [TV21, Paragraph 3.8] to characterize that S is the inductive limit:

- (L0) We have $\varphi_\mu \circ \varphi_{\mu,\lambda} = \varphi_\lambda$ for all $\lambda \leq \mu$ in Λ;
- (L1) If $x_\lambda \in S_\lambda$ and $x_\mu \in S_\mu$ satisfy $\varphi_\lambda(x_\lambda) \ll \varphi_\mu(x_\mu)$, then there exists μ with $\lambda, \mu \leq \nu$ such that $\varphi_{\nu,\lambda}(x_\lambda) \ll \varphi_{\nu,\mu}(x_\mu)$;
- (L2) For all $x', x \in S$ satisfying $x' \ll x$ there exist $\lambda \in \Lambda$ and $x_\lambda \in S_\lambda$ such that $x' \ll \varphi_\lambda(x_\lambda) \ll x$.

Let J and K be finite sets, let $x_j', x_j \in S$ satisfy $x_j' \ll x_j$ for $j \in J$, and let $m_k, n_k \colon J \to \mathbb{N}$ such that

$$\sum_{j\in J} m_k(j) x_j \ll \sum_{j\in J} n_k(j) x_j'$$

for all $k \in K$.

For each $j \in J$, applying (L2), we obtain $\lambda_j \in \Lambda$ and $z_j \in S_{\lambda_j}$ such that

$$x_j' \ll \varphi_{\lambda_j}(z_j) \ll x_j.$$

Choose $\lambda \in \Lambda$ with $\lambda_j \leq \lambda$ for all j, and set $\bar{z}_j := \varphi_{\lambda,\lambda_j}(z_j) \in S_\lambda$ for each j.

Given $k \in K$, we have

$$\varphi_\lambda\left(\sum_{j\in J} m_k(j)\bar{z}_j\right) \ll \sum_{j\in J} m_k(j) x_j \ll \sum_{j\in J} n_k(j) x_j' \ll \varphi_\lambda\left(\sum_{j\in J} n_k(j)\bar{z}_j\right).$$

Applying (L1), we obtain $\nu_k \in \Lambda$ with $\lambda \leq \nu_k$ such that

$$\varphi_{\nu_k,\lambda}\left(\sum_{j\in J} m_k(j)\bar{z}_j\right) \ll \varphi_{\nu_k,\lambda}\left(\sum_{j\in J} n_k(j)\bar{z}_j\right). \tag{3.1}$$

Choose $\nu \in \Lambda$ with $\nu_k \leq \nu$ for all k, and set $y_j := \varphi_{\nu,\lambda}(\bar{z}_j) \in S_\nu$ for each j.

For each j, we have

$$\varphi_\nu(y_j) = \varphi_\lambda(\bar{z}_j) = \varphi_{\lambda_j}(z_j)$$

and therefore $x_j' \ll \varphi_\nu(y_j) \ll x_j$. Further, for $k \in K$, using (3.1), we obtain

$$\begin{aligned}\sum_{j\in J} m_k(j) y_j &= \varphi_{\nu,\nu_k}\left(\varphi_{\nu_k,\lambda}\left(\sum_{j\in J} m_k(j)\bar{z}_j\right)\right) \\ &\ll \varphi_{\nu,\nu_k}\left(\varphi_{\nu_k,\lambda}\left(\sum_{j\in J} n_k(j)\bar{z}_j\right)\right) = \sum_{j\in J} n_k(j) y_j,\end{aligned}$$

as desired. □

The next result recovers [APT18, Theorem 4.5] and [TV21, Proposition 3.9], and is in fact new for (O7).

Corollary 3.6. *Let $S = \varinjlim_{\lambda\in\Lambda} S_\lambda$ be an inductive limit of* Cu*-semigroups. If each S_λ is weakly cancellative, then so is S. Similarly, if each S_λ satisfies (O5) (respectively, (O6) or (O7)), then so does S. Further, given $n \in \mathbb{N}$ such that $\dim(S_\lambda) \leq n$ for each λ, then $\dim(S) \leq n$.*

Proof. This follows by combining Proposition 3.5 with Propositions 3.3 and 3.4. □

A C^*-algebra A is said to be *approximated* by a collection of sub-C^*-algebras $A_\lambda \subseteq A$, for $\lambda \in \Lambda$, if for every finitely many elements $a_1, \ldots, a_n \in A$ and every $\varepsilon > 0$ there exist $\lambda \in \Lambda$ and $b_1, \ldots, b_n \in A_\lambda$ such that $\|b_j - a_j\| < \varepsilon$ for $j = 1, \ldots, n$.

Proposition 3.7. *Let A be a C^*-algebra that is approximated by a family of sub-C^*-algebras $A_\lambda \subseteq A$, and let $i_\lambda \colon A_\lambda \to A$ be the inclusion maps for $\lambda \in \Lambda$. Then, the system $(\mathrm{Cu}(A_\lambda), \mathrm{Cu}(i_\lambda))_{\lambda\in\Lambda}$ approximates $\mathrm{Cu}(A)$.*

Proof. We may assume that A and A_λ are stable for every $\lambda \in \Lambda$. We begin with three claims. Since they are simple computations, we omit their proof (for Claim 1, one can approximate the function $(t - \varepsilon)_+$ by a polynomial).

Claim 1. *For any $\varepsilon, \delta > 0$ and $a \in A_+$, there exists $\sigma > 0$ such that, whenever $b \in A_+$ satisfies $\|a - b\| \leq \sigma$, we have $\|(a - \varepsilon)_+ - (b - \varepsilon)_+\| \leq \delta$.*

Claim 2. *Let $\varepsilon > 0$ and let $a, b, r \in A$ be such that $\|a - rbr^*\| < \varepsilon$. Then, there exists $\delta > 0$ such that for every $c, d, s \in A$ with*

$$\|c - a\| < \delta, \quad \|d - b\| < \delta, \quad \textit{and} \quad \|s - r\| < \delta$$

one has

$$\|c - sds^*\| < 2\varepsilon.$$

Claim 3. *Given $a \in A_+$ and $\varepsilon > 0$, there exists $\delta > 0$ such that for every $b \in A$ with $\|b - a\| < \delta$ we have $\|(b^*b)^{1/2} - a\| < \varepsilon$.*

Now let J and K be finite sets, and take elements $x'_j, x_j \in \mathrm{Cu}(A)$ for $j \in J$, and functions $m_k, n_k \colon J \to \mathbb{N}$ for $k \in K$, such that $x'_j \ll x_j$ for all $j \in J$ and such that

$$\sum_{j\in J} m_k(j) x_j \ll \sum_{j\in J} n_k(j) x'_j$$

for all $k \in K$. We may assume that $(\sum_{j\in J} m_k(j))(\sum_{j\in J} n_k(j)) \neq 0$ for every k.

For each $j \in J$, let $a_j \in A_+$ be such that $[a_j] = x_j$. Since J is finite, there exists $\varepsilon > 0$ such that

$$x'_j \ll [(a_j - 2\varepsilon)_+] \ll [a_j] = x_j$$

for every $j \in J$.

For each $k \in K$, we have

$$\left[\bigoplus_{j\in J} a_j^{\oplus m_k(j)}\right] = \sum_{j\in J} m_k(j) x_j \ll \sum_{j\in J} n_k(j) x'_j \leq \left[\bigoplus_{j\in J} (a_j - 2\varepsilon)_+^{\oplus n_k(j)}\right],$$

which allows us to take $r_k \in A$ satisfying

$$\left\| \bigoplus_{j\in J} a_j^{\oplus m_k(j)} - r_k \left(\bigoplus_{j\in J} (a_j - 2\varepsilon)_+^{\oplus n_k(j)} \right) r_k^* \right\| < \frac{\varepsilon}{2}.$$

For each k, let δ_k be the bound given by Claim 2 for the previous inequality, and take $\delta > 0$ such that

$$\delta < \min_{k\in K} \frac{\delta_k}{(\sum_{j\in J} m_k(j))(\sum_{j\in J} n_k(j))}, \quad \text{and} \quad \delta < \varepsilon.$$

Then, using Claim 1, let $\sigma > 0$ satisfy $\sigma \leq \delta$ and such that for every $j \in J$ and $b \in A_+$ with $\|a_j - b\| \leq \sigma$, we have $\|(a_j - 2\varepsilon)_+ - (b - 2\varepsilon)_+\| \leq \delta$.

Since the sub-C^*-algebras A_λ approximate A, and using Claim 3, there exist $\lambda \in \Lambda$ and elements $s_k \in A_\lambda$ and $b_j \in (A_\lambda)_+$ such that

$$\|s_k - r_k\| \leq \sigma, \quad \text{and} \quad \|b_j - a_j\| \leq \sigma$$

for each $k \in K$ and $j \in J$.

By the choice of σ, note that we also have $\|(b_j - 2\varepsilon)_+ - (a_j - 2\varepsilon)_+\| \leq \delta$ for each $j \in J$. Using that $\|(b_j - \varepsilon)_+ - a_j\| < 2\varepsilon$ in the first step, and that $\|b_j - a_j\| < \varepsilon$ in the second step, we note that the element $[(b_j - \varepsilon)_+] \in \mathrm{Cu}(A)$ satisfies

$$[(a_j - 2\varepsilon)_+] \ll [(b_j - \varepsilon)_+] \ll [a_j]$$

for every $j \in J$.

Further, for each $k \in K$, we have

$$\left\| \bigoplus_{j \in J} a_j^{\oplus m_k(j)} - \bigoplus_{j \in J} b_j^{\oplus m_k(j)} \right\| \leq \sum_{j \in J} m_k(j) \|a_j - b_j\| \leq \sum_{j \in J} m_k(j) \delta < \delta_k$$

and, similarly,

$$\left\| \bigoplus_{j \in J} (a_j - 2\varepsilon)_+^{\oplus n_k(j)} - \bigoplus_{j \in J} (b_j - 2\varepsilon)_+^{\oplus n_k(j)} \right\| \leq \sum_{j \in J} n_k(j) \delta < \delta_k.$$

Thus, it follows from Claim 2 that, for every $k \in K$, we get

$$\left\| \bigoplus_{j \in J} b_j^{\oplus m_k(j)} - c_k \left(\bigoplus_{j \in J} (b_j - 2\varepsilon)_+^{\oplus n_k(j)} \right) c_k^* \right\| < 2\frac{\varepsilon}{2} = \varepsilon$$

and, consequently,

$$\sum_{j \in J} m_k(j) [(b_j - \varepsilon)_+] \leq \sum_{j \in J} n_k(j) [(b_j - 2\varepsilon)_+]$$

in $\mathrm{Cu}(A_\lambda)$.

Recall that $i_\lambda \colon A_\lambda \to A$ denotes the inclusion map. Using that $[(a_j - 2\varepsilon)_+] \ll [(b_j - \varepsilon)_+] \ll [a_j]$ in $\mathrm{Cu}(A)$ and $[(b_j - 2\varepsilon)_+] \ll [(b_j - \varepsilon)_+]$ in $\mathrm{Cu}(A_\lambda)$, one notes that the elements $[(b_j - \varepsilon)_+] \in \mathrm{Cu}(A_\lambda)$ satisfy

$$x_j' \ll [(a_j - 2\varepsilon)_+] \ll \mathrm{Cu}(i_\lambda)([(b_j - \varepsilon)_+]) \ll [a_j] = x_j$$

for every $j \in J$, and

$$\sum_{j \in J} m_k(j) [(b_j - \varepsilon)_+] \leq \sum_{j \in J} n_k(j) [(b_j - 2\varepsilon)_+] \ll \sum_{j \in J} n_k(j) [(b_j - \varepsilon)_+]$$

for every $k \in K$, as desired. □

Theorem 3.8. *Let A be a C^*-algebra that is approximated by a family of sub-C^*-algebras $A_\lambda \subseteq A$, for $\lambda \in \Lambda$. Then* $\dim(\mathrm{Cu}(A)) \leq \sup_{\lambda \in \Lambda} \dim(\mathrm{Cu}(A_\lambda))$.

Proof. By Proposition 3.7, we know that the system $(\mathrm{Cu}(A_\lambda), \mathrm{Cu}(i_\lambda))_{\lambda \in \Lambda}$ approximates $\mathrm{Cu}(A)$. Thus, the result follows from Proposition 3.4. □

4. The lattice of sub-Cu-semigroups

In this section, we provide characterizations for when a submonoid of a Cu-semigroup is a sub-Cu-semigroup, which will be used in Section 5. In particular, given a Cu-semigroup S, we construct for every submonoid $T \subseteq S$ an associated sup-closed submonoid $\overline{T}^{\sup} \subseteq S$ (see Definition 4.6) and a 'derived' submonoid $T' \subseteq S$ (see Definition 4.8). We show that a submonoid $T \subseteq S$ is a sub-Cu-semigroup if and only if $T = T'$; see Proposition 4.11.

We also describe, for every submonoid $T \subseteq S$, the largest sub-Cu-semigroup contained in $\overline{T}^{\mathrm{sup}}$. This construction is used in Theorem 4.15 to prove that the collection of sub-Cu-semigroups of a Cu-semigroup is a complete lattice.

Definition 4.1. Given a Cu-semigroup S, we say that a submonoid $T \subseteq S$ is a *sub*-Cu-*semigroup* if T is a Cu-semigroup with respect to the order induced by S and if the inclusion map $T \to S$ is a Cu-morphism.

The next results provide characterizations of sub-Cu-semigroups. We omit the straightforward proofs.

Lemma 4.2. *Let S be a* Cu-*semigroup. Then a submonoid $T \subseteq S$ is a sub-*Cu-*semigroup if and only if it is closed under passing to suprema of increasing sequences and for every $x' \in S$ and $x \in T$ with $x' \ll x$ there exists $y \in T$ such that $x' \ll y \ll x$.*

Lemma 4.3. *Let S, T be* Cu-*semigroups, and let $\varphi\colon T \to S$ be a* Cu-*morphism. Then the following are equivalent:*

(1) φ is an order-embedding, that is, $x, y \in T$ satisfy $x \leq y$ if (and only if) $\varphi(x) \leq \varphi(y)$;
(2) $x, y \in T$ satisfy $x \ll y$ if (and only if) $\varphi(x) \ll \varphi(y)$;
*(3) $\varphi(T) \subseteq S$ is a sub-*Cu-*semigroup and $\varphi\colon T \to \varphi(T)$ is an isomorphism.*

Remark 4.4. Let us recall the notion of subobjects from category theory. Let $\mathcal{C}$ be a category, and let X be an object in $\mathcal{C}$. Given monomorphisms $\alpha\colon Y \to X$ and $\beta\colon Z \to X$, one sets $\alpha \sim \beta$ if there exists an isomorphism $\gamma\colon Y \to Z$ such that $\beta \circ \gamma = \alpha$. This defines an equivalence relation on the class of monomorphisms to X, and a subobject of X is defined as an equivalence class of this relation. We refer to [Bor94, Section 4.1] for details.

Let S be a Cu-semigroup, and let $T \subseteq S$ be a sub-Cu-semigroup. It is easy to verify that the inclusion map $T \to S$ is a monomorphism in the category Cu, whence every sub-Cu-semigroup of S naturally is a subobject. The converse holds if and only if the following question has a positive answer.

Question 4.5. Is every monomorphism in the category Cu an order-embedding?

Definition 4.6. Let S be a Cu-semigroup, and let $T \subseteq S$ be a subset. We set

$$\overline{T}^{\mathrm{seq}} := \left\{ \sup_n x_n \in S : (x_n)_n \text{ is an increasing sequence in } T \right\}.$$

We define $\overline{T}^{(\alpha)}$ for every ordinal α by setting $T^{(0)} := T$, $T^{(1)} := \overline{T}^{\mathrm{seq}}$, and by using (transfinite) induction:

$$\overline{T}^{(\alpha+1)} := \overline{T^{(\alpha)}}^{\mathrm{seq}},$$
$$\overline{T}^{(\lambda)} := \bigcup_{\alpha<\lambda} \overline{T}^{(\alpha)}, \quad \text{if } \lambda \text{ is a limit ordinal.}$$

We define the *sup-closure* of T as $\overline{T}^{\mathrm{sup}} := \bigcup_{\alpha \geq 1} \overline{T}^{(\alpha)}$. We say that T is *sup-closed* if $T = \overline{T}^{\mathrm{sup}}$.

Remark 4.7. Let S be a Cu-semigroup, and let $T \subseteq S$ be a subset. Then $(\overline{T}^{(\alpha)})_\alpha$ is an increasing family of subsets of S, which therefore stabilizes eventually. Thus, there exists an ordinal α_0 such that $\overline{T}^{(\alpha)} = \overline{T}^{(\alpha_0)}$ for all $\alpha \geq \alpha_0$. Then $\overline{T}^{\mathrm{sup}} = \overline{T}^{(\alpha_0)}$, and we get $\overline{\overline{T}^{\mathrm{sup}}}^{\mathrm{seq}} = \overline{T}^{\mathrm{sup}}$. It follows that $\overline{T}^{\mathrm{sup}}$ is sup-closed, as expected.

We also note that T is sup-closed if and only if $T = \overline{T}^{\mathrm{seq}}$.

Definition 4.8. Let S be a Cu-semigroup, and let $T \subseteq S$ be a subset. We set

$$T' := \left\{ \sup_n x_n \in S : (x_n)_n \text{ is a } \ll\text{-increasing sequence in } T \right\}.$$

Remark 4.9. Given a topological space X and a subset $Y \subseteq X$, the *derived set* of Y, denoted by Y', is defined as the set of limit points of Y.

Let S be a Cu-semigroup and let T be a subset of S. We think of suprema of $\ll$-increasing sequences of elements in T as the limit points of T. Therefore, one may view T' as the *derived set* of T. Further, the derived subsets of a Cu-semigroup satisfy the following properties, which are analogs of well known facts satisfied by the derived subsets of a topological space:

(i) If $x \in T'$ and if x is not compact (that is, $x \not\ll x$), then x also belongs to $(T - \{x\})'$.
(ii) We have $(T \cup H)' = T' \cup H'$.
(iii) If $T \subseteq H$, then $T' \subseteq H'$.

To push the previous analogy even further, recall that a subset Y of a topological space is said to be *perfect* if $Y = Y'$. Proposition 4.11 below shows that we may think of sub-Cu-semigroups as the perfect submonoids of a Cu-semigroup.

Lemma 4.10. *Let S be a* Cu*-semigroup, and let $T \subseteq S$ be a submonoid. Then T' is a sup-closed submonoid of S.*

Proof. Using that the way-below relation is additive and that 0 is way-below itself, it follows that T' is a submonoid. It remains to verify that T' is closed under suprema of increasing sequences.

Let $(x_n)_n$ be an increasing sequence in T'. For each n, by definition of T', there exists a $\ll$-increasing sequence $(x_{n,k})_k$ with supremum x_n. Set $k(0) := 0$. Then $x_{0,k(0)+1} \ll x_0 \leq x_1$. Choose $k(1) \in \mathbb{N}$ such that $x_{0,k(0)+1} \ll x_{1,k(1)}$. Using that $x_{0,k(0)+2}$ and $x_{1,k(1)+2}$ are way-below x_2, we can choose $k(2) \in \mathbb{N}$ such that $x_{0,k(0)+2}, x_{1,k(1)+2} \ll x_{2,k(2)}$. We inductively choose indices $k(n) \in \mathbb{N}$ such that

$$x_{0,k(0)+n}, x_{1,k(1)+n}, \ldots, x_{n-1,k(n-1)+n} \ll x_{n,k(n)}.$$

For each $n \in \mathbb{N}$ set $y_n := x_{n,k(n)}$. Then $(y_n)_n$ is a $\ll$-increasing sequence with $\sup_n y_n = \sup_n x_n$, and consequently $\sup_n x_n$ belongs to T'. ☐

Proposition 4.11. *Let S be a* Cu*-semigroup. Then a submonoid $T \subseteq S$ is a sub-*Cu*-semigroup if and only if $T = T'$.*

Proof. The forward implication is clear. To show the converse, assume that $T \subseteq S$ is a submonoid satisfying $T = T'$. By Lemma 4.10, T is sup-closed. Hence, we can apply Lemma 4.2 to deduce that T is a sub-Cu-semigroup. ☐

The next result recovers [APT18, Lemma 5.3.17].

Corollary 4.12. *Let S be a* Cu*-semigroup, and let $T \subseteq S$ be a submonoid such that every element in T is the supremum of a $\ll$-increasing sequence in T. Then $T' = \overline{T}^{\mathrm{seq}} = \overline{T}^{\mathrm{sup}}$, which is a sub-*Cu*-semigroup of S.*

Proof. The inclusions $T' \subseteq \overline{T}^{\mathrm{seq}} \subseteq \overline{T}^{\mathrm{sup}}$ hold in general. By assumption, we have $T \subseteq T'$. Using Lemma 4.10 at the second step, we get

$$\overline{T}^{\mathrm{sup}} \subseteq \overline{T'}^{\mathrm{sup}} = T'.$$

Since T' is sup-closed, we have $T'' \subseteq T'$. On the other hand, using again that $T \subseteq T'$, we have $T' \subseteq T''$. Thus, $T' = T''$, which by Proposition 4.11 implies that $T' \subseteq S$ is a sub-Cu-semigroup. ☐

Let α be an ordinal number. Continuing with the analogy from Remark 4.9, we now define what may be seen as the Cu-counterpart of the α-th Cantor-Bendixson derivative.

Definition 4.13. Let S be a Cu-semigroup, and let $T \subseteq S$ be a submonoid. We define $T^{(\alpha)}$ for every ordinal α by setting $T^{(0)} := T$, $T^{(1)} := T'$, and by using (transfinite) induction:

$$T^{(\alpha+1)} := \left(T^{(\alpha)}\right)',$$
$$T^{(\lambda)} := \bigcap_{\alpha<\lambda} T^{(\alpha)}, \quad \text{if } \lambda \text{ is a limit ordinal.}$$

We set
$$\delta(T) := \bigcap_{\alpha\geq 1} T^{(\alpha)}.$$

Theorem 4.14. *Let S be a* Cu*-semigroup, and let $T \subseteq S$ be a submonoid. Then $\delta(T) \subseteq S$ is a sub-*Cu*-semigroup.*

We always have $\delta(T) \subseteq \overline{T}^{\mathrm{sup}}$. Thus, if T is sup-closed, then $\delta(T) \subseteq T$.

Proof. Using transfinite induction, we prove that $T^{(\alpha)}$ is a sup-closed submonoid for each ordinal $\alpha \geq 1$. For $\alpha = 1$ and the successor case, this follows from Lemma 4.10. The limit case follows directly from the definition.

Thus, $\delta(T)$ is a submonoid. Further, we deduce that the $T^{(\alpha)}$, for $\alpha \geq 1$, form a decreasing family of submonoids, which therefore stabilizes. Hence, there exists $\alpha \geq 1$ such that $\delta(T) = T^{(\alpha)}$. It follows that
$$\delta(T) = T^{(\alpha)} = T^{(\alpha+1)} = \delta(T)',$$
which by Proposition 4.11 implies that $\delta(T)$ is a sub-Cu-semigroup.

It is clear from the definition that $T' \subseteq \overline{T}^{\mathrm{seq}}$, which shows that $\delta(T) \subseteq \overline{T}^{\mathrm{sup}}$. $\square$

Let S be a Cu-semigroup. Let $\mathcal{P}$ be the collection of all subsets of S; let $\mathcal{C}$ be the collection of all sup-closed submonoids of S; and let $\mathcal{S}$ be the collection of sub-Cu-semigroups of S. We equip each of these collections with the partial order given by inclusion of subsets.

Let $\alpha\colon \mathcal{P} \to \mathcal{C}$ be the map that sends a subset of S to the sup-closure of the submonoid it generates. Then α is order-preserving. Further, considering α as a map $\mathcal{P} \to \mathcal{P}$, we see that α is idempotent and satisfies $X \subseteq \alpha(X)$ for every $X \in \mathcal{P}$.

Therefore, $\alpha\colon \mathcal{P} \to \mathcal{P}$ is a closure operator in the sense of [GHK$^+$03, Definition 0-3.8(ii)]. Using that $\mathcal{P}$ is a complete lattice, it follows that $\mathcal{C}$ is a complete lattice as well, that α preserves arbitrary suprema, and that the inclusion map $\iota\colon \mathcal{C} \to \mathcal{P}$ preserves arbitrary infima. In particular, given a subset $C \subseteq \mathcal{C}$, the supremum of C in $\mathcal{C}$ is $\sup_{\mathcal{C}} C = \alpha(\bigcup C)$ and the infimum is $\inf_{\mathcal{C}} C = \bigcap C$. (The intersection of a family of sup-closed submonoids of S is again a sup-closed submonoid.)

Let $\delta\colon \mathcal{C} \to \mathcal{S}$ be the map that sends $T \in \mathcal{C}$ to $\delta(T)$ as defined in Definition 4.13. It follows from Theorem 4.14 that δ is well-defined and order-preserving. Using also Proposition 4.11, we see that δ as map $\mathcal{C} \to \mathcal{C}$ is idempotent and satisfies $\delta(T) \subseteq T$ for every $T \in \mathcal{C}$. Thus, $\delta\colon \mathcal{C} \to \mathcal{C}$ is a kernel operator in the sense of [GHK$^+$03, Definition 0-3.8(iii)]. It follows that $\mathcal{S}$ is a complete lattice, that δ preserves arbitrary infima, and that the inclusion map $\iota\colon \mathcal{S} \to \mathcal{C}$ preserves arbitrary suprema.

The considered maps are shown in the following diagram:

$$\begin{tikzcd} \mathcal{S} \arrow[r, hook, "\iota"'] & \mathcal{C} \arrow[l, bend right, "\delta"'] \arrow[r, hook, "\iota"'] & \mathcal{P}. \arrow[l, bend right, "\alpha"'] \end{tikzcd}$$

Theorem 4.15. *Let S be a* Cu*-semigroup. Then the collection of sub*-Cu*-semigroups of S is a complete lattice when ordered by inclusion.*

Given a collection $(T_j)_{j\in J}$ of sub-Cu*-semigroups of S, their supremum is the sup-closure of the submonoid of S generated by $\bigcup_j T_j$, while their infimum is $\delta(\bigcap_j T_j)$.*

5. Reduction to countably based Cu-semigroups

In this section, we show that the dimension of a Cu-semigroup is determined by its countably based sub-Cu-semigroups; see Theorem 5.7. We then generalize some results from [TV21] by dropping the countably based assumption; see Propositions 5.8 and 5.9.

Lemma 5.1. *Let S be a* Cu*-semigroup, and let $T_0 \subseteq S$ be a countable subset. Then there exists a countably based sub*-Cu*-semigroup $T \subseteq S$ such that $T_0 \subseteq T$.*

Proof. We may assume that T_0 is a submonoid. For each $x \in T_0$ choose a $\ll$-increasing sequence in S with supremum x, and let T_1 be the submonoid of S generated by T_0 and the elements in each of the chosen sequences. Repeating this process, we successively obtain an increasing sequence $(T_k)_k$ of countable submonoids of S such that for each $k \in \mathbb{N}$ and $x \in T_k$ there exists a $\ll$-increasing sequence in T_{k+1} with supremum x. Then $T_\infty := \bigcup_k T_k$ is a countable submonoid of S such that every element in T_∞ is the supremum of a $\ll$-increasing sequence in T_∞. Set $T := \overline{T_\infty}^{\mathrm{seq}}$. By Corollary 4.12, T is a sub-Cu-semigroup of S. It is straightforward to verify that T_∞ is a countable basis for T. $\square$

5.2. Given a Cu-semigroup S, we let $\mathrm{Sub}_{\mathrm{ctbl}}(S)$ denote the collection of countably based sub-Cu-semigroups of S. If $\mathcal{T} \subseteq \mathrm{Sub}_{\mathrm{ctbl}}(S)$ is a countable, directed family, then $\bigcup\mathcal{T}$ is a submonoid of S such that every element is the supremum of a $\ll$-increasing sequence in $\bigcup\mathcal{T}$, whence it follows from Corollary 4.12 that the sup-closure $\overline{\bigcup\mathcal{T}}^{\mathrm{sup}}$ is a (countably based) sub-Cu-semigroup. Note that $\overline{\bigcup\mathcal{T}}^{\mathrm{sup}}$ is the supremum of $\mathcal{T}$ in the complete lattice of sub-Cu-semigroups; see Theorem 4.15.

A collection $\mathcal{R} \subseteq \mathrm{Sub}_{\mathrm{ctbl}}(S)$ is said to be *σ-complete* if $\overline{\bigcup\mathcal{T}}^{\mathrm{sup}}$ belongs to $\mathcal{R}$ for every countable, directed subset $\mathcal{T} \subseteq \mathcal{R}$. Further, $\mathcal{R} \subseteq \mathrm{Sub}_{\mathrm{ctbl}}(S)$ is said to be *cofinal* if for every $T_0 \in \mathrm{Sub}_{\mathrm{ctbl}}(S)$ there is $T \in \mathcal{R}$ satisfying $T_0 \subseteq T$.

We say that a property $\mathcal{P}$ of Cu-semigroups satisfies the *Löwenheim-Skolem condition* if for every Cu-semigroup S satisfying $\mathcal{P}$, there exists a σ-complete, cofinal subcollection $\mathcal{R} \subseteq \mathrm{Sub}_{\mathrm{ctbl}}(S)$ such that every $R \in \mathcal{R}$ satisfies $\mathcal{P}$. In Propositions 5.3, 5.4 and 5.6 below, we show that (O5), (O6), (O7), simplicity, weak cancellation and '$\dim(\_) \leq n$' (for fixed $n \in \mathbb{N}$) each satisfy the Löwenheim-Skolem condition.

Proposition 5.3. *Given a* Cu*-semigroup S satisfying (O5) (satisfying (O6), satisfying (O7)), the countably based sub*-Cu*-semigroups satisfying (O5) (satisfying (O6), satisfying (O7)) form a σ-complete, cofinal subset of* $\mathrm{Sub}_{\mathrm{ctbl}}(S)$.

In particular, the properties (O5), (O6) and (O7) each satisfy the Löwenheim-Skolem condition.

Proof. Let S be a Cu-semigroup satisfying (O5). Set

$$\mathcal{R} := \big\{R \in \mathrm{Sub}_{\mathrm{ctbl}}(S) : R \text{ satisfies (O5)}\big\}.$$

To show that $\mathcal{R}$ is σ-complete, let $\mathcal{T} \subseteq \mathcal{R}$ be a countable, directed subset. Then $\overline{\bigcup\mathcal{T}}^{\mathrm{sup}}$ is the inductive limit of the system $\mathcal{T}$. By [APT18, Theorem 4.5] (see also Corollary 3.6), (O5) passes to inductive limits, whence $\overline{\bigcup\mathcal{T}}^{\mathrm{sup}}$ belongs to $\mathcal{R}$.

To show that $\mathcal{R}$ is cofinal, let $R_0 \in \mathrm{Sub}_{\mathrm{ctbl}}(S)$. We need to find $R \in \mathcal{R}$ satisfying $R_0 \subseteq R$. Choose a countable basis $B_0 \subseteq R_0$.

We will inductively choose an increasing sequence $(R_n)_n$ in $\mathrm{Sub}_{\mathrm{ctbl}}(S)$ and a countable basis $B_n \subseteq R_n$ such that for each n the following holds:

For every $x', x, y', y, z', z \in B_n$ satisfying $x + y \ll z' \ll z$, $x' \ll x$ and $y' \ll y$, there exists $c \in B_{n+1}$ such that $x' + c \ll z$, $z' \ll x + c$ and $y' \ll c$.

We have already obtained R_0 and B_0. Let $n \in \mathbb{N}$ and assume that we have chosen R_k and B_k for all $k \leq n$. Consider the countable set

$$I_n := \big\{(x', x, y', y, z', z) \in B_n^6 : x + y \ll z' \ll z, x' \ll x, y' \ll y\big\}.$$

Since S satisfies (O5), we obtain for each $i = (x', x, y', y, z', z) \in I_n$ an element $c_i \in S$ such that $x' + c_i \ll z$, $z' \ll x + c_i$ and $y' \ll c_i$. Applying Lemma 5.1, we obtain $R_{n+1} \in \mathrm{Sub}_{\mathrm{ctbl}}(S)$ containing $B_n \cup \{c_i : i \in I_n\}$. Since B_n is a basis for R_n, we have $R_n \subseteq R_{n+1}$. Choose a countable basis B_{n+1} for R_{n+1} that contains B_n and each c_i for $i \in I_n$. This completes the induction step.

Now set $R := \overline{\bigcup_n R_n}^{\mathrm{sup}}$ and $B := \bigcup_n B_n$. Then R is a sub-Cu-semigroup of S containing R_0. Further, B is a countable basis of R. By construction, B satisfies the condition from Proposition 2.4, showing that R satisfies (O5). Thus, R belongs to $\mathcal{R}$, as desired.

A similar argument, using Corollary 3.6 (to show σ-completeness) and Propositions 2.5 and 2.6 (to show cofinality) proves that (O6) and (O7) satisfy the Löwenheim-Skolem condition. □

A Cu-semigroup S is *simple* if for all $x, y \in S$ with $y \neq 0$ we have $x \leq \infty y$.

Proposition 5.4. *Given a simple (weakly cancellative)* Cu*-semigroup S, every sub*-Cu*-semigroup of S is simple (weakly cancellative).*

In particular, simplicity and weak cancellation each satisfy the Löwenheim-Skolem condition.

Proof. Let S be a simple Cu-semigroup, and let $T \subseteq S$ be a sub-Cu-semigroup. To verify that T is simple, let $x, y \in T$ with $y \neq 0$. Since S is simple, we have $x \leq \infty y$ in S, and since the inclusion $T \to S$ is an order-embedding, we get $x \leq \infty y$ in T.

Let us now assume that S is a weakly cancellative Cu-semigroup, and let $T \subseteq S$ be a sub-Cu-semigroup. To show that T is weakly cancellative, let $x, y, z \in T$ satisfy $x + z \ll y + z$. This implies that $x \ll y$ in S, and thus $x \ll y$ in T. □

Lemma 5.5. *Let S be a* Cu*-semigroup, and let $T_0 \subseteq S$ be a countably based sub-*Cu*-semigroup. Then there exists a countably based sub-*Cu*-semigroup $T \subseteq S$ such that $T_0 \subseteq T$ and $\dim(T) \leq \dim(S)$.*

Proof. Set $n := \dim(S)$. If $n = \infty$, then $T := T_0$ has the desired properties. Thus, we may assume that n is finite.

Claim: *Let $P \subseteq S$ be a countably based sub-*Cu*-semigroup. Then there exists a countably based sub-*Cu*-semigroup $Q \subseteq S$ satisfying $P \subseteq Q$ and with the following property: Whenever $x' \ll x \ll y_1 + \ldots + y_r$ in P, then there exist $z_{j,k} \in Q$ for $j = 1, \ldots, r$ and $k = 0, \ldots, n$ satisfying (i)-(iii) from Definition 2.8.*

To prove the claim, choose a countable basis $B \subseteq P$. For each $r \geq 1$, consider the countable set

$$I_r := \big\{(x', x, y_1, \ldots, y_r) \in B^{r+2} : x' \ll x \ll y_1 + \ldots + y_r\big\}.$$

For each $i = (x', x, y_1, \ldots, y_r) \in I_r$, we apply $\dim(S) \leq n$ for $x' \ll x \ll y_1 + \ldots + y_r$ to obtain elements $z_{i,j,k} \in S$ for $j = 1, \ldots, r$ and $k = 0, \ldots, n$ satisfying (i)-(iii) from Definition 2.8. Applying Lemma 5.1, we obtain a countably based sub-Cu-semigroup $Q \subseteq S$ that contains B and each $z_{i,j,k}$ for $r \geq 1$, $i \in I_r$, $j = 1, \ldots, r$, and $k = 0, \ldots, n$. Since B is a basis for P, we have $P \subseteq Q$.

To verify that Q has the claimed property, let $x' \ll x \ll y_1 + \ldots + y_r$ in P. Using that B is a basis, we can choose $c', c, d_1, \ldots, d_r \in B$ such that

$$x' \ll c' \ll c \ll x \ll d_1 + \ldots + d_r, \quad d_1 \ll y_1, \quad \ldots, \quad d_r \ll y_r.$$

Then $i := (c', c, d_1, \dots, d_r)$ belongs to I_r. By construction, Q contains the elements $z_{i,j,k}$, which satisfy (i)-(iii) from Definition 2.8 for $c' \ll c \ll d_1 + \dots + d_r$, and it is easy to see that these same elements satisfy (i)-(iii) from Definition 2.8 for $x' \ll x \ll y_1 + \dots + y_r$. This proves the claim.

Now, we successively apply the claim to obtain an increasing sequence $(T_k)_k$ of countably based sub-Cu-semigroups $T_k \subseteq S$ such that for every $k \in \mathbb{N}$ and $x' \ll x \ll y_1 + \dots + y_r$ in T_k there exist $z_{j,k} \in T_{k+1}$ for $j = 1, \dots, r$ and $k = 0, \dots, n$ satisfying (i)-(iii) from Definition 2.8.

Let $T_\infty := \bigcup_k T_k$, which by construction is a submonoid of S such that every element in T_∞ is the supremum of a $\ll$-increasing sequence in T_∞. Set $T := \overline{T_\infty}^{\mathrm{seq}}$. By Corollary 4.12, T is a sub-Cu-semigroup of S satisfying $T_0 \subseteq T$. It is now straightforward to verify that T is countably based and satisfies $\dim(T) \leq n$. $\square$

Proposition 5.6. *Let $n \in \mathbb{N}$. Given a* Cu*-semigroup S satisfying* $\dim(S) \leq n$*, the countably based sub*-Cu*-semigroups $T \subseteq S$ satisfying* $\dim(T) \leq n$ *form a σ-complete, cofinal subset of* $\mathrm{Sub}_{\mathrm{ctbl}}(S)$*.*

In particular, the property of Cu*-semigroups of having dimension at most n satisfies the Löwenheim-Skolem condition.*

Proof. Let $\mathcal{R}$ be the collection of sub-Cu-semigroups $T \subseteq S$ satisfying $\dim(T) \leq n$. By Proposition 2.9, the property of having dimension at most n passes to inductive limits, which shows that $\mathcal{R}$ is σ-complete. Further, $\mathcal{R}$ is cofinal by Lemma 5.5. $\square$

Theorem 5.7. *Let S be a* Cu*-semigroup, and let $n \in \mathbb{N}$. Then the following are equivalent:*

(1) $\dim(S) \leq n$*;*
(2) every countable subset of S is contained in a countably based sub-Cu*-semigroup $T \subseteq S$ satisfying* $\dim(T) \leq n$*;*
(3) every finite subset of S is contained in a sub-Cu*-semigroup $T \subseteq S$ satisfying* $\dim(T) \leq n$*.*

Proof. It follows from Lemmas 5.1 and 5.5 that (1) implies (2). It is clear that (2) implies (3). To show that (3) implies (1), let $\mathcal{T}$ be the collection of sub-Cu-semigroups with dimension at most n. For each $T \in \mathcal{T}$, let $\iota_T \colon T \to S$ denote the inclusion map. It follows from the assumption that the family $(T, \iota_T)_{T \in \mathcal{T}}$ approximates S. Hence, we have $\dim(S) \leq n$ by Proposition 3.4. $\square$

As an application of the methods developed in this section, we can remove the assumption of being countably based in several results from [TV21]. We first generalize [TV21, Proposition 7.14].

Proposition 5.8. *Let S be a zero-dimensional, simple, weakly cancellative* Cu*-semigroup satisfying (O5). Then S has the Riesz interpolation property. If we additionally assume that S is nonelementary, then S is almost divisible.*

Proof. If S is elementary, then S is isomorphic to $\{0, 1, \dots, \infty\}$, or to $\{0, 1, \dots, n\}$ for some $n \in \mathbb{N}$; see [APT18, Proposition 5.1.19]. In either case, S has the Riesz interpolation property. So we may assume from now on that S is nonelementary. This allows us to choose a sequence $(s_n)_n$ in S with $s_0 > s_1 > \dots$.

Let $\mathcal{R}_{\mathrm{O5}}$, $\mathcal{R}_{\mathrm{simple}}$, $\mathcal{R}_{\mathrm{canc}}$, and $\mathcal{R}_{\mathrm{dim0}}$ be the collections of countably generated sub-Cu-semigroups of S that satisfy (O5), or that are simple, weakly cancellative, or zero-dimensional, respectively. By Propositions 5.3, 5.4, and 5.6, each of these collections are σ-complete and cofinal. Set $\mathcal{R} := \mathcal{R}_{\mathrm{O5}} \cap \mathcal{R}_{\mathrm{simple}} \cap \mathcal{R}_{\mathrm{canc}} \cap \mathcal{R}_{\mathrm{dim0}}$. Then $\mathcal{R}$ is σ-complete and cofinal.

To verify that S has the Riesz interpolation property, let $x_0, x_1, y_0, y_1 \in S$ satisfy $x_j \leq y_k$ for all $j, k \in \{0, 1\}$. We need to find $z \in S$ such that $x_j \leq z \leq y_k$ for all

$j,k \in \{0,1\}$. Using Lemma 5.1 and that $\mathcal{R}$ is cofinal, we obtain $R \in \mathcal{R}$ containing x_0, x_1, y_0, y_1 and containing $s_0, s_1, \ldots$, which forces R to be nonelementary.

Note that R is a zero-dimensional, countably based, simple, weakly cancellative, nonelementary Cu-semigroup satisfying (O5). By [TV21, Proposition 7.14], R has the Riesz interpolation property. We therefore obtain z with the desired properties in $R \subseteq S$.

To verify that S is almost divisible, let $n \geq 1$, and let $x', x \in S$ satisfy $x' \ll x$. We need to find $z \in S$ such that $nz \ll x$ and $x' \ll (n+1)z$. As above, we obtain $R \in \mathcal{R}$ containing $x', x, s_0, s_1, \ldots$. By [TV21, Proposition 7.14], R is almost divisible, which allows us to find z with the desired properties in R. □

The next result generalizes [TV21, Proposition 3.17]. Recall that an element x in a Cu-semigroup S is said to be *soft* if for every $x' \ll x$ there exists $k \in \mathbb{N}$ such that $(k+1)x' \ll kx$. The set of soft elements, denoted by S_{soft}, is a sub-Cu-semigroup of S satisfying (O5) and (O6) whenever S is simple, weakly cancellative and satisfies (O5) and (O6); see [APT18, Proposition 5.3.18].

Proposition 5.9. *Let S be a simple, weakly cancellative* Cu*-semigroup satisfying (O5) and (O6). Then*

$$\dim(S_{\mathrm{soft}}) \leq \dim(S) \leq \dim(S_{\mathrm{soft}}) + 1.$$

Proof. To prove the first inequality, set $n := \dim(S)$, which we may assume to be finite. To verify condition (3) of Theorem 5.7, let H be a finite subset of S_{soft}.

Proceeding as in the proof of Proposition 5.8, and using Lemma 5.1 and Propositions 5.3, 5.4 and 5.6, there exists a simple, weakly cancellative, countably based sub-Cu-semigroup $T \subseteq S$ satisfying (O5) and (O6) with $\dim(T) \leq n$ and $H \subseteq T$.

It follows from [TV21, Proposition 3.17] that $\dim(T_{\mathrm{soft}}) \leq n$. We note that T_{soft} is a sub-Cu-semigroup of S_{soft} containing H. Thus, every finite subset of S_{soft} is contained in a sub-Cu-semigroup of dimension at most n. This shows that $\dim(S_{\mathrm{soft}}) \leq n$ by Theorem 5.7.

To prove the second inequality, set $m := \dim(S_{\mathrm{soft}})$, which we may assume to be finite. Let H be a finite subset of S. Using once again Lemma 5.1 and Proposition 5.3, one finds a countably based sub-Cu-semigroup $T^{(1)} \subseteq S$ satisfying (O5) and (O6) with $H \subseteq T^{(1)}$.

By Proposition 5.6, there exists a countably based sub-Cu-semigroup $R^{(1)}$ of S_{soft} such that $T^{(1)}_{\mathrm{soft}} \subset R^{(1)}$ and $\dim(R^{(1)}) \leq m$. Since $R^{(1)}$ and $T^{(1)}$ are countably based, there exists by Lemma 5.1 and Proposition 5.3 a countably based sub-Cu-semigroup $T^{(2)} \subseteq S$ satisfying (O5) and (O6) with $R^{(1)}, T^{(1)} \subseteq T^{(2)}$.

Proceeding in this manner, one obtains an increasing sequence of countably based sub-Cu-semigroups $T^{(k)} \subseteq S$ satisfying (O5) and (O6) and an increasing sequence of sub-Cu-semigroup $R^{(k)}$ of S_{soft} with dimension at most m such that

$$T^{(k)}_{\mathrm{soft}} \subseteq R^{(k)} \subseteq T^{(k+1)}.$$

Set $T := \overline{\bigcup_k T^{(k)}}^{\mathrm{sup}}$ and $R := \overline{\bigcup_k R^{(k)}}^{\mathrm{sup}}$. Then $T \subseteq S$ is a countably based sub-Cu-semigroup. Since each $T^{(k)}$ satisfies (O5) and (O6), it follows from Proposition 5.3 that T also satisfies them. Moreover, using that T is a sub-Cu-semigroup of S, it follows from Proposition 5.4 that T is also simple and weakly cancellative.

Since $\dim(R^{(k)}) \leq m$ for every k, we have by Theorem 5.7 (or using Proposition 2.9) that $\dim(R) \leq m$, and it is easy to check that $T_{\mathrm{soft}} = R$. Applying [TV21, Proposition 3.17], we get

$$\dim(T) \leq \dim(T_{\mathrm{soft}}) + 1 = \dim(R) + 1 \leq m + 1.$$

Thus, every finite subset of S is contained in a sub-Cu-semigroup with dimension at most $m+1$. This shows, by Theorem 5.7, that $\dim(S) \leq m+1$, as desired. □

6. Dimension of the Cuntz semigroup as a noncommutative dimension theory

In this section, we show that associating to a C^*-algebra the dimension of its Cuntz semigroup satisfies the Löwenheim-Skolem condition; see Theorem 6.2. It follows that this association is a well-behaved invariant that satisfies most of the axioms of a noncommutative dimension theory in the sense of [Thi13, Definition 1]; see Paragraph 6.4. It remains open if the dimension of the Cuntz semigroup is compatible with minimal unitizations; see Question 6.6.

If $B \subseteq A$ is a sub-C^*-algebra, then the inclusion map $B \to A$ induces a Cu-morphism $\mathrm{Cu}(B) \to \mathrm{Cu}(A)$ which in general is not an order-embedding. Thus, the Cuntz semigroup of a sub-C^*-algebra is not necessarily a sub-Cu-semigroup. However, the next results shows that there are sufficiently many separable sub-C^*-algebras whose Cuntz semigroups are sub-Cu-semigroups.

Given a C^*-algebra A, we let $\mathrm{Sub}_{\mathrm{sep}}(A)$ denote the collection of separable sub-C^*-algebras of A. See [Thi20a, Paragraph 3.1] for details.

Proposition 6.1. *Let A be a C^*-algebra. Then*

$$\mathcal{S} := \big\{ B \in \mathrm{Sub}_{\mathrm{sep}}(A) : \mathrm{Cu}(B) \to \mathrm{Cu}(A) \text{ is an order-embedding} \big\}$$

*is σ-complete and cofinal. Each $B \in \mathcal{S}$ induces a countably based sub-*Cu*-semigroup* $\mathrm{Cu}(B) \subseteq \mathrm{Cu}(A)$*. Let $\alpha \colon \mathcal{S} \to \mathrm{Sub}_{\mathrm{ctbl}}(\mathrm{Cu}(A))$ be the map that sends $B \in \mathcal{S}$ to the sub-*Cu*-semigroup* $\mathrm{Cu}(B) \subseteq \mathrm{Cu}(A)$*. Then α preserves the order and the suprema of countable directed subsets, and the image of α is a cofinal subset of* $\mathrm{Sub}_{\mathrm{ctbl}}(\mathrm{Cu}(A))$.

Proof. To prove that $\mathcal{S}$ is σ-complete, let $\mathcal{T} \subseteq \mathcal{S}$ be a countable, directed subfamily. Set $D := \overline{\bigcup \mathcal{T}}$. We need to verify $D \in \mathcal{S}$. Let $\varphi_{A,D} \colon \mathrm{Cu}(D) \to \mathrm{Cu}(A)$ denote the Cu-morphism induced by the inclusion map $D \to A$. Similarly, we define $\varphi_{D,B} \colon \mathrm{Cu}(B) \to \mathrm{Cu}(D)$ and $\varphi_{A,B} \colon \mathrm{Cu}(B) \to \mathrm{Cu}(A)$ for each $B \in \mathcal{T}$.

To verify that $\varphi_{A,D}$ is an order-embedding, let $x, y \in \mathrm{Cu}(D)$ satisfy

$$\varphi_{A,D}(x) \leq \varphi_{A,D}(y).$$

Let $x', x'' \in \mathrm{Cu}(D)$ be such that $x' \ll x'' \ll x$. Then $\varphi_{A,D}(x'') \ll \varphi_{A,D}(x) \leq \varphi_{A,D}(y)$, which allows us to choose $y' \in \mathrm{Cu}(D)$ with

$$\varphi_{A,D}(x'') \ll \varphi_{A,D}(y'), \quad \text{and} \quad y' \ll y.$$

Using that $D \cong \varinjlim_{B \in \mathcal{T}} B$, we have $\mathrm{Cu}(D) \cong \varinjlim_{B \in \mathcal{T}} \mathrm{Cu}(B)$ by [APT18, Corollary 3.2.9]. Applying (L2) from [TV21, Paragraph 3.8] (see also the proof of Proposition 3.5), we obtain $B \in \mathcal{T}$ and $c, d \in \mathrm{Cu}(B)$ such that

$$x' \ll \varphi_{D,B}(c) \ll x'', \quad \text{and} \quad y' \ll \varphi_{D,B}(d) \ll y.$$

Then

$$\varphi_{A,B}(c) = \varphi_{A,D}(\varphi_{D,B}(c)) \ll \varphi_{A,D}(x'') \ll \varphi_{A,D}(y') \ll \varphi_{A,D}(\varphi_{D,B}(d)) = \varphi_{A,B}(d).$$

Using that $\varphi_{A,B}$ is an order-embedding, we obtain $c \ll d$ in $\mathrm{Cu}(B)$, and thus

$$x' \ll \varphi_{D,B}(c) \ll \varphi_{D,B}(d) \ll y.$$

Thus, $x' \ll y$ for every x' way-below x, which implies $x \leq y$.

To verify that $\mathcal{S}$ is cofinal, let $B_0 \in \mathrm{Sub}_{\mathrm{sep}}(A)$. By [FHL+16, Theorem 2.6.2], there exists $B \in \mathrm{Sub}_{\mathrm{sep}}(A)$ such that $B_0 \subseteq B$ and such that B is an elementary submodel of A. By [FHL+16, Lemma 8.1.3], $\mathrm{Cu}(B) \to \mathrm{Cu}(A)$ is an order-embedding. Thus, B belongs to $\mathcal{S}$, as desired.

Given $B \in \mathcal{S}$, it follows from Lemma 4.3 that we can identify $\mathrm{Cu}(B)$ with a sub-Cu-semigroup of $\mathrm{Cu}(A)$. Since B is separable, $\mathrm{Cu}(B)$ is countably based. It is then straightforward to see that the map α is order-preserving. Next, let $\mathcal{T} \subseteq \mathcal{S}$ be a countable, directed subset, and set $D := \overline{\bigcup \mathcal{T}}$. We identify $\mathrm{Cu}(D)$ and $\mathrm{Cu}(B)$

(for each $B \in \mathcal{T}$) with sub-Cu-semigroups of $\mathrm{Cu}(A)$. Then $(\mathrm{Cu}(B))_{B\in\mathcal{T}}$ is a countable, directed family in $\mathrm{Sub}_{\mathrm{ctbl}}(\mathrm{Cu}(A))$, with supremum given by $\sup_{B\in\mathcal{T}} \mathrm{Cu}(B) = \overline{\bigcup_{B\in\mathcal{T}} \mathrm{Cu}(B)}^{\mathrm{sup}}$; see Theorem 4.15. Since $\mathrm{Cu}(B)$ is contained in $\mathrm{Cu}(D)$ for each $B \in \mathcal{T}$, we have

$$\overline{\bigcup_{B\in\mathcal{T}} \mathrm{Cu}(B)}^{\mathrm{sup}} \subseteq \mathrm{Cu}(D).$$

The other inclusion follows using that $\mathrm{Cu}(D)$ is the inductive limit of $(\mathrm{Cu}(B))_{B\in\mathcal{T}}$. This shows that α preserves suprema of countable directed subsets.

Finally, to show that the image of α is cofinal, let $T \in \mathrm{Sub}_{\mathrm{ctbl}}(\mathrm{Cu}(A))$. Choose a countable basis $D \subseteq T$. For each $x \in D$ choose $a_x \in (A\otimes\mathcal{K})_+$ with $x = [a_x]$. We can then choose a separable sub-C^*-algebra $B_0 \subseteq A$ such that each a_x is contained in $B_0 \otimes \mathcal{K} \subseteq A\otimes\mathcal{K}$. Using that $\mathcal{S}$ is cofinal, we obtain $B \in \mathcal{S}$ containing B_0. Then the sub-Cu-semigroup $\mathrm{Cu}(B) \subseteq \mathrm{Cu}(A)$ contains each $x \in D$, which implies $T \subseteq \mathrm{Cu}(B)$ as required. □

Theorem 6.2. *Let $n \in \mathbb{N}$, and let A be a C^*-algebra satisfying* $\dim(\mathrm{Cu}(A)) \leq n$. *Then*

$$\mathcal{S} := \big\{B \in \mathrm{Sub}_{\mathrm{sep}}(A) : \mathrm{Cu}(B) \to \mathrm{Cu}(A) \text{ order-embedding}, \dim(\mathrm{Cu}(B)) \leq n\big\}$$

is σ-complete and cofinal.

In particular, for every separable sub-C^*-algebra $B_0 \subseteq A$ there exists a separable sub-C^*-algebra $B \subseteq A$ such that $B_0 \subseteq B$ and $\dim(\mathrm{Cu}(B)) \leq n$.

Proof. Set

$$\mathcal{S}_0 := \big\{B \in \mathrm{Sub}_{\mathrm{sep}}(A) : \mathrm{Cu}(B) \to \mathrm{Cu}(A) \text{ is an order-embedding}\big\},$$
$$\mathcal{T} := \big\{T \in \mathrm{Sub}_{\mathrm{ctbl}}(\mathrm{Cu}(A)) : \dim(T) \leq n\big\}.$$

By Proposition 6.1, $\mathcal{S}_0$ is a σ-complete and cofinal subfamily of $\mathrm{Sub}_{\mathrm{sep}}(A)$. Similarly, by Proposition 5.6, $\mathcal{T}$ is a σ-complete and cofinal subset of $\mathrm{Sub}_{\mathrm{ctbl}}(\mathrm{Cu}(B))$. Let $\alpha\colon \mathcal{S}_0 \to \mathrm{Sub}_{\mathrm{ctbl}}(\mathrm{Cu}(A))$ be the map that sends $B \in \mathcal{S}_0$ to the sub-Cu-semigroup $\mathrm{Cu}(B) \subseteq \mathrm{Cu}(A)$, as in Proposition 6.1. Then

$$\mathcal{S} = \big\{B \in \mathcal{S}_0 : \alpha(B) \in \mathcal{T}\big\}.$$

Using that $\mathcal{S}_0$ and $\mathcal{T}$ are σ-complete, and using that α preserves suprema of countable, directed sets, it follows that $\mathcal{S}$ is σ-complete. To show that $\mathcal{S}$ is cofinal, let $B_0 \in \mathrm{Sub}_{\mathrm{sep}}(A)$. Using that $\mathcal{T}$ is cofinal, we obtain $T_0 \in \mathcal{T}$ such that $\alpha(B_0) \subseteq T_0$. Using that the image of α is cofinal, we find $B_1 \in \mathcal{S}_0$ such that $T_0 \subseteq \alpha(B_1)$. Continuing successively, we obtain an increasing sequence $(T_k)_{k\in\mathbb{N}}$ in $\mathcal{T}$ and an increasing sequence $(B_k)_{k\geq 1}$ in $\mathcal{S}_0$ such that

$$\alpha(B_0) \subseteq T_0 \subseteq \alpha(B_1) \subseteq T_1 \subseteq \alpha(B_2) \subseteq T_2 \subseteq \ldots.$$

Set $B := \overline{\bigcup_k B_k}$ and $T := \overline{\bigcup_k T_k}^{\mathrm{sup}}$. Then $B_0 \subseteq B$, $B \in \mathcal{S}_0$ and $T \in \mathcal{T}$. Using that α preserves suprema of countable, directed sets, we get $\alpha(B) = T$, and thus $B \in \mathcal{S}$, as desired. □

Corollary 6.3. *Let A be a C^*-algebra, and let $n \in \mathbb{N}$. Then* $\dim(\mathrm{Cu}(A)) \leq n$ *if and only if every finite (or countable) subset of A is contained in a separable sub-C^*-algebra $B \subseteq A$ satisfying* $\dim(\mathrm{Cu}(B)) \leq n$.

Proof. The forward implication is Theorem 6.2, and the backward implication follows from Theorem 3.8. □

6.4. Following [Thi13, Definition 1], we say that an assignment that to each C^*-algebra A associates a number (the dimension) $d(A) \in \{0, 1, 2, \ldots, \infty\}$ is a *(noncommutative) dimension theory* if the following conditions are satisfied:

(D1) $d(I) \leq d(A)$ whenever $I \subseteq A$ is an ideal in a C^*-algebra A;
(D2) $d(A/I) \leq d(A)$ whenever $I \subseteq A$ is an ideal in a C^*-algebra A;
(D3) $d(A \oplus B) = \max\{d(A), d(B)\}$, whenever A and B are C^*-algebras;
(D4) $d(\widetilde{A}) = d(A)$ for every C^*-algebra A;
(D5) If $n \in \mathbb{N}$ and if A is a C^*-algebra that is approximated by sub-C^*-algebras $A_\lambda \subseteq A$ with $d(A_\lambda) \leq n$, then $d(A) \leq n$;
(D6) Given a C^*-algebra A and a separable sub-C^*-algebra $B_0 \subseteq A$, there exists a separable sub-C^*-algebra $B \subseteq A$ such that $B_0 \subseteq B$ and $d(B) \leq d(A)$.

Assigning to a C^*-algebra the dimension of its Cuntz semigroup satisfies conditions (D1), (D2) and (D3) (by Proposition 3.10 of [TV21]), (D5) (by Theorem 3.8) and (D6) (by Theorem 6.2). However, Example 6.5 shows that (D4) does not hold.

Using [Thi13, Proposition 3], one can also see that this assignemnt is in fact *Morita-invariant*, that is, $\dim(\mathrm{Cu}(A)) = \dim(\mathrm{Cu}(B))$ whenever A and B are Morita equivalent.

Example 6.5. Let $\mathcal{W}$ denote the Jacelon-Razac algebra. Then

$$\dim(\mathrm{Cu}(\mathcal{W})) = 0, \quad \text{and} \quad \dim(\mathrm{Cu}(\widetilde{\mathcal{W}})) = 1.$$

Indeed, we have $\mathrm{Cu}(\mathcal{W}) \cong [0, \infty]$, which is easily seen to be zero-dimensional. (See also [TV21, Proposition 3.22].)

Further, since the nuclear dimension of $\widetilde{\mathcal{W}}$ is 1, we have $\dim(\mathrm{Cu}(\widetilde{\mathcal{W}})) \leq 1$ by [TV21, Theorem 4.1]. On the other hand, since $\mathcal{W}$ has stable rank one, so does $\widetilde{\mathcal{W}}$ (by definition); but $\mathcal{W}$ does not have real rank zero, and hence neither does $\widetilde{\mathcal{W}}$. Thus, it follows from [TV21, Corollary 5.8] that $\mathrm{Cu}(\widetilde{\mathcal{W}})$ is not zero-dimensional.

Question 6.6. Let $I \subseteq A$ be an ideal in a unital C^*-algebra A. Do we have $\dim(\mathrm{Cu}(\widetilde{I})) \leq \dim(\mathrm{Cu}(A))$?

If the above question has a positive answer, then associating to a C^*-algebra the dimension of the Cuntz semigroup of its minimal unitization is a noncommutative dimension theory. Indeed, one can verify that this assignment satisfies (D2)-(D6), and Question 6.6 is asking if (D1) holds.

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