# Uniqueness of Bounded Solutions to Aggregation Equations by Optimal Transport Methods

J. A. Carrillo* J. Rosado†

February 9, 2009

**Abstract**

We show how to extend the method used in [22] to prove uniqueness of solutions to a family of several nonlocal equations containing aggregation terms and aggregation/diffusion competition. They contain several mathematical biology models proposed in macroscopic descriptions of swarming and chemotaxis for the evolution of mass densities of individuals or cells. Uniqueness is shown for bounded nonnegative mass-preserving weak solutions without diffusion. In diffusive cases, we use a coupling method [16, 33] and thus, we need an stochastic representation of the solution to hold. In summary, our results show, modulo certain technical hypotheses, that nonnegative mass-preserving solutions remain unique as long as their $L^\infty$-norm is controlled in time.

## 1 Introduction

We aim to study the uniqueness of solutions to continuity equations evolving a nonnegative density $\rho(t,x)$ at position $x \in \mathbb{R}^N$ and time $t > 0$ by the equation

$$\begin{cases} \dfrac{\partial \rho}{\partial t}(t,x) + \operatorname{div}\left[\rho(t,x)u(t,x)\right] = 0 & t > 0\,,\ x \in \mathbb{R}^N\,, \\ u(t,x) := -\nabla K * \rho(t,x) & t > 0\,,\ x \in \mathbb{R}^N\,, \\ \rho(0,x) = \rho_0(x) \geq 0 & x \in \mathbb{R}^N\,, \end{cases} \tag{1.1}$$

where $u(t,x) := -\nabla K * \rho(t,x)$ is the velocity field. We will also deal with this uniqueness issue for the associated equations in which a linear diffusion term is added, i.e.,

$$\begin{cases} \dfrac{\partial \rho}{\partial t}(t,x) + \operatorname{div}\left[\rho(t,x)u(t,x)\right] = \Delta\rho & t > 0\,,\ x \in \mathbb{R}^N\,, \\ \rho(0,x) = \rho_0(x) \geq 0 & x \in \mathbb{R}^N\,. \end{cases} \tag{1.2}$$

The initial data is assumed to have total finite mass, $\rho_0 \in L^1(\mathbb{R}^N)$. Moreover, since solutions of (1.1) formally preserves the total mass of the system

$$\int_{\mathbb{R}^N} \rho(t,x)\,dx = \int_{\mathbb{R}^N} \rho_0(y)\,dy := M, \tag{1.3}$$

we can assume, without loss of generality, that we work with probability measures, i.e. $M = 1$, by suitable scalings of the equation. A further assumption that will be done through this work is the boundedness of the initial data, i.e., $\rho_0 \in L^\infty(\mathbb{R}^N)$.

This kind of nonlocal interaction equations have been proposed as models for velocity distributions of inelastic colliding particles [3, 4, 31, 14, 15]. Here, typical interaction kernels $K(x)$ are convex and increasing algebraically at infinity. Convexity gives rates of expansion/contraction of distances between solutions, see also [1], and thus uniqueness.

*ICREA - Departament de Matemàtiques, Universitat Autònoma de Barcelona, 08193 Bellaterra, Spain.
†Departament de Matemàtiques, Universitat Autònoma de Barcelona, 08193 Bellaterra, Spain.

Other source of these models is in the field of collective animal behavior. One of the mathematical problems arising there is the analysis of the long time behavior of a collection of self-interacting individuals via pairwise potentials leading to patterns such as flocks, schools or swarms formed by insects, fishes and birds. The simplest models based on ODE/SDEs systems, for instance [10, 27], led to continuum descriptions [13, 12, 29, 30] for the evolution of densities of individuals. Here, one of the typical potentials used is the Morse potential, which is radial $K(x) = k(|x|)$ and given by

$$k(r) = -C_a e^{-r/\ell_a} + C_r e^{-r/\ell_r}, \tag{1.4}$$

with $C_a, C_r$ attractive and repulsive strengths and $\ell_a, \ell_r$ their respective length scales. Typically, these interaction potentials are not convex, and they are composed of an attraction part usually in a certain annular region and a repulsive region closed to the origin while the interaction gets asymptotically zero for large distances, see [12]. Global existence and uniqueness of weak solutions in Sobolev spaces when the potential is well-behaved and smooth, say $K \in C^2(\mathbb{R}^N)$ with bounded second derivatives, were established in [29, 21]. Uniqueness results in the smooth potential case also follows from the general theory developed in [1] as used in [13].

One of the interesting mathematical difficulties in these problems relates to the case of only attractive potentials with a Lipschitz point at the origin as the Morse Potential with $C_r = 0$. In this particular case, finite time blow-up for $L^1 - L^\infty$ solutions have been proved for compactly supported initial data, see [21, 6, 5, 7] for a series of results in this direction. In this particular case, a result of uniqueness of $L^1 - L^\infty$ solutions under some additional technical hypotheses was obtained in [5] inspired by ideas from 2D-incompressible Euler equations in fluid mechanics [34].

Finally, another source of problems of this form is the so-called Patlak-Keller-Segel (PKS) model [28, 20] for chemotaxis in the parabolic-elliptic approximation. This equation corresponds to the case in which the potential is the fundamental solution of the operator $-\Delta$ in any dimension. Originally, this model was written in two dimensions with linear diffusion, see [18, 9, 8] for a state of the art in two dimensions and [17] in larger dimensions. Therefore, in the rest we will refer as PKS equation without diffusion and the PKS equation. In the case without diffusion, it is know that bounded solutions will exist locally in time and that smooth fast-decaying solutions cannot exist globally. In the classical PKS system in 2D dimensions, the mass is a critical quantity and thus there are global solutions below a critical mass and local in time solutions that may blow-up in time for mass values larger than the critical one. In more dimensions, this dichotomy is not so well-known and there are criteria for both situations.

Here, we will essentially work with the three type of interaction potentials above: bounded second derivatives, pointy potentials and Poisson kernels, to show uniqueness of bounded weak solutions on a given time interval $[0, T]$. The idea is based on G. Loeper's work [22] who showed the uniqueness of bounded weak solutions for the Vlasov-Poisson system and the 2D incompressible Euler equations using as "distance" an estimate on the euclidean optimal transport distance between probability measures. We handle the case without diffusion in the next section. Finally, Section 3 will present the adaptation of this idea using a coupling method [16, 33] to the case with diffusion by assuming we have an stochastic representation formula.

# 2 Uniqueness for Aggregation Equations

## 2.1 Notion of Solution

Let us start by working with the continuity equation (1.1) with a given velocity field $u : [0, T] \times \mathbb{R}^N \longrightarrow \mathbb{R}^N$. The continuity equation comes from the assumption that the mass density of individuals in a set is preserved by the flow map or characteristics associated to the ODE system determined by

$$\begin{cases} \dfrac{dX(t,\alpha)}{dt} & = u(t, X(t,\alpha)) \qquad\qquad t \geq 0 \,, \\ X(0,\alpha) & = \alpha \qquad\qquad\qquad\qquad\quad \alpha \in \mathbb{R}^N \,. \end{cases}$$

Let us assume that the given velocity field $u$ is such that the solutions to the ODE system are globally defined in $[0,T]$ and unique. Moreover, let us assume that the flow map $X(t) : \mathbb{R}^N \to \mathbb{R}^N$ for all $t \geq 0$ associated to the velocity field $u(t,x)$, $X(t)(\alpha) := X(t,\alpha)$ for all $\alpha \in \mathbb{R}^N$, is a family of homeomorphisms from $\mathbb{R}^N$ onto $\mathbb{R}^N$. Typically in our cases, $u \in C([0,T] \times \mathbb{R}^N; \mathbb{R}^N)$ and is either Lipschitz or Log-Lipschitz in space, which implies the above statements on the ODE system, see for instance [23, 24].

Given $\rho \in C_w([0,T], L^1_+(\mathbb{R}^N))$, we will say that it is a distributional solution to the continuity equation (1.1) with the given velocity field $u$ and initial data $\rho_0 \in L^1_+(\mathbb{R}^N)$, if it verifies

$$\int_0^T \int_{\mathbb{R}^N} \left( \frac{\partial \varphi}{\partial t}(t,x) + u(t,x) \cdot \nabla \varphi(t,x) \right) \rho(t,x) \, dx \, dt = \int_{\mathbb{R}^N} \varphi(0,x) \rho_0(x) \, dx$$

for all $\varphi \in C_0^\infty([0,T) \times \mathbb{R}^N))$. Here, the symbol $C_w$ means continuity with the weak-* topology of measures. Let us point out that under the above hypotheses the term $(u \cdot \nabla\varphi)\rho$ makes perfect sense as duality $L^1 - L^\infty$.

In fact, the distributional solution of the continuity equation with initial data $\rho_0 \in L^1_+(\mathbb{R}^N)$ is uniquely characterized by

$$\int_B \rho(t,x) \, dx = \int_{X(t)^{-1}(B)} \rho_0(x) \, dx$$

for any measurable set $B \subset \mathbb{R}^N$, see [1]. In the optimal transport terminology, this is equivalent to say that $X(t)$ *transports* the measure $\rho_0$ onto $\rho(t)$ and we denote it by $\rho(t) = X(t)\#\rho_0$ defined by

$$\int_{\mathbb{R}^N} \zeta(x) \rho(t,x) \, dx = \int_{\mathbb{R}^N} \zeta(X(t,x)) \rho_0(x) \, dx \quad \forall \zeta \in \mathcal{C}_b^0(\mathbb{R}^N) \,. \tag{2.1}$$

With these ingredients, we can define the notion of solution for which we will prove its uniqueness.

**Definition 2.1** *A function $\rho$ is a bounded weak solution of* (1.1) *on [0,T] for a nonnegative initial data $\rho_0 \in L^1(\mathbb{R}^N)$, if it satisfies*

1. $\rho \in C_w([0,T], L^1_+(\mathbb{R}^N))$.
2. *The solutions of the ODE system $X'(t,\alpha) = u(t, X(t,\alpha))$ with the velocity field $u(t,x) := -\nabla K * \rho(t,x)$ are uniquely defined in $[0,T]$ for any initial data $\alpha \in \mathbb{R}^N$.*
3. *$\rho(t) = X(t)\#\rho_0$ is the unique distributional solution to the continuity equation with given velocity field $u$.*
4. $\rho \in L^\infty(0,T; L^\infty(\mathbb{R}^N))$.

**Remark 2.2**

1. *Let us point out that even if the solution to the continuity equation with given velocity field $u$ is unique, the uniqueness issue for* (1.1) *is not settled due to the nonlinear coupling through $u = -\nabla K * \rho$.*
2. *In order to show that bounded weak solutions exist, one usually needs more assumptions on the initial data depending on the particular choices of the kernel $K$. Typically for initial data $\rho_0 \in L^1_+ \cap L^\infty(\mathbb{R}^N)$, we will have a time $T > 0$ possibly depending on the initial data and a bounded weak solution on the time interval $[0,T]$.*

## 2.2 Ingredients about optimal transport

Given two probability measures $\rho_1$ and $\rho_2$ probability measures with bounded second moment, the Euclidean Wasserstein Distance is defined as

$$W_2(\rho_1, \rho_2) = \inf_{\Pi \in \Gamma} \left\{ \iint_{\mathbb{R}^N \times \mathbb{R}^N} |x-y|^2 \, d\Pi(x,y) \right\}^{1/2} \tag{2.2}$$

where $\Pi$ runs over the set of transference plans $\Gamma$, that is, the set of joint probability measures on $\mathbb{R}^N \times \mathbb{R}^N$ with marginals $\rho_1$ and $\rho_2$, i.e.,

$$\iint_{\mathbb{R}^N \times \mathbb{R}^N} \varphi(x) \, d\Pi(x,y) = \int_{\mathbb{R}^N} \varphi(x) \, \rho_1(x) \, dx$$

and

$$\iint_{\mathbb{R}^N \times \mathbb{R}^N} \varphi(y) \, d\Pi(x,y) = \int_{\mathbb{R}^N} \varphi(y) \, \rho_2(y) \, dy$$

for all $\varphi \in C_b(\mathbb{R}^N)$, the set of continuous and bounded functions on $\mathbb{R}^N$.

**Remark 2.3** *As it will be used below, given two maps $X_1, X_2 : \mathbb{R}^N \longrightarrow \mathbb{R}^N$ and a given probability measure with bounded second moment $\rho_0$, then*

$$W_2^2(X_1 \# \rho_0, X_2 \# \rho_0) \leq \int_{\mathbb{R}^N} |X_1(x) - X_2(x)|^2 \mathrm{d}\rho_0(x)$$

*just by using $\Pi = (X_1 \times X_2)\#\rho_0$ in the definition of the distance $W_2$.*

First, we recall a result, already used in [22], for estimating the displacement interpolation between two absolutely continuous measures $\rho_1$ and $\rho_2$ with respect to Lebesgue. Let us define the displacement interpolation between these measures as

$$\rho_\theta = ((\theta - 1)T + (2-\theta)\mathbb{I}_{\mathbb{R}^N})_{\#}\rho_1 \tag{2.3}$$

for $\theta \in [1,2]$, where $T$ is the optimal transport map between $\rho_1$ and $\rho_2$ due to Brenier's theorem [11] and $\mathbb{I}_{\mathbb{R}^N}$ is the identity map.

**Theorem 2.4** [2, 25, 19, 1] *Let $\rho_1$ and $\rho_2$ be two probability measures on $\mathbb{R}^N$, such that they are absolutely continuous with respect to the Lebesgue measure and $W_2(\rho_1, \rho_2) < \infty$. Then there exists a vector field $\nu_\theta \in L^2(\mathbb{R}^N, \rho_\theta \, dx)$ such that*

*i.* $\dfrac{d}{d\theta}\rho_\theta + \mathrm{div}(\rho_\theta \nu_\theta) = 0$ *for all $\theta \in [1,2]$.*

*ii.* $\displaystyle\int_{\mathbb{R}^N} \rho_\theta |\nu_\theta|^2 \, dx = W_2^2(\rho_1, \rho_2)$ *for all $\theta \in [1,2]$.*

*iii. We have the $L^\infty$-interpolation estimate*

$$\|\rho_\theta\|_{L^\infty(\mathbb{R}^N)} \leq \max\left\{ \|\rho_1\|_{L^\infty(\mathbb{R}^N)}, \|\rho_2\|_{L^\infty(\mathbb{R}^N)} \right\}$$

*for all $\theta \in [1,2]$.*

One of the ingredients in the proof of Loeper for the Vlasov-Poisson and the 2D incompressible Euler equations is an interpolation estimate between the associated Newtonian potentials.

**Proposition 2.5** [22] *Let $\rho_1$ and $\rho_2$ be two probability measures on $\mathbb{R}^N$ with $L^\infty$ densities with respect to the Lebesgue measure. Let $c_i$ the solution of the Poisson's equation $-\Delta c_i = \rho_i$ in $\mathbb{R}^N$ given by $c_i = \Gamma_N * \rho_i$ with $\Gamma_N$ the fundamental solution of $-\Delta$ in $\mathbb{R}^N$. Then,*

$$\|\nabla c_1 - \nabla c_2\|_{L^2(\mathbb{R}^N)} \leq \max(\|\rho_1\|_{L^\infty(\mathbb{R}^N)}, \|\rho_2\|_{L^\infty(\mathbb{R}^N)})^{1/2} \, W_2(\rho_1, \rho_2)$$

We will proceed similarly to the proof of the previous proposition to get the following interpolation in smoother situations where the kernel has integrable Hessian.

**Proposition 2.6** *Let $\rho_1$ and $\rho_2$ be two probability measures on $\mathbb{R}^N$ with $L^\infty$ densities with respect to the Lebesgue measure. Let $c_i = K * \rho_i$ with $\nabla K \in L^2(\mathbb{R}^N)$ and $|D^2K| \in L^1(\mathbb{R}^N)$. Then,*

$$\|\nabla c_1 - \nabla c_2\|_{L^2(\mathbb{R}^N)} \leq \max(\|\rho_1\|_{L^\infty(\mathbb{R}^N)}, \|\rho_2\|_{L^\infty(\mathbb{R}^N)})^{1/2}\, W_2(\rho_1, \rho_2).$$

*Proof.-* Let us first point out that $\nabla c_i \in L^2(\mathbb{R}^N)$ due to the assumption $\nabla K \in L^2(\mathbb{R}^N)$. By using Theorem 2.4, we can write

$$\begin{aligned}
\|\nabla c_1 - \nabla c_2\|^2_{L^2(\mathbb{R}^N)} &= \int_{\mathbb{R}^N} \left|\nabla K * (\rho_1 - \rho_2)(x)\right|^2 \mathrm{d}x \\
&= \int_{\mathbb{R}^N} \left|\left(\nabla K * \int_1^2 \frac{\partial}{\partial\theta}\rho_\theta \,\mathrm{d}\theta\right)(x)\right|^2 \mathrm{d}x \\
&= \int_{\mathbb{R}^N} \left|\left(\nabla K * \int_1^2 \mathrm{div}(\rho_\theta \nu_\theta)\,\mathrm{d}\theta\right)(x)\right|^2 \mathrm{d}x
\end{aligned}$$

Applying Jensen's inequality, integrating-by-parts and estimating the modulus and the $L^2$-norm of the convolution, we get

$$\begin{aligned}
\|\nabla c_1 - \nabla c_2\|^2_{L^2(\mathbb{R}^N)} &\leq \int_1^2 \int_{\mathbb{R}^N} \left||D^2K| * (\rho_\theta|\nu_\theta|)\right|^2 \mathrm{d}x\,\mathrm{d}\theta \\
&\leq \|\rho_\theta\|_{L^\infty(\mathbb{R}^N)} \||D^2K|\|_{\mathrm{L}^1(\mathbb{R}^N)} \int_1^2 \int_{\mathbb{R}^N} \rho_\theta|\nu_\theta|^2 \mathrm{d}x\,\mathrm{d}\theta
\end{aligned}$$

from which the desired inequality follows using the other information from Theorem 2.4. □

## 2.3 Uniqueness for bounded weak solutions

Let us set some notation for the rest of this section. Let us assume that $\rho_1$ and $\rho_2$ are two bounded weak solutions to (1.1), we look at the two characteristics flow maps, $X_1$ and $X_2$, such that $\rho_i = X_i\#\rho_0$, $i = 1, 2$, and provide a bound for the distance between them at time $t$ in terms of its distance at time $t = 0$.

In the following, we will address the uniqueness with $u = -\nabla K * \rho$, first providing the details of the computation for a regular smooth kernel $K \in C^2(\mathbb{R}^N)$ and with $L^\infty$-bounded Hessian and then modify it in order to include a more general family of kernels with possibly Lipschitz point at the origin, namely, for kernels with the Hessian bounded in $L^1(\mathbb{R}^N)$. Let us remark that the potential $K(x) = e^{-|x|}$ belongs to this class for $N \geq 2$. Finally, we look at the Keller-Segel model without diffusion, i.e. taking $u = \nabla c = -\nabla\Gamma_N * \rho$. The main theorem is summarized as:

**Theorem 2.7** *Let $\rho_1$, $\rho_2$ be two bounded weak solutions of equation* (1.1) *in the interval $[0, T]$ with initial data $\rho_0 \in L^1_+(\mathbb{R}^N)$ and assume that either:*

- *$u$ is given by $u = -\nabla K * \rho$, with $K$ such that $K \in C^2(\mathbb{R}^N)$ and $|D^2K| \in L^\infty(\mathbb{R}^N)$.*
- *$u$ is given by $u = -\nabla K * \rho$, with $K$ such that $\nabla K \in L^2(\mathbb{R}^N)$ and $|D^2K| \in L^1(\mathbb{R}^N)$.*
- *$u = -\nabla\Gamma_N * \rho$.*

*Then $\rho_1(t) = \rho_2(t)$ for all $0 \leq t \leq T$.*

*Idea of the proof.-* Given the two bounded weak solutions to (1.1), let us define the quantity

$$Q(t) := \frac{1}{2}\int_{\mathbb{R}^N} |X_1(t) - X_2(t)|^2 \rho_0(x)\,\mathrm{d}x, \tag{2.4}$$

with $X_i$ the flow map associated to each solution, $\rho_i(t) = X_i\#\rho_0$, $i = 1, 2$. Taking into account the remarks to the definition of the $W_2$-distance, we have $W_2^2(\rho_1(t), \rho_2(t)) \leq 2Q(t)$. It is clear then that $Q(t) \equiv 0$ would imply that $\rho_1 = \rho_2$.

1. *Regular kernel case.*- In this case, the velocity field is continuous and Lipschitz in space, therefore the characteristics are globally defined and unique. Now, by taking the derivative of $Q$ w.r.t. time, we get

$$\begin{aligned}\frac{\partial Q}{\partial t} &= \int_{\mathbb{R}^N} \langle X_1 - X_2, u_1(x_1) - u_2(x_2)\rangle \rho_0(x)\mathrm{d}x \\ &= \int_{\mathbb{R}^N} \langle X_1 - X_2, u_1(x_1) - u_1(x_2)\rangle \rho_0(x)\mathrm{d}x + \\ &\quad \int_{\mathbb{R}^N} \langle X_1 - X_2, u_1(x_2) - u_2(x_2)\rangle \rho_0(x)\mathrm{d}x \end{aligned} \tag{2.5}$$

where the time variable has been ellipsed for clarity. Now, taking into account the Lipschitz properties of $u$ into the first integral and using Hölder inequality in the second one, we can write

$$\begin{aligned}\frac{\partial Q}{\partial t} &\le CQ(t) + Q(t)^{\frac{1}{2}} \left( \int_{\mathbb{R}^N} |u_1\left(X_2(t,x)\right) - u_2\left(X_2(t,x)\right)|^2 \rho_0(x)\mathrm{d}x \right)^{\frac{1}{2}} \\ &= CQ(t) + Q(t)^{\frac{1}{2}} I(t)^{\frac{1}{2}}. \end{aligned} \tag{2.6}$$

Now, let us work in the term $I(t)$. By using that the solutions are constructed transporting the initial data through their flow maps, we deduce

$$\begin{aligned} I(t) &= \int_{\mathbb{R}^N} |\nabla K * (\rho_1 - \rho_2)\left[X_2(t,x)\right]|^2 \rho_0(x)\,\mathrm{d}x \\ &= \int_{\mathbb{R}^N} \left| \int_{\mathbb{R}^N} \nabla K(X_2(x) - y)\rho_1(y)\,\mathrm{d}y - \int_{\mathbb{R}^N} \nabla K(X_2(x) - y)\rho_2(y)\,\mathrm{d}y \right|^2 \rho_0(x)\,\mathrm{d}x \\ &= \int_{\mathbb{R}^N} \left| \int_{\mathbb{R}^N} \left[\nabla K(X_2(x) - X_1(y)) - \nabla K(X_2(x) - X_2(y))\right] \rho_0(y)\,\mathrm{d}y \right|^2 \rho_0(x)\,\mathrm{d}x \\ &\le \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \left| \nabla K(X_2(x) - X_1(y)) - \nabla K(X_2(x) - X_2(y)) \right|^2 \rho_0(y)\rho_0(x)\,\mathrm{d}y\,\mathrm{d}x, \end{aligned}$$

the last step holding due to Jensen's inequality. Using Taylor's theorem, since $K$ is twice differentiable, we deduce

$$\nabla K(A) = \nabla K(B) + \int_0^1 (D^2 K)\left[X_2(x) - X_1(y) + \zeta(X_1(y) - X_2(y))\right](X_1(y) - X_2(y))\,\mathrm{d}\zeta$$

with $A = X_2(x) - X_2(y)$ and $B = X_2(x) - X_1(y)$, and thus $|\nabla K(A) - \nabla K(B)| \le C|X_1(y) - X_2(y)|$ since $|D^2 K| \in L^\infty(\mathbb{R}^N)$. This finally gives that $I(t) \le CQ(t)$. Going back to (2.6), we recover $\frac{\partial Q}{\partial t} \le CQ(t)$, and hence we can conclude that if $Q(0) = 0$ then $Q(t) \equiv 0$, implying $\rho_1 = \rho_2$.

2. *Kernels allowing Lipschitz singularity.*- Under the assumptions on the kernel $K$ and the properties of bounded weak solutions, it was shown in [5, Lemma 4.2] that the velocity field $u$ is Lipschitz continuous in space and time. Therefore, we can recover exactly the relation (2.6) again. Now, in order to estimate $I(t)$, we write it as:

$$I(t) = \int_{\mathbb{R}^N} |\nabla K * (\rho_1 - \rho_2)\left[X_2(t,x)\right]|^2 \rho_0(x)\,\mathrm{d}x = \int_{\mathbb{R}^N} |\nabla K * (\rho_1 - \rho_2)(x)|^2 \rho_2(x)\,\mathrm{d}x$$

Using Proposition 2.6, we deduce that

$$I(t) \le \|\rho_2\|_{L^\infty(\mathbb{R}^N)} \max(\|\rho_1\|_{L^\infty(\mathbb{R}^N)}, \|\rho_2\|_{L^\infty(\mathbb{R}^N)})\, W_2^2(\rho_1,\rho_2) \le CQ(t),$$

and we can conclude similarly as in the previous case.

3. *PKS model without diffusion.*- Under the assumptions on bounded weak solutions, the velocity field in our case is Log-Lipschitz in space. This is a classical result used in 2D incompressible Euler equations and easily generalized to any dimension [23, 24, 22].

More precisely, $u \in L^\infty((0,T)\times\mathbb{R}^N;\mathbb{R}^N)$ and there exists a constant $C$ depending on the $L^1$ and $L^\infty$ norms of $\rho(t)$ such that

$$|u(t,x)-u(t,y)| \le C|x-y|\log\frac{1}{|x-y|} \text{ when } |x-y|\le\frac{1}{2}$$

for any $t\in[0,T]$. The flow map under these conditions can be uniquely defined and it is a Hölder homeomorphism.

The uniqueness proof follows estimating the second term in (2.5) as in the previous case. More precisely, we use Proposition 2.5 to infer that

$$I(t) \le \|\rho_2\|_{L^\infty(\mathbb{R}^N)} \max(\|\rho_1\|_{L^\infty(\mathbb{R}^N)}, \|\rho_2\|_{L^\infty(\mathbb{R}^N)})\, W_2^2(\rho_1,\rho_2) \le CQ(t),$$

implying

$$\int_{\mathbb{R}^N} \langle X_1 - X_2, u_1(x_2) - u_2(x_2)\rangle \rho_0(x)\mathrm{d}x \le Q(t)^{\frac12} I(t)^{\frac12} \le CQ(t).$$

Now, let us concentrate in the first term of (2.5), we just repeat the standard arguments in [22] to get that by taking $T$ small enough then

$$\int_{\mathbb{R}^N} \langle X_1 - X_2, u_1(x_1) - u_1(x_2)\rangle \rho_0(x)\mathrm{d}x \le CQ(t)\log^2(2Q(t))$$

where the log-Lipschitz property of $u$ was used. This finally gives the differential inequality

$$\frac{d}{dt}Q(t) \le CQ(t)\left(1+\log\frac{1}{Q(t)}\right),$$

for $0\le t\le T$ with $T$ small enough. Standard Gronwall-like arguments as in [23] imply $Q(t)=0$, and thus, the uniqueness. □

# 3 Uniqueness for aggregation/diffusion competition

We start by defining the concept of solution, we will work with.

**Definition 3.1** *A function $\rho$ is a bounded weak solution of* (1.2) *on [0,T] for a nonnegative initial data $\rho_0\in L^1(\mathbb{R}^N)$, if it satisfies*

1. $\rho\in C_w([0,T],L^1_+(\mathbb{R}^N))$.
2. *The SDE system*
$$dX(t) = u(t,X(t))\,dt + \sqrt{2}\,dW_t$$
*with the velocity field $u(t,x) := -\nabla K * \rho(t,x)$ and initial data $X(0)$ with law $\rho_0(x)$ has a solution given by a Markov process $X(t)$ of law $\rho(t,x)$. Here $W_t$ is the standard Wiener process.*
3. *$\rho(t)$ is a distributional solution to* (1.2).
4. $\rho\in L^\infty(0,T;L^\infty(\mathbb{R}^N))$.

We point out that again more additional assumptions on the kernel $K$ and the initial data $\rho_0$ are needed to prove the existence of such solutions. Solutions of this form have been obtained for particular cases of $K$ in [26, 16]. Moreover, stability estimates, leading in particular to uniqueness of solutions, are obtained under convexity assumptions on the kernel $K$ in [16]. Here, we will assume the existence of bounded weak solutions for the three models introduced in the previous section with diffusion. The existence theory seems a challenging problem to be tackled in the PKS system. The main theorem for these models with diffusion can be summarized as:

**Theorem 3.2** *Let $\rho_1$, $\rho_2$ be two bounded weak solutions of equation (1.2) in the interval $[0,T]$ with initial data $\rho_0 \in L^1_+(\mathbb{R}^N)$ and assume that either:*

- *$u$ is given by $u = -\nabla K * \rho$, with $K$ such that $K \in C^2(\mathbb{R}^N)$ and $|D^2K| \in L^\infty(\mathbb{R}^N)$.*
- *$u$ is given by $u = -\nabla K * \rho$, with $K$ such that $\nabla K \in L^2(\mathbb{R}^N)$ and $|D^2K| \in L^1(\mathbb{R}^N)$.*
- *$u = -\nabla \Gamma_N * \rho$.*

*Then $\rho_1(t) = \rho_2(t)$ for all $0 \le t \le T$.*

*Proof.-* Given the two bounded weak solutions to (1.2), let us consider that the solutions of the SDE systems:

$$dX_i(t) = u(t, X_i(t))\, dt + \sqrt{2} dW_t$$

with initial data $X_1(0) = X_2(0)$ a random variable with law $\rho_0$, are constructed based upon the same Wiener process as in [16], see also [33, Chapter 2]. Then, the stochastic process $X_1(t) - X_2(t)$ follow a deterministic equation:

$$\frac{d}{dt}(X_1(t) - X_2(t)) = u(t, X_1(t)) - u(t, X_2(t)).$$

Therefore, the quantity used in this case will be

$$Q(t) := \frac{1}{2}\, \mathbb{E}\left[|X_1(t) - X_2(t)|^2\right]. \tag{3.1}$$

It is easy also to check that $W_2^2(\rho_1(t), \rho_2(t)) \le 2Q(t)$ by defining an admissible plan $\pi$ transporting $\rho_1(t)$ to $\rho_2(t)$ by

$$\int_{\mathbb{R}^N \times \mathbb{R}^N} \varphi(x,y) \mathrm{d}\pi(x,y) = \mathbb{E}\left[\varphi(X_1(t), X_2(t))\right]$$

for all $\varphi \in C_b(\mathbb{R}^N)$. This plan has the right marginals since the law of $X_i(t)$ is given by $\rho_i(t,x)$ meaning that

$$\int_{\mathbb{R}^N} \varphi(x) \rho_i(t,x)\, \mathrm{d}x = \mathbb{E}\left[\varphi(X_i(t))\right].$$

It is clear then that $Q(t) \equiv 0$ would imply that $X_1(t) = X_2(t)$, and thus their laws $\rho_1 = \rho_2$. With this new quantity the proof now follow exactly the same steps as in Theorem 2.7. We make a quick summary of the new ingredients to consider. We first compute the time derivative of $Q(t)$ as

$$\frac{dQ}{dt} = \mathbb{E}\left[\langle X_1 - X_2, u_1(X_1) - u_1(X_2)\rangle\right] + \mathbb{E}\left[\langle X_1 - X_2, u_1(X_2) - u_2(X_2)\rangle\right],$$

with abuse of notation since an integrated in time version of it would give full rigor. Now, the proof of the smooth case can be really copied directly to this case by replacing integration with respect to the measure $\rho_0$ by expectations. The second and third cases can be also adapted by using the following ingredients:

1. The interpolation results in Propositions 2.6 and 2.5 can be used by realizing:

   $$I(t) = \mathbb{E}\left[|\nabla K * (\rho_1 - \rho_2)\,[X_2(t)]\,|^2\right] = \int_{\mathbb{R}^N} |\nabla K * (\rho_1 - \rho_2)\,(x)|^2 \rho_2(x)\, \mathrm{d}x$$

   since $\rho_2(t,x)$ is the law of $X_2(t)$.

2. In the case of the PKS system, one of the ingredients used by G. Loeper in his proof in [22] was the continuity in time of the solutions of the ODE system for small time. This step was not detailed in the previous section since the proof coincided with the one in [22]. This needed continuity can be also proved in the present case since being the two SDE systems solved with the same Brownian motion, then

   $$\frac{d}{dt}(X_1(t) - X_2(t)) = u_1(t, X_1(t)) - u_2(t, X_2(t)).$$

Using that under the assumptions of bounded densities the velocity fields are bounded and Log-Lipschitz and since the initial data is the same, we deduce $|X_1(t)-X_2(t)| \leq Ct$ for all $0 \leq t \leq T$ a.e. in the probability space.

All the rest of the details are left to the interested reader. ∎


## Acknowledgements

We acknowledge support from the project MTM2008-06349-C03-03 DGI-MICINN (Spain) and IPAM–UCLA where part of this work was done.


# References


[1] L.A. Ambrosio, N. Gigli, G. Savaré, *Gradient flows in metric spaces and in the space of probability measures*, Lectures in Mathematics, Birkhäuser, 2005.

[2] J.D. Benamou, Y. Brenier, *A computational Fluid Mechanics solution to the Monge-Kantorovich mass transfer problem*, Numer. Math., 84 (2000), pp. 375–393.

[3] D. Benedetto, E. Caglioti, M. Pulvirenti, *A kinetic equation for granular media*, RAIRO Modél. Math. Anal. Numér., 31, (1997), pp. 615–641.

[4] D. Benedetto, E. Caglioti, J.A. Carrillo, M. Pulvirenti, *A non-maxwellian steady distribution for one-dimensional granular media*, J. Stat. Phys., 91, (1998), pp. 979-990.

[5] A. Bertozzi, J. Brandman, *Finite-time blow-up of* $\mathrm{L}^\infty$*-weak solutions of an aggregation equation*, to appear in Comm. Math. Sci., special issue in honor of Andrew Majda's 60th birthday.

[6] A. Bertozzi, T. Laurent, *Finite-time blow-up of solutions of an aggregation equation in* $\mathbb{R}^n$, Comm. Math. Phys., 274 (2007), pp. 717–735.

[7] A. Bertozzi, J.A. Carrillo, T. Laurent, *Blowup in multidimensional aggregation equations with mildly singular interaction kernels*, to appear in Nonlinearity.

[8] A. Blanchet, J. A. Carrillo, N. Masmoudi, *Infinite Time Aggregation for the Critical Patlak-Keller-Segel model in* $\mathbb{R}^2$, Comm. Pure Appl. Math. 61 (2008), pp. 1449–1481.

[9] A. Blanchet, J. Dolbeault, B. Perthame, *Two-dimensional Keller-Segel model: optimal critical mass and qualitative properties of the solutions*, Electron. J. Differential Equations, (2006), No. 44, 32 pp. (electronic).

[10] S. Boi, V. Capasso, D. Morale, *Modeling the aggregative behavior of ants of the species Polyergus rufescens*, Spatial heterogeneity in ecological models (Alcalá de Henares, 1998), Nonlinear Anal. Real World Appl., 1 (2000), pp. 163–176.

[11] Y. Brenier, *Polar factorization and monotone rearrangement of vector-valued functions*, Comm. Pure Appl. Math. 44, (1991), pp. 375–417.

[12] M. Burger, V. Capasso, D. Morale, *On an aggregation model with long and short range interactions*, Nonlinear Analysis. Real World Applications. An International Multidisciplinary Journal, 8 (2007), pp. 939–958.

[13] M. Burger, M. Di Francesco, *Large time behavior of nonlocal aggregation models with nonlinear diffusion*, to appear in Networks and Heterogenous Media.

[14] J.A. Carrillo, R.J. McCann, C. Villani, *Kinetic equilibration rates for granular media and related equations: entropy dissipation and mass transportation estimates*, Rev. Matemática Iberoamericana, 19 (2003), pp. 1-48.

[15] J.A. Carrillo, R.J. McCann, C. Villani, *Contractions in the 2-Wasserstein length space and thermalization of granular media*, Arch. Rat. Mech. Anal., 179 (2006), pp. 217–263.

[16] P. Cattiaux, A. Guillin, F. Malrieu, *Probabilistic approach for granular media equations in the non-uniformly convex case*, Probab. Theory Related Fields, 140 (2008), pp. 19–40.

[17] L. Corrias, B. Perthame, H. Zaag, *Global solutions of some chemotaxis and angiogenesis systems in high space dimensions*, Milan J. Math., 72 (2004), pp. 1–28.

[18] J. Dolbeault and B. Perthame, *Optimal critical mass in the two-dimensional Keller-Segel model in* $\mathbb{R}^2$, C. R. Math. Acad. Sci. Paris, 339 (2004), pp. 611–616.

[19] W. Gangbo, R.J. McCann, *The geometry of optimal transportation*, Acta Math., 177 (1996), pp. 113161.

[20] E.F. Keller and L.A. Segel, *Initiation of slide mold aggregation viewed as an instability*, J. Theor. Biol., 26 (1970), pp. 399–415.

[21] T. Laurent, *Local and Global Existence for an Aggregation Equation*, Communications in Partial Differential Equations, 32 (2007), pp. 1941-1964.

[22] G. Loeper, *Uniqueness of the solution to the Vlasov-Poisson system with bounded density*, J. Math. Pures Appl. 86 (2006), pp. 68–79.

[23] A. Majda, A.L. Bertozzi, *Vorticity and Incompressible Flow*, Cambridge Texts in Applied Mathematics, Cambridge University Press, United Kingdom, 2002.

[24] C. Marchioro, M. Pulvirenti, *Mathematical theory of incompressible nonviscous fluids*, Applied Mathematical Sciences 96, Springer-Verlag, New York, 1994.

[25] R. J. McCann, *A convexity principle for interacting gases*, Adv. Math., 128 (1997), pp. 153–179.

[26] S. Méléard, *Asymptotic behaviour of some interacting particle systems: McKean-Vlasov and Boltzmann models*, in: Probabilistic Models for Nonlinear Partial Differential Equations, Talay and Tubaro Editors, Lecture Notes in Mathematics, vol 1627, pp. 42–95 (1995).

[27] D. Morale, V. Capasso and K. Oelschläger, *An interacting particle system modelling aggregation behavior: from individuals to populations*, J. Math. Biol., 50 (2005), pp. 49–66.

[28] C.S. Patlak, *Random walk with persistence and external bias*, Bull. Math. Biophys., 15 (1953), pp. 311–338.

[29] C.M. Topaz and A.L. Bertozzi, *Swarming patterns in a two-dimensional kinematic model for biological groups*, SIAM J. Appl. Math., 65 (2004), pp. 152–174.

[30] C.M. Topaz, A.L. Bertozzi, and M.A. Lewis, *A nonlocal continuum model for biological aggregation. Bulletin of Mathematical Biology*, 68(7), pp. 1601-1623, 2006.

[31] G. Toscani, *One-dimensional kinetic models of granular flows*, RAIRO Modél. Math. Anal. Numér., 34, 6 (2000), pp. 1277–1291.

[32] C. Villani, *Topics in optimal transportation*, volume 58 of *Graduate Studies in Mathematics*, American Mathematical Society, Providence, RI, 2003.

[33] C. Villani, *Optimal transport, old and new*, Lecture Notes for the 2005 Saint-Flour summer school, to appear, Springer 2008.

[34] V. I. Yudovich, *Non-stationary flow of an incompressible liquid.*, Zh. Vychisl. Mat. Mat. Fiz, 3 (1963), pp. 1032-1066.