Integral Geometry about the visual angle of a convex set

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Abstract

In this paper we deal with a general type of integral formulas of the visual angle, among them those of Crofton, Hurwitz and Masotti, from the point of view of Integral Geometry. The purpose is twofold: to provide an interpretation of these formulas in terms of integrals of densities with respect to the canonical measure in the space of pairs of lines and to give new simpler proofs of them.

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1 Introduction

Throughout this paper K will be a compact convex set in $\mathbb{R}^2$ with boundary of class $\mathcal{C}^2$. We will denote by F the area of K and by L the length of its boundary.

In 1868 Crofton showed ([1]), using arguments that nowadays belong to Integral Geometry, the well known formula

$$2\int_{P\notin K}(\omega-\sin\omega)\,dP+2\pi F=L^2, \tag{1}$$

where $\omega=\omega(P)$ is the *visual angle* of K from the point P, that is the angle between the two tangents from P to the boundary of K. In terms of Integral Geometry both sides of this formula represent the measure of pairs of lines meeting K. In fact the measure of all pairs of lines meeting K is L^2, twice the integral of $\omega-\sin\omega$ with respect to the area element dP is the measure of those pairs of lines intersecting themselves outside K and $2\pi F$ is the measure of those intersecting themselves in K.

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Later on, Hurwitz in 1902, in his celebrated paper [4] on the application of Fourier series to geometric problems, considers the integral of some new functions of the visual angle. Concretely he proves

$$\int_{P\notin K} f_k(\omega)\, dP = L^2 + (-1)^k \pi^2 (k^2-1) c_k^2, \tag{2}$$

where

$$f_k(\omega) = -2\sin\omega + \frac{k+1}{k-1}\sin((k-1)\omega) - \frac{k-1}{k+1}\sin((k+1)\omega), \quad k \geq 2, \tag{3}$$

and $c_k^2 = a_k^2 + b_k^2$, with a_k, b_k the Fourier coefficients of the support function of K.

In the particular case $k = 2$ formula (2) gives

$$\int_{P\notin K} \sin^3\omega\, dP = \frac{3}{4}L^2 + \frac{9}{4}\pi^2 c_2^2. \tag{4}$$

Masotti in 1955 ([5]) states without proof the following Crofton's type formula

$$\int_{P\notin K} (\omega^2 - \sin^2\omega)\, dP = -\pi^2 F + \frac{4L^2}{\pi} + 8\pi \sum_{k\geq 1} \left(\frac{1}{1-4k^2}\right) c_{2k}^2. \tag{5}$$

In [2] a unified approach that encompasses the previous results is provided. As well the following formula for the integral of any power of the sine function of the visual angle, that generalises (4), is given:

$$\int_{P\notin K} \sin^m(\omega)\, dP = \frac{m!}{2^m (m-2)\Gamma(\frac{m-1}{2})^2}\, L^2 + \frac{m!\pi^2}{2^{m-1}(m-2)} \sum_{k\geq 2,\ \text{even}} \frac{(-1)^{\frac{k}{2}+1}(k^2-1)}{\Gamma(\frac{m+1+k}{2})\Gamma(\frac{m+1-k}{2})} c_k^2. \tag{6}$$

In this paper we deal with a general type of integral formulas of the visual angle including those we have just commented above, from the point of view of Integral Geometry according to Crofton and Santaló [6]. The purpose is twofold: to provide an interpretation of these formulas in terms of integrals of densities with respect to the canonical measure in the space of pairs of lines and to give new simpler proofs of them.

For each straight line G of the plane that does not pass through the origin let P be the point of G at a minimum distance from the origin. We take as coordinates for G the polar coordinates (p, φ) of the point P, with $p > 0$ and $0 \leq \varphi < 2\pi$. The invariant measure in the set of lines of the plane not containing the origin is given by a constant multiple of $dG = dp\, d\varphi$. In the space of ordered pairs of lines we consider the canonical measure $dG_1\, dG_2$. This measure is, except for a constant factor, the only one invariant under Euclidean motions (see [6]). For every function $\tilde{f}(G_1, G_2)$ integrable with respect to $dG_1\, dG_2$ we can consider the measure with density $\tilde{f}$, that is $\tilde{f}(G_1, G_2)\, dG_1\, dG_2$. We prove in Proposition 1 that this measure is invariant under Euclidean motions if and only if $\tilde{f}(G_1, G_2) = f(\varphi_2 - \varphi_1)$ with f a π-periodic function on $\mathbb{R}$.

For such densities, and under some additional hypothesis, it follows from Theorem 1 and Corollary 2 that

$$A_0 L^2 + \pi^2 \sum_{n\geq 1} c_{2n}^2 A_{2n} = \int_{G_i \cap K \neq \emptyset} f(\varphi_2 - \varphi_1)\, dG_1\, dG_2$$
$$= 2H(\pi)F + 2\int_{P\notin K} H(\omega)\, dP, \quad (7)$$

where A_k, $k \geq 0$, are the Fourier coefficients of f corresponding to $\cos(k\varphi)$, and $H(x)$ is a $\mathcal{C}^2$ function on $[0, \pi]$ satisfying $H''(x) = f(x)\sin(x)$, $x \in [0, \pi]$, and $H(0) = H'(0) = 0$.

The above two equalities are the main tools to obtain both new proofs of the formulas discussed above and their interpretation as integrals of densities with respect to the canonical measure in the space of pairs of lines. As concerning to this second point, in section 3.3 one obtains the following formulas.

- *Crofton's formula*

$$\int_{P\notin K} (\omega - \sin\omega)\, dP = -\pi F + \frac{1}{2}\int_{G_i \cap K \neq \emptyset} dG_1\, dG_2.$$

- *Hurwitz's formula*

$$\int_{P\notin K} f_k(\omega)\, dP = \int_{G_i \cap K \neq \emptyset} (1 + (-1)^k (k^2 - 1)\cos(k(\varphi_2 - \varphi_1)))\, dG_1\, dG_2.$$

- *Masotti's formula*

$$\int_{P\notin K} (\omega^2 - \sin^2\omega)\, dP = -\pi^2 F + 2\int_{G_i \cap K \neq \emptyset} |\sin(\varphi_2 - \varphi_1)|\, dG_1\, dG_2.$$

- *Power sine formula*

$$\int_{P\notin K} \sin^m \omega\, dP$$
$$= \frac{1}{2}\int_{G_i \cap K \neq \emptyset} \left(m(m-1)|\sin^{m-3}(\varphi_2 - \varphi_1)| - m^2 |\sin^{m-1}(\varphi_2 - \varphi_1)| \right)\, dG_1\, dG_2.$$

Moreover using the first equality in (7) one gets the announced new proofs of formulas (1), (5) and (6).

Concerning Hurwitz's integral, when we apply the methods here developed, it appears a different behavior according to k is either even or odd. For k even using (7) one gets a new proof of (2). Nevertheless when k is odd the density associated to the Hurwitz integral is not π-periodic since the function $\cos(kx)$ is not, and so we cannot use (7). In this case appealing to Proposition 5 one obtains a new result that involves a decomposition of the visual angle ω into two parts $\omega = \omega_1 + \omega_2$ that also have a geometrical interpretation.

In this setting it plays a role the function $f_k(\omega) + 2(\sin\omega - \omega)$, that is the sum of the functions of Hurwitz and Crofton. In spite of $\int_{P\notin K}(f_k(\omega) + 2(\sin\omega - \omega))\,dP$ depends on k, the surprising fact is that, for k odd, decomposing the visual angle ω into the two parts ω_1, ω_2 and adding the corresponding integrals leads to

$$\int_{P\notin K} (f_k(\omega_1) + 2(\sin\omega_1 - \omega_1) + f_k(\omega_2) + 2(\sin\omega_2 - \omega_2))\; dP = 2\pi F,$$

for each $k \geq 3$ odd, as a consequence of Proposition 9.

Moreover it will appear that the functions of Crofton and Hurwitz are in some sense a basis for the integral of any π-periodic or anti π-periodic density with respect to the measure $dG_1\, dG_2$ over the set of pairs of lines meeting a given compact convex set.

2 Densities in the space of pairs of lines

For every function $\tilde f(G_1, G_2)$ defined on the space of pairs of lines integrable with respect to the measure $dG_1\, dG_2$ we consider the measure with density $\tilde f$, that is the measure $\tilde f(G_1, G_2)\, dG_1\, dG_2$. The measure of a set A of pairs of lines in the plane is then given by

$$\int_A \tilde f(G_1, G_2)\, dG_1\, dG_2.$$

We want now to determine when this measure is invariant under Euclidean motions.

Proposition 1. *The measure $\tilde f(G_1, G_2)\, dG_1\, dG_2$ is invariant under the group of Euclidean motions if and only if $\tilde f(G_1, G_2) = \tilde f(p_1, \varphi_1, p_2, \varphi_2) = f(\varphi_2 - \varphi_1)$ with f a π-periodic function on $\mathbb{R}$, where (p_i, φ_i) are the coordinates of G_i.*

Proof. The invariance of the measure is equivalent to the equality $\tilde f(p_1,\varphi_1,p_2,\varphi_2) = \tilde f(p_1', \varphi_1', p_2', \varphi_2')$ for each Euclidean motion sending the lines with coordinates $(p_1, \varphi_1, p_2, \varphi_2)$ to the lines with coordinates $(p_1', \varphi_1', p_2', \varphi_2')$. First of all let us show that $\tilde f$ does not depend on p_1, p_2. In fact, for every straight line $G = G(p, \varphi)$ and an arbitrary $a > 0$ there is a parallel line to G with coordinates (a, φ). Given two straight lines $G_1 = G(p_1, \varphi_1)$, $G_2 = G(p_2, \varphi_2)$ and two numbers $a_1, a_2 > 0$ let G_1' and G_2' be the corresponding parallel lines with coordinates (a_1, φ_1), (a_2, φ_2). Performing the translation that sends the point $G_1 \cap G_2$ to the point $G_1' \cap G_2'$

we have that $\tilde{f}(p_1, \varphi_1, p_2, \varphi_2) = \tilde{f}(a_1, \varphi_1, a_2, \varphi_2)$ and so $\tilde{f}$ does not depend on p_1 and p_2.

Given now the line $G(p, \varphi)$ if we perform, for instance, the translation given by the vector $-(p+\epsilon)(\cos\varphi, \sin\varphi)$, $\epsilon > 0$, the translated line has coordinates $(\epsilon, \varphi+\pi)$. Therefore the function $\tilde{f}$ must be π-periodic with respect to the arguments φ_1, φ_2. Finally due to the invariance under rotations it follows that $\tilde{f}(p_1, \varphi_1, p_2, \varphi_2) = \tilde{f}(p_1, 0, p_2, \varphi_2 - \varphi_1)$ and so $\tilde{f}(p_1, \varphi_1, p_2, \varphi_2) = f(\varphi_2 - \varphi_1)$ with f a π-periodic function. □

Our goal is now to integrate a measure given by a density over the set of pairs of lines meeting K. In view of Proposition 1 we shall only consider densities which depend on the angle of the two lines, that is of the form $\tilde{f}(G_1, G_2) = f(\varphi_2 - \varphi_1)$, with $G_i = G_i(p_i, \varphi_i)$, $i = 1, 2$. Note that $\varphi_2 - \varphi_1$ gives one of the two angles between the lines G_1 and G_2.

We give a formula to compute the integral of the measure $\tilde{f}(G_1, G_2)\, dG_1\, dG_2 = f(\varphi_2 - \varphi_1)\, dG_1\, dG_2$, with f a 2π-periodic function extended to the pairs of lines meeting K in terms of both the Fourier coefficients of f and of the support function of K. Recall that when the origin of coordinates is an interior point of K, a hypothesis that we will assume from now on, the support function $p(\varphi)$ is given by the distance to the origin of the tangent to K whose normal makes and angle φ with the positive part of the real axis (see [6]).

Theorem 1. *Let K be a compact convex set with boundary of length L. Let f be a 2π-periodic continuous function on $\mathbb{R}$ with Fourier expansion*

$$f(\varphi) = \sum_{n\geq 0} A_n \cos(n\varphi) + B_n \sin(n\varphi).$$

Then

$$\int_{G_i\cap K\neq\emptyset} f(\varphi_2 - \varphi_1)\, dG_1\, dG_2 = A_0 L^2 + \pi^2 \sum_{n\geq 1} c_n^2 A_n, \tag{8}$$

with $c_n^2 = a_n^2 + b_n^2$ where a_n, b_n are the Fourier coefficients of the support function $p(\varphi)$ of K.

Proof. We have

$$\begin{aligned}\int_{G_i\cap K\neq\emptyset} f(\varphi_2 - \varphi_1)\, dG_1\, dG_2 &= \int_0^{2\pi}\int_0^{2\pi}\int_0^{p(\varphi_1)}\int_0^{p(\varphi_2)} f(\varphi_2 - \varphi_1)\, dp_1\, dp_2\, d\varphi_1\, d\varphi_2 \\ &= \int_0^{2\pi}\int_0^{2\pi} p(\varphi_1)p(\varphi_2) f(\varphi_2 - \varphi_1)\, d\varphi_1\, d\varphi_2.\end{aligned} \tag{9}$$

Performing the change of variables $\varphi_2 - \varphi_1 = w$, $\varphi_1 = u$ the integral (9) becomes

$$\int_0^{2\pi} p(u) \int_{-u}^{2\pi-u} p(u+w) f(w)\, dw\, du. \tag{10}$$

The Fourier development of $p(u+w)$ in terms of the Fourier coefficients a_n, b_n of $p(u)$ is

$$p(u+w) = a_0 + \sum_{n\geq 1}\Big((a_n \cos(nu) + b_n \sin(nu)) \cos(nw) \\ + (-a_n \sin(nu) + b_n \cos(nu)) \sin(nw)\Big).$$

By the Plancherel identity the integral (10) is equal to

$$\int_0^{2\pi} p(u)\Big[2\pi\, A_0\, a_0 + \pi \sum_{n\geq 1} A_n(a_n \cos(nu) + b_n \sin(nu)) \\ + B_n(-a_n \sin(nu) + b_n \cos(nu))\Big]\, du$$

$$= \int_0^{2\pi} p(u)\Big[2\pi\, A_0\, a_0 + \pi \sum_{n\geq 1} (A_n a_n + B_n b_n) \cos(nu) + (A_n b_n - B_n a_n) \sin(nu))\Big]\, du$$

$$= 4\pi^2 A_0\, a_0^2 + \pi^2 \sum_{n\geq 1} (A_n a_n + B_n b_n) a_n + (A_n b_n - B_n a_n) b_n$$

$$= 4\pi^2 A_0\, a_0^2 + \pi^2 \sum_{n\geq 1} A_n (a_n^2 + b_n^2) = A_0 L^2 + \pi^2 \sum_{n\geq 1} A_n c_n^2,$$

where we have used that $L = 2\pi a_0$, which is a consequence of the equality $L = \int_0^{2\pi} p(\varphi)\, d\varphi$ (see for instance [6]), and the Theorem is proved. □

As it is well known (see [4]) the quantities $c_k^2 = a_k^2 + b_k^2$, $k \geq 2$, are invariant under Euclidean motions of K. However c_1^2 changes when moving K. So the integral in (8) is invariant under Euclidean motions of K if and only if $A_1 = 0$. In particular this is the case when f is π-periodic.

For a density given by a π-periodic function f and a compact set of constant width the measure of the pairs of lines that intersect K is proportional to L^2. More precisely we have

Corollary 1. *Let K be a compact convex set of constant width and f a continuous π-periodic function. Then*

$$\int_{G_i \cap K \neq \emptyset} f(\varphi_2 - \varphi_1)\, dG_1\, dG_2 = \lambda L^2,$$

where $\lambda = (1/\pi) \int_0^{\pi} f(\varphi)\, d\varphi$.

Proof. Since K is of constant width the Fourier development of $p(\varphi)$ has only odd terms (see for instance §2 of [2]). Moreover the Fourier development of f has only even terms because it is π-periodic. Hence (8) gives

$$\int_{G_i \cap K \neq \emptyset} f(\varphi_2 - \varphi_1)\, dG_1\, dG_2 = A_0 L^2,$$

with $A_0 = (1/\pi) \int_0^\pi f(\varphi)\, d\varphi$. □

3 Integral formulas of the visual angle in terms of densities in the space of pairs of lines

In [2] there is a unified approach to several classical formulas involving integrals of functions of the visual angle of a compact convex set K. Among them one can find the integrals of Crofton, Masotti, powers of sine, and Hurwitz.

The original proof of Crofton's formula, via Integral Geometry, involves a measure on the space of pairs of lines. The aim of this section is to interpret the formulas in [2] in terms of integrals of measures given by densities in the space of pairs of lines.

To begin with let us consider Hurwitz's formula

$$\int_{P \notin K} f_k(\omega)\, dP = L^2 + (-1)^k \pi^2 (k^2 - 1) c_k^2, \tag{11}$$

where $f_k(\omega)$ is given in (3). For a proof of (11) see [4] or [2].

Comparing this equality with (8) one gets immediately the following result.

Proposition 2. *Let f_k be the Hurwitz function defined in* (3)*. Then*

$$\int_{P \notin K} f_k(\omega)\, dP = \int_{G_i \cap K \neq \emptyset} (1 + (-1)^k (k^2 - 1) \cos(k(\varphi_2 - \varphi_1)))\, dG_1\, dG_2.$$

Nevertheless for the other quoted integral formulas it is not clear at all what density must be chosen. We shall provide a general method to find the densities corresponding to integrals of general functions of the visual angle.

3.1 A change of variables

The classical proof of Crofton's formula is based on the change of variables in the space of pairs of lines given by

$$(p_1, \varphi_1, p_2, \varphi_2) \longrightarrow (P, \alpha_1, \alpha_2),$$

where P is the intersection point of the two straight lines and $\alpha_i \in [0, \pi]$ are the angles which determine the directions of the lines. More precisely the angle α associated to a line through a given point P is defined in the following way. Let

$\vec{u}$ be a unitary vector orthogonal to $\overrightarrow{OP}$ where O is the origin of coordinates, and such that the basis $(\vec{u}, \overrightarrow{OP})$ is positively oriented. Let G be a line through P with unitary director vector $\vec{v}$ such that the basis $(\vec{u}, \vec{v})$ is positively oriented. Then $\alpha = \alpha(G)$ is defined by $\cos\alpha = \vec{u} \cdot \vec{v}$ and $0 < \alpha < \pi$. From now on we shall say that α is the *direction* of the line G.

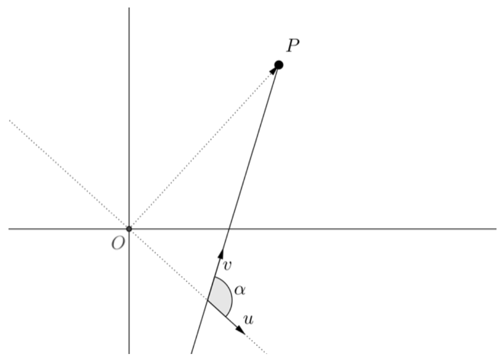

Figure 1: *Direction* of a line.

With these new coordinates, proceeding as in [6], one has

$$dG_1\, dG_2 = |\sin(\alpha_2 - \alpha_1)|\, d\alpha_1\, d\alpha_2\, dP. \tag{12}$$

We have used the fact that $\varphi_2 - \varphi_1 = \alpha_2 - \alpha_1 + \epsilon\pi$ where $\epsilon = \epsilon(P, \alpha_1, \alpha_2) = 0, \pm 1$, according to the position with respect to the origin of the lines G_1, G_2. As a consequence if f is a π-periodic function we have

$$f(\varphi_2 - \varphi_1)\, dG_1\, dG_2 = f(\alpha_2 - \alpha_1)|\sin(\alpha_2 - \alpha_1)|\, d\alpha_1\, d\alpha_2\, dP. \tag{13}$$

3.2 Integrals of functions of pairs of lines meeting a convex set

For a point $P \notin K$ let α, β be the directions we have introduced corresponding to the support lines of K through P, with $0 < \alpha < \pi/2$ and $\pi/2 < \beta < \pi$. Then $\omega = \beta - \alpha$ is the visual angle of K from P. This is the reason why we have slightly modified the definition of the direction angle given by Santaló in [6] as the angle between the line through P and the positive x axis, because with this definition one could have $\omega = \beta - \alpha$ or $\omega = \pi - (\beta - \alpha)$; see Figure 2. We shall provide now a general formula to calculate the integral of the right-hand side of (13).

Proposition 3. *Let f be a 2π-periodic continuous function on $\mathbb{R}$, and H a $\mathcal{C}^2$ function on $[-\pi, \pi]$ satisfying the conditions $H''(x) = f(x) \cdot \sin(x)$, $x \in [-\pi, \pi]$, and $H(0) = H'(0) = 0$. Denote by α_i the direction of the line G_i. Then*

$$\int_{G_i \cap K \neq \emptyset} f(\alpha_2 - \alpha_1)|\sin(\alpha_2 - \alpha_1)|\, d\alpha_1\, d\alpha_2\, dP$$
$$= (H(\pi) - H(-\pi))F + \int_{P \notin K} (H(\omega) - H(-\omega))\, dP,$$

where $\omega = \omega(P)$ is the visual angle of K from P.

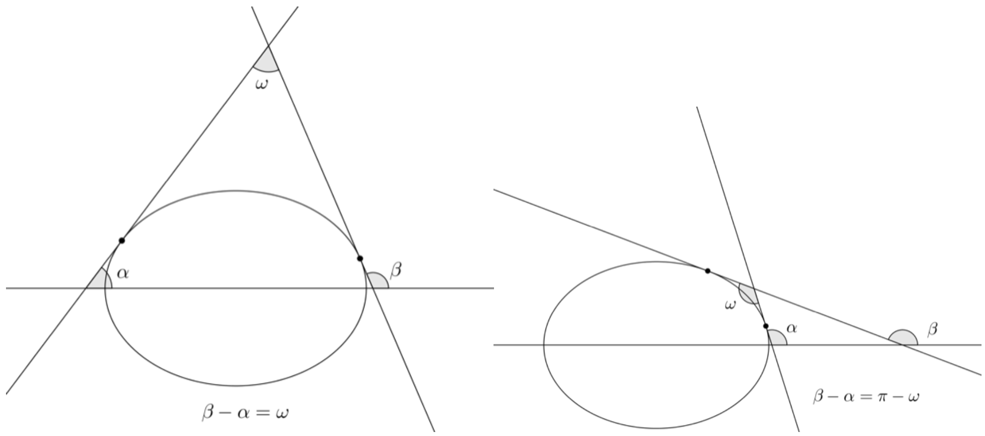

Figure 2: Visual angle of a convex set.

Proof. For a given point P in the plane there are angles $\alpha(P)$, $\beta(P)$ such that the pairs of lines G_1, G_2 through P that intersect the convex set K are those satisfying $\alpha(P) \leq \alpha_i \leq \beta(P)$, where $\alpha_i = \alpha(G_i)$. When $P \in K$ we have $\alpha(P) = 0$ and $\beta(P) = \pi$.

We need to integrate the function $f(\alpha_2 - \alpha_1)|\sin(\alpha_2 - \alpha_1)|$ over $[\alpha, \beta]^2$ with $\alpha = \alpha(P)$ and $\beta = \beta(P)$. In order to perform this integral we divide $[\alpha, \beta]^2$ into the union of the regions $\mathcal{R}_1 = \{(\alpha_1, \alpha_2) \in [\alpha, \beta]^2 : \alpha_2 \geq \alpha_1\}$ and $\mathcal{R}_2 = \{(\alpha_1, \alpha_2) \in [\alpha, \beta]^2 : \alpha_2 < \alpha_1\}$. Therefore

$$
\begin{aligned}
&\int_{[\alpha,\beta]^2} f(\alpha_2 - \alpha_1)|\sin(\alpha_2 - \alpha_1)|\, d\alpha_1\, d\alpha_2 \\
&= \int_{\mathcal{R}_1} f(\alpha_2 - \alpha_1)\sin(\alpha_2 - \alpha_1)\, d\alpha_1\, d\alpha_2 - \int_{\mathcal{R}_2} f(\alpha_2 - \alpha_1)\sin(\alpha_2 - \alpha_1)\, d\alpha_1\, d\alpha_2 \\
&\qquad = \int_\alpha^\beta \left(\int_\alpha^{\alpha_2} f(\alpha_2 - \alpha_1)\sin(\alpha_2 - \alpha_1)\, d\alpha_1 \right) d\alpha_2 \\
&\qquad\quad - \int_\alpha^\beta \left(\int_\alpha^{\alpha_1} f(\alpha_2 - \alpha_1)\sin(\alpha_2 - \alpha_1)\, d\alpha_2 \right) d\alpha_1 \\
&\qquad = \int_\alpha^\beta \left[-H'(\alpha_2 - \alpha_1) \right]_\alpha^{\alpha_2} d\alpha_2 - \int_\alpha^\beta \left[H'(\alpha_2 - \alpha_1) \right]_\alpha^{\alpha_1} d\alpha_1 \\
&\qquad\quad = [H(\alpha_2 - \alpha)]_\alpha^\beta - [H(\alpha - \alpha_1)]_\alpha^\beta = H(\beta - \alpha) - H(\alpha - \beta).
\end{aligned}
$$

Hence

$$
\begin{aligned}
&\int_{G_i \cap K \neq \emptyset} f(\alpha_2 - \alpha_1)|\sin(\alpha_2 - \alpha_1)|\, d\alpha_1\, d\alpha_2\, dP \\
&\qquad\qquad = \left(\int_{P \in K} + \int_{P \notin K} \right) (H(\beta - \alpha) - H(\alpha - \beta))\, dP.
\end{aligned}
$$

Taking into account that the visual angle $\omega(P)$ is given by $\beta(P)-\alpha(P)$ the result follows. $\square$

In the next result we assume the additional hypothesis that $f(x)$ is an even function.

Proposition 4. *Let f be a 2π-periodic continuous function on $\mathbb{R}$, with $f(-x)=f(x)$, $x\in\mathbb{R}$, and H a $\mathcal{C}^2$ function on $[0,\pi]$ satisfying the conditions $H''(x)=f(x)\cdot\sin(x)$, $x\in[0,\pi]$, and $H(0)=H'(0)=0$. Denote by α_i the direction of the line G_i. Then*

$$\int_{G_i\cap K\neq\emptyset} f(\alpha_2-\alpha_1)|\sin(\alpha_2-\alpha_1)|\,d\alpha_1\,d\alpha_2\,dP = 2H(\pi)F + 2\int_{P\notin K} H(\omega)\,dP,$$

where $\omega=\omega(P)$ is the visual angle of K from P.

Proof. Just proceed as in the above proof taking into account that

$$\int_{[\alpha,\beta]^2} f(\alpha_2-\alpha_1)|\sin(\alpha_2-\alpha_1)|\,d\alpha_1\,d\alpha_2$$
$$= 2\int_\alpha^\beta\left(\int_\alpha^{\alpha_2} f(\alpha_2-\alpha_1)\sin(\alpha_2-\alpha_1)\,d\alpha_1\right)d\alpha_2.$$

$\square$

For the special case where f is a π-periodic function one has

Corollary 2. *Let f be a π-periodic continuous function on $\mathbb{R}$, and H a $\mathcal{C}^2$ function on $[-\pi,\pi]$ satisfying the conditions $H''(x)=f(x)\cdot\sin(x)$, $x\in[-\pi,\pi]$, and $H(0)=H'(0)=0$. Then*

$$\int_{G_i\cap K\neq\emptyset} f(\varphi_2-\varphi_1)\,dG_1\,dG_2 = \left((H(\pi)-H(-\pi))F + \int_{P\notin K}(H(\omega)-H(-\omega))\,dP\right).$$

If moreover $f(-x)=f(x)$ and $H(x)$ is $\mathcal{C}^2$ on $[0,\pi]$ with $H''(x)=f(x)\cdot\sin(x)$, $x\in[0,\pi]$, and $H(0)=H'(0)=0$, one has

$$\int_{G_i\cap K\neq\emptyset} f(\varphi_2-\varphi_1)\,dG_1\,dG_2 = 2H(\pi)F + 2\int_{P\notin K} H(\omega)\,dP. \tag{14}$$

Proof. When f is a π -periodic function we have equality (13) and the result is then a consequence of Proposition (3) and Proposition (4). $\square$

Integral formulas as those given in (8) and (14) open the possibility to prove interesting relations for quantities linked to convex sets. For instance when applied to the function $f(x)=\cos kx$ they give Hurwitz's formula (11) for k even (see section 4).

For odd values of k the Corollary 2 does not apply because $f(x) = \cos kx$ is not a π-periodic function. In this case we have $f(x+\pi) = -f(x)$ and we say that f is an *anti π-periodic* function. For this type of functions we can modify the above proofs to obtain a new result that involve a decomposition of the visual angle ω into $\omega = \omega_1 + \omega_2$ where ω_1 and ω_2 are defined in the following way. Given a point $P \notin K$ we have considered in section 3.2 the directions $0 < \alpha < \pi/2 < \beta < \pi$ of the support lines of K through P and the visual angle $\omega = \beta - \alpha$. Let us take $\omega_1 = \pi/2 - \alpha$ and $\omega_2 = \beta - \pi/2$. Then we have

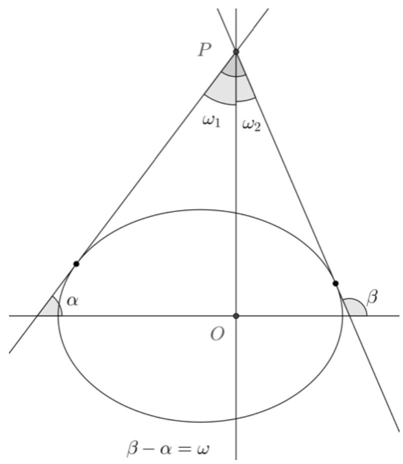

Figure 3: Angles ω_1 and ω_2.

Proposition 5. *Let f be an anti π-periodic continuous function on $\mathbb{R}$ such that $f(x) = f(-x)$ and H a $\mathcal{C}^2$ function on $[0,\pi]$ with $H''(x) = f(x)\cdot\sin(x)$, $x \in [0,\pi]$, and $H(0) = H'(0) = 0$. Then*

$$\int_{G_i \cap K \neq \emptyset} f(\varphi_2 - \varphi_1)\, dG_1\, dG_2$$
$$= 2(2H(\pi/2) - H(\pi))F + 2\int_{P \notin K} (2H(\omega_1) + 2H(\omega_2) - H(\omega))\, dP. \quad (15)$$

Proof. In section 3.1 we have seen that $\varphi_2 - \varphi_1 = \alpha_2 - \alpha_1 + \epsilon\pi$ where $\epsilon = \epsilon(P, \alpha_1, \alpha_2) = 0, \pm 1$. Then

$$\int_{G_i \cap K \neq \emptyset} f(\varphi_2 - \varphi_1)\, dG_1\, dG_2$$
$$= \int_{P \in \mathbb{R}^2} \int_{[\alpha(P), \beta(P)]^2} (-1)^\epsilon f(\alpha_2 - \alpha_1) |\sin(\alpha_2 - \alpha_1)|\, d\alpha_1\, d\alpha_2\, dP.$$

If $P \notin K$ we consider the regions

$$\begin{aligned}
\mathcal{R}_1 &= \{(\alpha_1, \alpha_2) : \alpha \leq \alpha_1 < \alpha_2 \leq \pi/2\}, \\
\mathcal{R}_2 &= \{(\alpha_1, \alpha_2) : \pi/2 \leq \alpha_1 < \alpha_2 \leq \beta\}, \\
\mathcal{R}_3 &= \{(\alpha_1, \alpha_2) : \alpha \leq \alpha_1 < \pi/2 < \alpha_2 \leq \beta\}.
\end{aligned}$$

In $\mathcal{R}_1$ and $\mathcal{R}_2$ we have $\epsilon = 1$ and $\epsilon = -1$ in region $\mathcal{R}_3$. Therefore, for $P \notin K$

$$\begin{aligned}
&\int_\alpha^\beta \int_\alpha^\beta (-1)^\epsilon f(\alpha_2 - \alpha_1) |\sin(\alpha_2 - \alpha_1)| \, d\alpha_1 \, d\alpha_2 \\
&= 2 \left(\int_{\mathcal{R}_1} f(\alpha_2 - \alpha_1) \sin(\alpha_2 - \alpha_1) \, d\alpha_1 \, d\alpha_2 + \int_{\mathcal{R}_2} f(\alpha_2 - \alpha_1) \sin(\alpha_2 - \alpha_1) \, d\alpha_1 \, d\alpha_2 \right. \\
&\qquad\qquad\qquad\qquad \left. = - \int_{\mathcal{R}_3} f(\alpha_2 - \alpha_1) \sin(\alpha_2 - \alpha_1) \, d\alpha_1 \, d\alpha_2 \right).
\end{aligned}$$

The integrals over $\mathcal{R}_1$ and $\mathcal{R}_2$ are easily computed and their values are $2H(\omega_1)$ and $2H(\omega_2)$ respectively. Let us compute the third integral.

$$\begin{aligned}
&\int_{\mathcal{R}_3} f(\alpha_2 - \alpha_1) \sin(\alpha_2 - \alpha_1) \, d\alpha_1 \, d\alpha_2 \\
&\qquad = \int_{\pi/2}^\beta \int_\alpha^{\pi/2} f(\alpha_2 - \alpha_1) \sin(\alpha_2 - \alpha_1) \, d\alpha_1 \, d\alpha_2) \\
&= 2 \int_{\pi/2}^\beta \left[-H'(\alpha_2 - \alpha_1) \right]_{\alpha_1 = \alpha}^{\alpha_1 = \pi/2} \, d\alpha_2 = 2 \int_{\pi/2}^\beta (H'(\alpha_2 - \alpha) - H'(\alpha_2 - \pi/2)) \, d\alpha_2 \\
&= 2 \left[H(\alpha_2 - \alpha) - H(\alpha_2 - \pi/2) \right]_{\alpha_2 = \pi/2}^{\alpha_2 = \beta} = 2 \left(H(\omega) - H(\beta - \pi/2) - H(\pi/2 - \alpha) \right) \\
&\qquad = 2 \left(H(\omega) - H(\omega_2) - H(\omega_1) \right).
\end{aligned}$$

Finally, for $G_1 \cap G_2 = P \notin K$ we have

$$\int_{P \notin K} f(\varphi_2 - \varphi_1) \, dG_1 \, dG_2 = 2 \int_{P \notin K} (2H(\omega_1) + 2H(\omega_2) - H(\omega)) \, dP. \tag{16}$$

When $P \in K$ we do the same computations but now $\alpha = 0$, $\beta = \pi$ and $\omega = \beta - \alpha = \pi$ and so $\omega_1 = \pi/2 = \omega_2$. Thus

$$\int_{G_1 \cap G_2 \in K} f(\varphi_2 - \varphi_1) \, dG_1 \, dG_2 = 2(4H(\pi/2) - H(\pi))F. \tag{17}$$

Joining (16) and (17) the Proposition follows. $\square$

3.3 Interpretation in terms of densities of the formulas of Crofton, Masotti and powers of sine

In this section we will give an interpretation of the integrals of the visual angle appearing in the formulas of Crofton, Masotti and power of sine in terms of integrals of densities in the space of pairs of lines. For Hurwitz's formula this was done in Proposition 2.

Crofton's formula

Taking $H(x) = x - \sin(x)$ it follows that $f = 1$ in Corollary 2 and since $H(\pi) = \pi$ using (14) we get

Proposition 6. *The following equality holds.*

$$\int_{G_i \cap K \neq \emptyset} dG_1\, dG_2 = 2\pi F + 2 \int_{P \notin K} (\omega - \sin\omega)\, dP.$$

Masotti's formula

Taking $H(x) = x^2 - \sin^2(x)$ one gets $H''(x)/\sin(x) = 4\sin(x)$. So the function $f(x) = 4|\sin(x)|$, $x \in \mathbb{R}$, satisfies the hypothesis of Corollary 2 and equation (14) gives

Proposition 7. *The following equality holds*

$$2 \int_{G_i \cap K \neq \emptyset} |\sin(\varphi_2 - \varphi_1)|\, dG_1\, dG_2 = \pi^2 F + \int_{P \notin K} (\omega^2 - \sin^2\omega)\, dP.$$

Powers of sine formula

Finally, in an analogous way we can interpretate the integral of any power of the sine of the visual angle. Effectively for $H(x) = \sin^m(x)$ it follows that

$$H''(x)/\sin(x) = m(m-1)\sin^{m-3}(x) - m^2 \sin^{m-1}(x).$$

So taking $f(x) = m(m-1)|\sin^{n-3}(x)| - m^2|\sin^{m-1}(x)|$ the hypothesis of Corollary 2 are satisfied and by (14) we have

Proposition 8. *The following equality holds*

$$\begin{aligned} &2 \int_{P \notin K} \sin^m(\omega)\, dP \\ &\quad = \int_{G_i \cap K \neq \emptyset} \left(m(m-1)|\sin^{m-3}(\varphi_2 - \varphi_1)| - m^2 |\sin^{m-1}(\varphi_2 - \varphi_1)| \right) dG_1\, dG_2. \end{aligned}$$

4 New proofs of classical formulas

Combining the results of the previous section with Theorem 1 new proofs of the formulas of Masotti and the powers of sine can be obtained, in the spirit of the classical proof of Crofton's formula via Integral Geometry

To begin with we note that Theorem 1 implies the equality $\int_{G_i \cap K \neq \emptyset} dG_1 \, dG_2 = L^2$ which is also an immediate consequence of the well known Cauchy–Crofton's formula (see [6]). Now this equality together with Proposition 6 gives Crofton's formula

$$L^2 = 2\pi F + 2 \int_{P \notin K} (\omega - \sin\omega) \, dP. \tag{18}$$

Masotti's formula

A simple calculation shows that the Fourier expansion of the function $|\sin(t)|$ is

$$|\sin(t)| = \frac{2}{\pi} + \frac{4}{\pi} \sum_{n \geq 1} \frac{\cos(2nt)}{1 - 4n^2}. \tag{19}$$

So by Theorem 1,

$$\int_{G_i \cap K \neq \emptyset} |\sin(\varphi_2 - \varphi_1)| \, dG_1 \, dG_2 = \frac{2L^2}{\pi} + 4\pi \sum_{n \geq 1} \frac{c_{2n}^2}{1 - 4n^2},$$

and using Proposition 7 one gets

$$\int_{P \notin K} (\omega^2 - \sin^2\omega) \, dP = -\pi^2 F + \frac{4L^2}{\pi} + 8\pi \sum_{n \geq 1} \frac{c_{2n}^2}{1 - 4n^2},$$

which is Masotti's formula (5).

Another example

In the preceding sections we have interpreted integral formulas of some functions of the visual angle in terms of densities in the space of pairs of lines. But one can also proceed in the reverse sense, that is to start from a density and to look for the corresponding function of the visual angle.

For instance the proof of Masotti's formula leads to compute $\int_{G_i \cap K} |\sin(\varphi_2 - \varphi_1)| \, dG_1 dG_2$. If we consider now the density function $|\cos(\varphi_2 - \varphi_1)|$, using Theorem 1 and that

$$|\cos(t)| = \frac{2}{\pi} + \frac{4}{\pi} \sum_{n \geq 1} \frac{(-1)^n \cos(2nt)}{1 - 4n^2}$$

we get

$$\int_{G_i \cap K \neq \emptyset} |\cos(\varphi_2 - \varphi_1)| \, dG_1 \, dG_2 = \frac{2L^2}{\pi} + 4\pi \sum_{n \geq 1} \frac{(-1)^n c_{2n}^2}{1 - 4n^2}.$$

The function H appearing in Corollary 2 is in this case

$$H(\omega) = \begin{cases} \frac{1}{4}(\omega - \sin\omega\cos\omega) & 0 \le \omega \le \pi/2, \\ \frac{1}{4}(3\omega - \pi + \sin\omega\cos\omega) & \pi/2 \le \omega \le \pi. \end{cases}$$

Hence, by (14) we have

$$\int_{G_i \cap K \neq \emptyset} |\cos(\varphi_2 - \varphi_1)| \, dG_1 \, dG_2 = \pi F + 2 \int_{P \notin K} H(\omega) \, dP.$$

Powers of sine formula

In order to apply Theorem 1 to the right-hand side of the equality in Proposition 8 we need to compute the Fourier coefficients of the function $f(x) = m(m-1)|\sin^{m-3}(x)| - m^2|\sin^{m-1}(x)|$. It is clear that $A_k = 0$ for k odd. For k even we have

$$\begin{aligned} A_k &= \frac{1}{\pi} \int_0^{2\pi} f(x) \cos(kx) \, dx \\ &= \frac{1}{\pi} \left[2m(m-1) \int_0^{\pi} \sin^{m-3} x \cos(kx) \, dx - 2m^2 \int_0^{\pi} \sin^{m-1} x \cos(kx) \, dx \right] \\ &= \frac{1}{\pi} [2m(m-1) I_{m-3,k} - 2m^2 I_{m-1,k}], \end{aligned} \tag{20}$$

where

$$I_{m,k} = \int_0^{\pi} \sin^m(x) \cos(kx) \, dx = (-1)^{k/2} \frac{2^{-m} m! \pi}{\Gamma(1 + \frac{m-k}{2}) \Gamma(1 + \frac{m+k}{2})},$$

(see, for instance, [3], p. 372). Substituting this expression in (20) it follows

$$A_k = \frac{m!}{2^{m-2}(m-2)} \frac{(-1)^{\frac{k}{2}+1}(k^2-1)}{\Gamma(\frac{m+1+k}{2}) \Gamma(\frac{m+1-k}{2})}.$$

Finally using Theorem 1 we get

$$\begin{aligned} \int_{P \notin K} \sin^m(\omega) \, dP = {} & \frac{m!}{2^m (m-2) \Gamma(\frac{m-1}{2})^2} L^2 \\ & + \frac{m! \pi^2}{2^{m-1}(m-2)} \sum_{k \ge 2, even} \frac{(-1)^{\frac{k}{2}+1}(k^2-1)}{\Gamma(\frac{m+1+k}{2}) \Gamma(\frac{m+1-k}{2})} c_k^2. \end{aligned}$$

Note that for m odd the index k in the sum runs only from 2 to $m-1$.

This formula, which was first obtained by a different method in [2], provides an interpretation of the coefficients of c_k^2 as the Fourier coefficients of the above function f.

Crofton-Hurwitz's integral

In the above two previous sections we have strongly used equality (14) of Corollary 2 that depends on the fact that the function $f(x)$ is π-periodic, a fact that is crucial in order that equality (13) holds.

Consider now the function $f(x) = \cos kx$ with $k > 1$. This function satisfies the hypothesis of Corollary 2 for k even and the hypothesis of Proposition 5 for k odd. We have that

$$H_k(x) = \frac{1}{2(k^2-1)} \left(f_k(x) + 2(\sin x - x) \right), \tag{21}$$

with $f_k(x)$ the Hurwitz's function given in (3), satisfies the equation $H_k''(x) = \cos kx \cdot \sin x$, $x \in [0, \pi]$, and $H_k(0) = H_k'(0) = 0$. Therefore, for k even, equalities (8) and (14) give

$$\pi^2 c_k^2 = \int_{G_i \cap K \neq \emptyset} \cos(k(\varphi_2 - \varphi_1))\, dG_1\, dG_2 = -\frac{\pi F}{k^2-1} + 2 \int_{P \notin K} H_k(\omega)\, dP,$$

and using Crofton's formula (18) one gets a new proof of Hurwitz's formula (11) for k even.

When k is odd equation (15) gives

$$\begin{aligned} &\int_{G_i \cap K \neq \emptyset} \cos k(\varphi_2 - \varphi_1)\, dG_1\, dG_2 = \\ &\qquad = -\frac{2\pi F}{k^2-1} + 2 \int_{P \notin K} \left(2H_k(\omega_1) + 2H_k(\omega_2) - H_k(\omega) \right) dP. \end{aligned}$$

Using the equality (8) one deduces that

$$\int_{P \notin K} H_k(\omega)\, dP = -\frac{\pi^2 c_k^2}{2} - \frac{2\pi F}{k^2-1} + \int_{P \notin K} (H_k(\omega_1) + H_k(\omega_2)\, dP.$$

Now by (21) and Crofton's formula we obtain

$$\int_{P \notin K} f_k(\omega)\, dP = L^2 - \pi^2(k^2-1)c_k^2 - 2\pi F + 2(k^2-1) \int_{P \notin K} (H_k(\omega_1) + H_k(\omega_2))\, dP. \tag{22}$$

Since we do not know the value of $\int_{P \notin K} (H_k(\omega_1) + H_k(\omega_2))\, dP$ we are not able to prove Hurwitz formula in the case of k odd. But from (11) we get the following result.

Proposition 9. *Let K be a compact convex set of area F. Then*

$$(k^2-1) \int_{P \notin K} (H_k(\omega_1) + H_k(\omega_2)\, dP = \pi F \tag{23}$$

for each $k \geq 3$ odd, where H_k is given in (21).

Notice that the above equation is equivalent to

$$\int_{P\notin K} (f_k(\omega_1) + 2(\sin\omega_1 - \omega_1) + f_k(\omega_2) + 2(\sin\omega_2 - \omega_2))\; dP = 2\pi F. \tag{24}$$

The function H_k is the sum, except for a constant, of Hurwitz's function and Crofton's function and so are the terms in the above integrand. The integral of the sum of Crofton's and Hurwitz's functions of the visual angle is

$$\int_{P\notin K} (f_k(\omega) + 2(\sin\omega - \omega))\, dP = 2\pi F + (-1)^k \pi^2 (k^2 - 1) c_k^2, \quad k \geq 2.$$

The surprising fact is that, for k odd, decomposing the visual angle ω into the two parts $\omega = \omega_1 + \omega_2$ and adding the corresponding integrals one gets (24) in which the right-hand side does not depend on k.

In concluding we make the following remark. Theorem 1 states that the integral $\int_{G_i\cap K\neq\emptyset} f(\varphi_2 - \varphi_1)\, dG_1\, dG_2$ depends only on the integrals $\int_{G_i\cap K\neq\emptyset} \cos k(\varphi_2 - \varphi_1)\, dG_1\, dG_2$. So, by the results of section 3.1 we are lead to calculate the functions $H_k(x)$ such that $H_k''(x) = \cos(kx)\sin(x)$ with $H_k(0) = H_k'(0) = 0$. These functions appear to be the sum of the functions of Hurtwitz and Crofton given in (21), that is

$$H_k(x) = \frac{1}{2(k^2 - 1)} \left(f_k(x) + 2(\sin x - x)\right), \quad k \geq 2,$$

and $H_1(x) = (1/8)(2x - \sin(2x))$.

As a consequence when f is a π-periodic density, according to Corollary 2, the integral $\int_{G_i\cap K\neq\emptyset} f(\varphi_2 - \varphi_1)\, dG_1\, dG_2$ is a linear combination of integrals extended outside K of the functions of the visual angle $H_k(\omega)$. Likewise when the density f is anti π-periodic, according to Proposition 5, the corresponding integral of the density is a linear combination of integrals extended outside K of the functions $H_k(\omega)$, $H_k(\omega_1)$ and $H_k(\omega_2)$.

Summarizing, it appears that the functions of Crofton and Hurwitz are some kind of basis for the integral of any π-periodic or anti π-periodic density with respect to the measure $dG_1\, dG_2$ over the set of pairs of lines meeting a given compact convex set.

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