THE RATIONAL COHOMOLOGY OF A p-LOCAL COMPACT GROUP

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Let p be a prime number. In [BLO3], we developed the theory of p-local compact groups. The theory is modelled on the p-local homotopy theory of classifying spaces of compact Lie groups and p-compact groups, and generalises the earlier concept of p-local finite groups [BLO2]. It provides a coherent context in which classifying spaces of compact Lie groups and p-compact groups [DW] can be studied, and also gives rise to many exotic examples. In this paper, we study the rational p-adic cohomology

$$H^*_{\mathbb{Q}_p}(-) \stackrel{\text{def}}{=} H^*(-,\mathbb{Z}_p)\otimes\mathbb{Q}_p$$

of a p-local compact group. Our main result here is that, as one would expect, the p-adic rational cohomology of p-local compact groups behaves the same way as that of a compact Lie group.

Theorem A. *Let $\mathcal{G}=(S,\mathcal{F},\mathcal{L})$ be a p-local compact group. Let $S_0\le S$ be its maximal torus, and let $W(\mathcal{G})\stackrel{\text{def}}{=}\mathrm{Aut}_{\mathcal{F}}(S_0)$ be its Weyl group. Then*

$$H^*_{\mathbb{Q}_p}(B\mathcal{G})\cong H^*_{\mathbb{Q}_p}(BS_0)^{W(\mathcal{G})}.$$

Of course, the Weyl group of a p-local compact group need not be a pseudo-reflection group, and hence the rational cohomology of the classifying space is not in general a polynomial algebra.

Like compact Lie groups and p-compact groups, p-local compact groups admit unstable Adams operations, which are defined in [JLL], using the internal structure of the p-local group in question, rather than its rational cohomology. One application of Theorem A is Proposition 3.2, which states that under a mild condition, the obvious cohomological definition of an unstable Adams operation characterises the same family of maps as the one referred to in [JLL] as *"geometric unstable Adams operations"*.

Another easy application of Theorem A is the observation that if $\mathcal{G}$ is a p-local compact group with maximal torus S_0, then the inclusion in $\mathcal{G}$ of the p-local subgroup given by the normaliser $N_{\mathcal{G}}(S_0)$ induces a rational p-adic cohomology isomorphism.

In Section 1, we recall the basic concepts in the theory of p-local compact groups which will be needed to prove Theorem A. Section 2 is dedicated to the proof of the theorem. Finally in Section 3 we discuss the applications described above.

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1. Some basic concepts

We recall the definition and some basic properties of p-local compact groups. The reader is referred to [BLO3] for a comprehensive account of these objects.

We begin by defining discrete p-toral groups. By $\mathbb{Z}/p^\infty$ we mean the union of all $\mathbb{Z}/p^r$ with respect to the natural inclusions.

Definition 1.1. *A* discrete p-torus *is a group isomorphic to* $(\mathbb{Z}/p^\infty)^r$ *for some positive integer* r. *A* discrete p-toral group *is a group* S *which contains a normal discrete* p*-torus* S_0*, with* p*-power index. The normal subgroup* S_0 *will be referred to as the* maximal torus *or the* identity component *of* S*, and the quotient group* $\Gamma \cong S/S_0$ *will be called the* group of components *of* S.

The identity component S_0 of a discrete p-toral group S can be characterised as the subset of all infinitely p-divisible elements in S, and also as the unique minimal subgroup of finite index in S. Thus, S_0 is a characteristic subgroup. The rank of S is the number $r = \mathrm{rk}(S)$ such that $S_0 \cong (\mathbb{Z}/p^\infty)^r$.

Recall that for $P, Q \leq S$, the transporter set $T_S(P,Q)$ is the set of all elements $g \in S$ such that $gPg^{-1} \leq Q$. We denote by $\mathrm{Hom}_S(P,Q)$ the set of all homomorphisms $c_g\colon P \to Q$, which are restrictions of an inner automorphism of S, and by $\mathrm{Inj}(P,Q)$ denote the set of all the injective homomorphisms $P \to Q$. We are now ready to recall the definition of fusion systems over discrete p-toral groups.

Definition 1.2. *A* fusion system $\mathcal{F}$ *over a discrete* p*-toral group* S *is a category whose objects are the subgroups of* S*, and whose morphism sets* $\mathrm{Hom}_{\mathcal{F}}(P,Q)$ *satisfy the following conditions:*

(a) $\mathrm{Hom}_S(P,Q) \subseteq \mathrm{Hom}_{\mathcal{F}}(P,Q) \subseteq \mathrm{Inj}(P,Q)$ *for all* $P, Q \leq S$.

(b) *Every morphism in* $\mathcal{F}$ *factors as an isomorphism in* $\mathcal{F}$ *followed by an inclusion.*

Two subgroups $P, P' \leq S$ are called $\mathcal{F}$*-conjugate* if P and P' are isomorphic as objects in $\mathcal{F}$. A subgroup $P \leq S$ is said to be $\mathcal{F}$*-centric* if for every subgroup $P' \leq S$ which is $\mathcal{F}$-conjugate to P, $C_S(P') = Z(P')$.

All fusion systems considered in this paper are required to be *saturated* [BLO3, Definition 2.2]. Although the results we present here are based on properties of saturated fusion systems proved in [BLO3], we do not explicitly use the saturation axioms, and thus we will not repeat them here.

Next, we briefly recall what are centric linking systems and p-local compact groups. The full definition can be found in [BLO3, Definitions 4.1, 4.2]

Definition 1.3. *Let* $\mathcal{F}$ *be a fusion system over a discrete* p*-toral group* S*. A* centric linking system associated to $\mathcal{F}$ *is a category* $\mathcal{L}$ *whose objects are the* $\mathcal{F}$*-centric subgroups of* S*, together with a functor*

$$\pi\colon \mathcal{L} \longrightarrow \mathcal{F}^c,$$

and "distinguished" monomorphisms $P \xrightarrow{\delta_P} \mathrm{Aut}_{\mathcal{L}}(P)$ *for each* $\mathcal{F}$*-centric subgroup* $P \leq S$*, which satisfy the following conditions.*

(A) π *is the identity on objects and surjective on morphisms. More precisely, for each pair of objects* $P, Q \in \mathcal{L}$*, the centre* $Z(P)$ *acts freely on* $\mathrm{Mor}_{\mathcal{L}}(P,Q)$ *by composition (upon identifying* $Z(P)$ *with* $\delta_P(Z(P)) \leq \mathrm{Aut}_{\mathcal{L}}(P)$*), and* π *induces a bijection*

$$\mathrm{Mor}_{\mathcal{L}}(P,Q)/Z(P) \xrightarrow{\ \cong\ } \mathrm{Hom}_{\mathcal{F}}(P,Q).$$

(B) *For each $\mathcal{F}$-centric subgroup $P \leq S$ and each $g \in P$, π sends $\delta_P(g) \in \mathrm{Aut}_{\mathcal{L}}(P)$ to $c_g \in \mathrm{Aut}_{\mathcal{F}}(P)$.*

(C) *For each $f \in \mathrm{Mor}_{\mathcal{L}}(P, Q)$ and each $g \in P$, the following square commutes in $\mathcal{L}$:*

$$\begin{array}{ccc} P & \xrightarrow{\;f\;} & Q \\ {\scriptstyle \delta_P(g)}\downarrow & & \downarrow{\scriptstyle \delta_Q(\pi(f)(g))} \\ P & \xrightarrow{\;f\;} & Q. \end{array}$$

*A p-*local compact group *is a triple $\mathcal{G} = (S, \mathcal{F}, \mathcal{L})$, where S is a discrete p-toral group, $\mathcal{F}$ is a saturated fusion system over S, and $\mathcal{L}$ is a centric linking system associated to $\mathcal{F}$. The* classifying space *of $\mathcal{G}$ is the p-completed nerve $|\mathcal{L}|^\wedge_p$, which we will generally denote by $B\mathcal{G}$.*

In [BLO3], the authors show that compact Lie groups and p-compact groups give rise to particular examples of p-local compact groups. Another large family of examples arises from linear torsion groups. In each case, the respective classifying space coincides up to homotopy (after p-completion in the case of genuine groups) with the classifying space of the p-local compact group it gives rise to.

Definition 1.4. *Let $\mathcal{G} = (S, \mathcal{F}, \mathcal{L})$ be a p-local compact group. Then the Weyl group $W(\mathcal{G})$ of $\mathcal{G}$ is defined to be the automorphism group in $\mathcal{F}$ of the maximal torus $S_0 \leq S$.*

Notice that $H^*_{\mathbb{Q}_p}(X) \overset{\text{def}}{=} H^*(X, \mathbb{Z}_p) \otimes \mathbb{Q}$ is not in general isomorphic to $H^*(X, \mathbb{Q}_p)$. For instance if $X = B\mathbb{Z}/p^\infty$, then $H^*_{\mathbb{Q}_p}(X)$ is a polynomial ring over the p-adic rationals on a generator in degree 2, while $H^*(X, \mathbb{Q}_p)$ is trivial. The use of $H^*_{\mathbb{Q}_p}$ as the appropriate cohomology theory for our purpose goes back to Dwyer and Wilkerson [DW], in their first paper on p-compact groups.

2. The rational cohomology

Two preparatory lemmas are needed before we prove our main claim.

Lemma 2.1. *Let P be a discrete p-toral group with maximal torus $P_0 \leq P$. Then $H^*_{\mathbb{Q}_p}(BP) \cong H^*_{\mathbb{Q}_p}(BP_0)^{P/P_0}$.*

Proof. This is of course a particular case of a much more general statement. Up to homotopy, BP_0 is a covering space of BP with group P/P_0, and so one has the usual transfer map

$$\mathsf{Tr}\colon H^*(BP_0, \mathbb{Z}^\wedge_p) \longrightarrow H^*(BP, \mathbb{Z}^\wedge_p),$$

where $\mathsf{Tr} \circ \mathsf{Res}$ is multiplication by $|P/P_0|$. Hence after tensoring with $\mathbb{Q}$ this composite is an isomorphism. On the other hand, the composition the other way $\mathsf{Res} \circ \mathsf{Tr}$ is norm map for the action of P/P_0 on $H^*_{\mathbb{Q}_p}(BP_0)$, and hence the image of restriction is the subgroup of invariants $H^*_{\mathbb{Q}_p}(BP_0)^{P/P_0}$. $\square$

To prove the theorem, we will use the subgroup decomposition for p-local compact groups [BLO3, Proposition 4.6]. Hence the following lemma is an essential ingredient. In order to state it, we need to recall some notation and terminology.

For a fusion system $\mathcal{F}$ over a discrete p-toral group S, we denote by $\mathcal{O}(\mathcal{F})$ the orbit category associated to $\mathcal{F}$, i.e., the category with the same objects and with morphisms $\mathrm{Mor}_{\mathcal{O}(\mathcal{F})}(P,Q) = \mathrm{Rep}_{\mathcal{F}}(P,Q) \stackrel{\text{def}}{=} \mathrm{Hom}_{\mathcal{F}}(P,Q)/\mathrm{Inn}(Q)$. For $P,Q \in S$, let $N_S(P,Q)$ denote the transporter set consisting of all elements of S which conjugate P into Q. If $\mathcal{F}'$ is a full subcategory of $\mathcal{F}$, we denote by $\mathcal{O}(\mathcal{F}')$ the full subcategory of $\mathcal{O}(\mathcal{F})$ whose objects are those of $\mathcal{F}'$. If Γ is a finite group, we denote by $\mathcal{O}_p(\Gamma)$ the category whose objects are the p-subgroups of Γ and whose morphisms are $\mathrm{Mor}(P,Q) = C_\Gamma(P)\backslash N_\Gamma(P,Q)/\mathrm{Inn}(Q)$.

Lemma 2.2. *Let $\mathcal{F}$ be any saturated fusion system over a discrete p-toral group S. Define*

$$F^*\colon \mathcal{O}(\mathcal{F}^c)^{\mathrm{op}} \longrightarrow \mathbb{Q}\text{-}\mathsf{mod}$$

on objects by setting $F^*(P) = H^*_{\mathbb{Q}_p}(BP)$. On morphisms, F^* sends the class of $\varphi \in \mathrm{Hom}_{\mathcal{F}}(P,P')$ to the homomorphism induced by $B\varphi$. Then F^* is acyclic, namely $\varprojlim^i_{\mathcal{O}(\mathcal{F}^c)}(F^*) = 0$ for all $i > 0$.

Proof. Set $Q = C_S(S_0) \lhd S$, and $\Gamma = \mathrm{Out}_{\mathcal{F}}(Q)$. Then Q is $\mathcal{F}$-centric, and is weakly closed in $\mathcal{F}$ since S_0 is. Let $\mathcal{F}_{\geq Q}$ denote the full subcategory of $\mathcal{F}$ whose objects are those $P \leq S$ which contain Q, and let

$$\Theta\colon \mathcal{O}(\mathcal{F}_{\geq Q}) \longrightarrow \mathcal{O}_p(\Gamma)$$

be the functor which sends an object P to $\mathrm{Out}_P(Q) \leq \Gamma$, and a morphism $\varphi \in \mathrm{Rep}_{\mathcal{F}}(P,P')$ to the class of $\varphi|_Q \in N_\Gamma(\Theta(P),\Theta(P'))$ (see [BLO3, Lemma 5.7]). For each p-subgroup $\Pi \leq \Gamma$, regarded as a group of automorphisms of S_0, define

$$\Phi^*(\Pi) = H^*_{\mathbb{Q}_p}(BS_0)^{\Pi}.$$

This defines a graded functor $\Phi^*\colon \mathcal{O}_p(\Gamma)^{\mathrm{op}} \longrightarrow \mathbb{Q}\text{-}\mathsf{mod}$. Furthermore, for each $P \leq S$ which contains Q,

$$F^*(P) = H^*_{\mathbb{Q}_p}(BQ)^{P/Q} = \Phi^*(\Theta(P)).$$

Thus $\Phi^* \circ \Theta \cong F^*|_{\mathcal{O}(\mathcal{F}_{\geq Q})}$.

For each $P \leq S$, $\mathrm{Out}_Q(P)$ acts trivially on $F^*(P)$ since Q centralises P_0, and $F^*(P)$ is a subring of $H^*_{\mathbb{Q}_p}(BP_0)$. So by [BLO3, Lemma 5.7],

$$\varprojlim^*_{\mathcal{O}(\mathcal{F}^c)}(F^*) \cong \varprojlim^*_{\mathcal{O}_p(\Gamma)}(\Phi^*).$$

The functor Φ^* is a Mackey functor on $\mathcal{O}_p(\Gamma)$, and hence is acyclic (see [JM, Proposition 5.14] or [JMO, Proposition 5.2]). □

We are now ready to prove our main theorem.

Theorem 2.3. *Let $\mathcal{G} = (S,\mathcal{F},\mathcal{L})$ be a p-local compact group. Then*

$$H^*_{\mathbb{Q}_p}(B\mathcal{G}) \cong H^*_{\mathbb{Q}_p}(BS_0)^{W(\mathcal{G})}.$$

Proof. Let $\pi\colon \mathcal{L} \to \mathcal{O}(\mathcal{F}^c)$ be the projection, and let $\widetilde{B}\colon \mathcal{O}(\mathcal{F}^c) \to \mathsf{Top}$ denote the left homotopy Kan extension of the constant functor on $\mathcal{L}$ along π. Then there is a homotopy equivalence

$$\operatorname*{hocolim}_{\mathcal{O}(\mathcal{F}^c)} \widetilde{B} \longrightarrow |\mathcal{L}|,$$

and for each object $P \in \mathcal{O}(\mathcal{F}^c)$, $\widetilde{B}(P) \simeq BP$ [BLO3, Proposition 4.6]. Consider the Bousfield-Kan spectral sequence [BK] for cohomology of the homotopy colimit, with coefficients in the p-adic integers $\mathbb{Z}_p$. Since $\mathbb{Q}$ is flat as a $\mathbb{Z}$-module, one can tensor the spectral sequence with $\mathbb{Q}$ to get a spectral sequence for p-adic rational cohomology

$$E_2^{p,q} = \varprojlim_{\mathcal{O}(\mathcal{F}^c)}{}^p H^q_{\mathbb{Q}_p}(\widetilde{B}(-)) \Longrightarrow H^{p+q}_{\mathbb{Q}_p}(|\mathcal{L}|).$$

By Lemma 2.2, the higher limits all vanish and we obtain the formula

$$H^*_{\mathbb{Q}_p}(|\mathcal{L}|) \cong \varprojlim_{\mathcal{O}(\mathcal{F}^c)} H^*_{\mathbb{Q}_p}(\widetilde{B}(-)). \tag{1}$$

For each $\mathcal{F}$-centric subgroup $P \leq S$, let $\iota_P \colon P \to S$ denote the inclusion. The inverse limit in (1) consists of all elements $x \in H^*_{\mathbb{Q}_p}(BS)$ such that $\varphi^* \circ \iota_Q^*(x) = \iota_P^*(x)$ for all $\mathcal{F}$-centric subgroups $P, Q \leq S$, and all morphisms $\varphi \in \mathrm{Hom}_{\mathcal{F}}(P,Q)$.

Let $\varphi\colon P \to Q$ be any morphism in $\mathcal{F}$, where P and Q are $\mathcal{F}$-centric. Then by [BLO3, Lemma 2.4] the restriction $\varphi|_{P_0}$ coincides with the restriction to P_0 of some automorphism $\sigma \in W(\mathcal{G})$. Let $x \in H^*_{\mathbb{Q}_p}(BS_0)^{W(\mathcal{G})} \leq H^*_{\mathbb{Q}_p}(BS)$ be any element. Then $\iota_P^*(x) \in H^*_{\mathbb{Q}_p}(BP) \leq H^*_{\mathbb{Q}_p}(P_0)$, and $\iota_Q^*(x) \in H^*_{\mathbb{Q}_p}(BQ) \leq H^*_{\mathbb{Q}_p}(BQ_0)$, and

$$\varphi^*(\iota_Q^*(x)) = \sigma^*(\iota_Q^*(x)) = \iota_P^*\sigma^*(x) = \iota_P^*(x).$$

Hence

$$H^*_{\mathbb{Q}_p}(BS_0)^{W(\mathcal{G})} \leq \varprojlim_{\mathcal{O}(\mathcal{F}^c)} H^*_{\mathbb{Q}_p}(\widetilde{B}(-)).$$

Conversely, let $y \in H^*_{\mathbb{Q}_p}(BS) \leq H^*_{\mathbb{Q}_p}(BS_0)$ be an element which is stable under each morphism in $\mathcal{F}$ between centric subgroups, and let $\sigma \in W(\mathcal{G})$. By Alperin's fusion theorem, σ can be decomposed into a sequence $\sigma = \sigma_1 \circ \sigma_2 \circ \cdots \sigma_n$, where each $\sigma_i \in W(\mathcal{G})$ can be extended to an automorphism of some $\mathcal{F}$-centric subgroup $P_i \leq S$. But since y is stable under each of the σ_i^*, it is also stable under σ^*. This shows that

$$\varprojlim_{\mathcal{O}(\mathcal{F}^c)} H^*_{\mathbb{Q}_p}(\widetilde{B}(-)) \cong H^*_{\mathbb{Q}_p}(BS_0)^{W(\mathcal{G})}$$

and thus completes the proof of our claim. □

3. Applications

For a compact Lie group G, one defines an unstable Adams operation of degree ζ to be a selfmap of the classifying space inducing multiplication by ζ^i on rational cohomology in dimension $2i$, where ζ is an integer. An analogous definition is made for p-compact groups, except ζ is required to be a p-adic unit, and rational cohomology is replaced by p-adic rational cohomology. Unstable Adams operations are a very important concept in the homotopy theory of classifying spaces of compact Lie groups and p-compact groups.

In [JLL], it is shown that p-local compact groups also admit unstable Adams operations. Let $\mathcal{G} = (S, \mathcal{F}, \mathcal{L})$ be a p-local compact group and let ζ be a p-adic unit. A *normal Adams automorphism of degree* ζ on S is an automorphism $\phi \in \mathrm{Aut}(S)$ which restricts to the ζ-power map on S_0, and induces the identity on the group of components S/S_0. A *geometric unstable Adams operation of degree* ζ *on* $\mathcal{G}$ is a selfmap Ψ of $B\mathcal{G}$, such that there exist a normal Adams automorphism ϕ of degree ζ on S, with the property that $\iota \circ B\phi \simeq \Psi \circ \iota$. Here $\iota\colon BS \to B\mathcal{G}$ is the canonical inclusion. (See [JLL, Definitions 2.3, 3.4]) Theorem A allows us to define geometric unstable Adams operations of p-local compact groups, along the lines of the classical cohomological definition.

The following lemma is an analogue of a theorem of Notbohm [N, Proposition 4.1].

Lemma 3.1. *Let $\mathcal{G} = (S, \mathcal{F}, \mathcal{L})$ be a p-local compact group, and let T be a discrete p-torus. Then there is an isomorphism*

$$\mathrm{Hom}(T, S_0)/W(\mathcal{G}) \xrightarrow{\cong} [BT, B\mathcal{G}],$$

where $W(\mathcal{G})$ acts by left translation. Also, two maps $f, h\colon BT \to B\mathcal{G}$ are homotopic if and only if they induce the same homomorphism on $H^*_{\mathbb{Q}_p}(-)$.

Proof. By [BLO3, Theorem 6.3 (a)] there is an isomorphism of sets

$$\mathrm{Rep}(T, \mathcal{L}) \xrightarrow{\cong} [BT, B\mathcal{G}],$$

where $\mathrm{Rep}(T, \mathcal{L}) \overset{\text{def}}{=} \mathrm{Hom}(T, S)/\!\sim$, with $\alpha \sim \beta$ if and only if there is some $\varphi \in \mathrm{Hom}_{\mathcal{F}}(\alpha(T), \beta(T))$ such that $\varphi \circ \alpha = \beta$. Since T is a discrete p-torus, the image of every homomorphism from it to S is contained in S_0, and by [BLO3, Lemma 2.4 (b)], every homomorphism in $\mathcal{F}$ between subgroups of S_0 is the restriction of some element in $W(\mathcal{G})$. Thus

$$\mathrm{Rep}(T, \mathcal{L}) \cong \mathrm{Hom}(T, S_0)/W(\mathcal{G}),$$

as claimed.

It remains to prove the last statement. Two maps $f, h\colon BT \to B\mathcal{G}$ that are homotopic clearly induce the same map on cohomology. Conversely, assume that $f, h\colon BT \to B\mathcal{G}$ are two maps such that $f^* = g^*\colon H^*_{\mathbb{Q}_p}(B\mathcal{G}) \longrightarrow H^*_{\mathbb{Q}_p}(BT)$. Let $\alpha, \beta\colon T \longrightarrow S_0$ be homomorphisms such that $f = \iota \circ B\alpha$ and $g = \iota \circ B\beta$, where $\iota\colon BS_0 \longrightarrow B\mathcal{G}$ is the inclusion of the maximal torus.

We will show that there is $w \in W(\mathcal{G})$ such that $w \circ \alpha = \beta$, following the argument used by Adams and Mahmud to prove [AM, Theorem 1.7]: an argument based on the uniqueness of factorisation in the polynomial ring $H^*_{\mathbb{Q}_p}(BS_0)$. For simplicity, write $V = H^2_{\mathbb{Q}_p}(BS_0)$ and $V' = H^2_{\mathbb{Q}_p}(BT)$. For each $w \in W(\mathcal{G})$, define

$$V(w) = \{\, x \in V \mid B\beta^*(x) = B(w \circ \alpha)^*(x) \,\} = \mathrm{Ker}\big((B\beta^* - B(w \circ \alpha)^*)|_V\big)\,.$$

For each $x \in V$, set

$$\widehat{x} = \prod_{w \in W(\mathcal{G})} Bw^*(1 + x) \in S(V) \cong H^*_{\mathbb{Q}_p}(BS_0)$$

where $S(V)$ denotes the symmetric algebra on the $\mathbb{Q}_p$-vector space V. Since $\widehat{x}$ is $W(\mathcal{G})$-invariant, Theorem 2.3 implies that $\widehat{x} \in \mathrm{Im}(\iota^*)$, and hence that $B\alpha^*(\widehat{x}) = B\beta^*(\widehat{x})$. In other words,

$$\prod_{w \in W(\mathcal{G})} (1 + B\alpha^* Bw^* x) = \prod_{w \in W(\mathcal{G})} (1 + B\beta^* Bw^* x) \in S(V')\,.$$

Since $S(V')$ is a unique factorization domain, there is $w \in W(\mathcal{G})$ such that $(1 + B\beta^* x) = \lambda(1 + B\alpha^* Bw^* x)$, for some $\lambda \in \mathbb{Q}_p^\times$. Then $\lambda = 1$ and hence $B\beta^* x = B\alpha^* Bw^* x$. In particular, $x \in V(w)$.

This proves that $V = \bigcup_{w \in W(\mathcal{G})} V(w)$. Since $\mathbb{Q}_p$ is infinite, V finite dimensional, and $W(\mathcal{G})$ finite, there is $w \in W(\mathcal{G})$ such that $V = V(w)$ (cf. [AM, Lemma 3.1]). Hence $B\beta^* = B(w \circ \alpha)^*$. Since $\mathrm{Hom}(T, S_0)$ injects into $\mathrm{Hom}\big(H^*_{\mathbb{Q}_p}(BS_0), H^*_{\mathbb{Q}_p}(T)\big)$, it now follows that $w \circ \alpha = \beta \in \mathrm{Hom}(T, S_0)$, and hence that $f \simeq g$ as maps $BT \longrightarrow B\mathcal{G}$. $\square$

Proposition 3.2. *Let $\mathcal{G} = (S, \mathcal{F}, \mathcal{L})$ be a p-local compact group, and let ζ be a p-adic unit. Then any geometric unstable Adams operation Ψ of degree ζ induces multiplication by ζ^i on $H^{2i}_{\mathbb{Q}_p}(B\mathcal{G})$. If S_0 is self centralising in S, then any self equivalence Ψ of $B\mathcal{G}$ which induces multiplication by ζ^i on $H^{2i}_{\mathbb{Q}_p}(B\mathcal{G})$ for each i is a geometric unstable Adams operation on $\mathcal{F}$.*

Proof. Let $\iota\colon BS \to B\mathcal{G}$ be the canonical inclusion (induced by the distinguished monomorphism $\delta_S\colon S \to \operatorname{Aut}_{\mathcal{L}}(S)$), and set $\iota_0 = \iota|_{S_0}$. If ψ is a geometric unstable Adams operation on $\mathcal{G}$ of degree ζ, then by definition, there exists a normal Adams automorphism ϕ of S such that $\Psi\circ\iota \simeq \iota\circ B\phi$, and hence $\Psi\circ\iota_0 \simeq \iota_0\circ B(\phi|_{S_0})$. For each $i \geq 0$, $\phi|_{S_0}$ induces multiplication by ζ^i on $H^{2i}_{\mathbb{Q}_p}(BS_0)$, and hence Ψ does the same on $H^{2i}_{\mathbb{Q}_p}(B\mathcal{G})$.

Assume now that S_0 is self centralising in S. Let $\Psi\colon B\mathcal{G} \to B\mathcal{G}$ be a self equivalence such that Ψ^* is multiplication by ζ^i on $H^{2i}_{\mathbb{Q}_p}(B\mathcal{G})$. By [BLO3, Theorem 6.3(a)] and Lemma 3.1, the natural maps

$$\operatorname{End}(S)/\operatorname{Aut}_{\mathcal{F}}(S) \xrightarrow{\cong} [BS, B\mathcal{G}] \quad\text{and}\quad \operatorname{End}(S_0)/W(\mathcal{G}) \xrightarrow{\cong} [BS_0, B\mathcal{G}] \tag{2}$$

are bijections. Hence there is $\varphi \in \operatorname{End}(S)$ such that $\iota\circ B\varphi \simeq \Psi\circ\iota$, and $\varphi \in \operatorname{Aut}(S)$ since Ψ is a homotopy equivalence. Let $\varphi_0 \in \operatorname{Aut}(S_0)$ be the restriction of φ to S_0, let $\widehat{\zeta}$ denote the ζ-power map on S_0, and set $\rho = \widehat{\zeta}\circ\varphi_0^{-1} \in \operatorname{Aut}(S_0)$. Then

$$B\varphi_0^*\circ\iota_0^* = B\widehat{\zeta}^*\circ\iota_0^*\colon H^*_{\mathbb{Q}_p}(B\mathcal{G}) \longrightarrow H^*_{\mathbb{Q}_p}(BS_0),$$

and by Lemma 3.1, there is $w \in W(\mathcal{G})$ such that $w\circ\varphi_0 = \widehat{\zeta} \in \operatorname{Aut}(S_0)$.

Fix a morphism $\widehat{\iota} \in \operatorname{Mor}_{\mathcal{L}}(S_0, S)$ such that $\pi(\widehat{\iota})$ is the inclusion, and regard this as the inclusion of S_0 in S in the category $\mathcal{L}$. By [BLO3, Lemma 4.3(a)], for each $g \in S$, there is a unique restriction $\delta_{S_0}(g) \in \operatorname{Aut}_{\mathcal{L}}(S_0)$ of $\delta_S(g) \in \operatorname{Aut}_{\mathcal{L}}(S)$; i.e., a unique morphism such that $\widehat{\iota}\circ\delta_{S_0}(g) = \delta_S(g)\circ\widehat{\iota}$. Identify S and S_0 with their images in $\operatorname{Aut}_{\mathcal{L}}(S_0)$. Let $\alpha \in \operatorname{Aut}_{\mathcal{L}}(S_0)$ be a lift of w, i.e., $\pi(\alpha) = w$. By Axiom (C), for each $t \in S_0$, $\alpha\circ\delta_{S_0}(t) = \delta_{S_0}(w(t))\circ\alpha$. Hence, $c_\alpha|_{S_0} = w$, and so $\chi \stackrel{\text{def}}{=} c_\alpha|_S\circ\varphi\colon S \to \operatorname{Aut}_{\mathcal{L}}(S_0)$ restricts to $w\circ\varphi_0 = \widehat{\zeta}$ on S_0.

Now, for each $g \in S$, $\chi\circ c_g = c_{\chi(g)}\circ\chi$ as automorphisms of S_0, and since $\chi|_{S_0} = \widehat{\zeta}$ is central in $\operatorname{Aut}(S_0)$, $c_g|_{S_0} = c_{\chi(g)}|_{S_0}$. Since S_0 is self centralising in S, it follows that for each $g \in S$, $g \equiv \chi(g) \pmod{S_0}$. In particular, $\chi(S) = S$, and χ induces the identity on S/S_0. Thus χ is a normal Adams automorphism of S of degree ζ. Also,

$$\iota\circ B\chi \simeq \iota\circ B(c_\alpha|_S)\circ B\varphi \simeq \iota\circ B\varphi \simeq \Psi\circ\iota\,,$$

and thus Ψ is a geometric unstable Adams operation on $\mathcal{G}$ as claimed. □

If $\mathcal{G} = (S, \mathcal{F}, \mathcal{L})$ is a p-local compact group, and $P \leq S$ is a subgroup satisfying a certain mild condition (fully normalised), then one can define the *normaliser fusion system*, $\mathcal{N}_{\mathcal{F}}(P)$, which is shown in [BLO6, Theorem 2.3] to be a saturated fusion system. The normaliser linking system $N_{\mathcal{L}}(P)$ can be defined in exactly the same way as in [BLO2, Definition 6.1], and the proof of [BLO2, Lemma 6.2] applies verbatim to show that $N_{\mathcal{L}}(P)$ is a centric linking system associated to $N_{\mathcal{F}}(P)$. Thus in this case $N_{\mathcal{G}}(P) = (N_S(P), N_{\mathcal{F}}(P), N_{\mathcal{L}}(P))$ is a p-local compact subgroup of $\mathcal{G}$.

In particular, the maximal torus S_0, is fully normalised, since it is unique in its $\mathcal{F}$-conjugacy class, and we may consider the inclusion

$$\mathcal{N}_{\mathcal{G}}(S_0) \longrightarrow \mathcal{G}\,. \tag{3}$$

Then, S_0 is the maximal torus in both $\mathcal{G}$ and $\mathcal{N}_{\mathcal{G}}(S_0)$, and from the definition of morphisms in the normaliser fusion system [BLO6, Definition 2.1],

$$W(\mathcal{G}) = \mathrm{Aut}_{\mathcal{F}}(S_0) = \mathrm{Aut}_{\mathcal{N}_{\mathcal{F}}(S_0)}(S_0) = W(N_{\mathcal{G}}(S_0)).$$

Thus one obtains as an immediate corollary of Theorem A, that the inclusion (3) induces an isomorphism in p-adic rational cohomology. This is analogous to the corresponding statements for compact Lie groups and p-compact groups.

References

[AM] J.F. Adams & Z. Mahmud, Maps between classifying spaces, Inv. Math. 35 (1976), 1–41

[BK] P. Bousfield & D. Kan, Homotopy limits, completions, and localizations, Lecture notes in math. 304, Springer-Verlag (1972)

[BLO2] C. Broto, R. Levi, & B. Oliver, The homotopy theory of fusion systems, Journal Amer. Math. Soc. 16 (2003), 779–856

[BLO3] C. Broto, R. Levi, & B. Oliver, Discrete models for the p-local homotopy theory of compact Lie groups and p-compact groups, Geom. & Topol. 11 (2007), 315427.

[BLO6] C. Broto, R. Levi, & B. Oliver, In Preparation.

[DW] W. G. Dwyer, & C. W. Wilkerson, Homotopy fixed-point methods for Lie groups and finite loop spaces. Ann. of Math. (2) 139 (1994), no. 2, 395–442.

[JLL] F. Junod, R. Levi and A. Libman, Unstable Adams operations of p-local compact groups, Alg. and Geom. Topol. 12 (2012) 4974

[JM] Stefan Jackowski, J. McClure, Homotopy decomposition of classifying spaces via elementary abelian subgroups. Topology 31 (1992), no. 1, 113132.

[JMO] S. Jackowski, J. McClure, & B. Oliver, Homotopy classification of self-maps of BG via G-actions, Annals of Math. 135 (1992), 184–270

[N] D. Notbohm, Abbildungen zwischen klassifizierenden Räume, Dissertation, Göttingen, 1988

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