DENSITY OF CERTAIN POLYNOMIAL MODULES

A. D. BARANOV, J. J. CARMONA, K. YU. FEDOROVSKIY

Abstract. In this paper the problem of density in the space $C(X)$, for a compact set $X \subset \mathbb{C}$, of polynomial modules of the type $\{p+\overline{z}^d q :\ p, q \in \mathbb{C}[z]\}$ for integer $d > 1$, as well as several related problems are studied. We obtain approximability criteria for Carathéodory compact sets using the concept of a d-Nevanlinna domain, which is a new special analytic characteristic of planar simply connected domains. In connection with this concept we study the problem of taking roots in the model spaces, that is, in the subspaces of the Hardy space H^2 which are invariant under the backward shift operator.

1. Introduction

Through this paper let X be a compact set in the complex plane $\mathbb{C}$. Denote by $\mathcal{P}$ be the space of all polynomials in the complex variable z, by $\mathcal{R}(X)$ the space of all rational functions in the complex variable having their poles outside X, as well as by $C(X)$ the space of all continuous complex-valued functions on X endowed with the uniform norm $\|f\|_X = \max_{z\in X} |f(z)|$.

Take an integer $d \geqslant 1$ and define the following spaces of functions

$$\mathcal{P}(\bar{\mathfrak{z}}^d) = \left\{p_0 + \bar{\mathfrak{z}}^d p_1 :\ p_0, p_1 \in \mathcal{P}\right\};$$
$$\mathcal{R}(X, \bar{\mathfrak{z}}^d) = \left\{g_0 + \bar{\mathfrak{z}}^d g_1 :\ g_0, g_1 \in \mathcal{R}(X)\right\},$$

where $\bar{\mathfrak{z}}(z) = \overline{z}$. These spaces are modules over the rings $\mathcal{P}$ and $\mathcal{R}(X)$ respectively, generated by the function $\bar{\mathfrak{z}}^d$. This function is called the generator of $\mathcal{P}(\bar{\mathfrak{z}}^d)$ and $\mathcal{R}(X, \bar{\mathfrak{z}}^d)$. For instance, if $d = 1$, then these spaces consist, respectively, of all *bianalytic polynomials* and *bianalytic rational functions* with poles lying outside X (note that bianalytic rational functions are not quotients of bianalytic polynomials).

We are interested in questions about density in the space $C(X)$ of the modules $\mathcal{P}(\bar{\mathfrak{z}}^d)$ and $\mathcal{R}(Y, \bar{\mathfrak{z}}^d)$ for some specially chosen compact set $Y \supseteq X$ in the case, when $d > 1$ as well as in questions about density in $C(X)$ of polynomial and

Key words and phrases. Polyanalytic polynomials, rational modules, d-Nevanlinna domains, uniform approximation, Carathéodory sets, model spaces K_Θ.

The work of the first author is partially supported by the *Chebyshev Laboratory* (St. Petersburg State University) under RF Government grant 11.G34.31.0026 and by *Russian Foundation for Basic Research* (grant 12-01-00434-a).

The work of the first author is partially supported by the grants MTM2008-05561-C02-02 of the *Ministerio de Ciencia e Innovación, Spain* and 2009 SGR 1303 of the *Generalitat de Catalunya*.

The work of the third author is partially supported by *Russian Foundation for Basic Research* (grants nos. 12-01-00434-a and 13-01-00923-a).

rational modules generated by multiple degrees of the function $\bar{\mathfrak{z}}$:

$$\mathcal{P}(\bar{\mathfrak{z}}^{k_1},\dots,\bar{\mathfrak{z}}^{k_m}) = \left\{p_0 + \bar{\mathfrak{z}}^{k_1}p_1 + \cdots + \bar{\mathfrak{z}}^{k_m}p_m \colon p_0, p_1, \dots, p_m \in \mathcal{P}\right\};$$
$$\mathcal{R}(Y, \bar{\mathfrak{z}}^{k_1},\dots,\bar{\mathfrak{z}}^{k_m}) = \left\{g_0 + \bar{\mathfrak{z}}^{k_1}g_1 + \cdots + \bar{\mathfrak{z}}^{k_m}g_m \colon g_0, g_1, \dots, g_m \in \mathcal{R}(Y)\right\},$$

where $k_1, \dots, k_m$ are integers such that $0 < k_1 < \cdots < k_m$.

Of course, if any of the modules under consideration is dense in the space $C(X)$, then, clearly, the interior of X is empty. Thus, speaking about density of modules under consideration we will always mean their density in the respective appropriately defined subspaces of the space $C(X)$.

The roots of these questions can be traced back to 1970s–1980s when problems about density of rational modules $\mathcal{R}(Y, \bar{\mathfrak{z}}, \bar{\mathfrak{z}}^2, \dots, \bar{\mathfrak{z}}^n)$ for integer $n \geqslant 1$, were studied by O'Farrell [1], Verdera [2], Carmona [3, 4], Wang [5], Trent and Wang [6, 7].

Systematical studies of the question about density of polynomial modules $\mathcal{P}(\bar{\mathfrak{z}}, \bar{\mathfrak{z}}^2, \dots, \bar{\mathfrak{z}}^n)$, $n \geqslant 1$, started in mid-1990s and this question turned out fairly different from the corresponding question about rational approximation, see [8] and references therein. In particular, the approximability conditions in this case cannot be expressed only in terms of topological, metrical or capacitary properties of X. Moreover, several new phenomena related with special analytic properties of planar domains and with special properties of function belonging to model spaces (that is, subspaces of the Hardy space H^2 that are invariant under the backward shift operator) arose in connection with this question.

One ought to notice that in the present paper we allow the situation when the sequence of degrees of generators of our modules has *gaps*. In this situation new interesting connections of problems under consideration with theory of model spaces arise in addition to the initial polyanalytic case.

Structure and notation. The structure of the paper is as follows. In section 2 we state the approximation problems under consideration for modules with one generator explicitly and discuss the state-of-the-art of these problems. In section 3 we study the problem of uniform approximation by the module $\mathcal{P}(\bar{\mathfrak{z}}^d)$, $d > 1$, for Carathéodory compact sets. The respective approximability criterion (Theorem 1 below) is formulated in terms of a new analytic characteristic of planar sets. This characteristic is the concept of a d-Nevanlinna domain, which will be studied in section 4. We give description of d-Nevanlinna domains in terms of conformal mappings from the unit disk onto the domain under consideration and explore useful relations of this concept with several interesting problems in the theory of model spaces. In section 5 we are dealing with modules generated by multiple degrees of the function $\bar{\mathfrak{z}}$.

In what follows we will use the following notation. By a contour we mean a simple closed curve in $\mathbb{C}$ (not necessarily rectifiable). If $\mathit{\Gamma}$ is a contour, then $D(\mathit{\Gamma})$ is the (Jordan) domain, bounded by it. Let, as usual, $\mathbb{T} = \{\xi \in \mathbb{C} : |\xi| = 1\}$ be the unit circle and let $\mathbb{D} = \{z \in \mathbb{C} : \ |z| < 1\}$ be the unit disk. Let, moreover, $\mathfrak{z}(z) = z$ be the identity function. Also for a set E let E° be its interior, $\overline{E}$ be its closure and ∂E be its boundary.

Compact sets X possessing the property $\partial X = \partial \widehat{X}$, where $\widehat{X}$ stands for the union of X and all bounded connected components of $\mathbb{C} \setminus X$, are traditionally called *Carathéodory compact sets*.

For $1 \leqslant p \leqslant \infty$ we will denote by $H^p = H^p(\mathbb{D})$ the standard Hardy spaces in $\mathbb{D}$ (see [9, chap. IV]), as well as by $H^\infty(U)$, for an open set $U \subset \mathbb{C}$, the space of all bounded holomorphic functions in U. By the classical Fatou theorem, for each function $f \in H^p$ and for almost all points $\xi \in \mathbb{T}$ the angular boundary values $f(\xi)$ exist; these values define the function from the space $L^p = L^p(\mathbb{T})$ (with respect to the normalized Lebesgue measure on $\mathbb{T}$), which is called the *boundary function* for f. The map mapping each $f \in H^p$ to its boundary function is an isometric isomorphism from H^p to the image space $H^p(\mathbb{T})$. The set of points $\xi \in \mathbb{T}$ where $f(\xi)$ exists is traditionally called the Fatou set of f and denoted by $\mathcal{F}(f)$. By [10, prop. 6.5], the set $\mathcal{F}(f)$ is a Borel set.

We recall that a function $\Theta \in H^\infty$ is called an inner function, if $|\Theta(\xi)| = 1$ for almost all $\xi \in \mathbb{T}$. Finally, let $\mathcal{I}$ denotes the class of all inner functions.

2. Approximation problems for modules with one generator

We need to introduce several new spaces of functions. For a compact set X in $\mathbb{C}$ let us denote by $A(X, \overline{\mathfrak{z}}^d)$ the space consisting of all functions $f \in C(X)$ such that their restrictions to X° has the form

$$f_0 + \overline{\mathfrak{z}}^d f_1, \tag{1}$$

where f_0 and f_1 are holomorphic functions in X°.

Let $\overline{\partial} = \partial/\partial\overline{z}$ be the standard Cauchy–Riemann operator in $\mathbb{C}$. It is worthwhile to notice, that for an open set $U \subset \mathbb{C}$ with $0 \notin U$ any continuous function f on U satisfying the following second order elliptic partial differential equation

$$\overline{\partial}\left(\frac{\overline{\partial} f}{\overline{\partial}\overline{z}^d}\right) = 0, \tag{2}$$

has the form (1), where f_0 and f_1 are holomorphic functions in U.

Furthermore, let $P(X, \overline{\mathfrak{z}}^d)$ be the $C(X)$-closure of the subspace

$$\left\{ p|_X : \ p \in \mathcal{P}(\overline{\mathfrak{z}}^d) \right\}$$

and, for a compact set $Y \subset \mathbb{C}$ with $X \subseteq Y$, let $R(X, Y, \overline{\mathfrak{z}}^d)$ be the closure in $C(X)$ of the subspace

$$\left\{ g|_X : \ g \in \mathcal{R}(Y, \overline{\mathfrak{z}}^d) \right\}.$$

For brevity let $R(X, \overline{\mathfrak{z}}^d) := R(X, X, \overline{\mathfrak{z}}^d)$. It is clear, that

$$P(X, \overline{\mathfrak{z}}^d) \subset R(X, \overline{\mathfrak{z}}^d) \subset A(X, \overline{\mathfrak{z}}^d)$$

for any compact set X. Thus the following problems arise naturally:

Problem 1. *To find necessary and sufficient conditions on X in order that*

$$A(X, \overline{\mathfrak{z}}^d) = P(X, \overline{\mathfrak{z}}^d). \tag{3}$$

Problem 2. *To find necessary and sufficient conditions on X in order that*

$$A(X, \overline{\mathfrak{z}}^d) = R(X, \overline{\mathfrak{z}}^d). \tag{4}$$

Let us briefly discuss several basic results obtained in Problem 2 which are relevant for our further considerations. Firstly, it was proved in [3] that the equality (4) takes place for such X that the set $\mathbb{C}\setminus X$ has finitely many connected components. Moreover, the latter condition can be replaced by the following condition stated in terms of standard analytic capacity γ:

$$\gamma\big(\{w \in \mathbb{C}\colon |w-z| < r\} \setminus X\big) \geqslant Cr,$$

for some positive constant C, for all $z \in \partial X$ and for any sufficiently small $r > 0$ (see [4], as well as [11] for the definition of γ). In particular, the equality (4) holds for any Carathéodory compact set X.

At the same time one ought to notice that the description of compact sets for which the equality (4) holds is unknown, except for the case $d = 1$. In this case Problem 2 was recently solved by Mazalov in [12], where it is proved that the equality $R(X, \bar{\mathfrak{z}}) = A(X, \bar{\mathfrak{z}})$ holds for *any* compact set $X \subset \mathbb{C}$. In fact this very deep and difficult theorem may be regarded as an analogue of the classical Vitushkin's approximation theorem for rational functions [11, chap. V, sect. 3, th. 1].

In connection with Problem 2 we mention one more result due to Mazalov. Let L be an elliptic partial differential operator in $\mathbb{C}$ with constant complex coefficients and locally bounded fundamental solution. In [13] it is proved that for any compact set X each function $f \in C(X)$ such that $Lf = 0$ on X° can be uniformly on X approximated by functions satisfying the equation $LF = 0$ on some (depending on F) neighborhood of X.

The operator (2) is, clearly, not a constant coefficient operator. But our understanding of main ideas and technical lemmas of Mazalov's papers [12, 13] allow us to state the following result which we consider as 'almost' direct corollary of the aforesaid Mazalov's results:

Approximation criterion for rational modules with one generator. *Let X be an arbitrary compact set in $\mathbb{C}$. Then $R(X, \bar{\mathfrak{z}}^d) = A(X, \bar{\mathfrak{z}}^d)$.*

From now on we are going to study Problem 1, which is the main aim of the present paper. In the case when $d = 1$ this problem is the well-known problem of uniform approximability of functions by bianalytic polynomials. Let us mention now several references concerning this problem. The specific analytic nature of approximability conditions in this problem was discovered in [14]. In [15] the approximability criterion for Carathéodory compact sets was obtained as well as in [16] and [17] several necessary and sufficient conditions on a general compact set X in order that the equality $A(X, \bar{\mathfrak{z}}) = P(X, \bar{\mathfrak{z}})$ holds were established (see also [8] and references therein). This paper is the first, when Problem 1 in the case $d > 1$ is considered.

3. Approximation by polynomial modules on Carathéodory compact sets and the concept of a d-Nevanlinna domain

In this section we study Problem 1 for Carathéodory compact sets X. We need to recall, that a bounded domain Ω is called a *Carathéodory domain* if $\partial\Omega = \partial\Omega_\infty$, where Ω_∞ is the unbounded connected component of the set $\overline{\mathbb{C}} \setminus \overline{\Omega}$.

If fact any Carathéodory domain is simply connected and has the property that $\Omega = (\overline{\Omega})^\circ$.

In order to state the results of the paper we introduce, for an integer $d \geqslant 1$, the concept of a *d-Nevanlinna domain*. For $d = 1$ this concept coincides with the concept of a *Nevanlinna domain* which was introduced in [14, def. 3] and [15, def. 2.1] and which was fruitfully employed in studies of problem about approximation by bianalytic polynomials.

Let Ω be a bounded simply connected domain in $\mathbb{C}$, let φ be a (fixed) conformal mapping from $\mathbb{D}$ onto Ω, and let $d \geqslant 1$ be a (fixed) integer.

Definition 1. *A bounded simply connected domain Ω is called a d-Nevanlinna domain, if there exist two functions $u, v \in H^\infty(\Omega)$ such that the equality*

$$\overline{z}^d = \frac{u(z)}{v(z)} \tag{5}$$

holds almost everywhere on $\partial\Omega$ in the sense of conformal mappings. It means that the following equality of angular boundary values

$$\overline{\varphi(\xi)}^d = \frac{(u \circ \varphi)(\xi)}{(v \circ \varphi)(\xi)} \tag{6}$$

holds for almost all points $\xi \in \mathbb{T}$.

This is a consistent definition not depending on one's choice of φ. Moreover, in view of the Luzin–Privalov boundary uniqueness theorem (see, e.g., [9, chap. III]), the quotient u/v is uniquely determined in (a d-Nevanlinna domain) Ω. If Ω is a Jordan domain with rectifiable boundary, then the equality (5) can be understood directly as the equality of angular boundary values for almost all (with respect to the length on $\partial\Omega$) points $z \in \partial\Omega$.

Denote by ND_d the set of all (bounded simply connected) d-Nevanlinna domains. We also put $ND = ND_1$. Is as clear that $ND \subset ND_d$ for each positive integer d. But the inverse inclusion fails also for any integer $d > 1$. The correcponding examples will be constructed and discussed in Section 4.

We are going now to prove the following result:

Theorem 1. *Let X be a Carathéodory compact set in $\mathbb{C}$ and let $d \geqslant 2$ be an integer. Then $A(X, \overline{\mathfrak{z}}^d) = P(X, \overline{\mathfrak{z}}^d)$ if and only if each bounded connected component of the set $\mathbb{C} \setminus X$ is not a d-Nevanlinna domain.*

First of all let us formulate and prove two propositions having an independent interest.

Proposition 1. *If $\Omega \in ND_d$, then $C(\partial\Omega) \neq R(\partial\Omega, \overline{\Omega}, \overline{\mathfrak{z}}^d)$.*

Proof. Let φ be some conformal mapping from $\mathbb{D}$ onto Ω. Since $\Omega \in ND_d$, then there exist bounded holomorphic functions u and $v \not\equiv 0$ in Ω such that the equality (6) holds.

We choose $z_0 \in \Omega$ such that $0 < |u(z_0) - \overline{z}_0^d v(z_0)|$ and we claim that

$$\left.\frac{\overline{z}^d - \overline{z_0}^d}{z - z_0}\right|_{\partial\Omega} \notin R(\partial\Omega, \overline{\Omega}, \overline{\mathfrak{z}}^d).$$

Otherwise for each $\delta > 0$ one can find rational functions f_1 and f_2 with poles outside $\overline{\Omega}$ such that

$$\left| f_1(\zeta) + \overline{\zeta}^d f_2(\zeta) - \frac{\overline{\zeta}^d - \overline{z}_0^d}{\zeta - z_0} \right| < \delta$$

for $\zeta \in \partial\Omega$. Let $U := u \circ \varphi$, $V := v \circ \varphi$, and $F_j := f_j \circ \varphi$ for $j = 1, 2$. Then, for almost all $\xi \in \mathbb{T}$ such that (6) holds, we have

$$\left| F_1(\xi) + F_2(\xi) \frac{U(\xi)}{V(\xi)} - \frac{U(\xi) - \overline{z}_0^d V(\xi)}{V(\xi)\big(\varphi(\xi) - z_0\big)} \right| < \delta.$$

Hence

$$\big|\big(F_1(\xi)V(\xi) + F_2(\xi)U(\xi)\big)\big(\varphi(\xi) - z_0\big) - U(\xi) + \overline{z}_0^d V(\xi)\big| \leqslant \delta M, \tag{7}$$

for almost all $\xi \in \mathbb{T}$, where M is the essential supremum of the function $(\varphi - z_0)V$. Note that the functions under the signs of absolute value on the right-hand side of inequality (7) are the boundary values of the corresponding functions in H^∞, therefore, by the maximum modulus principle, we can substitute $\varphi^{-1}(z_0)$ for ξ in (7):

$$\big|u(z_0) - \overline{z}_0^d v(z_0)\big| \leqslant \delta M,$$

which leads to a contradiction for sufficiently small value of δ. □

Proposition 2. *Let Ω be a Carathéodory domain in $\mathbb{C}$ and $d \geqslant 2$ be an integer. If $C(\partial\Omega) \neq R(\partial\Omega, \overline{\Omega}, \overline{\mathfrak{z}}^d)$, then $\Omega \in ND_d$.*

Proof. Let φ be a conformal mapping from $\mathbb{D}$ onto Ω and let ψ be the respective inverse mapping. Let $\partial_a\Omega$ be the accessible part of $\partial\Omega$, that is, the set of all points in $\partial\Omega$ which are accessible from Ω by some curve. By virtue of [10, prop. 2.14, 2.17] one concludes that $\partial_a\Omega = \{\varphi(\xi) : \ \xi \in \mathcal{F}(\varphi)\}$. As it was shown in [17], $\partial_a\Omega$ is a Borel set. In view of [17, cor. 1] the functions φ and ψ can be extended to Borel measurable functions (denoted also by φ and ψ) on $\mathbb{D} \cup \mathcal{F}(\varphi)$ and $\Omega \cup \partial_a\Omega$ respectively in such a way, that $\varphi(\psi(\zeta)) = \zeta$ for all $\zeta \in \partial_a\Omega$ and $\psi(\varphi(\xi)) = \xi$ for all $\xi \in \mathcal{F}(\varphi)$.

Let ω be the measure on $\partial\Omega$ defined by $\omega := \varphi(d\xi)$ (see [17, sect. 3]). In fact ω is a measure on $\partial_a\Omega$ and has no atoms. Moreover, $|\omega(\cdot)| = 2\pi\omega(\varphi(0), \cdot, \Omega)$, where $\omega(\varphi(0), \cdot, \Omega)$ is the harmonic measure on $\partial\Omega$ evaluated with respect to $\varphi(0)$ and Ω.

If $C(\partial\Omega) \neq R(\partial\Omega, \overline{\Omega}, \overline{\mathfrak{z}}^d)$, then there exists a non-zero measure μ on $\partial\Omega$ such that $\mu \perp \mathcal{R}(\overline{\Omega})$ and $\overline{\mathfrak{z}}^d\mu \perp \mathcal{R}(\overline{\Omega})$. In view of [17, th. 2] there exists two functions $h_1, h_2 \in H^1$ such that $h_1 \not\equiv 0$ and

$$\mu = (h_1 \circ \psi)\,\omega, \qquad \overline{\mathfrak{z}}^d\mu = (h_2 \circ \psi)\,\omega.$$

Therefore, for almost all $\xi \in \mathbb{T}$ one has $\overline{\varphi(\xi)}^d h_1(\xi) = h_2(\xi)$. Going further and replacing the quotient h_2/h_1 by f_2/f_1 with $f_1, f_2 \in H^\infty$ and defining the functions u and v in Ω as follows $u(z) = f_2(\psi(z))$, $v(z) = f_1(\psi(z))$ one obtains that $\overline{z}^d = u(z)/v(z)$ almost everywhere on $\partial\Omega$ in the sense of conformal mappings, as it is demanded. □

Proof of Theorem 1. In view of the classical Mergelyan's theorem [18] about uniform approximation by polynomials in the complex variable, we have that $A(X,\bar{\mathfrak{z}}^d) = P(X,\bar{\mathfrak{z}}^d)$ whenever the set $\mathbb{C}\setminus X$ is connected. So, proving this theorem we are dealing with compact sets with disconnected complement.

Assume now that there exists a bounded component Ω of the set $\mathbb{C}\setminus X$ such that $\Omega\in ND_d$. Then, in view of proposition 1, there exists a nontrivial measure μ on $\partial\Omega$ such that $\mu\perp\mathcal{R}(\overline{\Omega})$. Therefore, $\mu\perp\mathcal{P}$ and there exists a point $z_0\in\Omega$ such that $\widehat{\mu}(z_0)\neq 0$ (here and in the sequel $\widehat{\nu}$ stands for the Cauchy transform of a measure ν). It means that $\mu\not\perp(z-z_0)^{-1}\in A(X,\bar{\mathfrak{z}}^d)$, that is $A(X,\bar{\mathfrak{z}}^d)\neq P(X,\bar{\mathfrak{z}}^d)$ and the first part of the theorem is established.

Let us prove the converse assertion. Assume that μ is nontrivial measure on X such that $\mu\perp\mathcal{P}(\bar{\mathfrak{z}}^d)$. Since $\mu\perp\mathcal{P}$ and X is a Carathéodory compact set, then μ has no atoms due to [17, th. 2]. Our first aim is to prove that $\mu\perp\mathcal{R}(X,\bar{\mathfrak{z}}^d)$.

Let Ω be a bounded component of the set $\mathbb{C}\setminus X$, let $U:=(\widehat{X})^\circ\setminus\Omega$, and let Ω'_∞ be the unbounded connected component of the set $\mathbb{C}\setminus X$.

If $U\neq\varnothing$, take a sequence $\{q_j\}_{j=1}^\infty\in\mathcal{P}$ of polynomials such that $q_j\to 1$ locally uniformly on Ω, $q_j\to 0$ locally uniformly on U and $\|q_j\|_{\widehat{X}}\leqslant C$, where C is some absolute constant. The existence of such sequence is the direct consequence of [19, lem. 7] and Runge's classical theorem. In the case, when $U=\varnothing$, let $q_j\equiv 1$ for $j=1,\dots,\infty$.

Let now μ_Ω be some limit point of the sequence of measures $\{\mu_j\}_{j=1}^\infty$, $\mu_j=q_j\mu$, in the weak-$*$ topology in the space of measures on X. We recall, that it means that there exists some partial sequence $\{j_t\}_{t=1}^\infty$ such that $j_t\to\infty$ and $\mu_{j_t}\xrightarrow{*}\mu_\Omega$ as $t\to\infty$. It is clear, that $\operatorname{Supp}(\mu_\Omega)\subset\partial X$.

Take $z_0\notin\operatorname{Supp}(\mu)$. For $s=0$ and $s=d$ one has

$$\begin{aligned}\widehat{\bar{\mathfrak{z}}^s\mu_\Omega}(z_0) &= \frac{1}{2\pi i}\int\frac{\bar{z}^s\,d\mu_\Omega(z)}{z-z_0} = \lim_{t\to\infty}\frac{1}{2\pi i}\int\frac{\bar{z}^s q_{j_t}(z)\,d\mu(z)}{z-z_0} = \\ &= \lim_{t\to\infty}\left(\frac{1}{2\pi i}\int\frac{(q_{j_t}(z)-q_{j_t}(z_0))\bar{z}^m\,d\mu(z)}{z-z_0}+\frac{q_{j_t}(z_0)}{2\pi i}\int\frac{\bar{z}^s\,d\mu(z)}{z-z_0}\right) = \\ &= \lim_{t\to\infty}\frac{q_{j_t}(z_0)}{2\pi i}\int\frac{\bar{z}^s\,d\mu(z)}{z-z_0} = \begin{cases}\widehat{\bar{\mathfrak{z}}^s\mu}(z_0) & \text{for } z_0\in\Omega,\\ 0 & \text{for } z_0\in(U\setminus\operatorname{Supp}(\mu))\cup\Omega'_\infty,\end{cases}\end{aligned}$$

It now follows that $\mu_\Omega\perp\mathcal{P}(\bar{\mathfrak{z}}^d)$. Take another sequence $\{q_j\}_{j=1}^\infty$ as it was mentioned above and let ν be some limit point of the sequence $\{q_j\mu_\Omega\}_{j=1}^\infty$ in the weak-$*$ topology in the space of measures on X. Then $\nu\perp\mathcal{P}(\bar{\mathfrak{z}}^d)$ and, moreover, for $s=0$ and $s=d$,

$$\widehat{\bar{\mathfrak{z}}^s\nu}(z_0) = \begin{cases}\widehat{\bar{\mathfrak{z}}^s\mu}(z_0) & \text{for } z_0\in\Omega,\\ 0 & \text{for } z_0\in U\cup\Omega'_\infty,\end{cases}$$

since $\operatorname{Supp}(\mu_\Omega)\subset\partial X$. Since μ has no atoms, then μ_Ω and ν have no atoms too. Taking into account the fact that the kernel $(\bar{z}^d-\bar{w}^d)/(z-w)$ is bounded we conclude, that the function

$$f_\Omega(w) = \frac{1}{\pi}\int\frac{(\bar{z}^d-\bar{w}^d)\,d\nu(z)}{z-w}$$

is continuous is $\mathbb{C}$ and vanishes on $U \cup \Omega'_\infty$, so that $f_\Omega(w) = 0$ outside $\overline{\Omega}$. Furthermore, in view of [4, lem. 1] and [3, lem. 2], it is true that on the open set $W = \mathbb{C} \setminus \{0\}$ one has

$$\overline{\partial}\left(\frac{\overline{\partial} f_\Omega}{\overline{\partial}\overline{z}^d}\right) = \nu|_W$$

in the sense of distributions, which yields $\operatorname{Supp}(\nu) \subset \partial\Omega \cup \{0\}$. Since the measure ν has not atoms, then $\operatorname{Supp}(\nu) \subset \partial\Omega$ and therefore the measures ν and $\overline{\mathfrak{z}}^d\nu$ are orthogonal to $\mathcal{R}(\overline{\Omega})$. Since $\Omega \notin ND_d$, it follows from proposition 2 that $\nu = 0$ and the required result is established.

Thus it is proved that for each connected component Ω of the complement $\mathbb{C} \setminus X$ we have $\widehat{\overline{\mathfrak{z}}^s\mu}(z) = 0$ for $z \in \Omega$ and $s = 0, d$, that is, $\mu \perp \mathcal{R}(X, \overline{\mathfrak{z}}^d)$.

In order to finish the proof of the theorem we need to use the fact that $A(X, \overline{\mathfrak{z}}^d) = R(X, \overline{\mathfrak{z}}^d)$ for Carathéodory compact set X, which was proved implicitly in [4] (we need to use [4, th. 1] in the case $0 \notin X$, as well as [4, th. 3] in the case $0 \in X$). □

Dealing with Problem 1 in the case of non-Carathéodory compact sets we can observe, that the following result takes place

Theorem 2. *The equality $A(X, \overline{\mathfrak{z}}^d) = P(X, \overline{\mathfrak{z}}^d)$ holds if and only if the equality $A(X \cap \overline{\Omega}, \overline{\mathfrak{z}}^d) = R(X \cap \overline{\Omega}, \overline{\Omega}, \overline{\mathfrak{z}}^d)$ is satisfied for any connected component Ω of $(\widehat{X})^\circ$ that is not contained in X.*

This theorem can be considered as a refinement of the result of [16, th. 3], where the respective approximability criterion was proved in the case $d = 1$. The proof of [16, th. 3] contains two main steps. At the first step it was proved that any measure on X which is orthogonal to $\mathcal{P}(\overline{\mathfrak{z}})$ is also orthogonal to $\mathcal{R}(X, \overline{\mathfrak{z}})$. In our case one can prove that any measure on X which is orthogonal to $\mathcal{P}(\overline{\mathfrak{z}}^d)$ is also orthogonal to $\mathcal{R}(X, \overline{\mathfrak{z}}^d)$ following essentially the respective arguments given in the proof of Theorem 1 with minor clear modifications (see the proof of [16, th. 3] for details). At the last step it was proved that $A(X, \overline{\mathfrak{z}}) = R(X, \overline{\mathfrak{z}})$. The proof of the respective fact that $A(X, \overline{\mathfrak{z}}^d) = R(X, \overline{\mathfrak{z}}^d)$ may be obtained following the scheme used in the proof of [16, th. 3] (the special version of the Vitushkin localization scheme), where all necessary facts concerning Vitushkin's localization operator for solutions of differential operator (2) should be taken from [3] and [4].

The assertion of Theorem 2 can be said to be of a local nature, though the problem of the equality (3) is certainly non-local in the standard sense. We note also that the conclusion of Theorem 2 is reductive in nature, since in it the problem of possibility of approximation on a given compact set X is reduced to the problem of possibility of approximation on specially chosen compact subsets of X, which may have a simpler structure than the original compact set.

4. Some properties and examples of d-Nevanlinna domains

In what follows in this section let $d \geqslant 1$ be an integer. We start constructing for $d > 1$ a class of examples of d-Nevanlinna domains which are not Nevanlinna domains. For this sake we take a real $a \geqslant 1$ and consider the function $\varphi_{a,d}(w) = \sqrt[d]{a - w}$ where the d-root means the principal branch of the multivalued function

$\sqrt[d]{a-w}$ defined outside the ray $[a,+\infty)$. It is clear that $\varphi_{a,d}$ is univalent in $\mathbb{D}$. Put $D_a^d := \varphi_{a,d}(\mathbb{D})$. Since for all points $\zeta \in \partial D_a^d$ one has

$$\overline{\zeta} = \sqrt[d]{\frac{a^2 - a\zeta^d - 1}{a - \zeta^d}},$$

and since the function in the right-hand side of this equality has branch points in the domain D_a^d, then D_a^d is not a Nevanlinna domain. At the same time D_a^d is a d-Nevanlinna domain because for all points $\zeta \in \partial D_a^d$ one has

$$\overline{\zeta}^d = \frac{a^2 - a\zeta^d - 1}{a - \zeta^d}.$$

Moreover, $D_a^d \in ND_k$ for any $k \in d\mathbb{Z}$, but $D_a^d \notin ND_k$ whenever $k \notin d\mathbb{Z}$.

We now present one example of a domain which does not belong to the set ND_k for any integer $k \geqslant 1$. Let $\Gamma_{a,b}$ be the ellipse defined by the equation $x^2/a^2 + y^2/b^2 = 1$, where $a > b > 0$ are real numbers. Define $c > 0$ by $a^2 - b^2 = c^2$. It can be easily verified that for all $\zeta \in \Gamma_{a,b}$ one has

$$\overline{\zeta} = \frac{a^2+b^2}{c^2}\zeta - \frac{2ab}{c^2}\sqrt{\zeta^2 - c^2}, \tag{8}$$

where we consider the holomorphic branch of the respective square root function defined outside some the segment $[-c, c]$ and satisfying the condition $\sqrt{a^2 - c^2} = b$. It follows from (8), that for any integer $k \geqslant 1$ and for any $\zeta \in \Gamma_{a,b}$ one has

$$\overline{\zeta}^k = P_k(\zeta) + Q_k(\zeta)\sqrt{\zeta^2 - c^2},$$

where $P_k, Q_k \in \mathcal{P}$ and hence (by virtue of the Luzin–Privalov boundary uniqueness theorem) the function $\overline{z}^k$ cannot coincide with the quotient of two bounded holomorphic in $D(\Gamma_{a,b})$ functions. Hence $D(\Gamma_{a,b}) \notin ND_k$ for any integer $k \geqslant 1$.

Furthermore, one ought to notice that $ND_k \cap ND_{k'} \subset ND_d$, where d is the greatest common divisor of k and k'.

For our further studies of d-Nevanlinna domains we need to recall the concept of a (Nevanlinna type) pseudocontinuation of H^∞-functions. Let us put $\mathbb{D}_e := \overline{\mathbb{C}} \setminus \overline{\mathbb{D}}$ and bring in the following definition (see [20, def. 2.1.2]):

Definition 2. *A function $f \in H^\infty$ admits a* pseudocontinuation *(or, more precisely, a Nevanlinna type pseudocontinuation), if there exists two functions $f_1, f_2 \in H^\infty(\mathbb{D}_e)$ such that $f_2 \not\equiv 0$ and for almost all points $\xi \in \mathbb{T}$ the equality $f(\xi) = f_1(\xi)/f_2(\xi)$ is satisfied, where $f_1(\xi)$ and $f_2(\xi)$ are angular boundary values of functions f_1 and f_2 being taken from the domain $\mathbb{D}_e$.*

We need now to invite the concept of a *model space*. Let $\Theta \in \mathcal{I}$, and

$$K_\Theta := (\Theta H^2)^\perp = H^2 \ominus \Theta H^2.$$

By virtue of the classical Beurling theorem (see [21, lect. I]) the spaces K_Θ (for all $\Theta \in \mathcal{I}$), and only these spaces, are invariant subspaces for the backward shift operator $f \mapsto (f(z) - f(0))/z$ in H^2. These spaces are usually called model spaces, or $*$-invariant subspaces. We refer to [21] and [22] for the systematic exposition of the theory of model spaces.

The following description of d-Nevanlinna domains in terms of conformal mappings takes place:

Theorem 3. *Let Ω be a bounded simply connected domain in $\mathbb{C}$, and φ be a conformal mapping from $\mathbb{D}$ onto Ω. Then $\Omega \in ND_d$ if and only if φ^d admits a pseudocontinuation.*

Moreover, if $\Omega \in ND_d$, then there exists a function $\Theta \in \mathcal{I}$ such that $\varphi^d \in K_\Theta$. Conversely, let $\Theta \in \mathcal{I}$. Then $\varphi(\mathbb{D}) \in ND_d$ for any φ which is bounded and conformal in $\mathbb{D}$, and such that $\varphi^d \in K_\Theta$.

Sketch of the proof. The proof of the fact that $\Omega \in ND_d$ if and only if φ^d admits a pseudocontinuation is absolutely similar to the proof of [15, prop. 3.1]. The aforesaid Beurling theorem together with [20, th. 2.2.1] give that *a function $g \in H^\infty$ admits a pseudocontinuation if and only if there exists a function $\Theta \in \mathcal{I}$ such that $g \in K_\Theta$* (see [21, lect. II] for details). Applying this fact to the function $g = \varphi^d$ we verify the remaining part of Theorem 3. □

We remark that the question whether bounded univalent functions belonging to K_Θ exist as well as the boundary behavior of such functions were studied in [23] and [24].

It follows from examples given at the beginning of this section that there exist bounded conformal mapping φ in $\mathbb{D}$ such that $\varphi^d \in K_\Theta$ for some $d \geqslant 2$ and $\Theta \in \mathcal{I}$, but $\varphi \notin K_\Theta$ for any $\Theta \in \mathcal{I}$. In view of this observation it seems interesting and important to consider the problem of d-root extraction in the space K_Θ in more detail.

We need to recall several facts about bounded analytic functions in the unit disk, which can be found in [9, chap. IV]. Given a function $f \in H^\infty$, than f can be uniquely factorized in the following way:

$$f = e^{ic} BSF,$$

where B is a Blaschke product, S is a singular inner function, F is an outer function and $c \in \mathbb{R}$. The function $\Theta = e^{ic}BS$ is called the inner factor of f as well as F is its outer factor. This decomposition is traditionally called an inner–outer factorization of f. Recall that a Blaschke product is a function of the form

$$B(z) = \prod_{n=1}^{\infty} \frac{\overline{a}_n}{|a_n|} \frac{z - a_n}{\overline{a}_n z - 1},$$

where $(a_n)_{n=1}^\infty$ is some Blaschke sequence in $\mathbb{D}$ (that is, $a_n \in \mathbb{D}$ for $n \in \mathbb{N}$ and $\sum_{n=1}^\infty (1 - |a_n|) < \infty$), while a singular inner function is a function of the form

$$S(z) = \exp\biggl(-\int_{\mathbb{T}} \frac{\zeta + z}{\zeta - z}\, d\mu_S(\zeta)\biggr),$$

where μ_S is a finite positive singular measure on $\mathbb{T}$ and an outer function is

$$F(z) = \exp\left(\frac{1}{2\pi} \int_0^{2\pi} \frac{e^{it} + z}{e^{it} - z} \log |f(e^{it})|\, dt\right).$$

Furthermore, we need several basic facts about model spaces K_Θ (see, for example, [21, lect. II]):

(1) $f \in K_\Theta$ if and only if there exists a function $g \in H^2$ such that $f = \bar{\mathfrak{z}}\Theta\overline{g}$.

(2) For $\Theta \in \mathcal{I}$ let us define the operator, acting in L^2 as follows

$$f \mapsto \widetilde{f}_\Theta := \bar{\mathfrak{z}}\Theta\overline{f}.$$

This operator is an antilinear isometric involution, and it commutes with the orthogonal projection from L^2 to K_Θ. Thus, for any function $f \in K_\Theta$, one has $\bar{\mathfrak{z}}\Theta\overline{f} \in K_\Theta$.

Now we are going to prove the following result.

Theorem 4. *Let $k \geqslant 1$ be an integer, let $f \in H^\infty$ and let $\Theta \in \mathcal{I}$. Suppose that $h = f^k \in K_\Theta$. The following two assertions are equivalent:*

(1) *the function f admits a pseudocontinuation;*
(2) *there exists $\Theta_1 \in \mathcal{I}$ such that $(\mathfrak{z}J)^{k-1}\Theta = \Theta_1^k$, where J is the inner factor of the function $\widetilde{h}_\Theta$.*

Moreover, if f admits a pseudocontinuation, then $f \in K_{I_1}$ for some function $I_1 \in \mathcal{I}$ such that I_1 divides Θ and Θ_1 in $\mathcal{I}$ (so that $f \in K_\Theta \cap K_{\Theta_1}$), and I_1^k divides $\mathfrak{z}^{k-1}\Theta$ in $\mathcal{I}$.

Proof. Let the function f admit a pseudocontinuation. Then there exists $\Theta_* \in \mathcal{I}$ such that $f \in K_{\Theta_*}$. Let J_1 be the inner factor of the function $\widetilde{f}_{\Theta_*}$ and F_f be the outer factor of f. Then we have

$$\bar{\mathfrak{z}}\Theta_*\overline{f} = F_f J_1,$$

where this equality, as well as all forthcoming equalities in this proof, mean the respective equality of functions in the space L^∞. Raising this equality to the power k and taking into account the fact that $F_h = F_f^k$ (where F_h is the outer factor of the function $h = f^k$) we obtain

$$\bar{\mathfrak{z}}^k\Theta_*^k\overline{f}^k = J_1^k F_h. \tag{9}$$

Since $h = f^k \in K_\Theta$, then, considering the inner–outer decomposition of the function $\widetilde{h}_\Theta$ we have

$$\bar{\mathfrak{z}}\Theta\overline{f}^k = \bar{\mathfrak{z}}\Theta\overline{h} = JF_h. \tag{10}$$

Dividing (9) and (10) by their common factors we conclude that

$$\bar{\mathfrak{z}}J_1^k\Theta = \bar{\mathfrak{z}}^k J\Theta_*^k.$$

Therefore, multiplying the last equality by $\mathfrak{z}^k J^{k-1}$ we obtain

$$J_1^k\Theta(\mathfrak{z}J)^{k-1} = J^k\Theta_*^k,$$

which implies

$$(\mathfrak{z}J)^{k-1}\Theta = \frac{J^k\Theta_*^k}{J_1^k} = \Theta_1^k,$$

where $\Theta_1 = J\Theta_*/J_1 \in \mathcal{I}$ in view of the boundary uniqueness theorem for H^p-functions (see [9, chap. III]).

Now we going to prove the converse assertion. It follows from our hypothesis and from (10) that

$$\overline{f}^k\Theta_1^k = \overline{f}^k\mathfrak{z}^{k-1}\Theta J^{k-1} = \mathfrak{z}^k F_h J^k = (\mathfrak{z}F_f J)^k,$$

which in their own turn implies that the equality

$$\overline{f(\xi)}\Theta_1(\xi) = \omega_k \xi F_f(\xi) J(\xi)$$

holds for all $\xi \in E$, where $E \subset \mathbb{T}$ is a set of positive measure and ω_k is some k-root of the unity. Applying again the boundary uniqueness theorem one concludes that

$$f = \bar{\mathfrak{z}}\Theta_1 \overline{(\omega_k F_f J)} \in K_{\Theta_1}.$$

Therefore, f admits a pseudocontinuation, which is demanded.

To complete the proof we assume that f admits a pseudocontinuation. It is already shown that $f \in K_{\Theta_1}$ in such a case. Let Θ_0 be the greatest common divisor of Θ_1 and J in the class $\mathcal{I}$, so that there exist two relatively prime functions $I_1, I_2 \in \mathcal{I}$ such that $\Theta_1 = I_1\Theta_0$ and $J = I_2\Theta_0$. We claim that I_1 is an inner divisor of Θ. Indeed, since $\Theta_1^k = (\mathfrak{z}J)^{k-1}\Theta$, then

$$I_1^k \Theta_0^k = \mathfrak{z}^{k-1} I_2^{k-1} \Theta_0^{k-1} \Theta,$$

which gives that $I_1^k \Theta_0 = \mathfrak{z}^{k-1} I_2^{k-1}\Theta$. Assume that $\mathfrak{z}$ does not divide I_1. Then, since I_1 and I_2 are relatively prime, I_1 divides Θ. Let now $I_1 = \mathfrak{z}^m I_3$ for some integer $m \geqslant 1$ and $I_3 \in \mathcal{I}$ such that $\mathfrak{z}$ and I_3 are relatively prime. Then $\mathfrak{z}^{mk} I_3^k \Theta_0 = \mathfrak{z}^{k-1} I_2^{k-1}\Theta$, which gives

$$\mathfrak{z}^{(m-1)(k-1)} I_1 I_3^{k-1} \Theta_0 = I_2^{k-1}\Theta,$$

so that I_1 divides Θ in $\mathcal{I}$ in the last case too. Finally, the equality $\bar{\mathfrak{z}}\Theta_1 \overline{f} = \omega_k J F_f$ gives that $f = \bar{\mathfrak{z}} I_1 \overline{\omega_k I_2 F_f}$. Hence $f \in K_{I_1}$, but $K_{I_1} \subset K_\Theta$ (since I_1 divides Θ in $\mathcal{I}$). □

In connection with Theorem 4 we notice that the possibility of taking k-root of the function $(\mathfrak{z}J)^{k-1}\Theta$ is essentially depends on orders of its zeros, because for any singular inner function its k-root is always well defined.

At the beginning of this section some examples of domains $\Omega \in ND_d \setminus ND$ were given. Let us revert to these examples and analyze them from the point of view of Theorem 4. Let, for integer $k > 1$, $f(w) = \varphi_{a,k}(w) = \sqrt[k]{a - w}$, $a \geqslant 1$, and let $h(w) = f(w)^k = a - w$. It is clear that $h \in K_{\mathfrak{z}^2}$. Since $\widetilde{h}_{\mathfrak{z}^2}(w) = aw - 1$, then $J(w) = (aw - 1)/(w - a)$. Thus, in order to have a pseudocontinuation property for f one needs that $(\mathfrak{z}J)^{k-1}\mathfrak{z}^2 = \mathfrak{z}^{k+1}J^{k-1} = \Theta_1^k$ for some $\Theta_1 \in \mathcal{I}$, which is clearly impossible.

5. Modules with multiple generators

Let $m > 1$ and $k_1, \dots, k_m$ be positive integers such that $k_1 < \dots < k_m$. In what follows let d be the greatest common divisor of $k_1, \dots, k_m$ and $\ell_j = k_j/d$ for $j = 1, \dots, m$.

For a compact set $X \subset \mathbb{C}$ let us define the following spaces of functions: $P(X, \bar{\mathfrak{z}}^{k_1}, \dots, \bar{\mathfrak{z}}^{k_m})$ to be the uniform closure on X of the subspace $\{p|_X : p \in \mathcal{P}(\bar{\mathfrak{z}}^{k_1}, \dots, \bar{\mathfrak{z}}^{k_m})\}$ and $A(X, \bar{\mathfrak{z}}^{k_1}, \dots, \bar{\mathfrak{z}}^{k_m})$ to be the space consisting of all function $f \in C(X)$ such that their restrictions to X° has the form $f_0 + \bar{\mathfrak{z}}^{k_1} f_1 + \dots + \bar{\mathfrak{z}}^{k_m} f_m$, where $f_0, \dots, f_m$ are holomorphic functions on X°. Furthermore, for compact sets X and $Y \supseteq X$ we define the space $R(X, Y, \bar{\mathfrak{z}}^{k_1}, \dots, \bar{\mathfrak{z}}^{k_m})$ to be the closure in $C(X)$ of the subspace $\{g|_X \colon g \in \mathcal{R}(Y, \bar{\mathfrak{z}}^{k_1}, \dots, \bar{\mathfrak{z}}^{k_m}\}$. One has

$$P(X, \bar{\mathfrak{z}}^{k_1}, \dots, \bar{\mathfrak{z}}^{k_m}) \subset R(X, \bar{\mathfrak{z}}^{k_1}, \dots, \bar{\mathfrak{z}}^{k_m}) \subset A(X, \bar{\mathfrak{z}}^{k_1}, \dots, \bar{\mathfrak{z}}^{k_m}),$$

where $R(X, \bar{\mathfrak{z}}^{k_1}, \dots, \bar{\mathfrak{z}}^{k_m}) = R(X, X, \bar{\mathfrak{z}}^{k_1}, \dots, \bar{\mathfrak{z}}^{k_m})$ and the following question arises quite naturally: *to describe such compact sets $X \subset \mathbb{C}$, for which*

$$P(X, \bar{\mathfrak{z}}^{k_1}, \dots, \bar{\mathfrak{z}}^{k_m}) = A(X, \bar{\mathfrak{z}}^{k_1}, \dots, \bar{\mathfrak{z}}^{k_m}).$$

Let now X be a Carathéodory compact set. If there exists an integer $s \geqslant 1$ such that $k_j = js$ for $j = 1, \dots, m$, then the answer for the question under consideration is the same as in Theorem 1: $P(X, \bar{\mathfrak{z}}^{s}, \dots, \bar{\mathfrak{z}}^{ms}) = A(X, \bar{\mathfrak{z}}^{s}, \dots, \bar{\mathfrak{z}}^{ms})$ if and only if any bounded connected component Ω of the set $\mathbb{C} \setminus X$ is not a s-Nevanlinna domain. The proof of this result is word-by-word repetition of the proof of Theorem 1 with minor clear modifications (see also the proof of [15, th. 2.2]).

In the case of general exponents (of $\bar{\mathfrak{z}}$) the following result is satisfied.

Theorem 5. *For a bounded simply connected domain Ω let us consider the following assertions:*

(1) $R(\partial\Omega, \overline{\Omega}, \bar{\mathfrak{z}}^{k_1}, \dots, \bar{\mathfrak{z}}^{k_m}) = C(\partial\Omega)$*;*
(2) $R(\partial\Omega, \overline{\Omega}, \bar{\mathfrak{z}}^{d}) = C(\partial\Omega)$*;*
(3) $\Omega \notin ND_d$*.*

Then (1) $\Rightarrow$ (3) *and* (2) $\Rightarrow$ (3)*. If Ω is a Carathéodory domain, all these assertions are equivalent.*

Moreover, let X be a Carathéodory compact set in $\mathbb{C}$. If there exists some bounded connected component Ω of the set $\mathbb{C} \setminus X$ such that $\Omega \in ND_d$, then

$$A(X, \bar{\mathfrak{z}}^{k_1}, \dots, \bar{\mathfrak{z}}^{k_m}) \neq P(X, \bar{\mathfrak{z}}^{k_1}, \dots, \bar{\mathfrak{z}}^{k_m}).$$

Conversely, if $\Omega \notin ND_d$ for any bounded connected component Ω of the set $\mathbb{C} \setminus X$, then

$$P(X, \bar{\mathfrak{z}}^{k_1}, \dots, \bar{\mathfrak{z}}^{k_m}) = R(X, \bar{\mathfrak{z}}^{k_1}, \dots, \bar{\mathfrak{z}}^{k_m}).$$

Proof. We already have proved that (2) $\Rightarrow$ (3) and (2) $\Leftrightarrow$ (3) in the case of Carathéodory domains (see Propositions 1 and 2).

If $\Omega \in ND_d$ it can be shown by the same way as it was done in the proof of Proposition 1, that

$$\left.\frac{\overline{z}^d - \overline{z_0}^d}{z - z_0}\right|_{\partial\Omega} \notin R(\partial\Omega, \overline{\Omega}, \bar{\mathfrak{z}}^{k_1}, \dots, \bar{\mathfrak{z}}^{k_m}),$$

where $z_0 \in \Omega$ is such point that $|u(z_0) - \overline{z_0}^d v(z_0)| > 0$ and $v(z_0) \neq 0$ (here $u, v \in H^\infty(\Omega)$ are taken from Definition 1). Therefore, (1) $\Rightarrow$ (3).

In order to prove the implication (3) $\Rightarrow$ (1) for Carathéodory domain Ω we observe that if $R(\partial\Omega, \overline{\Omega}, \bar{\mathfrak{z}}^{k_1}, \dots, \bar{\mathfrak{z}}^{k_m}) \neq C(\partial\Omega)$, then $R(\partial\Omega, \overline{\Omega}, \bar{\mathfrak{z}}^{k_j}) \neq C(\partial\Omega)$ for each $j = 1, \dots, m$. Applying Proposition 2 we conclude that there exist m couples of functions $u_j, v_j \in H^\infty(\Omega)$ such that the equalities $\bar{\mathfrak{z}}^{k_j} = u_j/v_j$ are satisfied almost everywhere on $\partial\Omega$ in the sense of conformal mappings. By the well-known property of greater common divisor (Bezout's identity) there exists the set $\{t_1, \dots, t_m\}$ of integers such that $\sum_{j=1}^{m} k_j t_j = d$. Therefore almost everywhere on $\partial\Omega$ in the sense of conformal mappings we have

$$\bar{\mathfrak{z}}^d = \frac{u}{v} := \prod_{j=1}^{m} \frac{u_j^{t_j}}{v_j^{t_j}},$$

which exactly means that $\Omega \in ND_d$.

The proof of the first assertion in the second part of the theorem can be obtained similarly as the respective assertion in Theorem 1.

Let us give a sketch of the proof of the remaining assertion. As in the proof of Theorem 1 we take a measure μ on X which is orthogonal to $\mathcal{P}(\bar{\mathfrak{z}}^{k_1}, \dots, \bar{\mathfrak{z}}^{k_m})$ and we need to show that $\mu \perp \mathcal{R}(X, \bar{\mathfrak{z}}^{k_1}, \dots, \bar{\mathfrak{z}}^{k_m})$. Let Ω be any bounded component of the set $\mathbb{C} \setminus X$. Starting with this measure μ we construct, as it was done in the proof of Theorem 1, new measure ν possessing the properties $\mathrm{Supp}(\nu) \subset \partial\Omega$ and $\nu \perp \mathcal{R}(\overline{\Omega}, \bar{\mathfrak{z}}^{k_1}, \dots, \bar{\mathfrak{z}}^{k_m})$, and such that $\widehat{\bar{\mathfrak{z}}^s \nu}(z) = \widehat{\bar{\mathfrak{z}}^s \mu}(z)$ for all $z \in \Omega$ and $s = k_1, \dots, k_m$. Since $\Omega \notin ND_d$ and (1) $\Leftrightarrow$ (3), then $\nu = 0$ and, hence, $\mu \perp \mathcal{R}(X, \bar{\mathfrak{z}}^{k_1}, \dots, \bar{\mathfrak{z}}^{k_m})$. □

Let us give several remarks concerning Theorem 5. If Ω is a Carathéodory domain such that the set $\mathbb{C} \setminus \overline{\Omega}$ is connected (for instance, if Ω is a Jordan domain), then the spaces $R(\partial\Omega, \Omega, \bar{\mathfrak{z}}^{k_1}, \dots, \bar{\mathfrak{z}}^{k_n})$ and $R(\partial\Omega, \Omega, \bar{\mathfrak{z}}^d)$ in Theorem 5 can be replaced by the spaces $P(\partial\Omega, \bar{\mathfrak{z}}^{k_1}, \dots, \bar{\mathfrak{z}}^{k_n})$ and $P(\partial\Omega, \bar{\mathfrak{z}}^d)$ respectively.

Actually it is an open question should the equality

$$A(X, \bar{\mathfrak{z}}^{k_1}, \dots, \bar{\mathfrak{z}}^{k_m}) = R(X, \bar{\mathfrak{z}}^{k_1}, \dots, \bar{\mathfrak{z}}^{k_m})$$

be true for any compact set or not. One ought to notice that the equality $A(X, \bar{\mathfrak{z}}^{k_1}, \dots, \bar{\mathfrak{z}}^{k_m}) = R(X, \bar{\mathfrak{z}}^{k_1}, \dots, \bar{\mathfrak{z}}^{k_m})$ holds for any nowhere dense compact sets. It follows from the fact, that $A(X, \bar{\mathfrak{z}}^d) = C(X) = R(X, \bar{\mathfrak{z}}^d)$ for such compact sets, which was proved in [3]

The affirmative answer to the above question in the case of Carathéodory compact sets (which seems rather plausible) will give the criterion for coincidence of modules $A(X, \bar{\mathfrak{z}}^{k_1}, \dots, \bar{\mathfrak{z}}^{k_m})$ and $P(X, \bar{\mathfrak{z}}^{k_1}, \dots, \bar{\mathfrak{z}}^{k_m})$ for such X.

The next result shows that in the case when $\Omega \in ND_d$ is a Carathéodory domain, the modules $R(\partial\Omega, \overline{\Omega}, \bar{\mathfrak{z}}^{k_1}, \dots, \bar{\mathfrak{z}}^{k_m})$ and $R(\partial\Omega, \overline{\Omega}, \bar{\mathfrak{z}}^d)$ are different.

Proposition 3. *Let $\Omega \in ND_d$ be a Carathéodory domain such that there exists at least one point $z_0 \in \Omega$ such that for the functions u and v being taken from (5) we have $v(z_0) = 0$ and $u(z_0) \neq 0$. Then*

$$R(\partial\Omega, \overline{\Omega}, \bar{\mathfrak{z}}^{k_1}, \dots, \bar{\mathfrak{z}}^{k_m}) \neq R(\partial\Omega, \overline{\Omega}, \bar{\mathfrak{z}}^d).$$

Proof. In order to prove the proposition we show that $\bar{\mathfrak{z}}^{k_m} \notin R(\overline{\Omega}, \overline{\Omega}, \bar{\mathfrak{z}}^d)$. Assume that it is not true, then for any $\varepsilon > 0$ there exist two rational functions $f_1, f_2 \in \mathcal{R}(\overline{\Omega})$ such that

$$\|f_1 + \bar{\mathfrak{z}}^d f_2 - \bar{\mathfrak{z}}^{k_m}\|_{\partial\Omega} < \varepsilon.$$

Notice that $\ell_m = k_m/d > 1$. Taking a conformal mapping φ from $\mathbb{D}$ onto Ω and letting $u_1 = u \circ \varphi$ and $v_1 = v \circ \varphi$, we obtain that, for almost all $\xi \in \mathbb{T}$, the following inequality is satisfied

$$\left| f_1(\varphi(\xi)) v_1(\xi)^{\ell_m} + f_2(\varphi(\xi)) u_1(\xi) v_1(\xi)^{\ell_m - 1} - u_1(\xi)^{\ell_m} \right| < \varepsilon M,$$

where M is the essential supremum of the function $v_1^{\ell_m}$ on $\mathbb{T}$. It follows from the maximum modulus principle for H^∞ functions that in the previous inequality we can put $\varphi^{-1}(z_0)$ instead of ξ. Then $|u(z_0)|^{\ell_m} < \varepsilon M$ which gives a contradiction for sufficiently small ε. □

We remark that the condition on v in Proposition 3 is of a technical nature. This condition is fulfilled in all known examples of d-Nevanlinna domains and it is fairly plausible that it holds for any such domains.

References

[1] A. G. O'Farrell, Annihilators of rational modules, J. Functional Analysis, 19 (1975), no. 4, 373–389.

[2] J. Verdera, Approximation by rational modules in Sobolev and Lipschitz norms, J. Functional Analysis, 58 (1984), 267–290.

[3] J. J. Carmona, A necessary and sufficient condition for uniform approximation by certain rational modules, Proc. Amer. Math. Soc., 86 (1982), no. 3, 487–490.

[4] J. J. Carmona, Mergelyan approximation theorem for rational modules, J. Approx. Theory, 44 (1985), 113–126.

[5] J. L.-M. Wang, A localization operator for rational modules, Rocky Mountain J. Math., 19 (1989), no. 4, 999–1002.

[6] T. Trent, J. L.-M. Wang, Uniform approximation by rational modules on nowhere dense sets, Proc. Amer. Math. Soc., 81 (1981), no. 1, 62–64.

[7] T. Trent, J. L.-M. Wang, The uniform closure of rational modules, Bull. London Math. Soc., 13 (1981), 415–420.

[8] M. Ya. Mazalov, P. V. Paramonov, K. Yu. Fedorovskiy. *Conditions for C^m-approximability of functions by solutions of elliptic equations*, Russian Math. Surveys, 67 (2012), no. 6, 1023–1068.

[9] P. Koosis, Introduction to H^p spaces, Cambridge University Press, Cambridge, 1998.

[10] Ch. Pommerenke, Boundary behavior of the conformal maps, Springer-Verlag, Berlin, 1992.

[11] A. G. Vitushkin, The analytic capacity of sets in problems of approximation theory, Russian Math. Surveys, 22 (1967), no. 6, 139–200.

[12] M. Ya. Mazalov, Uniform approximations by bianalytic functions on arbitrary compact subsets of $\mathbb{C}$, Sb. Math., 195 (2004), no. 5, 687–709.

[13] M. Ya. Mazalov, A criterion for uniform approximability on arbitrary compact sets for solutions of elliptic equations, Sb. Math., 199 (2008), no. 1, 13–44.

[14] K. Yu. Fedorovskii, Uniform n-analytic polynomial approximations of functions on rectifiable contours in $\mathbb{C}$, Math. Notes, 59 (1996), no. 4, 435–439.

[15] J. J. Carmona, K. Yu. Fedorovskiĭ, P. V. Paramonov, On uniform approximation by polyanalytic polynomials and the Dirichlet problem for bianalytic functions, Sb. Math., 193 (2002), no. 10, 1469–1492.

[16] A. Boivin, P. M. Gauthier, P. V. Paramonov, Uniform approximation on closed subsets of $\mathbb{C}$ by polyanalytic functions, Izv. Math., 68 (2004), no. 3, 447–459.

[17] J. J. Carmona, K. Yu. Fedorovskiy, Conformal maps and uniform approximation by polyanalytic functions, Oper. Th. Adv. Appl., 158 (2005), 109–130.

[18] S. N. Mergelyan, Uniform approximation to functions of a complex variable, Amer. Math. Soc., Transl., 101 (1954), 294–391.

[19] E. Bishop, Boundary measures of analytic differentials, Duke Math. J., 27 (1960), no. 3, 331–340.

[20] R. G. Douglas, H. S. Shapiro, A. L. Shields, Cyclic vectors and invariant subspaces for the backward shift operator, Annales de l'institut Fourier, 20 (1970), no. 1, 37–76.

[21] N. K. Nikol'skiĭ, Treatise on the shift operator, Grundlehren Math. Wiss., vol. 273, Springer-Verlag, Berlin–Heidelberg, 1986.

[22] N. K. Nikolski, Operators, functions, and systems: an easy reading, Vols. 1 and 2, Math. Surveys Monogr., vol. 92–93, American Mathematical Society, Providence, RI, 2002.
[23] K. Yu. Fedorovskiy, On some properties and examples of Nevanlinna domains, Proc. Steklov Inst. Math., 253 (2006), 186–194.
[24] A. D. Baranov, K. Yu. Fedorovskiy, Nevanlinna domain and univalent functions in model subspaces, Sb. Math., 202 (2011), no. 12, 1723–1740.

A. D. Baranov:
St. Petersburg State University, St. Petersburg, Russia
E-mail address: anton.d.baranov@gmail.com

J. J. Carmona:
Universitat Autonoma de Barcelona, Spain
E-mail address: carmona@mat.uab.cat

K. Yu. Fedorovskiy:
Bauman Moscow State Technical University, Moscow, Russia
E-mail address: kfedorovs@yandex.ru