Weak approximation of the complex Brownian sheet from a Lévy sheet and applications to SPDEs

Xavier Bardina * Juan Pablo Márquez * Lluís Quer-Sardanyons *†

July 18, 2019

Abstract

We consider a Lévy process in the plane and we use it to construct a family of complex-valued random fields that we show to converge in law, in the space of continuous functions, to a complex Brownian sheet. We apply this result to obtain weak approximations of the random field solution to a semilinear one-dimensional stochastic heat equation driven by the space-time white noise.

2000 *Mathematics Subject Classification*: 60F17; 60G15; 60H15.

Key words and phrases: Brownian sheet; Lévy sheet; stochastic heat equation; weak approximation.

1 Introduction

Let $\{N(x,y);\, x,y \geq 0\}$ be a Poisson process in the plane and $S,T>0$. For any $\varepsilon>0$, define the following random field:

$$x_\varepsilon(s,t) := \varepsilon \int_0^{\frac{t}{\varepsilon}} \int_0^{\frac{s}{\varepsilon}} \sqrt{xy}\,(-1)^{N(x,y)} dx dy, \qquad (s,t)\in[0,S]\times[0,T]. \tag{1}$$

Then, in [2] (see Theorem 1.1 therein) the authors proved that, as ε tends to zero, x_ε converges in law, in the Banach space $\mathcal{C}([0,S]\times[0,T])$ of continuous functions, to the Brownian sheet on $[0,S]\times[0,T]$. It is worth mentioning that this result was motivated by its one-dimensional counterpart, which was proved by Stroock in [10] and says the following: the family of processes

$$y_\varepsilon(t) := \varepsilon \int_0^{\frac{t}{\varepsilon^2}} (-1)^{N(s)} ds,\ \varepsilon>0,$$

where here N denotes a standard Poisson process, converges in law, in the space of continuous functions, to a standard Brownian motion. Note that this kind of processes had already been used by Kac in [9] in order to express the solution of the telegrapher's equation in terms of a Poisson process.

In the present paper, we aim to extend the above result of [2] to the case where the Poisson process is replaced by a Lévy sheet $\{L(x,y);\, x,y\geq 0\}$ (see Section 2 for the precise definition). Indeed, note that expression $(-1)^{N(x,y)}$ can be written in terms of the complex exponential as $e^{i\pi N(x,y)}$. Hence, when replacing N by L, we will use the form $e^{i\pi L(x,y)} = \cos(\pi L(x,y)) + i\sin(\pi L(x,y))$ since the expression $(-1)^{L(x,y)}$ may not be well-defined in $\mathbb{R}$. On the other hand, we will replace π by an arbitrary angle $\theta\in(0,2\pi)$. The main result of the paper is the following:

*Departament de Matemàtiques, Universitat Autònoma de Barcelona, 08193 Bellaterra (Barcelona), Catalonia. E-mail addresses: bardina@mat.uab.cat, juanpablo@mat.uab.cat, quer@mat.uab.cat. Research supported by the grant MTM2015-67802P of the Ministerio de Economía y Competitividad.

†Corresponding author.

Theorem 1.1. *Let $\{L(x,y);\, x,y \geq 0\}$ be a Lévy sheet and $\Psi(\xi) := a(\xi)+ib(\xi)$, $\xi \in \mathbb{R}$, its Lévy exponent. Let $\theta \in (0, 2\pi)$ and $S,T > 0$, and define, for any $\varepsilon > 0$ and $(s,t) \in [0,S] \times [0,T]$,*

$$X_\varepsilon(s,t) := \varepsilon K \int_0^{\frac{t}{\varepsilon}} \int_0^{\frac{s}{\varepsilon}} \sqrt{xy}\, \{\cos(\theta L(x,y)) + i \sin(\theta L(x,y))\} dxdy, \tag{2}$$

where the constant K is given by

$$K = \frac{1}{\sqrt{2}} \frac{a(\theta)^2 + b(\theta)^2}{\sqrt{a(\theta)^2 - b(\theta)^2}}. \tag{3}$$

Assume that $a(\theta)a(2\theta) \neq 0$ and $|b(\theta)| \neq a(\theta)$. Then, as ε tends to zero, X_ε converges in law, in the space of complex-valued continuous functions $\mathcal{C}([0,S] \times [0,T]; \mathbb{C})$, to a complex Brownian sheet.

We recall that, by definition, a complex Brownian sheet is a complex random field whose real and imaginary parts are independent Brownian sheets. Hence, in view of the above theorem, we observe that the real and imaginary parts of X_ε are clearly not independent, for any $\varepsilon > 0$, while in the limit they are. This phenomenon is not new, for it already appeared in the study of analogous problems in the one-parameter setting (see, e.g., [1, 4]). Indeed, in [1], a family of processes that converges in law to a complex Brownian motion was constructed from a unique Poisson process. This result was generalized in [4], where the Poisson process was replaced by processes with independent increments whose characteristic functions satisfy some properties. Lévy processes are one of the examples where the latter results may be applied.

The main strategy in order to prove the kind of weak convergence stated in Theorem 1.1 consists in proving that the underlying family of laws is relatively compact in the space of continuous functions (with the usual topology). By Prohorov's theorem, this is equivalent to proving the tightness property of this family of laws. Next, we will check that every weakly convergent partial sequence converges to the limit law that we want to obtain.

In the last part of the paper (see Section 5), we consider the following semilinear stochastic heat equation driven by the space-time white noise:

$$\frac{\partial U}{\partial t}(t,x) - \frac{\partial^2 U}{\partial x^2}(t,x) = b(U(t,x)) + \dot{W}(t,x), \quad (t,x) \in [0,T] \times [0,1], \tag{4}$$

where $T > 0$ and b is a globally Lipschitz function. We impose some initial datum and Dirichlet boundary conditions. In Theorem 5.1 below, we will prove that the random field solution U of (4) can be approximated in law, in the space of continuous functions, by a sequence of random fields $\{U_\varepsilon\}_\varepsilon$, where U_ε is the mild solution to a stochastic heat equation like (4) but driven by either the real or imaginary part of the noise X_ε. This result provides an example of a kind of weak continuity phenomenon in the path space, where convergence in law of the noisy inputs implies convergence in law of the corresponding solutions. Another example of this fact was provided by Walsh in [12], where a parabolic stochastic partial differential equation was used to model a discontinuous neurophysiological phenomenon.

The proof of Theorem 5.1 will follow from [3, Thm. 1.4]. More precisely, Theorem 1.4 of [3] establishes sufficient conditions on a family of random fields that approximate the Brownian sheet (in some sense) under which the solutions of (4) driven by this family converges in law, in the space of continuous functions, to the random field U. We refer to Section 5 for the precise statement of the above-mentioned conditions. In [3], the authors apply their main result to two important families of random fields that approximate the Brownian sheet: the Donsker kernels in the plane and the Kac-Stroock processes, where the latter are defined by

$$\theta_n(t,x) := n\sqrt{tx}\,(-1)^{N\left(\sqrt{n}\,t,\sqrt{n}\,x\right)},$$

where N denotes a standard Poisson process in the plane (indeed, this case corresponds to (1)). As it will be exhibited in Section 5, the proof of Theorem 5.1 is strongly based on the treatment of the Kac-Stroock processes in [3] (see Section 4 therein), and also on some technical estimates contained in the proof of the tightness result given in Proposition 3.1 of the present paper.

Finally, we note that the kind of convergence results that are obtained in the present paper assure that the limit processes, which in our case correspond to the complex Brownian sheet and the solution to the stochastic heat equation, are robust when used as models in practical situations. Moreover, the obtained results provide expressions that can be useful to study simulations of these limit processes.

The paper is organized as follows. Section 2 contains some preliminaries on two-parameter random fields and the definition of Lévy sheet. Section 3 is devoted to prove that the family of laws of $(X_\varepsilon)_{\varepsilon>0}$ is tight in the space of complex-valued continuous functions. The limit identification is addressed in Section 4. Finally, the result on weak convergence for the stochastic heat equation is obtained in Section 5.

2 Preliminaries

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space. We will use some notation introduced by Cairoli and Walsh in [6]. Namely, let $\{\mathcal{F}_{s,t};\, (s,t) \in [0,S]\times[0,T]\}$ be a family of sub-σ-algebras of $\mathcal{F}$ satisfying:

(i) $\mathcal{F}_{s,t} \subset \mathcal{F}_{s',t'}$, for all $s \le s'$ and $t \le t'$.

(ii) All zero sets of $\mathcal{F}$ are contained in $\mathcal{F}_{0,0}$.

(iii) For any $z \in [0,S]\times[0,T]$, $\mathcal{F}_z = \cap_{z<z'}\mathcal{F}_{z'}$, where $z=(s,t) < z' = (s',t')$ denotes the partial order in $[0,S]\times[0,T]$, which means that $s < s'$ and $t < t'$.

If $(s,t) < (s',t')$ and Y denotes any random field defined in $[0,S]\times[0,T]$, the increment of Y on the rectangle $[(s,t),(s',t')]$ is defined by

$$\Delta_{s,t} Y(s',t') := Y(s',t') - Y(s,t') - Y(s',t) + Y(s,t).$$

An adapted process $\{Y(s,t);\, (s,t)\in[0,S]\times[0,T]\}$ with respect to the filtration $\{\mathcal{F}_{s,t};\, (s,t)\in[0,S]\times[0,T]\}$ is called a martingale if $\mathbb{E}[|Y(s,t)|] < \infty$ for all $(s,t)\in[0,S]\times[0,T]$ and

$$\mathbb{E}[\Delta_{s,t}Y(s',t')|\mathcal{F}_{s,t}] = 0, \quad \text{for all} \quad (s,t) < (s',t').$$

It will be called a strong martingale if $\mathbb{E}[|Y(s,t)|] < \infty$ for all $(s,t)\in[0,S]\times[0,T]$, $Y(s,0) = Y(0,t) = 0$ for all s,t and

$$\mathbb{E}[\Delta_{s,t}Y(s',t')|\mathcal{F}_{S,t}\vee\mathcal{F}_{s,T}] = 0, \quad \text{for all} \quad (s,t) < (s',t').$$

We recall that a Brownian sheet is an adapted process $\{W(s,t);\, (s,t)\in[0,S]\times[0,T]\}$ such that $W(s,0) = W(0,t) = 0$ $\mathbb{P}$-a.s., the increment $\Delta_{s,t}W(s',t')$ is independent of $\mathcal{F}_{S,t}\vee\mathcal{F}_{s,T}$, for all $(s,t) < (s',t')$, and it is normally distributed with mean zero and variance $(s'-s)(t'-t)$. If no filtration is specified, we will consider the one generated by the process itself, namely $\mathcal{F}^W := \sigma\{W(s,t);\, (s,t)\in[0,S]\times[0,T]\}$ (conveniently completed).

A Lévy sheet is defined as follows. In general, if Q is any rectangle in $\mathbb{R}^2_+$ and Y any random field in $\mathbb{R}^2_+$, we will also denote by $\Delta_Q Y$ the increment of Y on Q. It is well-known that, for any negative definite function Ψ in $\mathbb{R}$, there exists a real-valued random field $L = \{L(s,t);\, s,t\ge 0\}$ such that

(i) For any family of disjoint rectangles $Q_1,\dots,Q_n$ in $\mathbb{R}^2_+$, the increments $\Delta_{Q_1}L,\dots,\Delta_{Q_n}L$ are independent random variables.

(ii) For any rectangle Q in $\mathbb{R}^2_+$, the characteristic function of the increment $\Delta_Q L$ is given by

$$\mathbb{E}\left[e^{i\xi\Delta_Q L}\right] = e^{-\lambda(Q)\Psi(\xi)}, \quad \xi\in\mathbb{R}, \tag{5}$$

where λ denotes the Lebesgue measure on $\mathbb{R}^2_+$.

Definition 2.1. *A random field $L = \{L(s,t);\ s,t \geq 0\}$ taking values in $\mathbb{R}$ that is continuous in probability and satisfies the above conditions (i) and (ii) is called a Lévy sheet with exponent Ψ.*

By the Lévy-Khintchine formula, we have

$$\Psi(\xi) = ia\xi + \frac{1}{2}\sigma^2\xi^2 + \int_{\mathbb{R}} \left[1 - e^{i\xi x} + \frac{ix\xi}{1+|x|^2}\right] \eta(dx), \quad \xi \in \mathbb{R},$$

where $a \in \mathbb{R}$, $\sigma \geq 0$ and η is the corresponding Lévy measure, that is a Borel measure on $\mathbb{R} \setminus \{0\}$ that satisfies

$$\int_{\mathbb{R}} \frac{|x|^2}{1+|x|^2} \eta(dx) < \infty.$$

We write $\Psi(\xi) = a(\xi) + ib(\xi)$, where

$$a(\xi) := \frac{1}{2}\sigma^2\xi^2 + \int_{\mathbb{R}} [1 - \cos(\xi x)]\eta(dx),$$

and

$$b(\xi) := a\xi + \int_{\mathbb{R}} \left[\frac{x\xi}{1+|x|^2} - \sin(\xi x)\right] \eta(dx).$$

Observe that $a(\xi) \geq 0$ and, if $\xi \neq 0$, $a(\xi) > 0$ whenever $\sigma > 0$ and/or η is nontrivial.

3 Tightness

This section is devoted to prove that the family of probability laws of $\{X_\varepsilon\}_{\varepsilon>0}$ is tight in $\mathcal{C}([0,S]\times[0,T];\mathbb{C})$. This will be a consequence of the next result and the tightness criterion [5, Thm. 3] (see also [7]).

Proposition 3.1. *Let $\{X_\varepsilon\}_{\varepsilon>0}$ be the family of random fields defined by (2). There exists a positive constant C such that, for all $(0,0) \leq (s,t) < (s',t') \leq (S,T)$,*

$$\sup_{\varepsilon>0} \mathbb{E}\left[|\Delta_{s,t} X_\varepsilon(s',t')|^4\right] \leq C(s'-s)^2(t'-t)^2.$$

This implies that the the family of probability laws of $(X_\varepsilon)_{\varepsilon>0}$ is tight in $\mathcal{C}([0,S]\times[0,T];\mathbb{C})$.

Proof. By definition of X_ε and the properties of the modulus $|\cdot|$, we have

$$
\begin{aligned}
&\mathbb{E}\left[|\Delta_{s,t}X_\varepsilon(s',t')|^4\right]\\
&= \varepsilon^4K^4\mathbb{E}\left[\left|\int_{\frac{t}{\varepsilon}}^{\frac{t'}{\varepsilon}}\int_{\frac{s}{\varepsilon}}^{\frac{s'}{\varepsilon}}\sqrt{xy}\left\{\cos(\theta L(x,y))+i\sin(\theta L(x,y))\right\}dxdy\right|^4\right]\\
&= \varepsilon^4K^4\mathbb{E}\left[\left|\int_{\frac{t}{\varepsilon}}^{\frac{t'}{\varepsilon}}\int_{\frac{s}{\varepsilon}}^{\frac{s'}{\varepsilon}}\sqrt{xy}\,e^{i\theta L(x,y)}dxdy\right|^4\right]\\
&= \varepsilon^4K^4\mathbb{E}\left[\left(\int_{\frac{t}{\varepsilon}}^{\frac{t'}{\varepsilon}}\int_{\frac{s}{\varepsilon}}^{\frac{s'}{\varepsilon}}\sqrt{x_1y_1}\,e^{i\theta L(x_1,y_1)}dx_1dy_1\int_{\frac{t}{\varepsilon}}^{\frac{t'}{\varepsilon}}\int_{\frac{s}{\varepsilon}}^{\frac{s'}{\varepsilon}}\sqrt{x_2y_2}\,e^{-i\theta L(x_2,y_2)}dx_2dy_2\right)^2\right]\\
&= \varepsilon^4K^4\mathbb{E}\left[\left(\int_{\frac{t}{\varepsilon}}^{\frac{t'}{\varepsilon}}\int_{\frac{s}{\varepsilon}}^{\frac{s'}{\varepsilon}}\int_{\frac{t}{\varepsilon}}^{\frac{t'}{\varepsilon}}\int_{\frac{s}{\varepsilon}}^{\frac{s'}{\varepsilon}}\sqrt{x_1x_2y_1y_2}\,e^{i\theta(L(x_2,y_2)-L(x_1,y_1))}dx_1dx_2dy_1dy_2\right)^2\right]\\
&= \varepsilon^4K^4\mathbb{E}\left[\int_{\frac{t}{\varepsilon}}^{\frac{t'}{\varepsilon}}\int_{\frac{s}{\varepsilon}}^{\frac{s'}{\varepsilon}}\cdots\int_{\frac{t}{\varepsilon}}^{\frac{t'}{\varepsilon}}\int_{\frac{s}{\varepsilon}}^{\frac{s'}{\varepsilon}}\sqrt{x_1x_2x_3x_4}\sqrt{y_1y_2y_3y_4}\right.\\
&\qquad\left.\times\, e^{i\theta(L(x_4,y_4)-L(x_3,y_3)+L(x_2,y_2)-L(x_1,y_1))}dx_1\dots dx_4dy_1\dots dy_4\right].
\end{aligned}
$$

Taking into account that we can write $e^{i\theta\sum_{j=1}^4(-1)^jL(x_j,y_j)}=e^{i\theta\sum_{j=1}^4(-1)^j\Delta_{0,0}L(x_j,y_j)}$, we obtain

$$
\begin{aligned}
\mathbb{E}\left[|\Delta_{s,t}X_\varepsilon(s',t')|^4\right] = \varepsilon^4K^4&\int_{\frac{t}{\varepsilon}}^{\frac{t'}{\varepsilon}}\int_{\frac{s}{\varepsilon}}^{\frac{s'}{\varepsilon}}\cdots\int_{\frac{t}{\varepsilon}}^{\frac{t'}{\varepsilon}}\int_{\frac{s}{\varepsilon}}^{\frac{s'}{\varepsilon}}\sqrt{x_1x_2x_3x_4}\sqrt{y_1y_2y_3y_4}\\
&\times\mathbb{E}\left[e^{i\theta\sum_{j=1}^4(-1)^j\Delta_{0,0}L(x_j,y_j)}\right]dx_1\dots dx_4dy_1\dots dy_4.
\end{aligned}
$$

In order to estimate the expectation inside the above term, we need to consider all 24 possible orders of the x-variables and all 24 possible orders of the y-variables. Altogether, this amounts to take into account 576 possibilities. Let $\mathcal{P}_4$ be the group of permutations of degree 4. Then,

$$
\begin{aligned}
&\mathbb{E}\left[|\Delta_{s,t}X_\varepsilon(s',t')|^4\right]\\
&= \sum_{\sigma\in\mathcal{P}_4}\sum_{\beta\in\mathcal{P}_4}\varepsilon^4K^4\int_{\frac{t}{\varepsilon}}^{\frac{t'}{\varepsilon}}\int_{\frac{s}{\varepsilon}}^{\frac{s'}{\varepsilon}}\cdots\int_{\frac{t}{\varepsilon}}^{\frac{t'}{\varepsilon}}\int_{\frac{s}{\varepsilon}}^{\frac{s'}{\varepsilon}}\sqrt{x_1x_2x_3x_4}\sqrt{y_1y_2y_3y_4}\,\mathbb{E}\left[e^{i\theta\sum_{j=1}^4(-1)^j\Delta_{0,0}L(x_j,y_j)}\right]\\
&\qquad\times I_{\{x_{\sigma(1)}<x_{\sigma(2)}<x_{\sigma(3)}<x_{\sigma(4)}\}}I_{\{y_{\beta(1)}<y_{\beta(2)}<y_{\beta(3)}<y_{\beta(4)}\}}dx_1\dots dx_4dy_1\dots dy_4\\
&\le \sum_{\sigma\in\mathcal{P}_4}\sum_{\beta\in\mathcal{P}_4}\varepsilon^4K^4\int_{\frac{t}{\varepsilon}}^{\frac{t'}{\varepsilon}}\int_{\frac{s}{\varepsilon}}^{\frac{s'}{\varepsilon}}\cdots\int_{\frac{t}{\varepsilon}}^{\frac{t'}{\varepsilon}}\int_{\frac{s}{\varepsilon}}^{\frac{s'}{\varepsilon}}\sqrt{x_1x_2x_3x_4}\sqrt{y_1y_2y_3y_4}\left|\mathbb{E}\left[e^{i\theta\sum_{j=1}^4(-1)^j\Delta_{0,0}L(x_j,y_j)}\right]\right|\\
&\qquad\times I_{\{x_{\sigma(1)}<x_{\sigma(2)}<x_{\sigma(3)}<x_{\sigma(4)}\}}I_{\{y_{\beta(1)}<y_{\beta(2)}<y_{\beta(3)}<y_{\beta(4)}\}}dx_1\dots dx_4dy_1\dots dy_4. \qquad (6)
\end{aligned}
$$

At this point, we observe that the geometric structure of the resulting increments of L in the expression $\sum_{j=1}^4(-1)^j\Delta_{0,0}L(x_j,y_j)$ of any of the 576 possibilities corresponds to one of the 24 cases drawn in Figure 1; we note that the latter corresponds to all 24 possible orders of the x-variables with $y_1<y_2<y_3<y_4$. In each of these 24 possible structures, the corresponding increments of L turn out to be multiplied by $c_1\in\{-1,1\}$ in the black regions, while they are multiplied by $c_2\in\{-2,0,2\}$ in the white regions.

Let us now fix two permutations $\sigma, \beta \in \mathcal{P}_4$, and we will focus on the term

$$\left| \mathbb{E}\left[e^{i\theta \sum_{j=1}^4 (-1)^j \Delta_{0,0} L(x_i, y_j)} \right] \right| I_{\{x_{\sigma(1)} < x_{\sigma(2)} < x_{\sigma(3)} < x_{\sigma(4)}\}} I_{\{y_{\beta(1)} < y_{\beta(2)} < y_{\beta(3)} < y_{\beta(4)}\}}. \tag{7}$$

We perform a change of variables in such a way that, making a harmless abuse of notation and using again the same one for the new variables, we have $x_1 < x_2 < x_3 < x_4$ and $y_1 < y_2 < y_3 < y_4$.

On the other hand, if we denote by Q the region of Figure 1 corresponding the above fixed variables order, we know that Q can be decomposed as a union of black rectangles and white rectangles. More precisely, we can write

$$Q = \left(\cup_k \bar{Q}_k \right) \cup \left(\cup_l \tilde{Q}_l \right),$$

where the increments $\Delta_{\bar{Q}_k} L$ are multiplied by $c_1^k \in \{-1, 1\}$ and $\Delta_{\tilde{Q}_l} L$ are multiplied by $c_2^l \in \{-2, 0, 2\}$. Hence, expression (7) is given by

$$\begin{aligned}
& \left| \mathbb{E}\left[e^{i\theta (\sum_k c_1^k \Delta_{\bar{Q}_k} L + \sum_l c_2^l \Delta_{\tilde{Q}_l} L)} \right] \right| I_{\{x_1 < x_2 < x_3 < x_4\}} I_{\{y_1 < y_2 < y_3 < y_4\}} \\
= & \left| \mathbb{E}\left[e^{i\theta \sum_k c_1^k \Delta_{\bar{Q}_k} L} \right] \mathbb{E}\left[e^{i\theta \sum_l c_2^l \Delta_{\tilde{Q}_l} L} \right] \right| I_{\{x_1 < x_2 < x_3 < x_4\}} I_{\{y_1 < y_2 < y_3 < y_4\}} \\
= & \left| \mathbb{E}\left[e^{i\theta \sum_k c_1^k \Delta_{\bar{Q}_k} L} \right] \right| \times \left| \mathbb{E}\left[e^{i\theta \sum_l c_2^l \Delta_{\tilde{Q}_l} L} \right] \right| I_{\{x_1 < x_2 < x_3 < x_4\}} I_{\{y_1 < y_2 < y_3 < y_4\}} \\
= & \prod_k \left| e^{-\lambda(\bar{Q}_k) \Psi(c_1^k \theta)} \right| \prod_l \left| e^{-\lambda(\tilde{Q}_l) \Psi(c_2^l \theta)} \right| I_{\{x_1 < x_2 < x_3 < x_4\}} I_{\{y_1 < y_2 < y_3 < y_4\}} \\
= & \prod_k e^{-\lambda(\bar{Q}_k) a(c_1^k \theta)} \prod_l e^{-\lambda(\tilde{Q}_l) a(c_2^l \theta)} I_{\{x_1 < x_2 < x_3 < x_4\}} I_{\{y_1 < y_2 < y_3 < y_4\}} \\
\leq & \prod_k e^{-\lambda(\bar{Q}_k) a(\theta)} I_{\{x_1 < x_2 < x_3 < x_4\}} I_{\{y_1 < y_2 < y_3 < y_4\}} \\
= & \ e^{-\lambda(\bar{Q}) a(\theta)} I_{\{x_1 < x_2 < x_3 < x_4\}} I_{\{y_1 < y_2 < y_3 < y_4\}},
\end{aligned} \tag{8}$$

where $\bar{Q} := \cup_k \bar{Q}_k$. In the above computations, we have used that the real part of the Lévy exponent Ψ is a nonnegative function and satisfies $a(-\theta) = a(\theta)$. We remark that, independently of the constants c_1^k and c_2^l, we have obtained an estimated of (7) which only involves the black regions multiplied by 1. Recall that λ denotes the Lebesgue measure on $\mathbb{R}^2$.

Taking into account estimate (8), it is readily checked that, among all 24 possibilities drawn in Figure 1, it suffices to deal with 4 of these cases (see Figure 2). This is because, in the rest of the cases, the area of $\bar{Q}$ is greater than or equal to the corresponding one of one of these 4 possibilities. Thus, since in (8) the area of $\bar{Q}$ appears with a negative sign, we can focus only on the cases of Figure 2.

Let us start tackling the case corresponding to i) in Figure 2. That is, by estimates (6) and (8), we need to find suitable upper bounds of the term

$$\begin{aligned}
\varepsilon^4 \int_I \int_J & \sqrt{x_1 x_2 x_3 x_4} \sqrt{y_1 y_2 y_3 y_4} \, \exp[-a(\theta)[(x_4 - x_3) y_3 + (y_4 - y_3) x_3 \\
& + (x_2 - x_1) y_1 + (y_2 - y_1) x_1]] \, dx_1 ... dx_4 dy_1 ... dy_4,
\end{aligned}$$

where $J = \{\frac{s}{\varepsilon} \leq x_1 \leq x_2 \leq x_3 \leq x_4 \leq \frac{s'}{\varepsilon}\}$ and $I = \{\frac{t}{\varepsilon} \leq y_1 \leq y_2 \leq y_3 \leq y_4 \leq \frac{t'}{\varepsilon}\}$.

First, estimate x_4 and y_4 in the square roots above by $\frac{s'}{\varepsilon}$ and $\frac{t'}{\varepsilon}$, respectively, and then integrate with respect to theses two variables. The resulting expression can be easily bounded by, up to some positive constant,

$$\sqrt{s'} \sqrt{t'} \varepsilon^3 \int_{I_1} \int_{J_1} \frac{\sqrt{x_1 x_2 x_3} \sqrt{y_1 y_2 y_3}}{x_3 y_3} \exp\left[-a(\theta)[(x_2 - x_1) y_1 + (y_2 - y_1) x_1] \right] dx_1 ... dx_3 dy_1 ... dy_3,$$

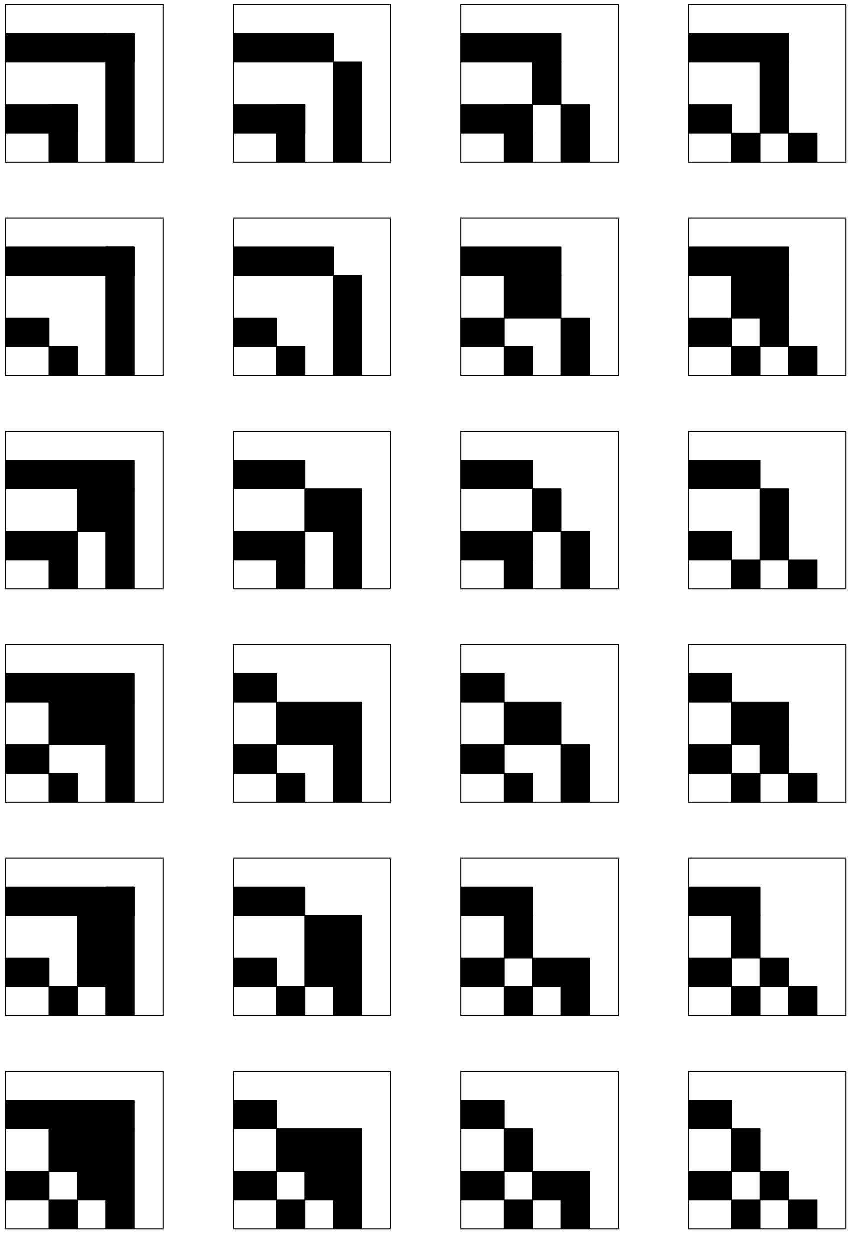

Figure 1: Each square represents the rectangle $[(s,t),(s',t')]$. Regions corresponding to $\sum_{j=1}^{4}(-1)^j \Delta_{0,0} L(x_j, y_j)$, for all possible 24 orders of the x-variables and $y_1 < y_2 < y_3 < y_4$, are drawn in each square. Black areas are regions where the corresponding increment of L appears an odd number of times. Note that, indeed, all areas are extended up to the plane axes.

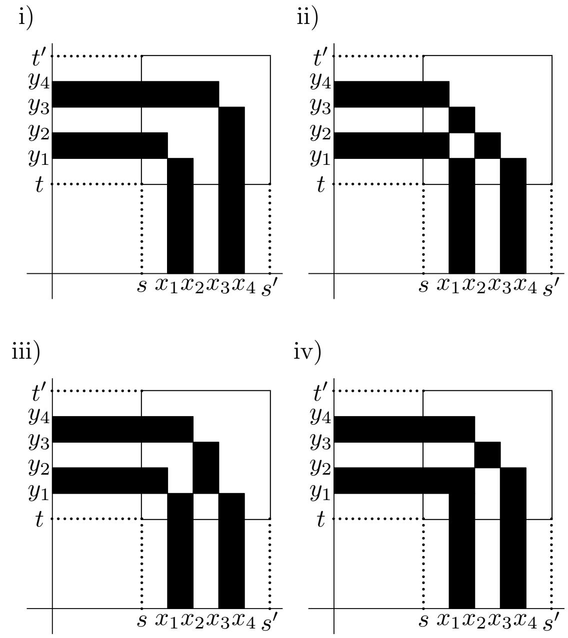

Figure 2: The 4 relevant cases of Figure 1

where $J_1 = \{\frac{s}{\varepsilon} \leq x_1 \leq x_2 \leq x_3 \leq \frac{s'}{\varepsilon}\}$ and $I_1 = \{\frac{t}{\varepsilon} \leq y_1 \leq y_2 \leq y_3 \leq \frac{t'}{\varepsilon}\}$. Now, we estimate x_2 and y_2 by $\frac{s'}{\varepsilon}$ and $\frac{t'}{\varepsilon}$, respectively, inside the square roots, and then integrate with respect to x_2 i y_2. Hence, up to some constant, we obtain an estimate for (9) of the form

$$\begin{aligned} s't'\varepsilon^2 \int_{\frac{s}{\varepsilon}}^{\frac{s'}{\varepsilon}} \frac{1}{\sqrt{x_1}}\, dx_1 \int_{\frac{s}{\varepsilon}}^{\frac{s'}{\varepsilon}} \frac{1}{\sqrt{x_3}}\, dx_3 \int_{\frac{t}{\varepsilon}}^{\frac{t'}{\varepsilon}} \frac{1}{\sqrt{y_1}}\, dy_1 \int_{\frac{t}{\varepsilon}}^{\frac{t'}{\varepsilon}} \frac{1}{\sqrt{y_3}}\, dy_3 \\ = C\big[\sqrt{s'}(\sqrt{s'} - \sqrt{s})\big]^2 \big[\sqrt{t'}(\sqrt{t'} - \sqrt{t})\big]^2 \\ \leq C(s' - s)^2 (t' - t)^2. \end{aligned}$$

This concludes the analysis of i) in Figure 2.

In the remaining three cases, the above-used argument does not directly work. Instead, we will add some small area in the corresponding drawing in such a way that we will be able to argue similarly as in case i). We remark that some of the integrand's estimates that will be obtained in the sequel will hold everywhere except of a zero Lebesgue measure set of $\mathbb{R}^8$.

Let us start with the analysis of the integral corresponding to ii). We need to bound the following term:

$$\varepsilon^4 \int_I \int_J \sqrt{x_1 x_2 x_3 x_4} \sqrt{y_1 y_2 y_3 y_4}\, e^{-\lambda(\bar{Q}) a(\theta)} dx_1 ... dx_4 dy_1 ... dy_4,$$

where $J = \{\frac{s}{\varepsilon} \leq x_1 \leq x_2 \leq x_3 \leq x_4 \leq \frac{s'}{\varepsilon}\}$, $I = \{\frac{t}{\varepsilon} \leq y_1 \leq y_2 \leq y_3 \leq y_4 \leq \frac{t'}{\varepsilon}\}$, and $\bar{Q}$ is the union of black rectangles corresponding to the case ii). Note that $A := \lambda(\bar{Q})$ is given by

$$A = (x_4 - x_3)y_1 + (y_4 - y_3)x_1 + (x_2 - x_1)y_1 + (y_2 - y_1)x_1 + (x_3 - x_2)(y_2 - y_1) + (y_3 - y_2)(x_2 - x_1).$$

We split the above integral into two terms:

$$
\begin{aligned}
&\varepsilon^4 \int_I \int_J \sqrt{x_1x_2x_3x_4}\sqrt{y_1y_2y_3y_4}\, e^{-a(\theta)A}\, I_{\{A\geq 2(x_2-x_1)(y_2-y_1)\}} dx_1...dx_4dy_1...dy_4 \\
&\quad + \varepsilon^4 \int_I \int_J \sqrt{x_1x_2x_3x_4}\sqrt{y_1y_2y_3y_4}\, e^{-a(\theta)A}\, I_{\{A< 2(x_2-x_1)(y_2-y_1)\}} dx_1...dx_4dy_1...dy_4. \qquad (9)
\end{aligned}
$$

When $A \geq 2(x_2-x_1)(y_2-y_1)$, we have

$$
\begin{aligned}
-a(\theta)A &\leq -\frac{a(\theta)}{2}A - a(\theta)(x_2-x_1)(y_2-y_1) \\
&= -\frac{a(\theta)}{2}[(x_4-x_3)y_1 + (y_4-y_3)x_1 + (x_2-x_1)y_3 + (y_2-y_1)x_3].
\end{aligned}
$$

Hence, the first integral in (9) is less or equal than

$$
\begin{aligned}
\varepsilon^4 \int_I \int_J \sqrt{x_1x_2x_3x_4}\sqrt{y_1y_2y_3y_4}\, \exp\Big\{ -\frac{a(\theta)}{2}[(x_4-x_3)y_1 + (y_4-y_3)x_1 \\
+ (x_2-x_1)y_3 + (y_2-y_1)x_3]\Big\} dx_1...dx_4dy_1...dy_4,
\end{aligned}
$$

and following the same arguments used in the case i), this term can be estimated by $(s'-s)^2(t'-t)^2$, up to some positive constant.

On the other hand, as far as the second integral in (9) is concerned, observe that we have

$$
\begin{aligned}
&\{A < 2(x_2-x_1)(y_2-y_1)\} \\
&\quad = \{(x_4-x_3)y_1 + (y_4-y_3)x_1 + (y_2-y_1)x_3 + (x_2-x_1)y_3 < 4(x_2-x_1)(y_2-y_1)\}.
\end{aligned}
$$

In particular, in this region we have

$$
\frac{1}{4}y_3 < (y_2-y_1) \quad \text{and} \quad \frac{1}{4}x_3 < (x_2-x_1),
$$

which implies that

$$
\begin{aligned}
A &\geq (x_4-x_3)y_1 + (y_4-y_3)x_1 + \frac{1}{4}x_3y_1 + \frac{1}{4}y_3x_1 + \frac{1}{4}(x_3-x_2)y_3 + \frac{1}{4}(y_3-y_2)x_3 \\
&\geq \frac{1}{4}\big[x_4y_1 + y_4x_1 + (x_3-x_2)y_3 + (y_3-y_2)x_3\big].
\end{aligned}
$$

Thus, the second integral in (9) can be bounded by

$$
\begin{aligned}
&\varepsilon^4 \int_I \int_J \sqrt{x_1x_2x_3x_4}\sqrt{y_1y_2y_3y_4} \\
&\qquad \times \exp\Big\{ -\frac{a(\theta)}{4}[x_4y_1 + y_4x_1 + (x_3-x_2)y_3 + (y_3-y_2)x_3]\Big\}\, dx_1...dx_4dy_1...dy_4,
\end{aligned}
$$

and here again the arguments of the case i) may be applied, yielding an estimate of the form $(s'-s)^2(t'-t)^2$, up to some positive constant.

The same idea can be used to deal with the integral corresponding to iii). Indeed, in this case the area A is given by

$$
A = (x_4-x_3)y_1 + (x_2-x_1)y_1 + (x_3-x_2)(y_3-y_1) + (y_4-y_3)x_2 + (y_2-y_1)x_1,
$$

and here one splits the underlying integral taking into account the regions $\{A \geq 2(x_3 - x_2)y_1\}$ and $\{A < 2(x_3 - x_2)y_1\}$. In the former, one has

$$\begin{aligned} -a(\theta)A &\leq -\frac{a(\theta)}{2}A - a(\theta)(x_3 - x_2)y_1 \\ &= -\frac{a(\theta)}{2}[(x_4 - x_1)y_1 + (y_2 - y_1)x_1 + (x_3 - x_2)y_3 + (y_4 - y_3)x_2 \end{aligned}$$

and the desired estimated is obtained by using the same computations as for the case i). Note that, in fact, variables which have to be bounded and integrated with respect to are x_4, y_4, y_2, x_3, following this specific order. On the other hand, in the region $\{A < 2(x_3 - x_2)y_1\}$, we get

$$\{A < 2(x_3 - x_2)y_1\} = \{(x_4 - x_1)y_1 + (y_2 - y_1)x_1 + (x_3 - x_2)y_3 + (y_4 - y_3)x_2 < 4(x_3 - x_2)y_1\}.$$

So, in particular, in this region we have

$$\frac{1}{4}y_3 < y_1 \quad \text{and} \quad \frac{1}{4}(x_4 - x_1) < (x_3 - x_2),$$

where we deduce

$$\begin{aligned} A &\geq \frac{1}{4}\big[(x_4 - x_3)y_3 + (x_2 - x_1)y_3 + (x_4 - x_1)(y_3 - y_1) + (y_4 - y_3)x_2 + (y_2 - y_1)x_1\big] \\ &\geq \frac{1}{4}\big[(y_4 - y_3)x_1 + (x_2 - x_1)y_3 + (x_4 - x_3)y_1 + (y_2 - y_1)x_3\big]. \end{aligned}$$

At this point, we can follow the arguments of the preceding cases.

Finally, it only remains to estimate the integral involving case iv) in Figure 2. In this case,

$$A = (x_4 - x_3)y_2 + (x_2 - x_1)y_2 + (y_4 - y_3)x_2 + (y_2 - y_1)x_1 + (x_3 - x_2)(y_3 - y_2)]$$

and the splitting regions are $\{A \geq 2(x_3 - x_2)y_2\} \cup \{A \geq 2(y_3 - y_2)x_2\}$ and the corresponding complement.

In the first region, condition $A \geq 2(x_3 - x_2)y_2$ turns out to be equivalent to

$$-a(\theta)A \leq -\frac{a(\theta)}{2}[(x_3 - x_2)y_3 + (y_4 - y_3)x_2 + (y_2 - y_1)x_1 + (x_4 - x_1)y_1],$$

so we will be able to mimic the arguments used so far. Moreover, note that this case is symmetric in x and y, which implies that the computations in the case $A \geq 2(y_3 - y_2)x_2$ will be the same just by exchanging x and y.

As far as the case $\{A < 2(x_3 - x_2)y_2\} \cap \{A < 2(y_3 - y_2)x_2\}$ is concerned, we have

$$\begin{aligned} &\{A < 2(x_3 - x_2)y_2\} \cap \{A < 2(y_3 - y_2)x_2\} \\ &\quad = \{(x_4 - x_1)y_2 + (y_4 - y_3)x_2 + (y_2 - y_1)x_1 + (x_3 - x_2)y_3 \leq 4(x_3 - x_2)y_2\} \\ &\qquad \cap \{(y_4 - y_1)x_2 + (x_4 - x_3)y_2 + (x_2 - x_1)y_1 + (y_3 - y_2)x_3 \leq 4(y_3 - y_2)x_2\}. \end{aligned}$$

In particular, one has

$$\begin{aligned} &\frac{1}{4}y_3 \leq y_2 \quad \text{and} \quad \frac{1}{4}(x_4 - x_1) \leq (x_3 - x_2), \\ &\frac{1}{4}x_3 \leq x_2 \quad \text{and} \quad \frac{1}{4}(y_4 - y_1) \leq (y_3 - y_2), \end{aligned}$$

which implies

$$A \geq \frac{1}{4}\big[(x_4 - x_3)y_3 + (x_2 - x_1)y_3 + (y_4 - y_3)x_3 + (y_2 - y_1)x_1\big]$$

$$\geq \quad \frac{1}{4}\big[(x_4-x_3)y_3+(y_4-y_3)x_3+(x_2-x_1)y_1+(y_2-y_1)x_1\big].$$

One can conclude the proof by following the same arguments as in the preceding cases. □

4 Limit identification

Let $\{\mathbb{P}_\varepsilon\}_{\varepsilon>0}$ be the family of probability laws in $\mathcal{C}([0,S]\times[0,T];\mathbb{C})$ corresponding to $\{X_\varepsilon\}_{\varepsilon>0}$. By Proposition 3.1, there exists a subsequence $\{\mathbb{P}_{\varepsilon_n}\}_{n\geq1}$ of $\{\mathbb{P}_\varepsilon\}_{\varepsilon>0}$ converging, in the weak sense in the space $\mathcal{C}([0,S]\times[0,T];\mathbb{C})$, to some probability measure $\mathbb{P}$. This section is devoted to prove that $\mathbb{P}$ is the law of a complex random field whose real and imaginary parts are independent Brownian sheets.

We will use the following characterization of the Brownian sheet, which is a quotation of [2, Thm. 4.1]. Note that other characterizations of Brownian sheet can be found, e.g., in [8, 11].

Theorem 4.1. *Let $Y=\{Y(s,t);\,(s,t)\in[0,S]\times[0,T]\}$ be a real-valued and continuous process such that $Y(s,0)=Y(0,t)=0$ for all $s\in[0,S]$ and $t\in[0,T]$. Let $\{\mathcal{F}_{s,t};\,(s,t)\in[0,S]\times[0,T]\}$ be the natural filtration associated to Y. Then, the following statements are equivalent:*

(i) Y is a Brownian sheet.

(ii) Y is a strong martingale and, for all $(0,0)<(s,t)\leq(s',t')\leq(S,T)$,

$$\mathbb{E}\big[\big(\Delta_{s,t}Y(s',t')\big)^2|\mathcal{F}_{s,T}\big]=(s'-s)(t'-t).$$

Owing to Theorem 4.1 and Proposition 3.1, the following two propositions will guarantee the validity of (almost all) the statement of Theorem 1.1.

Proposition 4.2. *Recall that $\mathbb{P}$ denotes the weak limit in $\mathcal{C}([0,S]\times[0,T];\mathbb{C})$ of a converging subsequence of the family $\{\mathbb{P}_\varepsilon\}_{\varepsilon>0}$. Let $X=\{X(s,t);\,(s,t)\in[0,S]\times[0,T]\}$ be the corresponding (complex-valued) canonical process and $\{\mathcal{F}_{s,t};\,(s,t)\in[0,S]\times[0,T]\}$ its associated natural filtration. Then, the real and imaginary parts of X define strong martingales under the probability $\mathbb{P}$.*

Proposition 4.3. *Let X be the canonical process defined in the previous proposition. Then, for all $(0,0)<(s,t)\leq(s',t')\leq(S,T)$, it holds:*

$$\mathbb{E}_{\mathbb{P}}\big[\big(\Delta_{s,t}\mathsf{Re}(X)(s',t')\big)^2|\mathcal{F}_{s,T}\big]=(s'-s)(t'-t),$$

and

$$\mathbb{E}_{\mathbb{P}}\big[\big(\Delta_{s,t}\mathsf{Im}(X)(s',t')\big)^2|\mathcal{F}_{s,T}\big]=(s'-s)(t'-t),$$

The proof of Proposition 4.2 is based on the following lemma.

Lemma 4.4. *Let $X_\varepsilon=\{X_\varepsilon(s,t);\,(s,t)\in[0,S]\times[0,T]\}$ be the (complex-valued) random field defined in (2) and $\{\mathcal{F}^\varepsilon_{s,t};\,(s,t)\in[0,S]\times[0,T]\}$ its natural filtration. Then, for all $(0,0)<(s,t)\leq(s',t')\leq(S,T)$,*

$$\lim_{\varepsilon\to0}\mathbb{E}[\Delta_{s,t}X_\varepsilon(s',t')|\mathcal{F}^\varepsilon_{S,t}\vee\mathcal{F}^\varepsilon_{s,T}]=0, \tag{10}$$

where the limit is understood in $L^2(\Omega)$.

Proof. We will use the notation $Y_\varepsilon:=\mathbb{E}[\Delta_{s,t}X_\varepsilon(s',t')|\mathcal{F}_{S,t}\vee\mathcal{F}_{s,T}]$. First, note that we can write

$$\Delta_{s,t}X_\varepsilon(s',t')=\varepsilon K\int_{\frac{t}{\varepsilon}}^{\frac{t'}{\varepsilon}}\int_{\frac{s}{\varepsilon}}^{\frac{s'}{\varepsilon}}\sqrt{xy}\,e^{i\theta\left(L(\frac{s}{\varepsilon},y)+L(x,\frac{t}{\varepsilon})-L(\frac{s}{\varepsilon},\frac{t}{\varepsilon})+\Delta_{\frac{s}{\varepsilon},\frac{t}{\varepsilon}}L(x,y)\right)}dxdy.$$

Thus, we have

$$Y_\varepsilon = \varepsilon K \int_{\frac{t}{\varepsilon}}^{\frac{t'}{\varepsilon}} \int_{\frac{s}{\varepsilon}}^{\frac{s'}{\varepsilon}} \sqrt{xy}\, e^{i\theta\left(L(\frac{s}{\varepsilon},y)+L(x,\frac{t}{\varepsilon})-L(\frac{s}{\varepsilon},\frac{t}{\varepsilon})\right)} \mathbb{E}\left[e^{i\theta\Delta_{\frac{s}{\varepsilon},\frac{t}{\varepsilon}}L(x,y)}\right] dxdy$$
$$= \varepsilon K \int_{\frac{t}{\varepsilon}}^{\frac{t'}{\varepsilon}} \int_{\frac{s}{\varepsilon}}^{\frac{s'}{\varepsilon}} \sqrt{xy}\, e^{i\theta\left(L(\frac{s}{\varepsilon},y)+L(x,\frac{t}{\varepsilon})-L(\frac{s}{\varepsilon},\frac{t}{\varepsilon})\right)} e^{-\Psi(\theta)\left(x-\frac{s}{\varepsilon}\right)\left(y-\frac{t}{\varepsilon}\right)} dxdy$$

and also

$$\mathbb{E}[Y_\varepsilon^2] = \varepsilon^2 K^2 \int_{\frac{t}{\varepsilon}}^{\frac{t'}{\varepsilon}} \int_{\frac{t}{\varepsilon}}^{\frac{t'}{\varepsilon}} \int_{\frac{s}{\varepsilon}}^{\frac{s'}{\varepsilon}} \int_{\frac{s}{\varepsilon}}^{\frac{s'}{\varepsilon}} \sqrt{x_1x_2}\sqrt{y_1y_2}\, e^{-\Psi(\theta)\left((x_1-\frac{s}{\varepsilon})(y_1-\frac{t}{\varepsilon})+(x_2-\frac{s}{\varepsilon})(y_2-\frac{t}{\varepsilon})\right)}$$
$$\times\, \mathbb{E}\left[e^{i\theta\left(L(\frac{s}{\varepsilon},y_1)+L(\frac{s}{\varepsilon},y_2)+L(x_1,\frac{t}{\varepsilon})+L(x_2,\frac{t}{\varepsilon})-2L(\frac{s}{\varepsilon},\frac{t}{\varepsilon})\right)}\right] dx_1dx_2dy_1dy_2.$$

At this point, we take into account the possible orders of x_1, x_2 and y_1, y_2, respectively, which amounts to consider 4 possibilities. Then, in each case we express the exponent in the complex exponential above as a suitable combination of rectangular increments of L, so that we can compute the corresponding expectation thanks to (5). Using this procedure, we end up with

$$\mathbb{E}[Y_\varepsilon^2] = 2(I_1 + I_2), \tag{11}$$

where

$$I_1 = \varepsilon^2 K^2 \int_{\frac{t}{\varepsilon}}^{\frac{t'}{\varepsilon}} \int_{\frac{t}{\varepsilon}}^{y_2} \int_{\frac{s}{\varepsilon}}^{\frac{s'}{\varepsilon}} \int_{\frac{s}{\varepsilon}}^{x_2} \sqrt{x_1x_2}\sqrt{y_1y_2}\, e^{-\Psi(\theta)\left((x_2-x_1)\frac{t}{\varepsilon}+(y_2-y_1)\frac{s}{\varepsilon}+(x_1-\frac{s}{\varepsilon})(y_1-\frac{t}{\varepsilon})+(x_2-\frac{s}{\varepsilon})(y_2-\frac{t}{\varepsilon})\right)}$$
$$\times\, e^{-\Psi(2\theta)\left((x_1-\frac{s}{\varepsilon})\frac{t}{\varepsilon}+(y_1-\frac{t}{\varepsilon})\frac{s}{\varepsilon}+\frac{st}{\varepsilon^2}\right)} dx_1dx_2dy_1dy_2$$

and

$$I_2 = \varepsilon^2 K^2 \int_{\frac{t}{\varepsilon}}^{\frac{t'}{\varepsilon}} \int_{\frac{t}{\varepsilon}}^{y_2} \int_{\frac{s}{\varepsilon}}^{\frac{s'}{\varepsilon}} \int_{\frac{s}{\varepsilon}}^{x_2} \sqrt{x_1x_2}\sqrt{y_1y_2}\, e^{-\Psi(\theta)\left((x_2-x_1)\frac{t}{\varepsilon}+(y_2-y_1)\frac{s}{\varepsilon}+(x_1-\frac{s}{\varepsilon})(y_2-\frac{t}{\varepsilon})+(x_2-\frac{s}{\varepsilon})(y_1-\frac{t}{\varepsilon})\right)}$$
$$\times\, e^{-\Psi(2\theta)\left((x_1-\frac{s}{\varepsilon})\frac{t}{\varepsilon}+(y_1-\frac{t}{\varepsilon})\frac{s}{\varepsilon}+\frac{st}{\varepsilon^2}\right)} dx_1dx_2dy_1dy_2.$$

We recall that $\Psi(\xi) = a(\xi) + ib(\xi)$ is the Lévy exponent. We take the modulus in (11) and estimate the modulus of the integrands in I_1 and I_2 simply by bounding by 1 the modulus of corresponding complex exponentials. This yields that $\mathbb{E}[Y_\varepsilon^2] \le 4I$, where

$$I = \varepsilon^2 K^2 \int_{\frac{t}{\varepsilon}}^{\frac{t'}{\varepsilon}} \int_{\frac{t}{\varepsilon}}^{y_2} \int_{\frac{s}{\varepsilon}}^{\frac{s'}{\varepsilon}} \int_{\frac{s}{\varepsilon}}^{x_2} \sqrt{x_1x_2}\sqrt{y_1y_2}\, e^{-a(\theta)\left((x_2-x_1)\frac{t}{\varepsilon}+(y_2-y_1)\frac{s}{\varepsilon}+(x_1-\frac{s}{\varepsilon})(y_2-\frac{t}{\varepsilon})+(x_2-\frac{s}{\varepsilon})(y_1-\frac{t}{\varepsilon})\right)}$$
$$\times\, e^{-a(2\theta)\left((x_1-\frac{s}{\varepsilon})\frac{t}{\varepsilon}+(y_1-\frac{t}{\varepsilon})\frac{s}{\varepsilon}+\frac{st}{\varepsilon^2}\right)} dx_1dx_2dy_1dy_2.$$

Taking into account the domain of integration in I above and applying Fubini theorem, we have that

$$I \le \varepsilon^2 K^2 \int_{\frac{t}{\varepsilon}}^{\frac{t'}{\varepsilon}} \int_{y_1}^{\frac{t'}{\varepsilon}} \int_{\frac{s}{\varepsilon}}^{\frac{s'}{\varepsilon}} \int_{\frac{s}{\varepsilon}}^{x_2} \sqrt{x_1x_2}\sqrt{y_1y_2}\, e^{-a(\theta)\left((x_2-x_1)\frac{t}{\varepsilon}+(y_2-y_1)\frac{s}{\varepsilon}\right)} e^{-a(2\theta)(y_1-\frac{t}{\varepsilon})\frac{s}{\varepsilon}} dx_1dx_2dy_2dy_1.$$

If we estimate $\sqrt{x_1}$ by $\sqrt{x_2}$ and $\sqrt{y_2}$ by $\sqrt{\frac{t'}{\varepsilon}}$, and integrate with respect to x_1, we get that

$$I \le C\varepsilon^2\sqrt{\varepsilon}K^2 \int_{\frac{t}{\varepsilon}}^{\frac{t'}{\varepsilon}} \int_{y_1}^{\frac{t'}{\varepsilon}} \int_{\frac{s}{\varepsilon}}^{\frac{s'}{\varepsilon}} x_2\sqrt{y_1} e^{-a(\theta)(y_2-y_1)\frac{s}{\varepsilon}} e^{-a(2\theta)(y_1-\frac{t}{\varepsilon})\frac{s}{\varepsilon}} dx_2dy_2dy_1,$$

where C is some positive constant. Integrating now with respect to y_2 and then bounding x_2 and y_1 by $\frac{s'}{\varepsilon}$ and $\frac{t'}{\varepsilon}$, respectively, and integrating in x_2 and y_1 (with this order), we finally prove that

$$I \leq C\varepsilon^2 K^2 \left(1 - e^{-a(2\theta)(\frac{t'}{\varepsilon} - \frac{t}{\varepsilon})\frac{s}{\varepsilon}}\right).$$

The latter expression converges to 0 as ε tends to 0, which proves the lemma's statement.

□

We are now in position to prove Proposition 4.2:

Proof of Proposition 4.2. It is very similar that of [2, Prop. 4.2]. Let $(0,0) < (s,t) < (s',t') \leq (S,T)$. It suffices to prove that, for any $n \geq 1$ and $(s_1,t_1),\dots,(s_n,t_n)$ such that either $s_i \leq S$ and $t_i \leq t$, or $s_i \leq s$ and $t_i \leq T$, $i = 1,\dots,n$, and for any continuous and bounded function $\varphi : \mathbb{C}^n \to \mathbb{R}$, it holds that

$$|\mathbb{E}_{\mathbb{P}}[\varphi(X(s_1,t_1),\dots,X(s_n,t_n))\Delta_{s,t}X(s',t')]| = 0.$$

We recall that the notation $|z|$ stands for the modulus of $z \in \mathbb{C}$. Without any loss of generality, the converging subsequence of probability measures to $\mathbb{P}$ will be simply denoted by $\{\mathbb{P}_\varepsilon\}_{\varepsilon>0}$. Thus, by Proposition 3.1, it suffices to check that

$$\lim_{\varepsilon\to 0} |\mathbb{E}_{\mathbb{P}_\varepsilon}[\varphi(X(s_1,t_1),\dots,X(s_n,t_n))\Delta_{s,t}X(s',t')]| = 0.$$

For this, we recall that, as in the statement of Lemma 4.4, $\{\mathcal{F}^\varepsilon_{s,t};\ (s,t) \in [0,S]\times[0,T]\}$ is the natural filtration associated to the (complex-valued) random field X^ε introduced in (2). Then, we can argue as follows:

$$\begin{aligned}
&|\mathbb{E}_{\mathbb{P}_\varepsilon}[\varphi(X(s_1,t_1),\dots,X(s_n,t_n))\Delta_{s,t}X(s',t')]| \\
&\quad = |\mathbb{E}[\varphi(X_\varepsilon(s_1,t_1),\dots,X_\varepsilon(s_n,t_n))\Delta_{s,t}X_\varepsilon(s',t')]| \\
&\quad \leq \left|\mathbb{E}\left[\varphi(X_\varepsilon(s_1,t_1),\dots,X_\varepsilon(s_n,t_n))\,\mathbb{E}[\Delta_{s,t}X_\varepsilon(s',t')|\mathcal{F}^\varepsilon_{S,t}\vee\mathcal{F}^\varepsilon_{s,T}]\right]\right| \\
&\quad \leq C\left(\mathbb{E}\left[\left|\mathbb{E}[\Delta_{s,t}X_\varepsilon(s',t')|\mathcal{F}^\varepsilon_{S,t}\vee\mathcal{F}^\varepsilon_{s,T}]\right|^2\right]\right)^{\frac{1}{2}}.
\end{aligned}$$

The latter term converges to zero as $\varepsilon \to 0$, by Lemma 4.4. □

In order to prove Proposition 4.3, we need two auxiliary results. The first one is the following.

Lemma 4.5. *For any* $(0,0) \leq (s,t) \leq (s',t') \leq (S,T)$*, it holds:*

$$\lim_{\varepsilon\to 0} \mathbb{E}\left[|\Delta_{s,t}X_\varepsilon(s',t')|^2\right] = 2(s'-s)(t'-t).$$

Proof. We split the proof in three steps.

Step 1. Owing to the definition of X_ε (see (2)) and applying Fubini theorem, we have

$$\begin{aligned}
&\mathbb{E}\left[|\Delta_{s,t}X_\varepsilon(s',t')|^2\right] \\
&\quad = \varepsilon^2 K^2 \int_{\frac{t}{\varepsilon}}^{\frac{t'}{\varepsilon}}\int_{\frac{t}{\varepsilon}}^{\frac{t'}{\varepsilon}}\int_{\frac{s}{\varepsilon}}^{\frac{s'}{\varepsilon}}\int_{\frac{s}{\varepsilon}}^{\frac{s'}{\varepsilon}} \sqrt{x_1x_2}\sqrt{y_1y_2}\;\mathbb{E}\left[e^{i\theta\left(\Delta_{0,0}L(x_2,y_2)-\Delta_{0,0}L(x_1,y_1)\right)}\right] dx_1dx_2dy_1dy_2.
\end{aligned}$$

As in the proof of Lemma 4.4, we need to take into account the possible orders of x_1, x_2 and y_1, y_2, respectively. Then, applying also some suitable changes of variables, we have

$$
\begin{aligned}
&\mathbb{E}\big[|\Delta_{s,t}X_\varepsilon(s',t')|^2\big]\\
&\quad= \varepsilon^2K^2\int_{\frac{t}{\varepsilon}}^{\frac{t'}{\varepsilon}}\int_{\frac{t}{\varepsilon}}^{y_2}\int_{\frac{s}{\varepsilon}}^{\frac{s'}{\varepsilon}}\int_{\frac{s}{\varepsilon}}^{x_2}\sqrt{x_1x_2y_1y_2}\,e^{-\Psi(\theta)((y_2-y_1)x_1+(x_2-x_1)y_2)}dx_1dx_2dy_1dy_2\\
&\qquad+\varepsilon^2K^2\int_{\frac{t}{\varepsilon}}^{\frac{t'}{\varepsilon}}\int_{\frac{t}{\varepsilon}}^{y_2}\int_{\frac{s}{\varepsilon}}^{\frac{s'}{\varepsilon}}\int_{\frac{s}{\varepsilon}}^{x_2}\sqrt{x_1x_2y_1y_2}\,e^{-\Psi(\theta)(x_2-x_1)y_1}e^{-\Psi(-\theta)(y_2-y_1)x_1}dx_1dx_2dy_1dy_2\\
&\qquad+\varepsilon^2K^2\int_{\frac{t}{\varepsilon}}^{\frac{t'}{\varepsilon}}\int_{\frac{t}{\varepsilon}}^{y_2}\int_{\frac{s}{\varepsilon}}^{\frac{s'}{\varepsilon}}\int_{\frac{s}{\varepsilon}}^{x_2}\sqrt{x_1x_2y_1y_2}\,e^{-\Psi(-\theta)(x_2-x_1)y_1}e^{-\Psi(\theta)(y_2-y_1)x_1}dx_1dx_2dy_1dy_2\\
&\qquad+\varepsilon^2K^2\int_{\frac{t}{\varepsilon}}^{\frac{t'}{\varepsilon}}\int_{\frac{t}{\varepsilon}}^{y_2}\int_{\frac{s}{\varepsilon}}^{\frac{s'}{\varepsilon}}\int_{\frac{s}{\varepsilon}}^{x_2}\sqrt{x_1x_2y_1y_2}\,e^{-\Psi(-\theta)((y_2-y_1)x_1+(x_2-x_1)y_2)}dx_1dx_2dy_1dy_2. \qquad (12)
\end{aligned}
$$

Recalling that $\Psi(\theta)=a(\theta)+ib(\theta)$, where $a(\theta)=a(-\theta)$ and $b(\theta)=-b(-\theta)$, we observe that

$$
\begin{aligned}
&e^{-\Psi(\theta)((y_2-y_1)x_1+(x_2-x_1)y_2)}+e^{-\Psi(\theta)(x_2-x_1)y_1}e^{-\Psi(-\theta)(y_2-y_1)x_1}\\
&\qquad+e^{-\Psi(-\theta)(x_2-x_1)y_1}e^{-\Psi(\theta)(y_2-y_1)x_1}+e^{-\Psi(-\theta)((y_2-y_1)x_1+(x_2-x_1)y_2)}\\
&\quad=e^{-a(\theta)(x_2y_2-x_1y_1)}2\cos\left(b(\theta)(x_2y_2-x_1y_1)\right)\\
&\qquad\quad+e^{-a(\theta)((y_2-y_1)x_1+(x_2-x_1)y_1)}2\cos\left(b(\theta)((y_2-y_1)x_1+(x_2-x_1)y_1)\right).
\end{aligned}
$$

As a consequence, we can infer that

$$
\mathbb{E}\big[|\Delta_{s,t}X_\varepsilon(s',t')|^2\big]=2(I_1^\varepsilon+I_2^\varepsilon), \qquad (13)
$$

where

$$
I_1^\varepsilon=\varepsilon^2K^2\int_{\frac{t}{\varepsilon}}^{\frac{t'}{\varepsilon}}\int_{\frac{t}{\varepsilon}}^{y_2}\int_{\frac{s}{\varepsilon}}^{\frac{s'}{\varepsilon}}\int_{\frac{s}{\varepsilon}}^{x_2}\sqrt{x_1x_2y_1y_2}\,e^{-a(\theta)(x_2y_2-x_1y_1)}\cos\left(b(\theta)(x_2y_2-x_1y_1)\right)dx_1dx_2dy_1dy_2 \qquad (14)
$$

and

$$
\begin{aligned}
I_2^\varepsilon=\varepsilon^2K^2&\int_{\frac{t}{\varepsilon}}^{\frac{t'}{\varepsilon}}\int_{\frac{t}{\varepsilon}}^{y_2}\int_{\frac{s}{\varepsilon}}^{\frac{s'}{\varepsilon}}\int_{\frac{s}{\varepsilon}}^{x_2}\sqrt{x_1x_2y_1y_2}\,e^{-a(\theta)((y_2-y_1)x_1+(x_2-x_1)y_1)}\\
&\times\cos\left(b(\theta)((y_2-y_1)x_1+(x_2-x_1)y_1)\right)dx_1dx_2dy_1dy_2. \qquad (15)
\end{aligned}
$$

Step 2. Let us consider the case $s=t=0$. In order to deal with I_1^ε, we make the changes of variables $z_i:=x_iy_i$ and $v_i:=\frac{\varepsilon}{s'}x_i$, $i=1,2$, and we define $u:=\frac{s't'}{\varepsilon^2}$. Thus, by l'Hôpital's rule, we have

$$
\lim_{\varepsilon\to0}I_1^\varepsilon=\lim_{u\to\infty}s't'K^2\int_0^1\int_0^{v_2}\int_0^{uv_1}\frac{\sqrt{z_1uv_2}}{v_1}\,e^{-a(\theta)uv_2+a(\theta)z_1}\cos\left(b(\theta)(uv_2-z_1)\right)dz_1dv_1dv_2.
$$

Applying now the changes of variables $v_2':=uv_2$ and $v_1':=uv_1$, and again l'Hôpital's rule, we obtain that the latter limit equals to

$$
\lim_{u\to\infty}s't'K^2\int_0^u\int_0^{v_1'}\frac{\sqrt{z_1u}}{v_1'}\,e^{-a(\theta)u+a(\theta)z_1}\cos\left(b(\theta)(u-z_1)\right)dz_1dv_1'. \qquad (16)
$$

In order to compute the above limit, we use the formula $\cos(\theta)=\frac{1}{2}(e^{i\theta}+e^{-i\theta})$. Hence, the expression inside the limit (16) can be written as the sum $\frac{1}{2}(A_u+B_u)$, where these terms are given by

$$
A_u:=s't'K^2\int_0^u\int_0^v\frac{\sqrt{zu}}{v}\,e^{-a(\theta)u+a(\theta)z}\,e^{ib(\theta)(u-z)}dzdv,
$$

$$B_u := s't'K^2 \int_0^u \int_0^v \frac{\sqrt{zu}}{v}\, e^{-a(\theta)u + a(\theta)z}\, e^{-ib(\theta)(u-z)} dz dv.$$

We will only deal with $\lim_{u\to\infty} B_u$, because $\lim_{u\to\infty} A_u$ can be treated in a similar way. Indeed, rewriting B_u as

$$B_u = s't'K^2 \frac{\int_0^u \int_0^v \frac{\sqrt{z}}{v}\, e^{(a(\theta)+ib(\theta))z} dz dv}{u^{-\frac{1}{2}}\, e^{(a(\theta)+ib(\theta))u}}$$

and applying l'Hôpital's rule twice, one easily proves that

$$\lim_{u\to\infty} B_u = \frac{s't'K^2}{(a(\theta)+ib(\theta))^2}.$$

Similarly, one gets

$$\lim_{u\to\infty} A_u = \frac{s't'K^2}{(a(\theta)-ib(\theta))^2}.$$

Thus,

$$\lim_{\varepsilon\to 0} I_1^\varepsilon = s't'K^2 \frac{a(\theta)^2 - b(\theta)^2}{(a(\theta)^2 + b(\theta)^2)^2}.$$

Now, we are going to compute $\lim_{\varepsilon\to 0} I_2^\varepsilon$. Recall that the latter term is given in (15). The strategy that we have followed to deal with I_1^ε cannot be applied here. More precisely, we have not been able to compute the limit of I_2^ε directly, but we will introduce an auxiliary term which will converge to some quantity, and we will prove that the remainder converges to zero.

To start with, we apply the same changes of variables that we performed for I_1^ε, we set $u := \frac{s't'}{\varepsilon}$ and apply l'Hôpital's rule, so $\lim_{\varepsilon\to 0} I_2^\varepsilon$ equals to

$$\lim_{u\to\infty} K^2 s't' \int_0^1 \int_0^{v_2} \int_0^{uv_1} \sqrt{uv_2} \frac{\sqrt{z_1}}{v_1}\, e^{a(\theta)\left(2z_1 - uv_1 - \frac{z_1 v_2}{v_1}\right)} \cos\left(b(\theta)\left(2z_1 - uv_1 - \frac{z_1 v_2}{v_1}\right)\right) dz_1 dv_1 dv_2.$$

Next, we make the changes of variables $\bar{v}_1 := uv_1$ and $\bar{v}_2 := uv_2$ and we apply again l'Hôpital's rule. Hence, the latter limit becomes

$$\lim_{u\to\infty} K^2 s't' \sqrt{u} \int_0^u \int_0^{\bar{v}_1} \frac{\sqrt{z_1}}{\bar{v}_1}\, e^{a(\theta)\left(2z_1 - \bar{v}_1 - \frac{z_1 u}{\bar{v}_1}\right)} \cos\left(b(\theta)\left(2z_1 - \bar{v}_1 - \frac{z_1 u}{\bar{v}_1}\right)\right) dz_1 d\bar{v}_1.$$

Finally, performing the changes $x := \frac{z_1}{\bar{v}_1}$ and $y := \frac{\bar{v}_1}{u}$, we end up with

$$\lim_{\varepsilon\to 0} I_2^\varepsilon = \lim_{u\to\infty} C_u,$$

with

$$C_u = K^2 s't'u^2 \int_0^1 \int_0^1 \sqrt{xy}\, e^{a(\theta)(2xy-y-x)u} \cos(b(\theta)(2xy-y-x)u) dx dy. \tag{17}$$

At this point, we introduce the auxiliary term mentioned above:

$$\tilde{C}_u := K^2 s't'u^2 \int_0^1 \int_0^1 \sqrt{y}\, e^{a(\theta)(2xy-y-x)u} \cos(b(\theta)(2xy-y-x)u) dx dy, \tag{18}$$

where we note that, compared to the right hand-side of (17), we have only replaced $\sqrt{xy}$ by $\sqrt{y}$. For the moment, assume that $\lim_{u\to\infty}(C_u - \tilde{C}_u) = 0$. Let us compute the limit of $\tilde{C}_u$, recalling that this term has been defined in (18). As in the analysis of the term I_1^ε, we use the formula $\cos(\theta) = \frac{1}{2}(e^{i\theta} + e^{-i\theta})$, so we split $\tilde{C}_u$ as the sum of two terms (multiplied by $\frac{1}{2}$), one of which is given by

$$K^2 s't'u^2 \int_0^1 \int_0^1 \sqrt{y}\, e^{u(2xy-y-x)(a(\theta)+ib(\theta))} dx dy,$$

and the other one is the same with $a(\theta) - ib(\theta)$ instead of $a(\theta) + ib(\theta)$. Integrating first with respect to x and then applying l'Hôpital's rule, one gets that the limit of the above term equals to

$$\lim_{u\to\infty} K^2 s' t' \frac{\int_0^u \sqrt{y}\, e^{(a(\theta)+ib(\theta))y} dy}{(a(\theta)+ib(\theta))\sqrt{u}\, e^{(a(\theta)+ib(\theta))u}}.$$

It is straightforward to check that the latter limit is $\frac{K^2 s' t'}{(a(\theta)+ib(\theta))^2}$. The limit of the term involving $a(\theta) - ib(\theta)$ will be given by $\frac{K^2 s' t'}{(a(\theta)-ib(\theta))^2}$. Therefore, we have that

$$\lim_{\varepsilon\to 0} I_2^\varepsilon = \lim_{u\to\infty} \tilde{C}_u = s' t' K^2 \frac{a(\theta)^2 - b(\theta)^2}{(a(\theta)^2 + b(\theta)^2)^2}.$$

Note that this is exactly the same limit that we obtained for I_1^ε. In conclusion, owing to (13) and the expression of K given in (3), the lemma's statement holds in the case $s = t = 0$.

In order to conclude the present step, we need to check that $\lim_{u\to\infty}(C_u - \tilde{C}_u) = 0$, that is

$$\lim_{u\to\infty} u^2 \int_0^1 \int_0^1 (\sqrt{xy} - \sqrt{y})\, e^{a(\theta)(2xy-y-x)u} \cos(b(\theta)(2xy - y - x)u) dx dy = 0.$$

Let us introduce the notation

$$D_u := u^2 \int_0^1 \int_0^1 (\sqrt{xy} - \sqrt{y})\, e^{a(\theta)(2xy-y-x)u} dx dy.$$

Then, it clearly holds that

$$-D_u \le C_u - \tilde{C}_u \le D_u.$$

In order to apply a sandwich type argument, we will prove that both $-D_u$ and D_u converge to zero as u tends to infinity. We will only tackle the term D_u, since the analysis of $-D_u$ is analogous. Note that $D_u = D_u^1 - D_u^2$, where

$$D_u^1 = u^2 \int_0^1 \int_0^1 \sqrt{xy}\, e^{a(\theta)(2xy-y-x)u} dx dy \quad \text{and} \quad D_u^2 = u^2 \int_0^1 \int_0^1 \sqrt{y}\, e^{a(\theta)(2xy-y-x)u} dx dy.$$

Regarding D_u^2, observe that the integral in x can be computed explicitly and we can argue as follows:

$$\begin{aligned}
\lim_{u\to\infty} D_u^2 &= \lim_{u\to\infty} u \int_0^1 \frac{\sqrt{y}}{a(\theta)(2y-1)} \left(e^{a(\theta)(y-1)u} - e^{-a(\theta)yu} \right) dy \\
&= \lim_{u\to\infty} \frac{1}{a(\theta)} \frac{\int_0^1 \frac{\sqrt{y}}{(2y-1)} \left(e^{(a(\theta)(y-1)+1)u} - e^{(-a(\theta)y+1)u} \right) dy}{\frac{e^u}{u}} \\
&= \lim_{u\to\infty} \frac{1}{a(\theta)} \frac{\int_0^1 \sqrt{y} \left(e^{(a(\theta)(y-1)+1)u} + e^{(-a(\theta)y+1)u} \right) dy}{\frac{ue^u - e^u}{u^2}}.
\end{aligned}$$

In the last inequality, we have applied l'Hôpital's rule. By performing a change of variable, the latter expression equals to

$$\lim_{u\to\infty} \frac{1}{a(\theta)} \left\{ \frac{\int_0^u \sqrt{y}\, e^{a(\theta)y} dy}{\sqrt{u}\, e^{a(\theta)u} - \frac{e^{a(\theta)u}}{\sqrt{u}}} + \frac{\int_0^u \sqrt{y}\, e^{-a(\theta)y} dy}{\sqrt{u} - \frac{1}{\sqrt{u}}} \right\}.$$

The second term in the above sum clearly converges to zero as $u \to \infty$, while the limit of the first one equals to, thanks to l'Hôpital's rule,

$$\lim_{u\to\infty} \frac{1}{a(\theta)} \frac{\sqrt{u}\, e^{a(\theta)u}}{e^{a(\theta)u}(a(\theta)\sqrt{u} + o(\sqrt{u}))} = \frac{1}{a(\theta)^2}.$$

Thus, we have proved that $\lim_{u\to\infty} D_u^2 = \frac{1}{a(\theta)^2}$. On the other hand, in order to deal with D_u^1 we will use again a sandwich type argument, as follows. First, note that we trivially have $D_u^1 \leq D_u^2$. Next, applying the changes of variables $v = uy$ and $z = \frac{xv}{u}$, we end up with

$$\begin{aligned}
\lim_{u\to\infty} D_u^1 &= \lim_{u\to\infty} \sqrt{u} \int_0^u \int_0^v \frac{\sqrt{z}}{v}\, e^{-a(\theta)(v+\frac{zu}{v}-2z)} dz dv \\
&\geq \lim_{u\to\infty} \sqrt{u} \int_0^u \int_0^v \frac{\sqrt{z}}{v}\, e^{-a(\theta)(u-z)} dz dv.
\end{aligned}$$

Observe that the latter limit equals to $\frac{1}{a(\theta)^2}$ because it corresponds to the limit of B_u defined above in the particular case of $s' = t' = K = 1$ and $b = 0$. Hence, we obtain that

$$\lim_{u\to\infty} D_u^1 = \frac{1}{a(\theta)^2}$$

and therefore $\lim_{u\to\infty} D_u = 0$.

Step 3. Assume that either $s \neq 0$ or $t \neq 0$. By step 1, recall that we have

$$\mathbb{E}\big[|\Delta_{s,t} X_\varepsilon(s', t')|^2\big] = 2(I_1^\varepsilon + I_2^\varepsilon),$$

where the terms on the right hand-side have been defined in (14) and (15), respectively. Set

$$F^\varepsilon(s,t) := \varepsilon^2 K^2 \int_0^{\frac{t}{\varepsilon}} \int_0^{\frac{s}{\varepsilon}} \int_0^{\frac{t}{\varepsilon}} \int_0^{\frac{s}{\varepsilon}} f(x_1,x_2,y_1,y_2) 1_{\{x_1\leq x_2, y_1\leq y_2\}} dx_1 dx_2 dy_1 dy_2,$$

where $f(x_1,x_2,y_1,y_2) := \sqrt{x_1x_2y_1y_2}\, e^{-a(\theta)(x_2y_2-x_1y_1)} \cos(b(\theta)(x_2y_2 - x_1y_1))$, and

$$G^\varepsilon(s,t) := \varepsilon^2 K^2 \int_0^{\frac{t}{\varepsilon}} \int_0^{\frac{s}{\varepsilon}} \int_0^{\frac{t}{\varepsilon}} \int_0^{\frac{s}{\varepsilon}} g(x_1,x_2,y_1,y_2) 1_{\{x_1\leq x_2, y_1\leq y_2\}} dx_1 dx_2 dy_1 dy_2,$$

where $g(x_1,x_2,y_1,y_2) := \sqrt{x_1x_2y_1y_2}\, e^{-a(\theta)((y_2-y_1)x_1+(x_2-x_1)y_1)} \cos(b(\theta)((y_2 - y_1)x_1 + (x_2 - x_1)y_1))$. Observe that I_1^ε and I_2^ε can be written as follows:

$$\begin{aligned}
I_1^\varepsilon &= \Delta_{s,t} F^\varepsilon(s',t') - \varepsilon^2 K^2 \int_{\frac{t}{\varepsilon}}^{\frac{t'}{\varepsilon}} \int_{\frac{s}{\varepsilon}}^{\frac{s'}{\varepsilon}} \int_{\frac{t}{\varepsilon}}^{\frac{t'}{\varepsilon}} \int_0^{\frac{s}{\varepsilon}} f(x_1,x_2,y_1,y_2) 1_{\{y_1\leq y_2\}} dx_1 dx_2 dy_1 dy_2 \\
&\quad - \varepsilon^2 K^2 \int_{\frac{t}{\varepsilon}}^{\frac{t'}{\varepsilon}} \int_{\frac{s}{\varepsilon}}^{\frac{s'}{\varepsilon}} \int_0^{\frac{t'}{\varepsilon}} \int_0^{\frac{s}{\varepsilon}} f(x_1,x_2,y_1,y_2) dx_1 dx_2 dy_1 dy_2 \\
&\quad - \varepsilon^2 K^2 \int_{\frac{t}{\varepsilon}}^{\frac{t'}{\varepsilon}} \int_{\frac{s}{\varepsilon}}^{\frac{s'}{\varepsilon}} \int_0^{\frac{t}{\varepsilon}} \int_{\frac{s}{\varepsilon}}^{\frac{s'}{\varepsilon}} f(x_1,x_2,y_1,y_2) 1_{\{x_1\leq x_2\}} dx_1 dx_2 dy_1 dy_2 \\
&=: \Delta_{s,t} F^\varepsilon(s',t') - I_{11}^\varepsilon - I_{12}^\varepsilon - I_{13}^\varepsilon,
\end{aligned}$$

and

$$I_2^\varepsilon = \Delta_{s,t} G^\varepsilon(s',t') - I_{21}^\varepsilon - I_{22}^\varepsilon - I_{23}^\varepsilon,$$

where I_{2i}^ε, $i = 1,2,3$, are defined analogously by using the function g. By step 1, one verifies that

$$\lim_{\varepsilon\to 0} \Delta_{s,t} F^\varepsilon(s',t') = \lim_{\varepsilon\to 0} \Delta_{s,t} G^\varepsilon(s',t') = \frac{1}{2}(s'-s)(t'-t).$$

In order to conclude the proof, it suffices to check that I_{ji}^ε converges to zero as $\varepsilon \to 0$, for all $j = 1,2$ and $i = 1,2,3$. For this, we estimate any I_{ji}^ε by $\tilde{I}_{ji}^\varepsilon$, where the latter are defined by simply bounding the cosinus by 1. Next, we note that $\tilde{I}^\varepsilon_{1,i} \leq \tilde{I}^\varepsilon_{2,i}$, for all $i = 1, 2, 3$, and that any of the $\tilde{I}^\varepsilon_{2,i}$ can be bounded by

$$\varepsilon^2 K^2 \int_0^{\frac{t'}{\varepsilon}} \int_0^{\frac{s}{\varepsilon}} \int_0^{y_2} \int_{\frac{s}{\varepsilon}}^{\frac{s'}{\varepsilon}} g(x_1, x_2, y_1, y_2) dx_2 dy_1 dx_1 dy_2. \tag{19}$$

In this integral, we use the explicit expression of g, we perform the changes of variables $\bar{x}_i := \varepsilon x_i$ and $\bar{y}_i := \varepsilon y_i$, $i = 1, 2$, we set $u := \frac{1}{\varepsilon^2}$, we use that $\bar{x}_2 \leq s'$ and we integrate with respect to $\bar{x}_2$. Thus, (19) can be bounded, up to some positive constant, by (using again the notation x_i and y_i for the variables)

$$u \int_0^{t'} \int_0^{s} \int_0^{y_2} \frac{\sqrt{x_1 y_2}}{\sqrt{y_1}}\, e^{-a(\theta)((y_2 - y_1)x_1 + (s - x_1)y_1)u} dy_1 dx_1 dy_2.$$

Estimating now y_2 by t' inside the square root and integrating in y_2, the above expression can be bounded by (up to some constant)

$$\int_0^{s} \int_0^{t'} \frac{1}{\sqrt{x_1 y_1}}\, e^{-a(\theta)u(s - x_1)y_1} dy_1 dx_1.$$

This expression converges to zero as $u \to \infty$, by the Monotone convergence theorem. $\square$

Here is the second auxiliary result needed to prove Proposition 4.3.

Lemma 4.6. *Let $(0, 0) \leq (s, t) \leq (s', t') \leq (S, T)$. Then, there exists a sequence $\{C_\varepsilon\}_{\varepsilon > 0}$ such that $\lim_{\varepsilon \to 0} C_\varepsilon = 4(s' - s)^2 (t' - t)^2$ and*

$$\mathbb{E}\left[\left(\mathbb{E}\left[|\Delta_{s,t} X^\varepsilon(s', t')|^2 \,|\mathcal{F}^\varepsilon_{s,T}\right]\right)^2\right] \leq C_\varepsilon.$$

Proof. We split the proof in four steps.

Step 1. By definition of the random field X^ε, we first observe that

$$\begin{aligned}
&\mathbb{E}\left[|\Delta_{s,t} X^\varepsilon(s', t')|^2 \,|\mathcal{F}^\varepsilon_{s,T}\right] \\
&\quad = K^2 \varepsilon^2 \int_{\frac{t}{\varepsilon}}^{\frac{t'}{\varepsilon}} \int_{\frac{s}{\varepsilon}}^{\frac{s'}{\varepsilon}} \int_{\frac{t}{\varepsilon}}^{\frac{t'}{\varepsilon}} \int_{\frac{s}{\varepsilon}}^{\frac{s'}{\varepsilon}} \sqrt{x_1 x_2 y_1 y_2}\, \mathbb{E}\left[e^{i\theta(L(x_2, y_2) - L(x_1, y_1))} |\mathcal{F}^\varepsilon_{s,T}\right] dx_1 dx_2 dy_1 dy_2.
\end{aligned}$$

In order to compute the above conditional expectation, we have to consider all possible orders of x_1, x_2 and y_1, y_2, respectively, which corresponds to a total of 4 possibilities. Hence,

$$\begin{aligned}
\mathbb{E}\left[|\Delta_{s,t} X^\varepsilon(s', t')|^2 \,|\mathcal{F}^\varepsilon_{s,T}\right] = {} & K^2 \varepsilon^2 \int_{\frac{t}{\varepsilon}}^{\frac{t'}{\varepsilon}} \int_{\frac{s}{\varepsilon}}^{\frac{s'}{\varepsilon}} \int_{\frac{t}{\varepsilon}}^{y_2} \int_{\frac{s}{\varepsilon}}^{x_2} \sqrt{x_1 x_2 y_1 y_2}\, e^{i\theta\left(L(\frac{s}{\varepsilon}, y_2) - L(\frac{s}{\varepsilon}, y_1)\right)} \\
& \quad \times e^{-\Psi(\theta)\left((x_2 - x_1)y_2 + (y_2 - y_1)(x_1 - \frac{s}{\varepsilon})\right)} dx_1 dy_1 dx_2 dy_2 \\
& + K^2 \varepsilon^2 \int_{\frac{t}{\varepsilon}}^{\frac{t'}{\varepsilon}} \int_{\frac{s}{\varepsilon}}^{\frac{s'}{\varepsilon}} \int_{\frac{t}{\varepsilon}}^{y_2} \int_{\frac{s}{\varepsilon}}^{x_2} \sqrt{x_1 x_2 y_1 y_2}\, e^{-i\theta\left(L(\frac{s}{\varepsilon}, y_2) - L(\frac{s}{\varepsilon}, y_1)\right)} \\
& \quad \times e^{-\Psi(\theta)(x_2 - x_1)y_1} e^{-\Psi(-\theta)(y_2 - y_1)(x_1 - \frac{s}{\varepsilon})} dx_1 dy_1 dx_2 dy_2 \\
& + K^2 \varepsilon^2 \int_{\frac{t}{\varepsilon}}^{\frac{t'}{\varepsilon}} \int_{\frac{s}{\varepsilon}}^{\frac{s'}{\varepsilon}} \int_{\frac{t}{\varepsilon}}^{y_2} \int_{\frac{s}{\varepsilon}}^{x_2} \sqrt{x_1 x_2 y_1 y_2}\, e^{i\theta\left(L(\frac{s}{\varepsilon}, y_2) - L(\frac{s}{\varepsilon}, y_1)\right)} \\
& \quad \times e^{-\Psi(-\theta)(x_2 - x_1)y_1} e^{-\Psi(\theta)(y_2 - y_1)(x_1 - \frac{s}{\varepsilon})} dx_1 dy_1 dx_2 dy_2 \\
& + K^2 \varepsilon^2 \int_{\frac{t}{\varepsilon}}^{\frac{t'}{\varepsilon}} \int_{\frac{s}{\varepsilon}}^{\frac{s'}{\varepsilon}} \int_{\frac{t}{\varepsilon}}^{y_2} \int_{\frac{s}{\varepsilon}}^{x_2} \sqrt{x_1 x_2 y_1 y_2}\, e^{-i\theta\left(L(\frac{s}{\varepsilon}, y_2) - L(\frac{s}{\varepsilon}, y_1)\right)} \\
& \quad \times e^{-\Psi(-\theta)\left((x_2 - x_1)y_2 + (y_2 - y_1)(x_1 - \frac{s}{\varepsilon})\right)} dx_1 dy_1 dx_2 dy_2.
\end{aligned}$$

We have also applied changes of variables in order to have $x_1 \le x_2$ and $y_1 \le y_2$ in all terms. We denote by A_i^ε, $i = 1, 2, 3, 4$, the above four terms, respectively. Thus, we have

$$\mathbb{E}\left[\left(\mathbb{E}\left[|\Delta_{s,t} X^\varepsilon(s',t')|^2 \,|\mathcal{F}^\varepsilon_{s,T}\right]\right)^2\right] = \sum_{i,j=1}^{4} \mathbb{E}\left[A_i^\varepsilon A_j^\varepsilon\right].$$

For the sake of clarity, we will only analyze one of the terms in the above sum, since the other ones can be treated exactly in the same way. So, we proceed to tackle the term $\mathbb{E}\left[(A_1^\varepsilon)^2\right]$. In fact, by Fubini theorem, we have that

$$\begin{aligned}\mathbb{E}\left[(A_1^\varepsilon)^2\right] = K^4\varepsilon^4 &\int_{\frac{t}{\varepsilon}}^{\frac{t'}{\varepsilon}}\int_{\frac{s}{\varepsilon}}^{\frac{s'}{\varepsilon}}\int_{\frac{t}{\varepsilon}}^{y_4}\int_{\frac{s}{\varepsilon}}^{x_4}\int_{\frac{t}{\varepsilon}}^{\frac{t'}{\varepsilon}}\int_{\frac{s}{\varepsilon}}^{\frac{s'}{\varepsilon}}\int_{\frac{t}{\varepsilon}}^{y_2}\int_{\frac{s}{\varepsilon}}^{x_2} \sqrt{x_1x_2x_3x_4}\sqrt{y_1y_2y_3y_4}\\
&\times \mathbb{E}\left[e^{i\theta\left(L(\frac{s}{\varepsilon},y_2)-L(\frac{s}{\varepsilon},y_1)+L(\frac{s}{\varepsilon},y_4)-L(\frac{s}{\varepsilon},y_3)\right)}\right]\\
&\times e^{-\Psi(\theta)\left((x_2-x_1)y_2+(y_2-y_1)(x_1-\frac{s}{\varepsilon})+(x_4-x_3)y_4+(y_4-y_3)(x_3-\frac{s}{\varepsilon})\right)}\\
&\times dx_1dy_1dx_2dy_2dx_3dy_3dx_4dy_4. \end{aligned} \tag{20}$$

Note that in the above integral we have $y_1 \le y_2$ and $y_3 \le y_4$. However, in order to compute the expectation in (20), we need to consider all possible orders of the variables y_1, y_2, y_3, y_4, with the restrictions $y_1 \le y_2$ and $y_3 \le y_4$. This amounts to take into account 6 different possibilities, which we split in two groups:

(i) $y_1 \le y_2 \le y_3 \le y_4$ and $y_3 \le y_4 \le y_1 \le y_2$,

(ii) $y_1 \le y_3 \le y_2 \le y_4$, $y_1 \le y_3 \le y_4 \le y_2$, $y_3 \le y_1 \le y_4 \le y_2$ and $y_3 \le y_1 \le y_2 \le y_4$.

Then, we have that

$$\mathbb{E}\left[(A_1^\varepsilon)^2\right] = \sum_{k=1}^{6} B_k^\varepsilon(1,1), \tag{21}$$

where $B_1^\varepsilon(1,1), B_2^\varepsilon(1,1)$ correspond to (20) with the orders of (i), respectively, while $B_k^\varepsilon(1,1)$, $k = 3, 4, 5, 6$, correspond to (20) with the orders of (ii), respectively. It turns out that we have a similar decomposition of any of the terms $\mathbb{E}\left[A_i^\varepsilon A_j^\varepsilon\right]$, which we denote by

$$\mathbb{E}\left[A_i^\varepsilon A_j^\varepsilon\right] = \sum_{k=1}^{6} B_k^\varepsilon(i,j).$$

Hence

$$\mathbb{E}\left[\left(\mathbb{E}\left[|\Delta_{s,t} X^\varepsilon(s',t')|^2 \,|\mathcal{F}^\varepsilon_{s,T}\right]\right)^2\right] = \sum_{i,j=1}^{4}\sum_{k=1}^{6} B_k^\varepsilon(i,j). \tag{22}$$

In the next two steps, we will focus on the analysis of (some of) the terms in the decomposition (21) of $\mathbb{E}\left[(A_1^\varepsilon)^2\right]$. As already mentioned, the terms arising from $\mathbb{E}\left[A_i^\varepsilon A_j^\varepsilon\right]$ can be treated analogously. We will come back to expansion (22) later in step 4.

Step 2. We claim that, for any $k = 3, 4, 5, 6$, it holds

$$\begin{aligned}|B_k^\varepsilon(1,1)| \le K^4\varepsilon^4 &\int_D \sqrt{x_1x_2x_3x_4}\sqrt{y_1y_2y_3y_4}\,1_{\{x_1\le x_2\}}1_{\{x_3\le x_4\}}1_{\{y_1\le y_2\le y_3\le y_4\}}\\
&\times e^{-a(\theta)\left((x_4-x_3)\frac{t}{\varepsilon}+(x_2-x_1)\frac{t}{\varepsilon}+(y_4-y_3)\frac{s}{\varepsilon}+(y_2-y_1)\frac{s}{\varepsilon}+(y_3-y_2)(x_1-\frac{s}{\varepsilon})\right)}\\
&\times dx_1dy_1dx_2dy_2dx_3dy_3dx_4dy_4,\end{aligned} \tag{23}$$

where $D := [\frac{s}{\varepsilon}, \frac{s'}{\varepsilon}]^4 \times [\frac{t}{\varepsilon}, \frac{t'}{\varepsilon}]^4$, and we recall that $a(\theta)$ is the real part of $\Psi(\theta)$. We prove this estimate for $B_3^\varepsilon(1,1)$. For the remaining terms the argument is completely analogous. So, let us assume that in (20) we have the order $y_1 \le y_3 \le y_2 \le y_4$. In this case, the expectation in (20) equals to

$$e^{-\Psi(\theta)\big((y_4-y_2)\frac{s}{\varepsilon}+(y_3-y_1)\frac{s}{\varepsilon}+2(y_2-y_3)\frac{s}{\varepsilon}\big)}.$$

Plugging this term in (20) and shifting the modulus inside the integral, we can infer that

$$\begin{aligned}|B_3^\varepsilon(1,1)| \le K^4\varepsilon^4 \int_D & \sqrt{x_1x_2x_3x_4}\sqrt{y_1y_2y_3y_4}1_{\{x_1\le x_2\}}1_{\{x_3\le x_4\}}1_{\{y_1\le y_3\le y_2\le y_4\}} \\ & \times e^{-a(\theta)\big((x_4-x_3)\frac{t}{\varepsilon}+(x_2-x_1)\frac{t}{\varepsilon}+(y_4-y_2)\frac{s}{\varepsilon}+(y_3-y_1)\frac{s}{\varepsilon}+(y_2-y_1)(x_1-\frac{s}{\varepsilon})\big)} \\ & \times e^{-a(\theta)\big(2(y_2-y_3)\frac{s}{\varepsilon}+(y_4-y_3)(x_3-\frac{s}{\varepsilon})\big)} dx_1dy_1dx_2dy_2dx_3dy_3dx_4dy_4 \\ \le K^4\varepsilon^4 \int_D & \sqrt{x_1x_2x_3x_4}\sqrt{y_1y_2y_3y_4}1_{\{x_1\le x_2\}}1_{\{x_3\le x_4\}}1_{\{y_1\le y_3\le y_2\le y_4\}} \\ & \times e^{-a(\theta)\big((x_4-x_3)\frac{t}{\varepsilon}+(x_2-x_1)\frac{t}{\varepsilon}+(y_4-y_2)\frac{s}{\varepsilon}+(y_3-y_1)\frac{s}{\varepsilon}+(y_2-y_1)(x_1-\frac{s}{\varepsilon})\big)} \\ & \times dx_1dy_1dx_2dy_2dx_3dy_3dx_4dy_4.\end{aligned}$$

Performing a change of variable, we obtain that the latter term equals to

$$\begin{aligned}K^4\varepsilon^4 \int_D & \sqrt{x_1x_2x_3x_4}\sqrt{y_1y_2y_3y_4}1_{\{x_1\le x_2\}}1_{\{x_3\le x_4\}}1_{\{y_1\le y_2\le y_3\le y_4\}} \\ & \times e^{-a(\theta)\big((x_4-x_3)\frac{t}{\varepsilon}+(x_2-x_1)\frac{t}{\varepsilon}+(y_4-y_3)\frac{s}{\varepsilon}+(y_2-y_1)\frac{s}{\varepsilon}+(y_3-y_1)(x_1-\frac{s}{\varepsilon})\big)} \\ & \times dx_1dy_1dx_2dy_2dx_3dy_3dx_4dy_4.\end{aligned}$$

In order to obtain (23), it suffices to observe that, in the domain of integration, it holds that $(y_3-y_1)(x_1-\frac{s}{\varepsilon}) \ge (y_3-y_2)(x_1-\frac{s}{\varepsilon})$.

Step 3. Here, we prove that the right hand-side of (23) converges to zero as $\varepsilon \to 0$. Let us introduce the following notation:

$$\begin{aligned}\beta^\varepsilon := \varepsilon^4 \int_D & \sqrt{x_1x_2x_3x_4}\sqrt{y_1y_2y_3y_4}\, 1_{\{x_1\le x_2\}}1_{\{x_3\le x_4\}}1_{\{y_1\le y_2\le y_3\le y_4\}} \\ & \times e^{-a(\theta)\big((x_4-x_3)\frac{t}{\varepsilon}+(x_2-x_1)\frac{t}{\varepsilon}+(y_4-y_3)\frac{s}{\varepsilon}+(y_2-y_1)\frac{s}{\varepsilon}+(y_3-y_2)(x_1-\frac{s}{\varepsilon})\big)} \\ & \times dx_1dy_1dx_2dy_2dx_3dy_3dx_4dy_4,\end{aligned}$$

so we want to check that $\lim_{\varepsilon\to 0}\beta^\varepsilon = 0$.

To start with, in the expression of β^ε we bound the two square roots by using the upper limit of any x_i and y_i. Next, we integrate with respect to x_4, x_3 and x_2. We also use the fact that, according to the statement of Proposition 4.3, we may assume that $t > 0$. Thus,

$$\beta^\varepsilon \le C\varepsilon \int_{\frac{s}{\varepsilon}}^{\frac{s'}{\varepsilon}} \int_{\frac{t}{\varepsilon}}^{\frac{t'}{\varepsilon}} \int_{\frac{t}{\varepsilon}}^{y_3} \int_{y_3}^{\frac{t'}{\varepsilon}} \int_{\frac{t}{\varepsilon}}^{y_2} e^{-a(\theta)\big((y_4-y_3)\frac{s}{\varepsilon}+(y_2-y_1)\frac{s}{\varepsilon}+(y_3-y_2)(x_1-\frac{s}{\varepsilon})\big)}\, dy_1dy_4dy_2dy_3dx_1.$$

At this point, we integrate with respect to y_1 and y_4, thus

$$\begin{aligned}\beta^\varepsilon &\le C\varepsilon^3 \int_{\frac{s}{\varepsilon}}^{\frac{s'}{\varepsilon}} \int_{\frac{t}{\varepsilon}}^{\frac{t'}{\varepsilon}} \int_{\frac{t}{\varepsilon}}^{y_3} e^{-a(\theta)(y_3-y_2)(x_1-\frac{s}{\varepsilon})} dy_2dy_3dx_1 \\ &= C\varepsilon^3 \int_{\varepsilon}^{\frac{s'-s}{\varepsilon}} \int_{\frac{t}{\varepsilon}}^{\frac{t'}{\varepsilon}} \int_{\frac{t}{\varepsilon}}^{y_3} e^{-a(\theta)(y_3-y_2)x} dy_2dy_3dx \\ &\quad + C\varepsilon^3 \int_0^{\varepsilon} \int_{\frac{t}{\varepsilon}}^{\frac{t'}{\varepsilon}} \int_{\frac{t}{\varepsilon}}^{y_3} e^{-a(\theta)(y_3-y_2)x} dy_2dy_3dx.\end{aligned}$$

Note that the second term in the latter sum may be bounded, up to some positive constant, by ε^2, which converges to zero. Regarding the first term, it can be bounded by

$$C\varepsilon^2 \int_{\varepsilon}^{\frac{s'-s}{\varepsilon}} \frac{1}{x}\, dx = C\varepsilon^2 \left(\ln(s'-s) - 2\ln(\varepsilon)\right),$$

which also converges to zero as $\varepsilon \to 0$.

Step 4. By (22) in step 1 and steps 2 and 3, we have that

$$\begin{aligned} \mathbb{E}\left[\left(\mathbb{E}\left[|\Delta_{s,t} X^{\varepsilon}(s',t')|^2 \,|\mathcal{F}^{\varepsilon}_{s,T}\right]\right)^2\right] &= \sum_{i,j=1}^{4} \sum_{k=1}^{6} B_k^{\varepsilon}(i,j) \\ &= \sum_{i,j=1}^{4} \sum_{k=1}^{2} B_k^{\varepsilon}(i,j) + \rho_{\varepsilon}, \end{aligned} \tag{24}$$

where we recall that $B_1^{\varepsilon}(i,j)$ and $B_2^{\varepsilon}(i,j)$ are the terms in the decomposition of $\mathbb{E}\left[A_i^{\varepsilon} A_j^{\varepsilon}\right]$ with the orders of (i), respectively, and $\lim_{\varepsilon\to 0} \rho_{\varepsilon} = 0$.

Focusing again (only) on the case $i = j = 1$, one easily verifies that

$$\begin{aligned} \sum_{k=1}^{2} B_k^{\varepsilon}(1,1) = K^4 \varepsilon^2 \int_D & \sqrt{x_1 x_2 x_3 x_4}\sqrt{y_1 y_2 y_3 y_4}\, 1_{\{x_1 \le x_2\}} 1_{\{x_3 \le x_4\}} 1_{\{\{y_1 \le y_2 \le y_3 \le y_4\} \cup \{y_3 \le y_4 \le y_1 \le y_2\}\}} \\ & \times e^{-\Psi(\theta)((x_2-x_1)y_2 + (x_4-x_3)y_4 + (y_2-y_1)x_1 + (y_4-y_3)x_3)} \\ & \times dx_1 dy_1 dx_2 dy_2 dx_3 dy_3 dx_4 dy_4, \end{aligned}$$

where we recall that $D := [\frac{s}{\varepsilon}, \frac{s'}{\varepsilon}]^4 \times [\frac{t}{\varepsilon}, \frac{t'}{\varepsilon}]^4$. Observing that

$$1_{\{\{y_1 \le y_2 \le y_3 \le y_4\} \cup \{y_3 \le y_4 \le y_1 \le y_2\}\}} \le 1_{\{y_1 \le y_2\}} 1_{\{y_3 \le y_4\}},$$

we end up with

$$\begin{aligned} \sum_{k=1}^{2} B_k^{\varepsilon}(1,1) \le K^4 \varepsilon^2 \int_D & \sqrt{x_1 x_2 x_3 x_4}\sqrt{y_1 y_2 y_3 y_4}\, 1_{\{x_1 \le x_2\}} 1_{\{x_3 \le x_4\}} 1_{\{y_1 \le y_2\}} 1_{\{y_3 \le y_4\}} \\ & \times e^{-\Psi(\theta)((x_2-x_1)y_2 + (x_4-x_3)y_4 + (y_2-y_1)x_1 + (y_4-y_3)x_3)} \\ & \times dx_1 dy_1 dx_2 dy_2 dx_3 dy_3 dx_4 dy_4. \end{aligned}$$

One can get similar estimates for $B_1^{\varepsilon}(i,j) + B_2^{\varepsilon}(i,j)$ with $i,j \neq 1$. Gathering all the resulting bounds together, it can be verified that

$$\mathbb{E}\left[\left(\mathbb{E}\left[|\Delta_{s,t} X^{\varepsilon}(s',t')|^2 \,|\mathcal{F}^{\varepsilon}_{s,T}\right]\right)^2\right] \le \Theta_{\varepsilon}^2 + \rho_{\varepsilon}, \tag{25}$$

where

$$\begin{aligned} \Theta_{\varepsilon} = {} & K^2 \varepsilon^2 \int_{\frac{t}{\varepsilon}}^{\frac{t'}{\varepsilon}} \int_{\frac{t}{\varepsilon}}^{y_2} \int_{\frac{s}{\varepsilon}}^{\frac{s'}{\varepsilon}} \int_{\frac{s}{\varepsilon}}^{x_2} \sqrt{x_1 x_2 y_1 y_2}\, e^{-\Psi(\theta)((x_2-x_1)y_2 + (y_2-y_1)x_1)} dx_1 dx_2 dy_1 dy_2 \\ & + K^2 \varepsilon^2 \int_{\frac{t}{\varepsilon}}^{\frac{t'}{\varepsilon}} \int_{\frac{t}{\varepsilon}}^{y_2} \int_{\frac{s}{\varepsilon}}^{\frac{s'}{\varepsilon}} \int_{\frac{s}{\varepsilon}}^{x_2} \sqrt{x_1 x_2 y_1 y_2}\, e^{-\Psi(\theta)(x_2-x_1)y_1} e^{-\Psi(-\theta)(y_2-y_1)x_1} dx_1 dx_2 dy_1 dy_2 \\ & + K^2 \varepsilon^2 \int_{\frac{t}{\varepsilon}}^{\frac{t'}{\varepsilon}} \int_{\frac{t}{\varepsilon}}^{y_2} \int_{\frac{s}{\varepsilon}}^{\frac{s'}{\varepsilon}} \int_{\frac{s}{\varepsilon}}^{x_2} \sqrt{x_1 x_2 y_1 y_2}\, e^{-\Psi(-\theta)(x_2-x_1)y_1} e^{-\Psi(\theta)(y_2-y_1)x_1} dx_1 dx_2 dy_1 dy_2 \\ & + K^2 \varepsilon^2 \int_{\frac{t}{\varepsilon}}^{\frac{t'}{\varepsilon}} \int_{\frac{t}{\varepsilon}}^{y_2} \int_{\frac{s}{\varepsilon}}^{\frac{s'}{\varepsilon}} \int_{\frac{s}{\varepsilon}}^{x_2} \sqrt{x_1 x_2 y_1 y_2}\, e^{-\Psi(-\theta)((x_2-x_1)y_2 + (y_2-y_1)x_1)} dx_1 dx_2 dy_1 dy_2. \end{aligned}$$

Note that Θ_ε coincides with the right hand-side of equality (12) in the proof of Lemma 4.5, where in the latter it was precisely proved that

$$\lim_{\varepsilon\to 0}\Theta_\varepsilon = 2(t'-t)(s'-s).$$

Therefore, by (25) and recalling that $\lim_{\varepsilon\to 0}\rho_\varepsilon = 0$, we conclude the proof by taking $C_\varepsilon := \Theta_\varepsilon^2 + \rho_\varepsilon$. $\square$

We can now provide the proof of Proposition 4.3.

Proof of Proposition 4.3. We prove that, for all $0 \le s_1 < \cdots < s_n \le s$ and $0 \le t_1 < \cdots < t_n \le T$, and any continuous and bounded function $\varphi : \mathbb{C} \to \mathbb{R}$, we have

$$\mathbb{E}_{\mathbb{P}}\left[\varphi(X(s_1,t_1),\ldots,X(s_n,t_n))\left(\left(\Delta_{s,t}\mathsf{Re}(X)(s',t')\right)^2 - (s'-s)(t'-t)\right)\right] = 0$$

and

$$\mathbb{E}_{\mathbb{P}}\left[\varphi(X(s_1,t_1),\ldots,X(s_n,t_n))\left(\left(\Delta_{s,t}\mathsf{Im}(X)(s',t')\right)^2 - (s'-s)(t'-t)\right)\right] = 0.$$

Since $\mathbb{P}_\varepsilon$ converges to $\mathbb{P}$ weakly in $\mathcal{C}([0,S]\times[0,T];\mathbb{C})$, it suffices to check that

$$\lim_{\varepsilon\to 0} A_\varepsilon = 0 \quad \text{and} \quad \lim_{\varepsilon\to 0} B_\varepsilon = 0, \tag{26}$$

where

$$A_\varepsilon := \mathbb{E}\left[\varphi(X_\varepsilon(s_1,t_1),\ldots,X_\varepsilon(s_n,t_n))\left(\left(\Delta_{s,t}\mathsf{Re}(X_\varepsilon)(s',t')\right)^2 - (s'-s)(t'-t)\right)\right]$$

and

$$B_\varepsilon := \mathbb{E}\left[\varphi(X_\varepsilon(s_1,t_1),\ldots,X_\varepsilon(s_n,t_n))\left(\left(\Delta_{s,t}\mathsf{Im}(X_\varepsilon)(s',t')\right)^2 - (s'-s)(t'-t)\right)\right].$$

Indeed, in order to check the validity of the limits in (26), we will prove that

$$\lim_{\varepsilon\to 0}(A_\varepsilon + B_\varepsilon) = 0 \quad \text{and} \quad \lim_{\varepsilon\to 0}(A_\varepsilon - B_\varepsilon) = 0.$$

We will first deal with the limit of $A_\varepsilon + B_\varepsilon$. More precisely, we have that

$$\begin{aligned} A_\varepsilon + B_\varepsilon &= \mathbb{E}\left[\varphi(X_\varepsilon(s_1,t_1),\ldots,X_\varepsilon(s_n,t_n))\left(|\Delta_{s,t}X^\varepsilon(s',t')|^2 - 2(s-s')(t-t')\right)\right] \\ &= \mathbb{E}\left[\varphi(X_\varepsilon(s_1,t_1),\ldots,X_\varepsilon(s_n,t_n))\left(\mathbb{E}\left[|\Delta_{s,t}X^\varepsilon(s',t')|^2\,|\mathcal{F}^\varepsilon_{s,T}\right] - 2(s-s')(t-t')\right)\right]. \end{aligned}$$

Hence, to prove that $\lim_{\varepsilon\to 0}(A_\varepsilon + B_\varepsilon) = 0$, it is enough to check that $\mathbb{E}\big[|\Delta_{s,t}X^\varepsilon(s',t')|^2\,|\mathcal{F}^\varepsilon_{s,T}\big]$ converges in $L^2(\Omega)$ to $2(s-s')(t-t')$, as $\varepsilon\to 0$. Indeed, by Lemma 4.6, we have:

$$\begin{aligned} &\mathbb{E}\left[\left(\mathbb{E}\big[|\Delta_{s,t}X^\varepsilon(s',t')|^2\,|\mathcal{F}^\varepsilon_{s,T}\big] - 2(s-s')(t-t')\right)^2\right] \\ &\qquad \le C_\varepsilon - 4(s-s')(t-t')\mathbb{E}\big[|\Delta_{s,t}X^\varepsilon(s',t')|^2\big] + 4(s-s')^2(t-t')^2, \end{aligned} \tag{27}$$

where $\lim_{\varepsilon\to 0} C_\varepsilon = 4(s'-s)^2(t'-t)^2$. So, by Lemma 4.5, the right hand-side of (27) converges to zero as $\varepsilon\to 0$.

Let us now deal with the limit of $A_\varepsilon - B_\varepsilon$. To start with, note that

$$\begin{aligned} A_\varepsilon - B_\varepsilon = \frac{1}{2}\,\mathbb{E}\Bigg[&\varphi(X_\varepsilon(s_1,t_1),\ldots,X_\varepsilon(s_n,t_n)) \\ &\times\left\{\left(K\varepsilon\int_{\frac{t}{\varepsilon}}^{\frac{t'}{\varepsilon}}\int_{\frac{s}{\varepsilon}}^{\frac{s'}{\varepsilon}}\sqrt{xy}\,e^{i\theta L(x,y)}dxdy\right)^2 + \left(K\varepsilon\int_{\frac{t}{\varepsilon}}^{\frac{t'}{\varepsilon}}\int_{\frac{s}{\varepsilon}}^{\frac{s'}{\varepsilon}}\sqrt{xy}\,e^{-i\theta L(x,y)}dxdy\right)^2\right\}\Bigg]. \end{aligned}$$

We are going to prove that $\lim_{\varepsilon\to 0}\Lambda_\varepsilon = 0$, where

$$\Lambda_\varepsilon := \mathbb{E}\left[\varphi(X_\varepsilon(s_1,t_1),\ldots,X_\varepsilon(s_n,t_n))\left(K\varepsilon\int_{\frac{t}{\varepsilon}}^{\frac{t'}{\varepsilon}}\int_{\frac{s}{\varepsilon}}^{\frac{s'}{\varepsilon}}\sqrt{xy}\,e^{i\theta L(x,y)}dxdy\right)^2\right].$$

The limit involving the complex conjugate $e^{-i\theta L(x,y)}$ can be tackled using analogous arguments. Expanding the squared integral of Λ_ε, we end up with

$$\begin{aligned}\Lambda_\varepsilon = \mathbb{E}\Bigg[&\varphi(X_\varepsilon(s_1,t_1),\ldots,X_\varepsilon(s_n,t_n))\\ &\times K^2\varepsilon^2\int_{\frac{t}{\varepsilon}}^{\frac{t'}{\varepsilon}}\int_{\frac{s}{\varepsilon}}^{\frac{s'}{\varepsilon}}\int_{\frac{t}{\varepsilon}}^{\frac{t'}{\varepsilon}}\int_{\frac{s}{\varepsilon}}^{\frac{s'}{\varepsilon}}\sqrt{x_1x_2y_1y_2}\,e^{i\theta(L(x_1,y_1)+L(x_2,y_2))}dx_1dx_2dy_1dy_2\Bigg].\end{aligned}$$

As we have already done several times throughout the paper, we consider the four possible orders of x_1,x_2 and y_1,y_2 and, in each of these terms, we apply a change of variables so that we have $x_1\le x_2$ and $y_1\le y_2$. Thus,

$$\begin{aligned}\Lambda_\varepsilon = \mathbb{E}\Bigg[&\varphi(X_\varepsilon(s_1,t_1),\ldots,X_\varepsilon(s_n,t_n))\\ &\times\Bigg(2K^2\varepsilon^2\int_{\frac{t}{\varepsilon}}^{\frac{t'}{\varepsilon}}\int_{\frac{s}{\varepsilon}}^{\frac{s'}{\varepsilon}}\int_{\frac{t}{\varepsilon}}^{y_2}\int_{\frac{s}{\varepsilon}}^{x_2}\sqrt{x_1x_2y_1y_2}\,e^{i\theta(L(x_1,y_1)+L(x_2,y_2))}dx_1dy_1dx_2dy_2\\ &\quad+2K^2\varepsilon^2\int_{\frac{t}{\varepsilon}}^{\frac{t'}{\varepsilon}}\int_{\frac{s}{\varepsilon}}^{\frac{s'}{\varepsilon}}\int_{\frac{t}{\varepsilon}}^{y_2}\int_{\frac{s}{\varepsilon}}^{x_2}\sqrt{x_1x_2y_1y_2}\,e^{i\theta(L(x_2,y_1)+L(x_1,y_2))}dx_1dy_1dx_2dy_2\Bigg)\Bigg].\end{aligned}$$

At this point, the idea is to write $L(x_1,y_1)+L(x_2,y_2)$ and $L(x_2,y_1)+L(x_1,y_2)$ as sums of suitable rectangular increments of L (which will be clearly specified in the next equation), and use the property of independent (rectangular) increments of L (see Definition2.1). Proceeding in this way, one obtains that

$$\begin{aligned}\Lambda_\varepsilon = {}&\mathbb{E}\left[\varphi(X_\varepsilon(s_1,t_1),\ldots,X_\varepsilon(s_n,t_n))\,e^{i2\theta L\left(\frac{s}{\varepsilon},\frac{t}{\varepsilon}\right)}\right]\\ &\times\Bigg(2K^2\varepsilon^2\int_{\frac{t}{\varepsilon}}^{\frac{t'}{\varepsilon}}\int_{\frac{s}{\varepsilon}}^{\frac{s'}{\varepsilon}}\int_{\frac{t}{\varepsilon}}^{y_2}\int_{\frac{s}{\varepsilon}}^{x_2}\sqrt{x_1x_2y_1y_2}\,\mathbb{E}\left[e^{i\theta\left(\Delta_{x_1,y_1}L(x_2,y_2)+\Delta_{0,y_1}L(x_1,y_2)+\Delta_{x_1,0}L(x_2,y_1)\right)}\right]\\ &\qquad\times\mathbb{E}\left[e^{i2\theta\left(\Delta_{\frac{s}{\varepsilon},\frac{t}{\varepsilon}}L(x_1,y_1)+\Delta_{0,\frac{t}{\varepsilon}}L(\frac{s}{\varepsilon},y_1)+\Delta_{\frac{s}{\varepsilon},0}L(x_1,\frac{t}{\varepsilon})\right)}\right]dx_1dy_1dx_2dy_2\\ &\quad+2K^2\varepsilon^2\int_{\frac{t}{\varepsilon}}^{\frac{t'}{\varepsilon}}\int_{\frac{s}{\varepsilon}}^{\frac{s'}{\varepsilon}}\int_{\frac{t}{\varepsilon}}^{y_2}\int_{\frac{s}{\varepsilon}}^{x_2}\sqrt{x_1x_2y_1y_2}\,\mathbb{E}\left[e^{i\theta\left(\Delta_{0,y_1}L(x_1,y_2)+\Delta_{x_1,0}L(x_2,y_1)\right)}\right]\\ &\qquad\times\mathbb{E}\left[e^{i2\theta\left(\Delta_{\frac{s}{\varepsilon},\frac{t}{\varepsilon}}L(x_1,y_1)+\Delta_{0,\frac{t}{\varepsilon}}L(\frac{s}{\varepsilon},y_1)+\Delta_{\frac{s}{\varepsilon},0}L(x_1,\frac{t}{\varepsilon})\right)}\right]dx_1dy_1dx_2dy_2\Bigg).\end{aligned}$$

Recalling that φ is a bounded function and computing the expectations of complex exponentials in terms of the Lévy exponent $\Psi(\xi)=a(\xi)+ib(\xi)$, one can easily obtain that $|\Lambda_\varepsilon|\le C\,\tilde{\Lambda}_\varepsilon$, where C is a positive constant and

$$\begin{aligned}\tilde{\Lambda}_\varepsilon := K^2\varepsilon^2\int_{\frac{t}{\varepsilon}}^{\frac{t'}{\varepsilon}}\int_{\frac{s}{\varepsilon}}^{\frac{s'}{\varepsilon}}\int_{\frac{t}{\varepsilon}}^{y_2}\int_{\frac{s}{\varepsilon}}^{x_2}&\sqrt{x_1x_2y_1y_2}\,e^{-a(\theta)((y_2-y_1)x_1+(x_2-x_1)y_1)}\\ &\times e^{-a(2\theta)\left((y_1-\frac{t}{\varepsilon})\frac{s}{\varepsilon}+(x_1-\frac{s}{\varepsilon})y_1\right)}dx_1dy_1dx_2dy_2.\end{aligned}$$

We finally prove that $\lim_{\varepsilon\to 0}\tilde{\Lambda}_\varepsilon = 0$. Indeed, taking into account the integration limits of all variables and applying Fubini theorem, we have

$$\tilde{\Lambda}_\varepsilon \leq C\int_{\frac{t}{\varepsilon}}^{\frac{t'}{\varepsilon}}\int_{\frac{s}{\varepsilon}}^{\frac{s'}{\varepsilon}}\int_{y_1}^{\frac{t'}{\varepsilon}}\int_{x_1}^{\frac{s'}{\varepsilon}} e^{-\min(a(\theta),a(2\theta))\left((y_2-\frac{t}{\varepsilon})\frac{s}{\varepsilon}+(x_2-\frac{s}{\varepsilon})\frac{t}{\varepsilon}\right)}dx_2dy_2dx_1dy_1.$$

We integrate with respect to x_2 and y_2, thus

$$\begin{aligned}\tilde{\Lambda}_\varepsilon &\leq C\,\varepsilon^2\int_{\frac{t}{\varepsilon}}^{\frac{t'}{\varepsilon}}\int_{\frac{s}{\varepsilon}}^{\frac{s'}{\varepsilon}} e^{-\min(a(\theta),a(2\theta))\left((y_1-\frac{t}{\varepsilon})\frac{s}{\varepsilon}+(x_1-\frac{s}{\varepsilon})\frac{t}{\varepsilon}\right)}dx_1dy_1\\ &\leq C\,\varepsilon^4.\end{aligned}$$

Hence, we have $\lim_{\varepsilon\to 0}\tilde{\Lambda}_\varepsilon = 0$, which implies that $\lim_{\varepsilon\to 0}\Lambda_\varepsilon = 0$, and so $\lim_{\varepsilon\to 0} A_\varepsilon - B_\varepsilon = 0$. The proof is complete. $\square$

We have all needed ingredients to prove the main result of the paper:

Proof of Theorem 1.1. The tightness result Proposition 3.1 and Propositions 4.2 and 4.3 imply, by Theorem 4.1, that X_ε converges in law, as $\varepsilon\to 0$ and in the space $\mathcal{C}([0,S]\times[0,T];\mathbb{C})$, to a complex random field $X=\{X(s,t);\,(s,t)\in[0,S]\times[0,T]\}$ whose real and imaginary parts are (real-valued) Brownian sheets. It only remains to check that those real and imaginary parts are independent, for which it suffices to prove that the corresponding covariance vanishes. For this, we will make use of the approximation sequence $(X_\varepsilon)_{\varepsilon>0}$, as follows:

Note that $\mathsf{Re}(X)$ and $\mathsf{Im}(X)$ are independent if, for any $(0,0)\leq(s,t)\leq(s',t')\leq(S,T)$, $0\leq s_1<\cdots<s_n\leq s$ and $0\leq t_1<\cdots<t_n\leq t$, and any continuous bounded function $\varphi:\mathbb{C}^n\to\mathbb{R}$, we have

$$\lim_{\varepsilon\to 0}\mathbb{E}\left[\varphi(X_\varepsilon(s_1,t_1),\ldots,X_\varepsilon(s_n,t_n))\,(\Delta_{s,t}\mathsf{Re}(X_\varepsilon)(s',t'))\,(\Delta_{s,t}\mathsf{Im}(X_\varepsilon)(s',t'))\right]=0.$$

Using the equality $\alpha\beta=\frac{i}{4}\left\{(\alpha-i\beta)^2-(\alpha+i\beta)^2\right\}$, we obtain that

$$\begin{aligned}&\mathbb{E}\left[\varphi(X_\varepsilon(s_1,t_1),\ldots,X_\varepsilon(s_n,t_n))\,(\Delta_{s,t}\mathsf{Re}(X_\varepsilon)(s',t'))\,(\Delta_{s,t}\mathsf{Im}(X_\varepsilon)(s',t'))\right]\\ &\quad=\frac{i}{4}\,\mathbb{E}\Bigg[\varphi(X_\varepsilon(s_1,t_1),\ldots,X_\varepsilon(s_n,t_n))\Bigg\{\left(K\varepsilon\int_{\frac{t}{\varepsilon}}^{\frac{t'}{\varepsilon}}\int_{\frac{s}{\varepsilon}}^{\frac{s'}{\varepsilon}}\sqrt{xy}\,e^{-i\theta L(x,y)}dxdy\right)^2\\ &\qquad\quad-\left(K\varepsilon\int_{\frac{t}{\varepsilon}}^{\frac{t'}{\varepsilon}}\int_{\frac{s}{\varepsilon}}^{\frac{s'}{\varepsilon}}\sqrt{xy}\,e^{i\theta L(x,y)}dxdy\right)^2\Bigg\}\Bigg].\end{aligned}\tag{28}$$

We observe that, in the analysis of the term $A_\varepsilon - B_\varepsilon$ in the proof of Proposition 4.3, we indeed proved that the two terms arising from the difference in (28) converge to zero as $\varepsilon\to 0$. So the proof is complete. $\square$

5 Weak convergence for the stochastic heat equation

We consider the following one-dimensional quasi-linear stochastic heat equation:

$$\frac{\partial U}{\partial t}(t,x)-\frac{\partial^2 U}{\partial x^2}(t,x)=b(U(t,x))+\dot{W}(t,x),\quad (t,x)\in[0,T]\times[0,1],\tag{29}$$

where $T>0$ stands for a fixed time horizon, $b:\mathbb{R}\to\mathbb{R}$ is a globally Lipschitz function and $\dot{W}(t,x)$ denotes the space-time white noise. We impose the initial condition $U(0,x)=u_0(x)$, $x\in[0,1]$, where $u_0:[0,1]\to\mathbb{R}$ is a continuous function, and boundary conditions of Dirichlet type:

$$U(t,0)=U(t,1)=0,\ t\in[0,T].$$

For simplicity's sake, throughout the section we will assume that $T = 1$. All results presented here can be easily extended to a general $T > 0$.

The solution to equation (29) is interpreted in the mild sense, as follows. Let $\{W(t,x);\ (t,x) \in [0,1]^2\}$ be a Brownian sheet defined on some probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and $\{\mathcal{F}_t;\ t \in [0,1]\}$ its natural filtration. A jointly measurable and adapted random field $U = \{U(t,x);\ (t,x) \in [0,1]^2\}$ is a solution of (29) if it holds that

$$\begin{aligned} U(t,x) = \int_0^1 G_t(x,y) u_0(y) dy + \int_0^t \int_0^1 G_{t-s}(x,y)\, b(U(s,y)) dy ds \\ + \int_0^t \int_0^1 G_{t-s}(x,y) W(ds,dy), \quad a.s. \end{aligned} \tag{30}$$

for all $(t,x) \in (0,1] \times (0,1)$, where G denotes the Green function associated to the heat equation in $[0,1]$ with Dirichlet boundary conditions. Existence, uniqueness and pathwise continuity of the solution to (30) are a consequence of [13, Thm 3.5]. For the reader's convenience, we recall that the Green function G is given by

$$G_t(x,y) = 2 \sum_{n=1}^{\infty} \sin(n\pi x) \sin(n\pi y) e^{-n^2\pi^2 t}.$$

In this section, we aim to apply [3, Thm. 1.4] in order to deduce that the above solution U can be approximated in law, in the space $\mathcal{C}([0,1]^2)$ of continuous functions, by the family of mild solutions $\{U_n\}_{n \geq 0}$, where U_n solves a stochastic heat equation perturbed by (the formal derivative of) either the real or imaginary part of the family introduced in (2):

$$X_\varepsilon(t,x) = \varepsilon K \int_0^{\frac{t}{\varepsilon}} \int_0^{\frac{x}{\varepsilon}} \sqrt{sy}\, \{\cos(\theta L(s,y)) + i \sin(\theta L(s,y))\}\, dy ds,$$

where we recall that $\{L(s,y);\ s,y \geq 0\}$ denotes a Lévy sheet and its Lévy exponent is given by $\Psi(\xi) = a(\xi) + ib(\xi)$. The constant K is given in (3) and $\theta \in (0, 2\pi)$, where we assume that $a(\theta)a(2\theta) \neq 0$. Note that, compared to (2), in the above expression of X_ε we have modified the variables' notation in order to properly match with the framework of stochastic partial differential equations.

Let us be more precise about the above statement. First, we rewrite X_ε in the following way:

$$X_\varepsilon(t,x) = nK \int_0^t \int_0^x \sqrt{sy}\, \left\{\cos\left(\theta L\left(\sqrt{n}\, s, \sqrt{n}\, y\right)\right) + i \sin\left(\theta L\left(\sqrt{n}\, s, \sqrt{n}\, y\right)\right)\right\}\, dy ds,$$

with $n = \varepsilon^{-2}$. Set

$$\theta_n^1(s,y) := nK\sqrt{sy}\, \cos\left(\theta L\left(\sqrt{n}\, s, \sqrt{n}\, y\right)\right) \quad \text{and} \quad \theta_n^2(s,y) := nK\sqrt{sy}\, \sin\left(\theta L\left(\sqrt{n}\, s, \sqrt{n}\, y\right)\right).$$

Let $i \in \{1,2\}$ and consider the stochastic heat equation

$$\frac{\partial U_n^i}{\partial t}(t,x) - \frac{\partial^2 U_n^i}{\partial x^2}(t,x) = b(U_n^i(t,x)) + \theta_n^i(t,x), \qquad (t,x) \in [0,1]^2,$$

with initial condition u_0 and Dirichlet boundary conditions. The mild form of this equation is given by

$$\begin{aligned} U_n^i(t,x) = \int_0^1 G_t(x,y) u_0(y) dy + \int_0^t \int_0^1 G_{t-s}(x,y)\, b(U_n^i(s,y)) dy ds \\ + \int_0^t \int_0^1 G_{t-s}(x,y) \theta_n^i(s,y) dy ds. \end{aligned} \tag{31}$$

Owing to [3, Sec. 3], equation (31) admits a unique solution U_n^i whose paths are continuous almost surely. Here is the main result of the section:

Theorem 5.1. *For any $i \in \{1,2\}$, the sequence $\{U_n^i\}_{n\geq 1}$ converges in law, as $n \to \infty$ and in the space $\mathcal{C}([0,1]^2)$, to the solution U of* (30).

The proof of this theorem is based on [3, Thm. 1.4], where sufficient conditions on a family of random fields $\{\theta_n\}_{n\geq 1}$ have been established such that the sequence of solutions to the stochastic heat equation driven by θ_n converges in law to U, in the space of continuous functions. The first requirement is that $\theta_n \in L^2([0,1]^2)$ a.s., and then there are the following conditions (see hypotheses 1.1, 1.2 and 1.3 in [3]):

(i) The finite dimensional distributions of the processes

$$\zeta_n(t,x) := \int_0^t \int_0^x \theta_n(s,y)dyds, \quad (t,x) \in [0,1]^2,$$

converge in law to those of the Brownian sheet.

(ii) For some $q \in [2,3)$, there exists a positive constant C_q such that, for any $f \in L^q([0,1]^2)$, it holds:

$$\sup_{n\geq 1} \mathbb{E}\left[\left(\int_0^1 \int_0^1 f(t,x)\theta_n(t,x)\, dxdt\right)^2\right] \leq C_q \left(\int_0^1 \int_0^1 |f(t,x)|^q\, dxdt\right)^{\frac{2}{q}}.$$

(iii) There exist $m > 8$ and a positive constant C such that the following is satisfied: for all s_0, $s_0' \in [0,1]$ and x_0, $x_0' \in [0,1]$ satisfying $0 < s_0 < s_0' < 2s_0$ and $0 < x_0 < x_0' < 2x_0$, and for any $f \in L^2([0,1]^2)$, it holds:

$$\sup_{n\geq 1} \mathbb{E}\left[\left|\int_{s_0}^{s_0'} \int_{x_0}^{x_0'} f(s,y)\, \theta_n(s,y)dyds\right|^m\right] \leq C \left(\int_{s_0}^{s_0'} \int_{x_0}^{x_0'} f(s,y)^2\, dyds\right)^{\frac{m}{2}}.$$

Hence, in the proof of Theorem 5.1 we will prove the validity of all above conditions in the case where θ_n is given by θ_n^i, for any $i \in \{1,2\}$. Indeed, as it will be explained below, we will use similar arguments as those used in one of the applications tackled in [3], namely the case where θ_n are given by the Kac-Stroock processes on the plane:

$$\theta_n(t,x) = n\sqrt{tx}\,(-1)^{N_n(t,x)},$$

where $N_n(t,x) := N(\sqrt{n}t, \sqrt{n}x)$, and $\{N(t,x);\, (t,x) \in [0,1]^2\}$ is a standard Poisson process in the plane.

We start with the following technical lemma, which is the analogous of [3, Lem. 4.2]:

Lemma 5.2. *Let $f \in L^2([0,1]^2)$ and $\alpha \geq 1$. Then, for any $u, u' \in (0,1)$ satisfying that $0 < u < u' \leq 2^\alpha u$, it holds*

$$\sup_{n\geq 1} \mathbb{E}\left[\left(\int_0^1 \int_u^{u'} f(t,x)\theta_n^i(t,x)\, dxdt\right)^2\right] \leq \frac{3}{a(\theta)^2}(2^{\alpha+1}-1)K^2 \int_0^1 \int_u^{u'} f(t,x)^2\, dxdt,$$

for any $i \in \{1,2\}$.

Proof. We will only deal with the case involving θ_n^1, since the result for θ_n^2 follows exactly in the same way. Note that we clearly have

$$\mathbb{E}\left[\left(\int_0^1 \int_u^{u'} f(t,x)\theta_n^i(t,x)\, dxdt\right)^2\right] \leq \mathbb{E}\left[\left|nK \int_0^1 \int_u^{u'} \sqrt{tx}\, e^{i\theta L(\sqrt{n}\,t, \sqrt{n}\,x)} f(t,x)\, dxdt\right|^2\right],$$

and the latter term equals to

$$n^2K^2\int_0^1\int_u^{u'}\int_0^1\int_u^{u'}\sqrt{t_1x_1t_2x_2}\,\mathbb{E}\left[e^{i\theta\left(\Delta_{0,0}L\left(\sqrt{n}\,t_1,\sqrt{n}\,x_1\right)-\Delta_{0,0}L\left(\sqrt{n}\,t_2,\sqrt{n}\,x_2\right)\right)}\right]$$
$$\times f(t_1,x_1)f(t_2,x_2)dx_1dt_1dx_2dt_2.$$

Observe that this expression is completely analogous as that at the beginning of the first step in the proof of Lemma 4.5. Thus, the same arguments used therein yield

$$\mathbb{E}\left[\left(\int_0^1\int_u^{u'} f(t,x)\theta_n^i(t,x)\,dxdt\right)^2\right]\le 2\left(I_1^n+I_2^n\right), \tag{32}$$

where

$$I_1^n=n^2K^2\int_0^1\int_0^{t_2}\int_u^{u'}\int_u^{x_2} f(t_1,x_1)f(t_2,x_2)\sqrt{t_1x_1t_2x_2}\,e^{-a(\theta)n(x_2t_2-x_1t_1)}$$
$$\times\cos(b(\theta)n(x_2t_2-x_1t_1))dx_1dx_2dt_1dt_2$$

and

$$I_2^n=n^2K^2\int_0^1\int_0^{t_2}\int_u^{u'}\int_u^{x_2} f(t_1,x_1)f(t_2,x_2)\sqrt{t_1x_1t_2x_2}\,e^{-a(\theta)n((t_2-t_1)x_1+(x_2-x_1)t_1)}$$
$$\times\cos\left(b(\theta)n\left((t_2-t_1)x_1+(x_2-x_1)t_1\right)\right)dx_1dx_2dt_1dt_2.$$

At this point, we apply the inequality $zw\le\frac{1}{2}(z^2+w^2)$ in such a way that

$$f(t_1,x_1)f(t_2,x_2)\sqrt{t_1x_1t_2x_2}\le\frac{1}{2}\left(t_1x_1f(t_1,x_1)^2+t_2x_2f(t_2,x_2)^2\right).$$

This makes that both I_1^n and I_2^n can be bounded by the sum of two terms of the form $I_{j,1}^n+I_{j,2}^n$, $j=1,2$, respectively, where $I_{j,1}^n$ involves $f(t_1,x_1)$ and $I_{j,2}^n$ involves $f(t_2,x_2)$. Then, once all cosinus are simply bounded by 1, one observe that the resulting four terms are completely analogous as those appearing in the proof of Lemma 4.2 in [3], and can be treated using the same kind of arguments. Thus, we obtain that

$$I_{1,1}^n\le\frac{1}{2}\frac{1}{a(\theta)^2}K^2\int_0^1\int_u^{u'} f(t_1,x_1)^2dx_1dt_1,$$
$$I_{1,2}^n\le\frac{1}{2}\frac{2^\alpha}{a(\theta)^2}K^2\int_0^1\int_u^{u'} f(t_2,x_2)^2dx_2dt_2,$$
$$I_{2,1}^n\le\frac{1}{2}\frac{2^\alpha}{a(\theta)^2}K^2\int_0^1\int_u^{u'} f(t_1,x_1)^2dx_1dt_1,$$
$$I_{2,2}^n\le\frac{1}{2}\frac{4(2^\alpha-1)}{a(\theta)^2}K^2\int_0^1\int_u^{u'} f(t_2,x_2)^2dx_2dt_2.$$

Plugging everything together and using (32), we conclude the proof. □

The above lemma allows us to prove the following proposition. In fact, its proof follows exactly the same lines as that of Proposition 4.1 in [3] and therefore will be omitted.

Proposition 5.3. *Let $p > 1$ and $f \in L^{2p}([0,1]^2)$. Then, there exists a positive constant C_p which does not depend on f such that*

$$\sup_{n\geq 1} \mathbb{E}\left[\left(\int_0^1\int_0^1 f(t,x)\theta_n^i(t,x)\,dxdt\right)^2\right] \leq C_p\left(\int_0^1\int_0^1 |f(t,x)|^{2p}\,dxdt\right)^{\frac{1}{p}},$$

for any $i \in \{1,2\}$.

The last needed ingredient for the proof of Theorem 5.1 is the following result, which is the analogous of [3, Prop. 4.4] in our setting.

Proposition 5.4. *Let $m \in \mathbb{N}$ be an even number and $f \in L^2([0,1]^2)$. Then, there exists a positive constant C_m which does not depend on f such that, for all s_0, s_0', x_0, $x_0' \in [0,1]$ satisfying $0 < s_0 < s_0' < 2s_0$ and $0 < x_0 < x_0' < 2x_0$, we have that*

$$\sup_{n\geq 1} \mathbb{E}\left[\left(\int_{s_0}^{s_0'}\int_{x_0}^{x_0'} f(s,y)\,\theta_n^i(s,y)dyds\right)^m\right] \leq C_m\left(\int_{s_0}^{s_0'}\int_{x_0}^{x_0'} f(s,y)^2\,dyds\right)^{\frac{m}{2}},$$

for any $i \in \{1,2\}$.

Proof. Let $i \in \{1,2\}$. For any $(s_0,x_0) \in [0,1]^2$, we define

$$Z_n^i(s_0,x_0) := \int_0^{s_0}\int_0^{x_0} f(s,y)\theta_n^i(s,y)dyds.$$

Observe that, for all $(0,0) \leq (s_0,x_0) < (s_0',x_0') \leq (1,1)$, we have

$$\mathbb{E}\left[\left(\Delta_{s_0,x_0} Z_n^i(s_0',x_0')\right)^m\right] \leq \mathbb{E}\left[\left|\Delta_{s_0,x_0}\bar{Z}_n(s_0',x_0')\right|^m\right], \tag{33}$$

where the random field $\bar{Z}_n$, which does not depend on i, is complex-valued and given by

$$\bar{Z}_n(s_0,x_0) := \int_0^{s_0}\int_0^{x_0} f(s,y)\left(\theta_n^1(s,y) + i\theta_n^2(s,y)\right)dyds$$

(here $i = \sqrt{-1}$). In order to bound the right hand-side of (33), we can proceed as in the first part of the proof of the tightness result Proposition 3.1, obtaining

$$\begin{aligned}\mathbb{E}\left[\left(\Delta_{s_0,x_0} Z_n^i(s_0',x_0')\right)^m\right] \leq n^m K^m &\int_{[s_0,s_0']^m}\int_{[x_0,x_0']^m}\prod_{j=1}^m f(s_j,y_j)\sqrt{s_jy_j}\\ &\times\left|\mathbb{E}\left[e^{i\theta\sum_{j=1}^m(-1)^j\Delta_{0,0}L(\sqrt{n}\,s_j,\sqrt{n}\,y_j)}\right]\right| dy_1\cdots dy_m ds_1\cdots ds_m.\end{aligned}$$

At this point, we apply that $y_j < x_0' < 2x_0$ and $s_j < s_0' < 2s_0$, and we compute the modulus of the expectation as it has been done in the proof of Proposition 3.1; more precisely, using the method set up therein in order to end up with the estimate (8). Thus, we can infer that

$$\begin{aligned}&\mathbb{E}\left[\left(\Delta_{s_0,x_0} Z_n^i(s_0',x_0')\right)^m\right]\\ &\quad\leq 2^m(s_0x_0)^{\frac{m}{2}} n^m K^m\int_{[s_0,s_0']^m}\int_{[x_0,x_0']^m}\prod_{j=1}^m f(s_j,y_j)\,e^{-a(\theta)ns_0\left((y_{(m)}-y_{(m-1)})+\cdots+(y_{(2)}-y_{(1)})\right)}\\ &\qquad\qquad\times e^{-a(\theta)nx_0\left((s_{(m)}-s_{(m-1)})+\cdots+(s_{(2)}-s_{(1)})\right)}\,dy_1\cdots dy_m ds_1\cdots ds_m\\ &\quad= 2^m(s_0x_0)^{\frac{m}{2}} m!\,n^m K^m\int_{[s_0,s_0']^m}\int_{[x_0,x_0']^m}\prod_{j=1}^m f(s_j,y_j)\,e^{-a(\theta)ns_0\left((y_{(m)}-y_{(m-1)})+\cdots+(y_{(2)}-y_{(1)})\right)}\\ &\qquad\qquad\times e^{-a(\theta)nx_0((s_m-s_{m-1})+\cdots+(s_2-s_1))}\,1_{\{s_1\leq\cdots\leq s_m\}}\,dy_1\cdots dy_m ds_1\cdots ds_m.\end{aligned}$$

Note that the latter expression is almost equal to that in the right hand-side of equation (31) in the proof of [3, Prop. 4.4]. Hence, we can conclude the proof exactly in the same way as in that result. □

Proof of Theorem 5.1. As explained above, we need that $\theta_n^i \in L^2([0,1]^2)$, a.s., which is clear by definition of the random fields θ_n^i, $i = 1, 2$, and that conditions (i), (ii) and (iii) are fulfilled.

First, note that (i) is a consequence of Theorem 1.1. Secondly, Proposition 5.3 implies that (ii) is satisfied and, finally, Proposition 5.4 assures the validity of condition (iii). □

References

[1] X. Bardina. The complex Brownian motion as a weak limit of processes constructed from a Poisson process. Stochastic analysis and related topics, VII (Kusadasi, 1998), 149-158, Progr. Probab. 48, Birkhäuser Boston, Boston, MA, 2001.

[2] X. Bardina and M. Jolis. Weak approximation of the Brownian sheet from a Poisson process in the plane. Bernoulli 6 (2000), no. 4, 653-665.

[3] X. Bardina, M. Jolis and L. Quer-Sardanyons. Weak convergence for the stochastic heat equation driven by Gaussian white noise. Electron. J. Probab. 15 (2010), no. 39, 1267-1295.

[4] X. Bardina and C. Rovira. Approximations of a complex Brownian motion by processes constructed from a Lévy process. Mediterr. J. Math. 13 (2016), no. 1, 469-482.

[5] P.J. Bickel and M.J. Wichura. Convergence criteria for multiparameter stochastic processes and applications. Ann. Math. Stat. 42 (1971), 1656-1670.

[6] R. Cairoli and J.B. Walsh. Stochastic integrals in the plane. Acta Math. 134 (1975), 111-183.

[7] N.N. Centsov. Limit theorems for certain classes of random functions. (Russian) 1960 Proc. All-Union Conf. Theory Prob. and Math. Statist. (Erevan, 1958) (Russian) pp. 280–285, Izdat. Akad. Nauk Armjan. SSR, Erevan. In: Selected Translations in Math. Statistics and Probability, 9 (1971), 37-42.

[8] C. Florit and D. Nualart. Diffusion aproximation for hyperbolic stochastic differential equations. Stoch. Proc. and their Appl. 65 (1996), 1-15.

[9] M. Kac. A stochastic model related to the telegrapher's equation. Reprinting o an article published in 1956. Rocky Mountain J. Math. 4 (1974) 497-509.

[10] D. Stroock. Lectures on Topics in Stochastic Differential Equations. Springer.Verlag, Berlin, 1982.

[11] C. Tudor. Remarks on the martingale problem in the two dimensional time parameter. Rev. Roum. Math. Pures Appl. (1980), 1551-1556.

[12] J.B. Walsh. A stochastic model of neural response. Adv. in Appl. Probab. 13 (1981), no. 2, 231-281.

[13] J.B. Walsh. An introduction to stochastic partial differential equations. In: Hennequin, P. L. (ed.) École d'été de probabilités de Saint-Flour XIV - 1984, Lect. Notes Math. vol. 1180, pp. 265-437. Springer, Berlin (1986).