PURE C*-ALGEBRAS

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Dedicated to Mikael Rørdam on the occasion of his 65th birthday

Abstract. We show that every C^*-algebra that is (m,n)-pure in the sense of Winter is already pure, that is, its Cuntz semigroup is almost unperforated and almost divisible. More generally, we show that even weaker comparison and divisibility properties automatically lead to pureness. We use this to show that, under a mild comparison assumption, pureness is automatic for C^*-algebras that are either nowhere scattered with real rank zero or stable rank one, or simple, unital, non-elementary, with a unique quasitrace.

As an application to the non-simple Toms-Winter conjecture, we show that every C^*-algebra with the Global Glimm Property and finite nuclear dimension is pure. It follows that a separable, locally subhomogeneous C^*-algebra with stable rank one and topological dimension zero is pure if and only if it is $\mathcal{Z}$-stable, if and only if it is nowhere scattered and has finite nuclear dimension.

1. Introduction

Pureness is a regularity property for C^*-algebras introduced by Winter in his seminal investigation into $\mathcal{Z}$-stability and finite nuclear dimension of simple, nuclear C^*-algebras [Win12]. Interest in these concepts was sparked by Toms' groundbreaking examples [Tom08], where he constructed two simple, nuclear C^*-algebras with the same K-theoretic and tracial data —one pure, $\mathcal{Z}$-stable (i.e., tensorially absorbing the Jiang-Su algebra $\mathcal{Z}$), and of finite nuclear dimension, and the other not. This led to a revision of the Elliott program, aiming at classifying separable, simple, nuclear C^*-algebras. Building on decades of research by numerous mathematicians, this was ultimately achieved in a series of remarkable results [Win14, TWW17, GLN20]. (See also the recent approach [CGS+23] and the survey [Whi23].) The classification theorem states that separable, simple, nuclear C^*-algebras that are $\mathcal{Z}$-stable and satisfy the universal coefficient theorem (UCT) can be distinguished up to isomorphism by their Elliott invariant, which essentially consists of topological K-Theory and the tracial simplex.

In his work, Winter defined the quantified notions of *(m,n)-pureness* for simple C^*-algebras, with m and n specifying types of comparison and divisibility properties among operators. For simple C^*-algebras with locally finite nuclear dimension (a strong form of nuclearity), Winter showed that (m,n)-pureness implies $\mathcal{Z}$-stability

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regardless of m and n. Further, it follows from results of Rørdam [Rør04] that every simple $\mathcal{Z}$-stable C^*-algebra is $(0,0)$-pure, usually abbreviated to just *pure*.

The concept of (m,n)-pureness does not depend on the ideal structure of the algebra, and hence may be stated for non-simple C^*-algebras; see Definition 5.1. Moreover, as we show in this paper, all versions of pureness coincide. Our first main result is:

Theorem A (5.7)**.** *If a C^*-algebra is (m,n)-pure for some $m,n \in \mathbb{N}$, then it is pure.*

This should be contrasted with the well-known examples of simple, nuclear C^*-algebras that do not have 0-comparison ([Vil98, Vil99]), or that do not have 0-almost divisibility ([DHTW09]).

Previously, Theorem A had been obtained for separable, simple, unital C^*-algebras with locally finite nuclear dimension [Win12], and more recently for arbitrary simple C^*-algebras [APRT24, Theorem 10.5]. Our proof of the general non-simple, non-nuclear case is based on the methods developed in [APRT24] on *controlled comparison* —a version of comparison that is weaker than m-comparison for any m (see Section 3 for details)— in combination with new techniques involving *functional divisibility* —a form of divisibility that is weaker than n-almost divisibility for any n (see Section 4 for details). Our main technical result shows that pureness follows from controlled comparison together with functional divisibility; see Proposition 5.5. The techniques on functional divisibility are partly inspired by results in [RT17].

We show in Section 4 that functional divisibility holds automatically for large classes of C^*-algebras. Combining this with results in Section 5, we prove:

Theorem B (5.12, 5.13)**.** *Let A be a C^*-algebra with controlled comparison. Moreover, assume that*

(i) *A is nowhere scattered with real rank zero or stable rank one, or*
(ii) *A is simple, unital, non-elementary, with a unique quasitracial state.*

Then A is pure.

In particular, Villadsen algebras are functionally divisible and thus do not enjoy controlled comparison, let alone m-comparison for any m; see Example 5.16.

A key ingredient in our approach is the Cuntz semigroup, a geometric refinement of K-theory that was introduced by Cuntz [Cun78] in his pioneering study of traces, and that was developed by Coward, Elliott and Ivanescu [CEI08] to its modern form (see Section 2 for the precise definition). The Cuntz semigroup features prominently in the structure and classification theory of C^*-algebras, and its power was demonstrated by Toms [Tom08] when he used it to distinguish simple, nuclear C^*-algebras with the same Elliott invariant. More recently, the Cuntz semigroup was used in [APRT22] to resolve three long-standing problems for C^*-algebras of stable rank one.

The Toms-Winter conjecture. This conjecture [Win18] aims at establishing a robust class of C^*-algebras covered by the classification theorem mentioned at the beginning of the introduction. Explicitly, it predicts that for a separable, simple, unital, non-elementary, nuclear C^*-algebra, the following properties are equivalent:

(1) The C^*-algebra has finite nuclear dimension, a noncommutative generalization of finite covering dimension introduced in [WZ10].
(2) The C^*-algebra is $\mathcal{Z}$-stable.
(3) The C^*-algebra has strict comparison of positive elements (see Section 3).

In two groundbreaking papers [Win12, CET+21], it was shown that conditions (1) and (2) are equivalent. Further, it is known that (2) implies (3) [Rør04]. The implication (3)⇒(2) has been verified under certain additional assumptions [Sat12, KR14, TWW15, Thi20], but remains open in general.

The Toms-Winter conjecture is closely related to pureness, which we view as a property lying between (2) and (3). Indeed, $\mathcal{Z}$-stability implies pureness, which in turn implies strict comparison of positive elements; see Proposition 5.2. Further, strict comparison of positive elements implies pureness under the additional assumption of a positive solution to the Rank Problem; see Section 2. Such a positive solution has been obtained for C^*-algebras of real rank zero [ER06] and of stable rank one [Thi20, APRT22].

The non-simple Toms-Winter conjecture. The regularity properties of the Toms-Winter conjecture are also relevant for non-simple C^*-algebras, and an investigation of their interplay was initiated by Robert and Tikuisis [RT17]. While the assumption of unitality in the original formulation of the Toms-Winter conjecture could be dispensed with, the condition of being non-elementary is crucial. Indeed, elementary C^*-algebras have nuclear dimension zero, but they are neither $\mathcal{Z}$-stable nor pure.

The natural generalization of non-elementariness to the non-simple setting is the notion of nowhere scateredness: a C^*-algebra is *nowhere scattered* if none of its quotients has a nonzero, elementary ideal, or equivalently if every hereditary sub-C^*-algebra is generated by nilpotent elements [TV24a]. This notion is tightly related to the Global Glimm Property: a C^*-algebra has the *Global Glimm Property* if every hereditary sub-C^*-algebra contains an approximately full, nilpotent element [TV23]. Clearly, every C^*-algebra with the Global Glimm Property is nowhere scattered. The converse remains open, and is known as the *Global Glimm Problem.* In light of these considerations, the following question is natural:

Question C. *Let A be a separable, nowhere scattered, nuclear C^*-algebra. Are the following properties equivalent?*

- *(1) A has finite nuclear dimension.*
- *(2) A is $\mathcal{Z}$-stable.*
- *(3a) A is pure.*
- *(3b) A has strict comparison of positive elements.*

As for simple C^*-algebras, the question whether strict comparison of positive elements implies pureness is closely related to the Rank Problem. From our discussion above, it follows that (3b) implies (3a) for C^*-algebras of real rank zero or stable rank one.

The implication (3a)⇒(2) holds for C^*-algebras that have locally finite nuclear dimension, no simple purely infinite ideal-quotients, and topological dimension zero (the primitive ideal space has a basis of compact-open subsets) or Hausdorff primitive ideal space; see [RT17, Theorems 7.10, 7.15].

The implication (2)⇒(1) holds for all locally subhomogeneous C^*-algebras by [ENST20], for $\mathcal{O}_\infty$-stable C^*-algebras by [BGSW22], and for twisted étale groupoid C^*-algebras with finite dynamic asymptotic dimension by [BL24].

The implications (2)⇒(3a)⇒(3b) hold in general, and pureness always implies the Global Glimm Property; see Proposition 5.2. Therefore, Question C contains in particular the Global Glimm Problem for C^*-algebras with finite nuclear dimension.

We provide a further partial answer to Question C:

Theorem D (6.5)**.** *Every C^*-algebra with the Global Glimm Property and finite nuclear dimension is pure.*

Combining Theorem D with the above mentioned results, we obtain an answer to Question C for a relevant class of C^*-algebras, thereby verifying the non-simple Toms-Winter conjecture in this setting.

Theorem E (6.6)**.** *Let A be a separable, locally subhomogeneous C^*-algebra that has stable rank one, topological dimension zero, and the Global Glimm Property. Then, the answer to Question C is positive.*

Conventions. Given a C^*-algebra, we let A_+ to denote its positive elements. We let $\mathbb{N}$ denote the set of natural numbers, including 0. Further, $\mathcal{K}$ denotes the C^*-algebra of compact operators on a separable, infinite-dimensional Hilbert space.

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2. Preliminaries

The techniques used in this paper mainly concern the structure of the Cuntz semigroup of a C^*-algebra whose definition and properties we describe below.

Given positive elements a, b in a C^*-algebra A, one says that a is *Cuntz subequivalent* to b, denoted by $a \precsim b$, if there is a sequence $(r_n)_n$ in A such that $\lim_{n\to\infty} \|a - r_n b r_n^*\| = 0$. Further, a and b are *Cuntz equivalent*, denoted $a \sim b$, if $a \precsim b$ and $b \precsim a$. These relations were introduced by Cuntz in [Cun78].

The *Cuntz semigroup* $\mathrm{Cu}(A)$ of A is defined as $\mathrm{Cu}(A) = (A \otimes \mathcal{K})_+/{\sim}$, equipped with the partial order induced by $\precsim$, and addition induced by addition of orthogonal positive elements.

It is proved in [CEI08] that $\mathrm{Cu}(A)$ satisfies the following properties (usually referred to as axioms):

- (O1) If $(x_n)_n$ is an increasing sequence in $\mathrm{Cu}(A)$, then $\sup_n x_n$ exists.
- (O2) For any $x \in \mathrm{Cu}(A)$ there exists a sequence $(x_n)_n$ such that $x_n \ll x_{n+1}$ for all n and $x = \sup_n x_n$. (We say that $(x_n)_n$ is a $\ll$-increasing sequence.)
- (O3) If $x_1 \ll x_2$ and $y_1 \ll y_2$ in $\mathrm{Cu}(A)$, then $x_1 + y_1 \ll x_2 + y_2$.
- (O4) If $(x_n)_n$ and $(y_n)_n$ are increasing sequences in $\mathrm{Cu}(A)$, then $\sup_n (x_n + y_n) = \sup_n x_n + \sup_n y_n$.

The relation $\ll$ in these axioms is defined as follows: $x \ll y$ if for every increasing sequence $(y_n)_n$ satisfying $y \leq \sup_n y_n$ there exists $n_0 \in \mathbb{N}$ such that $x \leq y_{n_0}$. The relation $\ll$ is called the *way-below relation*, or the *compact containment relation*, and one says that 'x is way-below y' (or that 'x is compactly contained in y') if $x \ll y$. An element $u \in S$ such that $u \ll u$ is termed *compact*.

A positively ordered monoid satisfying axioms (O1)-(O4) is called a Cu-*semigroup* and belongs to a category of semigroups that has thoroughly been studied (see, for example, [APT18, APT20c, APT20a, APT20b], among others; see also the recent survey [GP24]). Yet, when dealing with semigroups obtained as Cuntz semigroups of C^*-algebras, additional axioms and properties are satisfied:

- (O5) For all x', x, y with $x' \ll x \leq y$ there exists z such that $x' + z \leq y \leq x + z$. Moreover, if $x + w \leq y$ for some w, and $w' \ll w$, then z may be chosen such that $w' \ll z$.

This axiom is usually referred to as the axiom of *almost algebraic order*. For the Cuntz semigroup of an arbitrary C^*-algebra, the first part of the statement above was proved in [RW10, Lemma 7.1], and the second part can be found in [APT18, Proposition 4.6].

Since we will often use the following consequence of (O5), we add a proof here for completeness.

Lemma 2.1 ([TV23, Lemma 2.2]). *Let S be a* Cu*-semigroup satisfying (O5), let $k \in \mathbb{N}$, and let $z', z, x \in S$ satisfy $z' \ll z$ and $(k+1)z \leq x$. Then there exists $d \in S$ such that*

$$kz' + d \leq x \leq kz + d, \quad \textit{and} \quad x \leq (k+1)d.$$

Proof. Choose $z'' \in S$ such that $z' \ll z'' \ll z$. Applying (O5) for

$$kz' \ll kz'', \quad z'' \ll z, \quad \text{and} \quad kz'' + z \leq x,$$

we obtain $d \in S$ satisfying

$$kz' + d \leq x \leq kz'' + d, \quad \text{and} \quad z'' \ll d.$$

Then $x \leq (k+1)d$, which shows that d satisfies the desired conditions. □

Two more axioms are used very often in the theory of Cu-semigroups. The first one, (O6), is a weaker form of *Riesz decomposition* and the second one, (O7), is a weak form of interpolation. That the Cuntz semigroup of any C^*-algebra satisfies both (O6) and (O7) was proved in [Rob13, Proposition 5.1.1] and [APRT21, Proposition 2.2], respectively. We will not need the precise formulation of these axioms here, but rather a very useful consequence of them that allows to take infima with idempotent elements. We make this precise below.

Recall that a Cu-semigroup S is said to be *countably based* if all elements can be written as suprema of elements from a fixed countable subset. It is known that $\mathrm{Cu}(A)$ is countably based whenever A is separable; see [APS11, Lemma 1.3].

Theorem 2.2 ([APRT21, Theorems 2.4, 2.5]). *Let S be a countably based* Cu*-semigroup satisfying (O5)-(O7). Then each $x \in S$ and each idempotent element $w \in S$ have an infimum $x \wedge w$ in S. Further, the map $S \to S$ given by $x \mapsto x \wedge w$ is a monoid homomorphism preserving the order and the suprema of increasing sequences.*

Let S be a Cu-semigroup. A map $\lambda \colon S \to [0, \infty]$ is called a *functional* if λ preserves addition, the zero element, order and suprema of increasing sequences. We denote by $F(S)$ the set of functionals on S. This is a cone when endowed with the operations of pointwise addition and pointwise scalar multiplication by positive real numbers. When equipped with the appropriate topology, $F(S)$ becomes a compact Hausdorff cone (see [ERS11, Theorems 4.4, 4.8], [Rob13, Section 2.2]; see also [Kei17, Theorem 3.17]).

Given $x \in S$, we obtain a map $\widehat{x} \colon F(S) \to [0, \infty]$ defined by $\widehat{x}(\lambda) = \lambda(x)$, for $\lambda \in F(S)$. The map $\widehat{x}$ is linear and lower semicontinuous. Denote by $\mathrm{Lsc}(F(S))$ the set of all linear, lower semicontinuous maps on $F(S)$ with values in $[0, \infty]$.

Two main questions arise in this setting. The first concerns the extent to which the ordering in A can be inferred from the ordering in $\mathrm{Lsc}(F(S))$. Note that whilst $x \mapsto \widehat{x}$ is an additive map that preserves the order and suprema of increasing sequences, it typically does not preserve the way-below relation and, moreover, is generally not an order embedding. Still, the result below will be useful.

Given elements x and y in a partially ordered semigroup, one writes $x <_s y$ if there exists $k \in \mathbb{N}$ such that $(k+1)x \leq ky$.

Proposition 2.3 ([Rob13, Lemma 2.2.5], [APRT24, Lemma 5.3]). *Let S be a* Cu*-semigroup satisfying (O5), and let $x, y \in S$. Then:*

(i) *If $x \ll y$, and $s, t \in (0, \infty]$ satisfy $s < t$, we have $s\widehat{x} \ll t\widehat{y}$ in* $\mathrm{Lsc}(F(S))$.
(ii) *If $\widehat{x} \leq \gamma\widehat{y}$ for some $\gamma \in (0,1)$, then $x' <_s y$ for all $x' \in S$ with $x' \ll x$.*

The second question that arises consists of determining the range of the map $x \mapsto \widehat{x}$. This is referred to as the *Rank Problem* and was explored in detail in

[Thi20] and [APRT22]. For a Cu-semigroup S, let us denote by $\mathrm{L}(F(S))$ the smallest subsemigroup of $\mathrm{Lsc}(F(S))$ closed under suprema of increasing sequences and containing all elements of the form $\frac{1}{n}\widehat{x}$ (see [Rob13, Section 3]). Recall that an element $x \in S$ is termed *soft* if for every $x' \in S$ with $x' \ll x$ there exists $t \in S$ such that $x' + t \ll x$ and $x' \ll \infty t$ (see [TV24b, Definition 4.3, Remark 4.4]; for C^*-algebras of stable rank one, this agrees with [APT18, Definition 5.3.1]).

The following theorem summarizes the realization results from [APRT22], which will be required in Section 4. The existence of the map α is shown in [APRT22, Theorem 7.2], and the properties of α are proved in [APRT22, Proposition 7.4, Theorem 7.13, Proposition 8.2, Theorem 8.4, Corollary 8.5].

Theorem 2.4 ([APRT22])**.** *Let A be a separable, nowhere scattered C^*-algebra with stable rank one. Then there is a natural, additive, order-preserving map $\alpha\colon L(F(\mathrm{Cu}(A))) \to \mathrm{Cu}(A)$ that preserves suprema of increasing sequences and such that $f = \widehat{\alpha(f)}$ for every $f \in L(F(\mathrm{Cu}(A)))$.*

The image of α consists of soft elements, and we have

$$\alpha(f + \widehat{x}) = \alpha(f) + x$$

whenever $x \in \mathrm{Cu}(A)$ and $f \in L(F(\mathrm{Cu}(A)))$ satisfy $\widehat{x} \leq \infty f$.

3. Variants of comparison

In this section we deal with various comparison properties. In particular, we recall the definition of *m-comparison* (first considered in [TW09, Lemma 6.1] and later formalized in [OPR12, Definition 2.8]) and clarify the equivalence between 0-comparison (also known as almost unperforation) with the notion of strict comparison for C^*-algebras. We also recall the concept introduced in [APRT24, Definition 6.1], which in this paper is renamed as *controlled comparison.*

Definition 3.1. Let $m \in \mathbb{N}$. A Cu-semigroup S has *m-comparison* if for all $x, y_0, \ldots, y_m \in S$ satisfying $x <_s y_j$ for $j = 0, \ldots, m$, we have $x \leq y_0 + \ldots + y_m$.

We say that a C^*-algebra has *m-comparison* if its Cuntz semigroup does. A Cu-semigroup or C^*-algebra is said to be *almost unperforated* if it has 0-comparison.

Remarks 3.2. (1) Robert showed in [Rob11, Theorem 1.3] that a C^*-algebra has m-comparison whenever it has nuclear dimension at most m.

(2) Let $m \in \mathbb{N}$ and let S be a Cu-semigroup that has m-comparison. Then, $x <_s y$ implies $x \leq (m+1)y$, for all $x, y \in S$.

The notion of strict comparison was first considered in [Bla88]. For Cu-semigroups, the concept was considered in [ERS11, Section 6].

Definition 3.3. A Cu-semigroup S has *strict comparison* if for all $x, y \in S$ satisfying $x \leq \infty y$ and $\lambda(x) < \lambda(y)$ for any $\lambda \in F(S)$ with $\lambda(y) = 1$, one has $x \leq y$.

A C^*-algebra A has *strict comparison* of positive elements by quasitraces (specifically, $[0, \infty]$-valued, lower semicontinuous 2-quasitraces) if its Cuntz semigroup does.

3.4 (Strict comparison and almost unperforation)**.** We take the opportunity here to clarify the equivalence between the concepts of strict comparison and almost unperforation. Let S be a Cu-semigroup, and let $x, y \in S$. The following conditions relating the comparison of x, y and $\widehat{x}, \widehat{y}$ were considered in [APT18]:

(i) We have $x <_s y$, that is, there exists $k \in \mathbb{N}$ such that $(k+1)x \leq ky$.
(ii) We have $\widehat{x} <_s \widehat{y}$, that is, there exists $\varepsilon \in (0,1)$ such that $\widehat{x} \leq (1-\varepsilon)\widehat{y}$.
(iii) We have $x \leq \infty y$ and $\lambda(x) < \lambda(y)$ for any $\lambda \in F(S)$ with $\lambda(y) = 1$.
(iv) We have $x' <_s y$ for every $x' \in S$ with $x' \ll x$.

It was shown in [APT18, Theorem 5.2.18] that the implications (i)⇒(ii)⇒(iii)⇒(iv) always hold. Using this, it follows that the following conditions are equivalent for S. (This was first obtained in [ERS11, Proposition 6.2]; see also [APT18, Proposition 5.2.20].)

(i') S is almost unperforated.
(ii') If $x, y \in S$ satisfy $\widehat{x} \leq (1-\varepsilon)\widehat{y}$ for some $\varepsilon \in (0,1)$, then $x \leq y$.
(iii') S has strict comparison.

Now, let A be a C^*-algebra. There is a natural correspondence between functionals on $\mathrm{Cu}(A)$ and quasitraces on A. This is defined by sending a quasitrace $\tau\colon (A \otimes \mathcal{K})_+ \to [0,\infty]$ to the functional $d_\tau\colon \mathrm{Cu}(A) \to [0,\infty]$ given by

$$d_\tau([a]) = \lim_{n\to\infty} \tau(a^{1/n})$$

for $a \in (A \otimes \mathcal{K})_+$; see, for example, [ERS11, Proposition 4.2].

Using this, it follows that A has strict comparison if and only if for all $a, b \in (A \otimes \mathcal{K})_+$ such that a belongs to the closed ideal generated by b, and such that $d_\tau([a]) < d_\tau([b])$ for every quasitrace τ with $d_\tau([b]) = 1$, we have $a \precsim b$.

It also follows from the above considerations that a C^*-algebra A has strict comparison (of positive elements by quasitraces) if and only if it is almost unperforated.

Some of the confusion surrounding the notion of strict comparison arises since it is applied in a more restrictive setting, for example by considering comparison of projections (instead of positive elements), or by considering comparison by traces (instead of quasitraces); see [NR16, Section 3] for some clarifications.

Another source of confusion is that strict comparison is often used in the context of simple C^*-algebras, in which case every element is automatically contained in the closed ideal generated by a nonzero element. Therefore, a simple, unital C^*-algebra A has strict comparison if and only if for all nonzero elements $a, b \in (A\otimes\mathcal{K})_+$ such that $d_\tau([a]) < d_\tau([b])$ for every normalized quasitrace τ, we have $a \precsim b$.

For the definition below, we need to recall the notion of a scale in a Cu-semigroup.

3.5 (Scales)**.** Let S be a Cu-semigroup. Following [APT20c, Definition 4.1], a *scale* in S is a subset $\Sigma \subseteq S$ satisfying the following conditions:

(i) Σ is downward hereditary: If $x \leq y$ for some $x \in S$ and $y \in \Sigma$, then $x \in \Sigma$.
(ii) Σ is closed under suprema of increasing sequences.
(iii) Σ generates S as an ideal, that is, for $x', x \in S$ with $x' \ll x$, there exist $y_1, \ldots, y_n \in \Sigma$ such that $x' \leq y_1 + \ldots + y_n$.

We say that the pair (S, Σ) is a *scaled* Cu-*semigroup*.

Given a scale Σ and $d \in \mathbb{N}$, the *d-fold amplification* of Σ is defined as $\Sigma^{(0)} := \{0\}$, and for $d \geq 1$ as

$$\Sigma^{(d)} := \left\{ x \in S : \begin{array}{l} \text{for each } x' \in S \text{ with } x' \ll x \text{ there exist} \\ y_1, \ldots, y_d \in \Sigma \text{ such that } x' \leq y_1 + \ldots + y_d \end{array} \right\}.$$

If A is a C^*-algebra, then

$$\Sigma_A := \big\{ x \in \mathrm{Cu}(A) : x \leq [a] \text{ for some } a \in A_+ \big\}$$

is a scale in $\mathrm{Cu}(A)$, and the pair $(\mathrm{Cu}(A), \Sigma_A)$ is called the *scaled Cuntz semigroup* of A; see [APT20c, Paragraph 4.2] and [TV23, Lemma 3.3], and also [APRT24, Paragraph 4.5]. By [APRT24, Proposition 7.4], for $d \geq 1$ we have

$$\Sigma_A^{(d)} = \big\{ x \in \mathrm{Cu}(A) : x \leq [a] \text{ for some } a \in M_d(A)_+ \big\}.$$

Given $N \in \mathbb{N}$ and elements x and y in a partially ordered semigroup, we write $x \leq_N y$ if $nx \leq ny$ for all $n \in \mathbb{N}$ with $n \geq N$. The following was termed 'locally bounded comparison amplitude' in [APRT24, Definition 6.1].

Definition 3.6. A scaled Cu-semigroup (S, Σ) has *controlled comparison* if for every $\gamma \in (0,1)$ and $d \in \mathbb{N}$ there exists $N = N(\gamma, d)$ such that the following holds:

$$\widehat{x} \leq \gamma \widehat{y} \quad \text{implies} \quad x \leq_N y, \quad \text{for all } x, y \in \Sigma^{(d)}.$$

We say that a C^*-algebra A has controlled comparison if $(\mathrm{Cu}(A), \Sigma_A)$ does.

The next result follows easily from [APRT24, Proposition 10.3].

Proposition 3.7. *If a C^*-algebra has m-comparison for some $m \in \mathbb{N}$, then it has controlled comparison.*

Proof. In order to apply [APRT24, Proposition 10.3] it is required that the Cuntz semigroup of a C^*-algebra satisfies (O5), (O6), and the so-called Edward's condition. The first two are always satisfied, as mentioned in the preliminaries, and the third condition was shown to hold in [APRT21, Theorem 5.3]. □

Corollary 3.8. *Every C^*-algebra with finite nuclear dimension has controlled comparison.*

Proof. Let A be a C^*-algebra with nuclear dimension at most m for some $m \in \mathbb{N}$. Then A has m-comparison by [Rob11, Theorem 1.3], hence the result follows from Proposition 3.7. □

4. Variants of divisibility

In this section we introduce the concept of *functional divisibility* and show it is a natural generalization of n-almost divisibility (as introduced in [Win12, Definition 3.5] for simple, unital C^*-algebras, and later refined in [RT17, Section 2.3] for general C^*-algebras). We prove that functional divisibility is automatic for nowhere scattered C^*-algebras of stable rank one, and also for simple, unital, non-elementary C^*-algebras with a unique normalized quasitrace.

Definition 4.1. Let $n \in \mathbb{N}$. A Cu-semigroup S is *n-almost divisible* if for every $x' \ll x$ in S and every $k \in \mathbb{N}$ with $k \geq 1$, there exists $y \in S$ such that

$$ky \leq x, \quad \text{and} \quad x' \leq (k+1)(n+1)y.$$

We say that a C^*-algebra is *n-almost divisible* if its Cuntz semigroup is. A Cu-semigroup or C^*-algebra is said to be *almost divisible* if it is 0-almost divisible. In other words, if for all $x' \ll x$ in S and $k \in \mathbb{N}$ with $k \geq 1$ there exists $y \in S$ such that $ky \leq x$, and $x' \leq (k+1)y$.

4.2 (Connections with earlier divisibility notions)**.** In parts of the literature, 'almost divisible' is defined as the slightly stronger condition that for every $x \in S$ and $k \in \mathbb{N}$ with $k \geq 1$ there exists $y \in S$ such that

$$ky \leq x \leq (k+1)y.$$

This version of 'almost divisibility' implies the one from Definition 4.1, and for almost unperforated Cu-semigroups both versions agree by [APT18, Proposition 7.3.7]. The advantage of the version from Definition 4.1 is that it enjoys better permanence properties; see [APT18, Remark 7.3.5].

Almost divisibility is automatic in certain cases. For example, it was shown in [TV24a, Theorem 9.1] that every nowhere scattered C^*-algebra A of real rank zero is almost divisible. In fact, $\mathrm{Cu}(A)$ is even 'weakly divisible' in the sense that for every $x \in \mathrm{Cu}(A)$, there are $y, z \in \mathrm{Cu}(A)$ such that $x = 2y + 3z$; see also [APGPSM10, Section 6]. In the simple case, and for the Murray-von Neumann semigroup, this was obtained in [PR04, Proposition 5.3].

Inspired by the results in [RT17, Section 6] and also in [Win12], we introduce the following more general divisibility condition in terms of comparison by functionals.

Definition 4.3. We say that a Cu-semigroup S is *functionally divisible* if the following conditions are satisfied:

(L) For every $x' \ll x$ in S, every $k \in \mathbb{N}$ with $k \geq 1$, and every $\varepsilon \in (0,1)$, there exists $y \in S$ such that
$$ky \leq x, \quad \text{and} \quad (1-\varepsilon)\widehat{x'} \leq k\widehat{y}.$$

(U) For every $x' \ll x$ in S, every $k \in \mathbb{N}$ with $k \geq 1$, and every $\varepsilon \in (0,1)$, there exists $z \in S$ such that
$$z \leq x, \quad (1-\varepsilon)k\widehat{z} \leq \widehat{x}, \quad \text{and} \quad x' \leq kz.$$

We say that a C^*-algebra A is *functionally divisible* if $\mathrm{Cu}(A)$ is.

Remark 4.4. Let S be a Cu-semigroup satisfying (O5), and let $x \in S$. While Definition 4.1 captures notions of divisibility of x in S, one may also consider the divisibility of $\widehat{x}$ in $\mathrm{Lsc}(F(S))$. The straightforward concept would be to require that for every $x \in S$ and every $k \in \mathbb{N}$ with $k \geq 1$, there exists $y \in S$ such that $k\widehat{y} = \widehat{x}$. Taking also the way-below relation into account in order to obtain better permanence properties, one is led to the following notion:

(D) For every $x \in S$, every $f \in \mathrm{Lsc}(F(S))$ with $f \ll \widehat{x}$, and every $k \in \mathbb{N}$ with $k \geq 1$, there exists $y \in S$ such that
$$f \leq k\widehat{y} \leq \widehat{x}.$$

Using Proposition 2.3(i), we see that a function $f \in \mathrm{Lsc}(F(S))$ satisfies $f \ll \widehat{x}$ if and only if there exists $x' \in S$ with $x' \ll x$ and $\varepsilon > 0$ such that $f \leq (1-\varepsilon)\widehat{x'}$. Thus, condition (D) holds if and only if for every $x' \ll x$ in S, every $k \in \mathbb{N}$ with $k \geq 1$, and every $\varepsilon > 0$, there exists $y \in S$ such that
$$(1-\varepsilon)\widehat{x'} \leq k\widehat{y} \leq \widehat{x}.$$

Conditions (L) and (U) in Definition 4.3 are natural strengthenings of this, where one of the inequalities on functionals is upgraded to an inequality in S. The 'L' stands for 'lower', since in this condition the element ky is smaller than x, while the 'U' stands for 'upper' since here ky is larger than x'.

The next result shows that it suffices to consider $k = 2$ in order to verify condition (L) in Definition 4.3.

Proposition 4.5. *Let S be a* Cu*-semigroup satisfying (O5). Then the following statements are equivalent:*

(i) *S satisfies condition (L) in Definition 4.3.*

(ii) *For every $x' \ll x$ in S and $k \in \mathbb{N}$, there exists $y \in S$ such that*
$$ky \leq x, \quad \textit{and} \quad \widehat{x'} \leq (k+1)\,\widehat{y}.$$

(iii) *For every $x' \ll x$ in S and every $\varepsilon \in (0,1)$ there exists $y \in S$ such that*
$$2y \leq x, \quad \textit{and} \quad (1-\varepsilon)\widehat{x'} \leq 2\widehat{y}.$$

Proof. Assuming (i), let us verify (ii). For $k = 0$, we use $y = x$. For $k \geq 1$, we apply (i) with $\varepsilon > 0$ small enough such that $(1-\varepsilon)^{-1}k \leq k+1$. Similarly, to see that (ii) implies (iii), for a given $\varepsilon \in (0,1)$ apply (ii) with $2k$ where k is large enough such that $1 - \varepsilon \leq (1 + \frac{1}{2k})^{-1}$.

Next, assuming that (iii) holds, let us verify (i). First, by induction over m, we verify that the following statement holds:

(L_{2^m}) For every $x' \ll x$ in S and $\varepsilon \in (0,1)$ there exists $y \in S$ such that
$$2^m y \leq x, \quad \text{and} \quad (1-\varepsilon)\widehat{x'} \leq 2^m\widehat{y}.$$

The case $m=1$ holds by assumption. Assume that (L_{2^m}) holds for some $m\geq 1$. To verify $(\mathrm{L}_{2^{m+1}})$, let $x'\ll x$ in S and $\varepsilon\in(0,1)$. Pick $x''\in S$ and $\delta>0$ such that

$$x'\ll x''\ll x, \quad \text{and} \quad (1-\delta)^3=1-\varepsilon.$$

Applying (L_{2^m}) for $x''\ll x$ and δ, we obtain $w\in S$ such that

$$2^m w\leq x, \quad \text{and} \quad (1-\delta)\widehat{x''}\leq 2^m\widehat{w}.$$

By Proposition 2.3(i), we have $(1-\delta)^2\widehat{x'}\ll(1-\delta)\widehat{x''}$ in $\mathrm{Lsc}(F(S))$, hence also $(1-\delta)^2\widehat{x'}\ll 2^m\widehat{w}$. This allows us to choose $w'\in S$ such that

$$w'\ll w, \quad \text{and} \quad (1-\delta)^2\widehat{x'}\leq 2^m\widehat{w'}.$$

Next, applying (iii) for $w'\ll w$ and δ, we obtain $y\in S$ such that

$$2y\leq w, \quad \text{and} \quad (1-\delta)\widehat{w'}\leq 2\widehat{y}.$$

Then

$$2^{m+1}y\leq 2^m w\leq x, \quad \text{and} \quad (1-\varepsilon)\widehat{x'}=(1-\delta)^3\widehat{x'}\leq(1-\delta)2^m\widehat{w'}\leq 2^{m+1}\widehat{y},$$

which verifies $(\mathrm{L}_{2^{m+1}})$.

Now, to prove (i), let $x'\ll x$ in S, let $k\in\mathbb{N}$ with $k\geq 1$, and let $\varepsilon\in(0,1)$. Pick $x''\in S$ and $\delta>0$ such that

$$x'\ll x''\ll x, \quad \text{and} \quad (1-\delta)^2=1-\varepsilon.$$

Using that dyadic rationals are dense in $\mathbb{R}$, choose $n,m\geq 1$ such that

$$(1-\delta)\frac{1}{k}\leq\frac{n}{2^m}\leq\frac{1}{k}.$$

Then

$$(1-\delta)2^m\leq kn\leq 2^m.$$

Applying (L_{2^m}) for $x''\ll x$ and δ, we obtain $w\in S$ such that

$$2^m w\leq x, \quad \text{and} \quad (1-\delta)\widehat{x'}\leq 2^m\widehat{w}.$$

Set $y:=nw$. Then

$$ky=k(nw)\leq 2^m w\leq x, \quad \text{and} \quad (1-\varepsilon)\widehat{x'}=(1-\delta)^2\widehat{x'}\leq(1-\delta)2^m\widehat{w}\leq kn\widehat{w}=k\widehat{y},$$

which shows that y has the desired properties. □

Remark 4.6. Let S be a Cu-semigroup satisfying (O5). It is not clear if there exists a characterization of condition (U) in Definition 4.3 analogous to Proposition 4.5. The reason is that condition (U) has the additional assumption that the dividing element is dominated by x, which is automatic in condition (L).

On the other hand, for $m\geq 1$ let us consider the following condition:

(U_{2^m}) For every $x'\ll x$ in S and every $\varepsilon\in(0,1)$, there exists $y\in S$ such that

$$y\leq x, \quad (1-\varepsilon)2^m\widehat{y}\leq\widehat{x}, \quad \text{and} \quad x'\leq 2^m y.$$

An argument as in the proof of Proposition 4.5 shows that if S satisfies (U_2), then it satisfies (U_{2^m}) for all m. (See also Remark 5.6.)

To show that n-almost divisible Cuntz semigroups are functionally divisible (Proposition 4.9), we first need some preparatory technical results. We will use the concept of $(2,\omega)$-divisibility (see also Section 6). Given $k\in\mathbb{N}$ with $k\geq 2$, a Cu-semigroup S is said to be (k,ω)*-divisible* if for all $x'\ll x$ in S there exists $y\in S$ such that

$$ky\leq x, \quad \text{and} \quad x'\leq\infty y.$$

If S is (k,ω)-divisible for some $k \geq 2$, then it is $(2,\omega)$-divisible. The converse also holds by [TV23, Lemma 3.4]. Thus, a Cu-semigroup is $(2,\omega)$-divisible if and only if it is (k,ω)-divisible for all $k \geq 2$.

Condition (L') in the result below was considered in [RT17, Theorem 6.1].

Lemma 4.7. *Let S be a* Cu*-semigroup satisfying (O5)-(O7). Then S satisfies condition (L) in Definition 4.3 if and only if S is $(2,\omega)$-divisible and satisfies*

(L') For every $x' \ll x$ in S, every $k \in \mathbb{N}$ with $k \geq 1$, and every $\varepsilon > 0$, there exists $y \in S$ such that

$$ky \leq x, \quad \textit{and} \quad \widehat{x'} \leq k\widehat{y} + \varepsilon\widehat{x}.$$

Proof. To show the forward implication, assume that S satisfies

(L) For every $x' \ll x$ in S, every $k \in \mathbb{N} \setminus \{0\}$, and every $\varepsilon \in (0,1)$, there exists $y \in S$ such that

$$ky \leq x, \quad \text{and} \quad (1-\varepsilon)\widehat{x'} \leq k\widehat{y}.$$

To verify that S is $(2,\omega)$-divisible, let $x' \ll x$ in S. Applying (L) with the given x', x, and also $k = 2$ and $\varepsilon = \frac{1}{2}$, we obtain $y \in S$ such that

$$2y \leq x, \quad \text{and} \quad \frac{1}{2}\widehat{x'} \leq 2\widehat{y}.$$

The latter condition implies that $\widehat{x'} \leq 4\widehat{y}$, and hence $x' \leq \infty y$ (using, for example, [APRT22, Lemma 6.6]). This shows that S is $(2,\omega)$-divisible.

To verify (L'), let $x' \ll x$ in S, let $k \in \mathbb{N}$ with $k \geq 1$, and $\varepsilon > 0$. Pick $\delta \in (0,1)$ such that $(1-\delta)^{-1} = 1 + \varepsilon$. Applying (L) with the given x', x and k, and also δ, we obtain $y \in S$ such that

$$ky \leq x, \quad \text{and} \quad (1-\delta)\widehat{x'} \leq k\widehat{y}.$$

Using that $ky \leq x$, we get

$$\widehat{x'} \leq (1-\delta)^{-1}k\widehat{y} = (1+\varepsilon)k\widehat{y} \leq k\widehat{y} + \varepsilon\widehat{x},$$

as desired.

To show the backward implication, assume that S is $(2,\omega)$-divisible and satisfies (L'). We verify condition (ii) of Proposition 4.5. Let $x' \ll x$ in S, and let $k \in \mathbb{N}$. We need to find $y \in S$ such that

$$ky \leq x, \quad \text{and} \quad \widehat{x'} \leq (k+1)\widehat{y}.$$

For $k = 0$ and $k = 1$ use $y := x$. Thus, from now on we may assume that $k \geq 2$.

Using the methods from [TV21, Section 5], we find a countably based, $(2,\omega)$-divisible sub-Cu-semigroup $H \subseteq S$ that satisfies (O5)-(O7) and contains x', x.

Choose $x'' \in H$ and $n \in \mathbb{N}$ with $n \geq 1$ such that

$$x' \ll x'' \ll x, \quad k+1 \leq n, \quad \text{and} \quad \frac{k(k+1)+n}{n} \leq 1 + \frac{1}{k}.$$

Since H is $(2,\omega)$-divisible, it is also (n,ω)-divisible by the arguments in [TV23, Lemma 3.4]. Applied to $x'' \ll x$ and n, we obtain $c \in H$ such that

$$nc \leq x, \quad \text{and} \quad x'' \ll \infty c.$$

Using that $x'' \ll \infty c$, we can choose $c' \in H$ such that

$$x'' \ll \infty c', \quad \text{and} \quad c' \ll c.$$

We have $(k+1)c \leq nc \leq x$. Using Lemma 2.1, we obtain $d \in H$ such that

$$kc' + d \leq x \leq kc + d, \quad \text{and} \quad x \leq (k+1)d.$$

Thus, one has

$$nx'' \leq nx \leq nkc + nd \leq kx + nd \leq (k(k+1)+n)d.$$

Since H is countably based and satisfies (O5)-(O7), we may apply Theorem 2.2. Hence, for every $s \in H$ the infimum $s \wedge \infty c'$ exists (in H), and the map $s \mapsto s \wedge \infty c'$ is additive. Set $e := d \wedge \infty c'$. Since $x'' \ll \infty c'$, we have

$$\begin{aligned} nx' \ll nx'' = (nx'') \wedge (\infty c') &\leq (k(k+1)+n)d \wedge (\infty c') \\ &= \big(k(k+1)+n\big)\big(d \wedge (\infty c')\big) = (k(k+1)+n)e. \end{aligned}$$

This allows us to find $e' \in H$ satisfying

$$nx' \leq (k(k+1)+n)e', \quad \text{and} \quad e' \ll e.$$

Let us further choose $e'' \in H$ such that $e' \ll e'' \ll e$.

Since $e \leq \infty c'$ and $e' \ll e$, we may pick $N \in \mathbb{N}$ with $N \geq 1$ such that $e'' \leq Nc'$. Then choose $\varepsilon > 0$ small enough such that

$$\left(1+\frac{1}{k}\right)\varepsilon \leq \frac{1}{N}.$$

Now, working again in S, and applying (L') to $e' \ll e''$, k and ε, we obtain $f \in S$ such that

$$kf \leq e'', \quad \text{and} \quad \widehat{e'} \leq k\widehat{f} + \varepsilon\widehat{e''}.$$

Set $y := c' + f$. We have $kf \leq e \leq d$ and therefore

$$ky = kc' + kf \leq kc' + d \leq x.$$

Further, we have

$$\begin{aligned} \widehat{x'} &\leq \frac{k(k+1)+n}{n}\widehat{e'} \leq \left(1+\frac{1}{k}\right)\widehat{e'} \leq \left(1+\frac{1}{k}\right)\left(k\widehat{f} + \varepsilon\widehat{e''}\right) \\ &\leq (k+1)\widehat{f} + \frac{1}{N}\widehat{e''} \leq (k+1)\widehat{f} + \widehat{c'} \leq (k+1)\left(\widehat{f} + \widehat{c'}\right) = (k+1)\widehat{y}, \end{aligned}$$

as desired. □

The next result shows that if a *compact* element satisfies condition (L') in Lemma 4.7, then it also satisfies condition (U) in Definition 4.3.

Lemma 4.8. *Let S be a* Cu*-semigroup satisfying (O5), and let $x \in S$ be a compact element such that for every $k \in \mathbb{N}$ with $k \geq 1$, and every $\varepsilon > 0$, there exists $y \in S$ such that*

$$ky \leq x, \quad \text{and} \quad \widehat{x} \leq k\widehat{y} + \varepsilon\widehat{x}.$$

Then, for every $k \in \mathbb{N}$ with $k \geq 1$, and every $\delta > 0$, there exists $z \in S$ such that

$$z \leq x, \quad (1-\delta)k\widehat{z} \leq \widehat{x}, \quad \text{and} \quad x \leq kz.$$

Proof. Given $k \in \mathbb{N}$ with $k \geq 1$ and $\delta > 0$, pick $\varepsilon > 0$ such that

$$1 + 2(k-1)\varepsilon \leq (1-\delta)^{-1}.$$

Applying the assumption for k and ε, we obtain $y \in S$ such that

$$ky \leq x, \quad \text{and} \quad \widehat{x} \leq k\widehat{y} + \varepsilon\widehat{x}.$$

Now, using that $x \ll x$, it follows from Proposition 2.3(i) that

$$(1-\varepsilon)\widehat{x} \ll \widehat{x} \leq k\widehat{y} + \varepsilon\widehat{x}.$$

This allows us to choose $y' \in S$ such that $y' \ll y$ and

$$(1-\varepsilon)\widehat{x} \leq k\widehat{y'} + \varepsilon\widehat{x}.$$

Applying Lemma 2.1 for $ky \leq x$ and $y' \ll y$, we get $z \in S$ such that

$$(k-1)y' + z \leq x \leq kz.$$

We now show that z has the claimed properties. It is clear that $z \leq x$ and $x \leq kz$. It remains to verify that $(1-\delta)k\lambda(z) \leq \lambda(x)$ for every $\lambda \in F(S)$. This is clear if $\lambda(x) = \infty$. Let $\lambda \in F(S)$ with $\lambda(x) < \infty$. From the fact that $(1-\varepsilon)\widehat{x} \leq k\widehat{y'} + \varepsilon\widehat{x}$, as shown above, we deduce that

$$(1-2\varepsilon)\lambda(x) \leq k\lambda(y').$$

Since $ky \leq x$, it follows that $\lambda(y') < \infty$ and, consequently, using our choice of ε in the last step,

$$\begin{aligned} k\lambda(z) &\leq k\lambda(x) - k(k-1)\lambda(y') \leq k\lambda(x) - (1-2\varepsilon)(k-1)\lambda(x) \\ &= (1+2(k-1)\varepsilon)\lambda(x) \leq (1-\delta)^{-1}\lambda(x), \end{aligned}$$

as desired. □

Proposition 4.9. *If a C^*-algebra is n-almost divisible for some $n \in \mathbb{N}$, then it is functionally divisible.*

Proof. Let $n \in \mathbb{N}$, and let A be a C^*-algebra that is n-almost divisible. It is easy to deduce that $\mathrm{Cu}(A)$ is $(2,\omega)$-divisible. Therefore, by Lemma 4.7, to show that $\mathrm{Cu}(A)$ is functionally divisible it suffices to verify condition (L') in Lemma 4.7 and condition (U) in Definition 4.3.

Let $x' \ll x$ in $\mathrm{Cu}(A)$, let $k \in \mathbb{N}$ with $k \geq 1$, and let $\varepsilon > 0$. We may assume that $\varepsilon = \frac{1}{N}$ for some integer $N \geq 1$, and we need to find $y \in \mathrm{Cu}(A)$ such that

$$ky \leq x, \quad \text{and} \quad N\widehat{x'} \leq Nk\widehat{y} + \widehat{x},$$

and $z \in \mathrm{Cu}(A)$ such that

$$z \leq x, \quad (N-1)k\widehat{z} \leq N\widehat{x}, \quad \text{and} \quad x' \leq kz.$$

Without loss of generality, we may assume that A is stable, which allows us to pick $a \in A_+$ with $x = [a]$.

Let $\mathcal{U}$ be a free ultrafilter on $\mathbb{N}$. Denote by $A_{\mathcal{U}}$ the free ultrapower of A. Set $C = \{a\}' \cap A_{\mathcal{U}}$ and $I = \{a\}^{\perp} \cap A_{\mathcal{U}}$. Then, it follows from [RT17, Theorem 6.1] that the class of the unit [1] in $\mathrm{Cu}(C/I)$ satisfies condition (L') in Lemma 4.7, that is, for every $l \in \mathbb{N}$ with $l \geq 1$, and every $\delta > 0$, there exists $v \in \mathrm{Cu}(C/I)$ such that

$$lv \leq [1], \quad \text{and} \quad \widehat{[1]} \leq l\widehat{v} + \delta\widehat{[1]}.$$

Applied for $l = k$ and $\delta = \frac{1}{2N}$, we obtain $v \in \mathrm{Cu}(C/I)$ such that

$$kv \leq [1], \quad \text{and} \quad \widehat{[1]} \leq k\widehat{v} + \frac{1}{2N}\widehat{[1]}.$$

Adding $\frac{1}{2N}\widehat{[1]}$ and multiplying by N, we get

$$(1+\frac{1}{2N})N\widehat{[1]} \leq Nk\widehat{v} + \widehat{[1]}.$$

Now we apply Proposition 2.3(ii) to find a positive integer s such that

$$sN[1] \leq (s+1)N[1] \leq s(Nkv + [1]).$$

Since $sN[1]$ is compact, we can pick $v' \in \mathrm{Cu}(C/I)$ such that

$$v' \ll v, \quad \text{and} \quad sN[1] \leq s(Nkv' + [1]).$$

Using also that $v \leq [1]$ (since $kv \leq [1]$), we can apply [RR13, Lemma 2.3 (ii)] to find $e \in (C/I)_+$ such that $v' \leq [e] \leq v$, and so

$$k[e] \leq [1], \quad \text{and} \quad sN[1] \leq s(Nk[e] + [1]).$$

Separately, and using again that $[1] \in \mathrm{Cu}(C/I)$ satisfies condition (L'), we apply Lemma 4.8 to the compact element $[1]$ in $\mathrm{Cu}(C/I)$ and for the given k and $\frac{1}{2N}$ to obtain $w \in \mathrm{Cu}(C/I)$ such that

$$w \leq [1], \quad (1 - \tfrac{1}{2N})k\widehat{w} \leq \widehat{[1]}, \quad \text{and} \quad [1] \leq kw.$$

Multiplying by N, we get

$$(N - \tfrac{1}{2})k\widehat{w} \leq N\widehat{[1]}.$$

Using that $[1]$ is compact and $[1] \leq kw$, we find $w'', w' \in \mathrm{Cu}(C/I)$ such that

$$w'' \ll w' \ll w, \quad \text{and} \quad [1] \leq kw''.$$

First, applying Proposition 2.3(ii), we find a positive integer t such that

$$t(N-1)kw' \leq (t+1)(N-1)kw' \leq tN[1].$$

Then, arguing as above, we find $f \in (C/I)_+$ with $w'' \leq [f] \leq w'$, and thus

$$[f] \leq [1], \quad t(N-1)k[f] \leq tN[1], \quad \text{and} \quad [1] \leq k[f].$$

Let $\bar{e}, \bar{f} \in C_+$ be lifts of e, f respectively. Then, there exists $z \in \mathrm{Cu}(I)$ such that

$$k[\bar{e}] \leq [1] + z, \quad sN[1] \leq s(Nk[\bar{e}] + [1]) + z,$$

and

$$[\bar{f}] \leq [1] + z, \quad t(N-1)k[\bar{f}] \leq tN[1] + z, \quad [1] \leq k[\bar{f}] + z$$

in $\mathrm{Cu}(C)$.

Set $b = a\bar{e}$ and $c = a\bar{f}$ in $A_{\mathcal{U}}$. Using that every element in C commutes with a (in particular, those elements applying the Cuntz subequivalences above) and that every element in I is orthogonal to a, one obtains

$$k[b] \leq [a], \quad sN[a] \leq s(Nk[b] + [a]),$$

and

$$[c] \leq [a], \quad t(N-1)k[c] \leq tN[a], \quad [a] \leq k[c]$$

in $\mathrm{Cu}(A_{\mathcal{U}})$.

A finite collection of Cuntz subequivalences in $A_{\mathcal{U}}$ can be 'lifted' to A simultaneously up to a cut-down (see, for example, the proof of [APRT24, Theorem 10.5]). Thus, using that $x' \ll x = [a]$, we find $y, z \in \mathrm{Cu}(A)$ such that

$$ky \leq x, \quad sNx' \leq s(Nky + x),$$

and

$$z \leq x, \quad t(N-1)kz \leq tNx, \quad x' \leq kz$$

in $\mathrm{Cu}(A)$. Passing to functionals, we can remove the multiplication by s and t, and we obtain $N\widehat{x'} \leq Nk\widehat{y} + \widehat{x}$ and $(N-1)k\widehat{z} \leq N\widehat{x}$, which shows that y and z have the desired properties. □

It was proved in [APRT22, Theorem 3.8] that, if A is a separable C^*-algebra of stable rank one, then $\mathrm{Cu}(A)$ is an inf-semilattice ordered semigroup. That is, for each $x, y \in \mathrm{Cu}(A)$ their greatest lower bound $x \wedge y$ exists in $\mathrm{Cu}(A)$ and addition is distributive over the meet operation:

$$(x \wedge y) + z = (x + z) \wedge (y + z)$$

for all $x, y, z \in \mathrm{Cu}(A)$. This will be used below.

Proposition 4.10. *Every nowhere scattered C^*-algebra with stable rank one is functionally divisible.*

Proof. Let us first assume that every separable, nowhere scattered C^*-algebra with stable rank one is functionally divisible, and let A be an arbitrary nowhere scattered C^*-algebra with stable rank one. To show that A is functionally divisible, let $x' \ll x$ in $\mathrm{Cu}(A)$, let $k \in \mathbb{N}$ with $k \geq 1$ and let $\varepsilon \in (0,1)$ be given. We need to find $y, z \in \mathrm{Cu}(A)$ such that

$$ky \leq x, \quad (1-\varepsilon)\widehat{x'} \leq k\widehat{y}, \quad z \leq x, \quad (1-\varepsilon)k\widehat{z} \leq \widehat{x}, \quad \text{and} \quad x' \leq kz.$$

Using that stable rank one and nowhere scatteredness satisfy the Löwenheim-Skolem condition ([TV24a, Proposition 4.11]) and applying [TV21, Proposition 6.1], we can find a separable sub-C^*-algebra $B \subseteq A$ such that B is nowhere scattered, has stable rank one, and such that the inclusion $B \to A$ induces an order-embedding $\mathrm{Cu}(B) \to \mathrm{Cu}(A)$ whose image contains x' and x. Viewing x' and x as elements in $\mathrm{Cu}(B)$, and using that B is functionally divisible, we find y and z with the desired properties in $\mathrm{Cu}(B)$, and thus in $\mathrm{Cu}(A)$. (See also [APRT22, Lemma 9.2].)

By the argument above, we may assume that A is a separable, nowhere scattered C^*-algebra A with stable rank one, and we need to show that A is functionally divisible. We first verify condition (iii) in Proposition 4.5, which then implies that $\mathrm{Cu}(A)$ satisfies condition (L).

Let $x' \ll x$ in $\mathrm{Cu}(A)$, and let $\varepsilon > 0$. We need to find $y \in \mathrm{Cu}(A)$ such that

$$2y \leq x, \quad \text{and} \quad (1-\varepsilon)\widehat{x'} \leq 2\widehat{y}.$$

Using the solution to the rank problem (see Theorem 2.4) we can find $u \in \mathrm{Cu}(A)$ such that $\widehat{u} = \frac{1}{2}\widehat{x}$. Set $v := u \wedge x$, which exists by the comments preceding this proposition. Using [APRT22, Theorem 6.12] at the second step, we have

$$\widehat{v} = \widehat{u \wedge x} = \widehat{u} \wedge \widehat{x} = \frac{1}{2}\widehat{x}.$$

Using that $\mathrm{Cu}(A)$ is $(2,\omega)$-divisible by the results in [APRT22, Section 5] (see also Theorem 7.1 and Proposition 7.3 in [TV23]), we can apply [AVTV23, Theorem 5.10] to find a soft element $w \in \mathrm{Cu}(A)$ such that

$$w \leq v, \quad \text{and} \quad \widehat{w} = \widehat{v}.$$

By Proposition 2.3(i), we have

$$(1-\varepsilon)\frac{1}{2}\widehat{x'} \ll \frac{1}{2}\widehat{x} = \widehat{w},$$

which allows us to choose $w'', w' \in \mathrm{Cu}(A)$ such that

$$w'' \ll w' \ll w, \quad \text{and} \quad (1-\varepsilon)\frac{1}{2}\widehat{x'} \leq \widehat{w''}.$$

Since w is soft, there exists by [TV24b, Proposition 4.6] some $t \in \mathrm{Cu}(A)$ such that

$$w' + t \ll w, \quad \text{and} \quad w' \ll \infty t.$$

In particular, $w' + t \leq v \leq x$.

Choose $t' \in \mathrm{Cu}(A)$ such that

$$t' \ll t, \quad \text{and} \quad w' \ll \infty t'.$$

Applying (O5) for $w' + t \leq x$ and $w'' \ll w'$ and $t' \ll t$, we obtain $c \in \mathrm{Cu}(A)$ such that

$$w'' + c \leq x \leq w' + c, \quad \text{and} \quad t' \ll c.$$

Let us show that $\frac{1}{2}\widehat{x} \leq \widehat{c}$. Since $w' \ll \infty t'$, there is $m \geq 1$ such that $w' \leq mt'$, and thus $x \leq w' + c \leq mt' + t' = (m+1)t'$. Therefore, if $\lambda \in F(\mathrm{Cu}(A))$ satisfies

$\lambda(x) = \infty$, then $\lambda(c) = \infty \geq \frac{1}{2}\lambda(x)$. On the other hand, if $\lambda \in F(\mathrm{Cu}(A))$ satisfies $\lambda(x) < \infty$, then using that $w' \leq w$ and $\widehat{w} = \frac{1}{2}\widehat{x}$, we get

$$\lambda(c) \geq \lambda(x) - \lambda(w') \geq \lambda(x) - \frac{1}{2}\lambda(x) = \frac{1}{2}\lambda(x).$$

Set $y := w'' \wedge c$. Using that $(1-\varepsilon)\frac{1}{2}\widehat{x'} \leq \widehat{w''}$ and $\frac{1}{2}\widehat{x} \leq \widehat{c}$, and using [APRT22, Theorem 6.12] at the second step, we have

$$(1-\varepsilon)\tfrac{1}{2}\widehat{x'} \leq \widehat{w''} \wedge \widehat{c} = \widehat{w'' \wedge c} = \widehat{y},$$

and thus

$$2y \leq w'' + c \leq x, \quad \text{and} \quad (1-\varepsilon)\widehat{x'} \leq 2\widehat{y},$$

as desired.

Next, to verify condition (U) from Definition 4.3, let $x' \ll x$ in $\mathrm{Cu}(A)$, let $k \in \mathbb{N}$ with $k \geq 1$, and let $\varepsilon \in (0,1)$. We need to find $z \in S$ such that

$$z \leq x, \quad (1-\varepsilon)k\widehat{z} \leq \widehat{x}, \quad \text{and} \quad x' \leq kz.$$

Set $s := (1-\varepsilon)^{-1}\frac{1}{k}$. If $s \geq 1$ (that is, if $(1-\varepsilon)k \leq 1$), then we can use $z := x$. Thus, we may assume that $s < 1$. For $f \in L(F(\mathrm{Cu}(A)))$, let $\alpha(f) \in \mathrm{Cu}(A)$ be as in Theorem 2.4, hence $\widehat{\alpha(f)} = f$. Set

$$z := x \wedge \alpha\left(s\widehat{x}\right).$$

Then $z \leq x$. Further, using that $s < 1$, and using [APRT22, Theorem 6.12], we have

$$\widehat{z} = \widehat{x} \wedge s\widehat{x} = s\widehat{x} = \frac{(1-\varepsilon)^{-1}}{k}\widehat{x}$$

and thus $(1-\varepsilon)k\widehat{z} = \widehat{x}$.

Finally, let us verify that $x' \leq kz$. In fact we will see that $x \leq kz$. Since $\mathrm{Cu}(A)$ is inf-semilattice ordered, we have

$$kz = k\big(x \wedge \alpha(s\widehat{x})\big) = \bigwedge_{i=0}^{k} \big(ix + (k-i)\alpha(s\widehat{x})\big).$$

Clearly $x \leq ix + (k-i)\alpha(s\widehat{x})$ when $i > 0$. For $i = 0$, first note that $ks - 1 > 0$ and therefore $\widehat{x} \leq \infty(ks-1)\widehat{x}$, which allows us to apply the partial additivity of α stated in Theorem 2.4. Hence, we get

$$x \leq x + \alpha\big((ks-1)\widehat{x}\big) = \alpha\big(\widehat{x} + (ks-1)\widehat{x}\big) = \alpha(ks\widehat{x}) = k\alpha(s\widehat{x}),$$

and thus $x \leq kz$, as desired. □

Proposition 4.11. *Let A be a simple, unital, non-elementary C^*-algebra with a unique normalized quasitrace. Then* $\mathrm{Cu}(A)$ *is functionally divisible.*

Proof. By assumption, there exists a unique functional $\lambda\colon \mathrm{Cu}(A) \to [0,\infty]$ satisfying $\lambda([1]) = 1$. Note that any two elements $x, y \in \mathrm{Cu}(A)$ satisfy $\widehat{x} \leq \widehat{y}$ if and only if $\lambda(x) \leq \lambda(y)$.

We first verify condition (iii) in Proposition 4.5, which then implies that $\mathrm{Cu}(A)$ satisfies condition (L) in Definition 4.3. Let $x' \ll x$ in $\mathrm{Cu}(A)$, and let $\varepsilon \in (0,1)$. We need to find $y \in \mathrm{Cu}(A)$ such that

$$2y \leq x, \quad \text{and} \quad (1-\varepsilon)\lambda(x') \leq 2\lambda(y).$$

If $x' = 0$, then we can use $y = 0$. Therefore, we may from now on assume that $x' \neq 0$. Then $\lambda(x') \in (0,\infty)$. Since A is non-elementary, we can pick $v \in \mathrm{Cu}(A)$ such that $\lambda(v) = \frac{1}{2}\lambda(x')$. We have

$$(1-\varepsilon)\frac{1}{2}\lambda(x') < \frac{1}{2}\lambda(x') \leq \lambda(x), \lambda(v).$$

Using that $\mathrm{Cu}(A)$ satisfies Edwards' condition for λ (see Theorem 4.7 and Remark 4.2(3) in [Thi20], and [APRT21]), we obtain $w \in \mathrm{Cu}(A)$ such that

$$(1-\varepsilon)\frac{1}{2}\lambda(x') < \lambda(w), \quad \text{and} \quad w \leq v, x.$$

Choose $w'', w' \in \mathrm{Cu}(A)$ such that

$$(1-\varepsilon)\frac{1}{2}\lambda(x') < \lambda(w''), \quad \text{and} \quad w'' \ll w' \ll w.$$

Applying (O5) for $w'' \ll w' \leq x$, we obtain $c \in \mathrm{Cu}(A)$ such that

$$w'' + c \leq x \leq w' + c.$$

Using that $\lambda(x') \leq \lambda(x)$ and $\lambda(w') \leq \lambda(w) \leq \lambda(v) = \frac{1}{2}\lambda(x')$, we get

$$\frac{1}{2}\lambda(x') = \lambda(x') - \frac{1}{2}\lambda(x') \leq \lambda(x) - \lambda(w') \leq \lambda(c).$$

Then

$$(1-\varepsilon)\frac{1}{2}\lambda(x') < \lambda(w''), \quad \text{and} \quad (1-\varepsilon)\frac{1}{2}\lambda(x') < \frac{1}{2}\lambda(x') \leq \lambda(c).$$

Applying Edwards' condition again, we obtain $y \in \mathrm{Cu}(A)$ such that

$$(1-\varepsilon)\frac{1}{2}\lambda(x') < \lambda(y), \quad \text{and} \quad y \leq w'', c.$$

Then $2y \leq w'' + c \leq x$. Thus y has the desired properties.

Next, to verify condition (U) of Definition 4.3, let $x' \ll x$ in $\mathrm{Cu}(A)$, let $k \in \mathbb{N}$ with $k \geq 1$, and let $\varepsilon \in (0,1)$. We need to find $z \in \mathrm{Cu}(A)$ such that

$$z \leq x, \quad (1-\varepsilon)k\lambda(z) \leq \lambda(x), \quad \text{and} \quad x' \leq kz.$$

Choose $\delta > 0$ such that

$$1 + (2k-1)\delta \leq (1-\varepsilon)^{-1}.$$

Then pick $v'', v', v \in \mathrm{Cu}(A)$ such that

$$x' \ll v'' \ll v' \ll v \ll x, \quad \text{and} \quad \lambda(v) \leq (1+\delta)\lambda(v'').$$

We have already verified condition (L) of Definition 4.3. Applied for $v' \ll v$, for k and for $\frac{\delta}{2}$, we obtain $y \in \mathrm{Cu}(A)$ such that

$$ky \leq v, \quad \text{and} \quad \left(1 - \frac{\delta}{2}\right)\lambda(v') \leq k\lambda(y).$$

By Proposition 2.3(i), we have $(1-\delta)\widehat{v''} \ll (1-\frac{\delta}{2})\widehat{v'}$ in $\mathrm{Lsc}(F(\mathrm{Cu}(A)))$, which allows us to choose $y' \in \mathrm{Cu}(A)$ such that

$$y' \ll y, \quad \text{and} \quad (1-\delta)\lambda(v'') \leq k\lambda(y').$$

Applying Lemma 2.1 for $ky \leq v$ and $y' \ll y$, we obtain $z \in \mathrm{Cu}(A)$ such that

$$(k-1)y' + z \leq v \leq (k-1)y + z, \quad \text{and} \quad v \leq kz.$$

Using that $\lambda(y')$ is finite, we get

$$\begin{aligned} k\lambda(z) &\leq k\lambda(v) - (k-1)k\lambda(y') \\ &\leq k(1+\delta)\lambda(v'') - (k-1)(1-\delta)\lambda(v'') \\ &= (1+(2k-1)\delta)\lambda(v'') \leq (1-\varepsilon)^{-1}\lambda(v''), \end{aligned}$$

and thus

$$(1-\varepsilon)k\lambda(z) \leq \lambda(v'') \leq \lambda(x).$$

We further have

$$z \leq v \leq x, \quad \text{and} \quad x' \leq v \leq kz,$$

which shows that z has the claimed properties. $\square$

5. Pure C*-algebras

This section is devoted to proving Theorems A and B. We actually show that a C^*-algebra is pure if, and only if, it has controlled comparison and is functionally divisible; if, and only if, it is (m,n)-pure for some m and n. Building on the results from Section 4 concerning automatic functional divisibility, we show that a C^*-algebra with controlled comparison is pure if it is either nowhere scattered with real rank zero or stable rank one, or also if it is simple, unital, non-elementary with a unique normalized quasitrace.

Definition 5.1. Given $m,n\in\mathbb{N}$, we say that a Cu-semigroup is (m,n)-*pure* if it has m-comparison and is n-almost divisible. We say that a Cu-semigroup is *pure* if it is $(0,0)$-pure.

A C^*-algebra is (m,n)-*pure* if its Cuntz semigroup is. Similarly, a C^*-algebra is *pure* if it is $(0,0)$-pure.

Proposition 5.2. *Every $\mathcal{Z}$-stable C^*-algebra is pure. Every pure C^*-algebra has the Global Glimm Property and strict comparison (of positive elements by quasitraces).*

Proof. Let A be a $\mathcal{Z}$-stable C^*-algebra. Then the (classical, uncomplete) Cuntz semigroup $W(A)$ is almost unperforated by [Rør04, Theorem 4.5]. Further, by [APT11, Theorem 5.35], for every $x\in W(A)$ and every $k\geq 1$ there exists $y\in W(A)$ such that $ky\leq x\leq (k+1)y$. Using that $W(A)$ is order-dense in $\mathrm{Cu}(A)$ (see [APT18, Theorem 3.2.8]), it follows that $\mathrm{Cu}(A)$ is almost unperforated and almost divisible (see also Paragraph 4.2), and thus pure.

By definition, every pure C^*-algebra is almost divisible, and therefore $(2,\omega)$-divisible, which by [TV23, Theorem 3.6] is equivalent to the Global Glimm Property. Further, every pure C^*-algebra has 0-comparison, and thus enjoys strict comparison as noted in Paragraph 3.4. □

One can ask whether, for Cu-semigroups, $(0,0)$-pureness agrees with some sort of tensorial absorption. This was explored in [APT18], together with an extensive analysis of the tensor product in the category Cu.

Theorem 5.3 ([APT18, Theorem 7.3.11])**.** *A* Cu*-semigroup S is pure if and only if $S\cong \mathrm{Cu}(\mathcal{Z})\otimes S$.*

If the Cuntz semigroup of a C^*-algebra A is isomorphic to that of $\mathcal{Z}\otimes A$, then A is clearly pure. In the proof of [Tom11, Theorem 1.2], Toms shows that the converse holds for simple C^*-algebras. (See also [APP18, Section 7].) The following question is thus pertinent:

Question 5.4. Let A be a pure C^*-algebra. Is $\mathrm{Cu}(A)\cong\mathrm{Cu}(\mathcal{Z}\otimes A)$?

We start with an important technical result, which generalizes some of the Cu-semigroup techniques underlying Proposition 6.4 and Theorem 10.5 in [APRT24].

Proposition 5.5. *Let (S,Σ) be a scaled* Cu*-semigroup satisfying (O5). Assume that (S,Σ) has controlled comparison and is functionally divisible. Then S is pure.*

Proof. We first prove that S is almost unperforated. Let $x,y\in S$, $n\in\mathbb{N}$, and assume that $(n+1)x\leq ny$. We must show that $x\leq y$. This is equivalent to showing that $x'\leq y$ for any $x'\in S$ such that $x'\ll x$.

Therefore, let $x'\in S$ with $x'\ll x$, and pick $x''\in S$ such that $x'\ll x''\ll x$. Since we have $\widehat{x}\leq\frac{n}{n+1}\widehat{y}$, we may choose γ,γ' such that $\frac{n}{n+1}<\gamma<\gamma'<1$. Using Proposition 2.3(i) one gets $\widehat{x''}\ll\gamma\widehat{y}$, which allows us to pick $y'\in S$ such that

$$\widehat{x''}\ll\gamma\widehat{y'},\quad\text{and}\quad y'\ll y.$$

Take y'' such that $y' \ll y'' \ll y$, and choose $d \in \mathbb{N}$ such that $x'', y'' \in \Sigma^{(d)}$. Since (S, Σ) has controlled comparison, we obtain $N = N(\gamma', d) \in \mathbb{N}$ such that

$$\widehat{v} \leq \gamma' \widehat{w} \quad \text{implies} \quad v \leq_N w, \quad \text{for all } v, w \in \Sigma^{(d)}.$$

Pick $\varepsilon > 0$ such that $\gamma \leq (1-\varepsilon)^2 \gamma'$. Applying condition (U) in Definition 4.3 for $x' \ll x''$ and N, we obtain $v \in S$ satisfying

$$v \leq x'', \quad (1-\varepsilon) N \widehat{v} \leq \widehat{x''}, \quad \text{and} \quad x' \leq Nv.$$

Similarly, applying condition (L) in Definition 4.3 for $y' \ll y''$ and N, one gets $w \in S$ such that

$$Nw \leq y'', \quad \text{and} \quad (1-\varepsilon)\widehat{y'} \leq N\widehat{w}.$$

Combining the functional inequalities for $\widehat{v}, \widehat{w}$ and using that $\widehat{x''} \leq \gamma \widehat{y'}$ at the second step, we have

$$(1-\varepsilon)^2 N \widehat{v} \leq (1-\varepsilon)\widehat{x''} \leq (1-\varepsilon)\gamma \widehat{y'} \leq \gamma N \widehat{w} \leq (1-\varepsilon)^2 N \gamma' \widehat{w},$$

which implies that $\widehat{v} \leq \gamma' \widehat{w}$.

Note that both v and w belong to $\Sigma^{(d)}$, since $v \leq x''$, $Nw \leq y''$, and $x'', y'' \in \Sigma^{(d)}$. It follows that $v \leq_N w$, and thus

$$x' \leq Nv \leq Nw \leq y'' \leq y,$$

as desired.

To show that S is almost divisible, let $x' \ll x$ in S, and let $k \in \mathbb{N}$. Pick $\varepsilon \in (0,1)$ such that $(1-\varepsilon)^{-1} k < k+1$. Applying condition (L) in Definition 4.3 for $x'' \ll x$, and for k and ε, we obtain $y \in S$ such that

$$ky \leq x, \quad \text{and} \quad (1-\varepsilon)\widehat{x'} \leq k\widehat{y}.$$

It follows that $x' \leq \infty y$; see, for example, [APRT22, Lemma 6.6(iii)]. Since $(1-\varepsilon)^{-1} k < k+1$, we also have that $\lambda(x') < \lambda((k+1)y)$ for every functional normalized at y. Since S is almost unperforated, we obtain $x' \leq (k+1)y$ by [ERS11, Proposition 6.2]. □

Remark 5.6. It is possible to adapt the proof of Proposition 5.5 so that instead of (U) it suffices to assume the condition (U_{2^m}) from Remark 4.6 for all m. Namely, the following stronger form of Proposition 5.5 is available: If a scaled Cu-semigroup (S, Σ) satisfies (O5) and has controlled comparison, and for every $x' \ll x$ in S and every $\varepsilon \in (0,1)$ there exist $y, z \in S$ such that

$$2y \leq x, \quad (1-\varepsilon)\widehat{x'} \leq 2\widehat{y}, \quad z \leq x, \quad (1-\varepsilon)2\widehat{z} \leq \widehat{x}, \quad \text{and} \quad x' \leq 2z,$$

then S is pure.

Theorem 5.7. *Given a C^*-algebra A, the following statements are equivalent:*

(i) *A is pure, that is, A has strict comparison and is almost divisible.*
(ii) *A is (m,n)-pure for some $m, n \in \mathbb{N}$.*
(iii) *A has controlled comparison and is functionally divisible.*

Proof. It is clear that (i) implies (ii). Further, (ii) implies (iii) since m-comparison implies controlled comparison by Proposition 3.7, and n-almost divisibility implies functional divisibility by Proposition 4.9. Finally, (iii) implies (i) by Proposition 5.5 since the Cuntz semigroup of every C^*-algebra satisfies (O5). □

Remark 5.8. If follows from Theorem 5.7 (see also Paragraph 4.2) that A is (m,n)-pure if, and only if, $\mathrm{Cu}(A)$ has m-comparison and is n-almost divisible in the sense of [Win12, Definition 3.5], that is, given $x \in \mathrm{Cu}(A)$ and $k \geq 1$, there is $y \in \mathrm{Cu}(A)$ such that $ky \leq x \leq (n+1)(k+1)y$.

Remark 5.9. Theorem 5.7 shows that the combination of weak forms of comparison and divisibility implies strong forms of comparison and divisibility. One may wonder if this already holds individually for comparison and divisibility.

For the comparison properties, this is not the case. Indeed, there exist C^*-algebras that have controlled comparison (or even m-comparison for some $m \geq 1$) that do not have 0-comparison. For example, if $X = [0,1]^5$ is the five-dimensional cube, then $C(X)$ has nuclear dimension at most five, and therefore has 5-comparison by [Rob11, Theorem 1.3], but it does not have 0-comparison since the radius of comparison is at least one by [EN13, Theorem 1.1].

Similarly, we expect that the answer to the following question is positive.

Question 5.10. Does there exist a C^*-algebra that is functionally divisible (or even n-almost divisible for some n) but not almost divisible?

Proposition 5.11. *Let A be a functionally divisible C^*-algebra. Then the following statements are equivalent:*

(i) *A is pure.*
(ii) *A has strict comparison (of positive elements by quasitraces).*
(iii) *A has m-comparison for some m.*
(iv) *A has controlled comparison.*

Proof. By definition, pureness implies 0-comparison, which is equivalent to strict comparison by Paragraph 3.4. This shows that (i) implies (ii), which in turn implies (iii). Further, (iii) implies (iv) by Proposition 3.7. Finally, (iv) implies (i) by Theorem 5.7. □

Combining Proposition 5.11 with the results proving functional divisibility from Section 4, we can now verify pureness in various settings:

Theorem 5.12. *Let A be a nowhere scattered C^*-algebra that has real rank zero or stable rank one. Assume that A has controlled comparison (for example, m-comparison for some m). Then A is pure.*

Proof. If A is nowhere scattered and has real rank zero, then A is almost divisible as noted in Paragraph 4.2. If A is nowhere scattered and has stable rank one, then A is functionally divisible by Proposition 4.10. Now, in both cases the result follows from Proposition 5.11. □

Theorem 5.13. *Let A be a simple, unital, non-elementary C^*-algebra with a unique normalized quasitrace. Assume that A has controlled comparison (for example, m-comparison for some m). Then A is pure.*

Proof. By Proposition 4.11, $\mathrm{Cu}(A)$ is functionally divisible. Now, the result follows from Proposition 5.11. □

Example 5.14. Let G be an infinite, discrete group such that the reduced group C^*-algebra $C^*_{\mathrm{red}}(G)$ is simple. Then $C^*_{\mathrm{red}}(G)$ is simple, unital, non-elementary and has a unique trace by [BKKO17, Corollary 4.3]. If G is exact, then quasitraces on $C^*_{\mathrm{red}}(G)$ are traces by Haagerup's theorem ([Haa14]), and it follows that $C^*_{\mathrm{red}}(G)$ has a unique normalized quasitrace. In this case, Theorem 5.13 applies, and it follows that $C^*_{\mathrm{red}}(G)$ is pure if, and only if, it has controlled comparison.

It is a major open problem to determine if for $\mathbb{F}_2$, the free group on two generators, the reduced group C^*-algebra $C^*_{\mathrm{red}}(\mathbb{F}_2)$ has strict comparison (equivalently, if it is pure). Since $C^*_{\mathrm{red}}(\mathbb{F}_2)$ is exact and simple, by Theorem 5.13 it suffices to show that $C^*_{\mathrm{red}}(\mathbb{F}_2)$ has controlled comparison (for example, m-comparison for some m).

For non-exact groups, we ask the following question:

Question 5.15. Do simple reduced group C^*-algebras have a unique normalized quasitrace?

Example 5.16. In his groundbreaking work on regularity properties for simple, nuclear C^*-algebras, Villadsen constructed examples of such algebras that fail strict comparison, and consequently are not pure. The algebras of 'first type' from [Vil98] have stable rank one (but possibly a complicated trace simplex), while the algebras of 'second type' from [Vil99] have higher stable rank but a unique tracial state (and consequently also a unique quasitracial state). Applying Propositions 4.10 and 4.11, we see that both types of Villadsen algebras are automatically functionally divisible.

We deduce that Villadsen algebras that fail strict comparison also fail the much weaker property of controlled comparison (and in particular, they do not have m-comparison for any m).

6. C*-algebras with the Global Glimm Property and finite nuclear dimension

In this section, we prove Theorem D. For this, we use that a C^*-algebra has the Global Glimm Property precisely when its Cuntz semigroup is $(2,\omega)$-divisible; see [TV23, Theorem 3.6]. We then verify a particular statement of the non-simple Toms-Winter conjecture for separable, locally subhomogeneous C^*-algebras with stable rank one and topological dimension zero, which yields Theorem E.

The next result is a variation of [RT17, Proposition 3.2(ii)] for nowhere scattered C^*-algebras. This has essentially appeared in the argument of [Vil23, Proposition 4.1]. We use $\dim_{\mathrm{nuc}}(A)$ to denote the nuclear dimension of A.

Lemma 6.1. *Let $m \in \mathbb{N}$, and let A be a nowhere scattered C^*-algebra satisfying $\dim_{\mathrm{nuc}}(A) \leq m$. Then for every $x' \ll x$ in $\mathrm{Cu}(A)$ and every $k \in \mathbb{N}$ with $k \geq 1$, there exists $y \in \mathrm{Cu}(A)$ such that*

$$y \ll x, \quad \textit{and} \quad x' \ll ky \ll 2(m+1)x.$$

Proof. By [TV24a, Proposition 4.12] and [WZ10, Proposition 2.3], $A\otimes\mathcal{K}$ is nowhere scattered and of nuclear dimension at most m. We may thus assume that A is stable.

Choose $a \in A_+$ and $\varepsilon > 0$ such that

$$x' \ll [(a-\varepsilon)_+], \quad \text{and} \quad [a] \ll x.$$

Since A is nowhere scattered, the hereditary sub-C^*-algebra $\overline{aAa}$ has no finite-dimensional representations, by [TV24a, Theorem 3.1]. It also has nuclear dimension at most m (by [WZ10, Proposition 2.5]), whence we can apply [RT17, Proposition 3.2(ii)] for k (and for $\overline{aAa}$, a and ε) to obtain $b \in \overline{aAa}_+$ such that

$$[(a-\varepsilon)_+] \leq k[b] \leq 2(m+1)[a].$$

Then $y := [b] \in \mathrm{Cu}(A)$ has the desired properties. □

The next two results are inspired by [RT17, Lemma 3.4].

Lemma 6.2. *Let $m \in \mathbb{N}$, and let A be a C^*-algebra with the Global Glimm Property and with $\dim_{\mathrm{nuc}}(A) \leq m$. Set $L := 14m^2 + 6$. Then for every $x' \ll x$ in $\mathrm{Cu}(A)$, there exist $y_0, y_1 \in \mathrm{Cu}(A)$ such that*

$$y_0 + y_1 \leq x, \quad \textit{and} \quad x' \ll Ly_0, Ly_1.$$

Proof. Let $x' \ll x$ in $\mathrm{Cu}(A)$. We claim that there exists a separable sub-C^*-algebra $B \subseteq A$ such that

(i) B has the Global Glimm Property and $\dim_{\mathrm{nuc}}(B) \leq m$;
(ii) the induced inclusion $\mathrm{Cu}(\iota)\colon \mathrm{Cu}(B) \to \mathrm{Cu}(A)$ is an order-embedding;
(iii) x' and x are in the image of $\mathrm{Cu}(\iota)$.

Indeed, we know from Proposition 2.3(iii) and 2.6 in [WZ10] that the property '$\dim_{\mathrm{nuc}}(\_) \leq m$' is separably inheritable. Using the same methods from [TV21, Section 5], one can also see that the Global Glimm Property satisfies the same condition. Now [TV21, Proposition 6.1] allows us to choose a sub-C^*-algebra B with the required conditions.

Identifying $\mathrm{Cu}(B)$ with its image in $\mathrm{Cu}(A)$, we view x', x as elements in $\mathrm{Cu}(B)$, and it suffices to find $y_0, y_1 \in \mathrm{Cu}(B)$ with $y_0 + y_1 \leq x$ and $x' \ll Ly_0, Ly_1$. Therefore we may assume that A is separable.

Choose $x'' \in \mathrm{Cu}(A)$ such that $x' \ll x'' \ll x$. Since A has the Global Glimm Property, $\mathrm{Cu}(A)$ is $(2,\omega)$-divisible by [TV23, Theorem 3.6]. Applied for x'' and x, we obtain $c \in \mathrm{Cu}(A)$ such that

$$2c \leq x, \quad \text{and} \quad x'' \ll \infty c.$$

Choose $c' \in \mathrm{Cu}(A)$ such that $x'' \leq \infty c'$ and $c' \ll c$. Using Lemma 2.1 with $c' \ll c$ and $2c \leq x$, we find $d \in \mathrm{Cu}(A)$ such that

$$c' + d \leq x \leq 2d.$$

Set $e := d \wedge \infty c'$, which is possible by Theorem 2.2. Using at the first step that $x'' \leq \infty c'$ and $x'' \leq x \leq 2d$, we get

$$x'' \leq (2d) \wedge \infty c' = 2(d \wedge \infty c') = 2e.$$

Choose $e'', e' \in \mathrm{Cu}(A)$ such that

$$x' \ll 2e'', \quad \text{and} \quad e'' \ll e' \ll e.$$

Applying Lemma 6.1 for $k = 2m+3$ and $e'' \ll e'$, we obtain $f \in \mathrm{Cu}(A)$ such that

$$f \ll e', \quad \text{and} \quad e'' \ll (2m+3)f \ll 2(m+1)e'. \tag{1}$$

Choose $f' \in \mathrm{Cu}(A)$ such that

$$e'' \ll (2m+3)f', \quad \text{and} \quad f' \ll f.$$

Applying (O5) for $f' \ll f \leq e'$, we obtain $g \in \mathrm{Cu}(A)$ such that

$$f' + g \leq e' \leq f + g. \tag{2}$$

Set $y_0 := f'$ and $y_1 := g + c'$. Then

$$y_0 + y_1 = f' + g + c' \leq e' + c' \leq d + c' \leq x.$$

Further,

$$x' \ll 2e'' \ll 2(2m+3)f' = (4m+6)y_0 \leq (14m^2+6)y_0 = Ly_0.$$

Multiplying (2) by $2m+3$ at the first step, and using (1) at the second step, we deduce

$$(2m+3)e' \leq (2m+3)f + (2m+3)g \leq (2m+1)e' + (2m+3)g. \tag{3}$$

We claim that $\widehat{e'} \leq (2m+3)\widehat{y_1}$. To show this, let $\lambda \in F(S)$. We distinguish two cases:

First, assume that $\lambda(c') < \infty$. Using that $e' \ll e \leq \infty c'$, it follows that $\lambda(e') < \infty$ which allows, in (3), to cancel after applying λ to get

$$\lambda(e') \leq 2\lambda(e') \leq (2m+3)\lambda(g) \leq (2m+3)\lambda(y_1).$$

On the other hand, if $\lambda(c') = \infty$, then

$$\lambda(e') \leq \infty = (2m+3)\lambda(c') \leq (2m+3)\lambda(y_1).$$

Since $\mathrm{Cu}(A)$ has m-comparison (see Remarks 3.2), we get

$$e' \leq (m+1)(2m+3)y_1.$$

Using that $x' \ll 2e'$, one has

$$x' \ll 2e' \leq 2(m+1)(2m+3)y_1 = 2(2m^2+5m+3)y_1 \leq (14m^2+6)y_1 = Ly_1.$$

This shows that y_0 and y_1 have the desired properties. □

Lemma 6.3. *Given $m, l \in \mathbb{N}$, set $L := (14m^2+6)^l$. Then, for every C^*-algebra A with the Global Glimm Property and with $\dim_{\mathrm{nuc}}(A) \leq m$, and for every $x' \ll x$ in $\mathrm{Cu}(A)$, there exist $y_0, \ldots, y_l \in \mathrm{Cu}(A)$ such that*

$$y_0 + \ldots + y_l \leq x, \quad \textit{and} \quad x' \ll Ly_j$$

for $j = 0, \ldots, l$.

Proof. Fix some $m \in \mathbb{N}$, and let A be a C^*-algebra that has the Global Glimm Property and with $\dim_{\mathrm{nuc}}(A) \leq m$. Write $L(m,l) := (14m^2+6)^l$ for $l \geq 0$.

By induction over l, we verify that the following statement holds:

(D_l) For every $x' \ll x$ in $\mathrm{Cu}(A)$ there exist $y_0, \ldots, y_l \in \mathrm{Cu}(A)$ such that

$$y_0 + \ldots + y_l \leq x, \quad \text{and} \quad x' \ll L(m,l)y_j$$

for $j = 0, \ldots, l$.

To verify (D_0), given $x' \ll x$ in $\mathrm{Cu}(A)$ we simply use $y_0 = x$. Further, (D_1) was shown in Lemma 6.2. Now let $l \in \mathbb{N}$ with $l \geq 2$, and assume that we have proved (D_{l-1}). To verify (D_l), let $x' \ll x$ in $\mathrm{Cu}(A)$. Applying (D_{l-1}) for $x' \ll x$, we get $z_0, \ldots, z_{l-1} \in \mathrm{Cu}(A)$ such that

$$z_0 + \ldots + z_{l-1} \leq x, \quad \text{and} \quad x' \ll L(m,l-1)z_j$$

for $j = 0, \ldots, l-1$.

Set $y_j := z_j$ for $j = 0, \ldots, l-2$, and take $z'_{l-1} \in \mathrm{Cu}(A)$ such that

$$z'_{l-1} \ll z_{l-1}, \quad \text{and} \quad x' \ll L(m,l-1)z'_{l-1}.$$

Applying (D_1) for $z'_{l-1} \ll z_{l-1}$, we find $y_{l-1}, y_l \in \mathrm{Cu}(A)$ satisfying

$$y_{l-1} + y_l \leq z_{l-1}, \quad \text{and} \quad z'_{l-1} \ll L(m,1)y_{l-1}, L(m,1)y_l.$$

Then $y_0 + \ldots + y_l \leq x$, and

$$x' \ll L(m,l-1)z_j \leq L(m,l)z_j = L(m,l)y_j$$

for $j = 0, \ldots, l-2$ and

$$x' \ll L(m,l-1)z'_{l-1} \ll L(m,l-1)L(m,1)y_j = L(m,l)y_j$$

for $j = l-1, l$. This shows that $y_0, \ldots, y_l$ have the desired properties. □

Proposition 6.4. *Given $m \in \mathbb{N}$, there exists $N = N(m)$ with the following property: For every C^*-algebra A with the Global Glimm Property and $\dim_{\mathrm{nuc}}(A) \leq m$, the Cuntz semigroup $\mathrm{Cu}(A)$ is N-almost divisible.*

Proof. The proof is inspired by that of [RT17, Lemma 3.6]. Let $L = L(m,m)$ be the constant obtained from Lemma 6.3. We verify the statement for $N := 3L(m+1)$.

Let A be a C^*-algebra with the Global Glimm Property and $\dim_{\mathrm{nuc}}(A) \leq m$. To show that $\mathrm{Cu}(A)$ is N-almost divisible, let $x' \ll x$ in $\mathrm{Cu}(A)$, and let $k \geq 1$. We need to find $y \in \mathrm{Cu}(A)$ such that

$$ky \leq x, \quad \text{and} \quad x' \leq (k+1)(N+1)y.$$

Applying Lemma 6.1 for $3kL(m+1)$ and $x' \ll x$, we obtain $c \in \mathrm{Cu}(A)$ such that

$$c \ll x, \quad \text{and} \quad x' \ll 3kL(m+1)c \ll 2(m+1)x.$$

Choose $c' \in \mathrm{Cu}(A)$ such that

$$x' \ll 3kL(m+1)c', \quad \text{and} \quad c' \ll c.$$

Now choose $x'' \in \mathrm{Cu}(A)$ such that

$$3kL(m+1)c' \ll 2(m+1)x'', \quad \text{and} \quad x'' \ll x.$$

Applying Lemma 6.3 for $x'' \ll x$, we obtain $y_0, \ldots, y_m$ such that

$$y_0 + \ldots + y_m \leq x, \quad \text{and} \quad x'' \ll Ly_j \qquad \text{for } j = 0, \ldots, m.$$

For each j, we obtain

$$3kL(m+1)c' \ll 2(m+1)x'' \ll 2L(m+1)y_j.$$

and thus $kc' <_s y_j$.

As noted in Remarks 3.2, A has m-comparison, and we get

$$kc' \leq y_0 + \ldots + y_m \leq x.$$

Further,

$$x' \ll 3kL(m+1)c' \leq 3(k+1)L(m+1)c' = (k+1)Nc' \leq (k+1)(N+1)c',$$

which shows that $y = c'$ has the desired properties. □

Theorem 6.5. *Every C^*-algebra with the Global Glimm Property and finite nuclear dimension is pure.*

Proof. Let $m \in \mathbb{N}$, and let A be a C^*-algebra with the Global Glimm Property and $\dim_{\mathrm{nuc}}(A) \leq m$. Then A has m-comparison, as noted in Remarks 3.2. Further, by Proposition 6.4, we know that A is n-almost divisible for some n. Hence, A is (m, n)-pure, and the result now follows from Theorem 5.7. □

Theorem 6.6. *Let A be a separable, locally subhomogeneous C^*-algebra that has stable rank one and topological dimension zero. Then the following statements are equivalent:*

- *(1) A has the Global Glimm Property and finite nuclear dimension.*
- *(2) A is $\mathcal{Z}$-stable.*
- *(3a) A is pure.*
- *(3b) A is nowhere scattered and has strict comparison of positive elements.*

Proof. By Theorem 6.5, (1) implies (3a). By Proposition 5.2, (2) implies (3a), and (3a) implies (3b). By Theorem 5.12, (3b) implies (3a).

Every locally subhomogeneous C^*-algebra has locally finite nuclear dimension by [NW06]. Since locally subhomogeneous C^*-algebras have no simple, purely infinite ideal-quotients, it follows from [RT17, Theorem 7.10] that (3a) implies (2).

Finally, by [ENST20, Theorem A], every $\mathcal{Z}$-stable, locally subhomogeneous C^*-algebra has decomposition rank at most two, and therefore finite nuclear dimension. Further, by Proposition 5.2, $\mathcal{Z}$-stability implies the Global Glimm Property. This shows that (2) implies (1). □

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