COHOMOLOGICAL UNIQUENESS, MASSEY PRODUCTS AND THE MODULAR ISOMORPHISM PROBLEM FOR 2-GROUPS OF MAXIMAL NILPOTENCY CLASS

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Abstract. Let G be a finite 2-group of maximal nilpotency class, and let BG be its classifying space. We prove that iterated Massey products in $H^*(BG;\mathbb{F}_2)$ do characterize the homotopy type of BG among 2-complete spaces with the same cohomological structure. As a consequence we get an alternative proof of the modular isomorphism problem for 2-groups of maximal nilpotency class.

1. Introduction

Let G be a finite p-group, and let BG be its classifying space. In this work we consider the cohomological uniqueness of BG: choose some mod-p cohomological invariants and consider $\mathcal{X}$ a p-complete space such that BG and $\mathcal{X}$ agree on these cohomological invariants, does it mean that $\mathcal{X}$ is homotopy equivalent to BG? As one may expect, the answer to this question depends on the mod-p cohomological invariants chosen.

In [7], C. Broto and R. Levi initiated the study of the cohomological uniqueness of BG in terms of Steenrod operations and Bockstein spectral sequences. In this setting, they proved the cohomological uniqueness of the classifying space of every dihedral 2-groups [7], and every quaternion group [8]. Unfortunately, the available techniques seem not to be strong enough to decide the cohomological uniqueness of the classifying space of semidihedral 2-groups in terms of Steenrod operations and Bockstein spectral sequences.

In order to give a unified approach to the cohomological uniqueness of BG, when G is a finite 2-group of maximal nilpotency class (i.e. dihedral, quaternion and semidihedral 2-groups), we propose a different set of mod-p cohomology invariants: the algebra structure of $H^*(BG;\mathbb{F}_p)$ (not taking into account Steenrod operations), and iterated Massey products in this algebra (see Section 2 for precise definitions). In this setting we prove:

Theorem 1.1. *Let G be a finite 2-group of maximal nilpotency class. Let $\mathcal{X}$ be a 2-complete topological space having the homotopy type of a CW-complex such that $H^*(\mathcal{X};\mathbb{F}_2) \cong H^*(BG;\mathbb{F}_2)$ as algebras with iterated Massey products. Then $\mathcal{X} \simeq BG$.*

Proof. We first describe $H^*(BG;\mathbb{F}_2)$ as algebra with iterated Massey products in Section 4. This structure is used to construct a homotopy equivalence $\mathcal{X} \to BG$ whenever G is dihedral (Theorem 5.2), quaternion (Theorem 5.4) or semidihedral (Theorem 5.7). □

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Besides of its own topological interest, the study of the cohomological uniqueness of classifying spaces may have implications in a long standing algebraic problem: the modular isomorphism problem.

The modular isomorphism problem asks whether a p-group G is determined by $\mathbb{F}_p[G]$, its group algebra over the field of p-elements. That is, given another finite p-group H such that $\mathbb{F}_p[G] \cong \mathbb{F}_p[H]$ as rings, does it mean that $G \cong H$ as groups? The general question still remains open, a positive answer to this question is known for a few families of p-groups (see [2, Introduction] for an up to date list of results).

Then the interplay between cohomological uniqueness of classifying spaces and the modular isomorphism problem is clear: let G be a finite p-group such that BG is cohomology unique when considered a set of mod-p cohomological invariants that can be deduced from the group algebra $\mathbb{F}_p[G]$, then the modular isomorphism problem has a positive answer for G.

So we obtain a positive answer to the modular isomorphism problem for finite 2-groups of maximal nilpotency class (see [9], and [1] for other approaches to this result):

Corollary 1.2. *Let G be a finite 2-group of maximal nilpotency class, and let H be a finite 2-group such that $\mathbb{F}_2[G] \cong \mathbb{F}_2[H]$ as rings, then $G \cong H$ as groups.*

Proof. Given any finite p-group K, the algebra structure with iterated Massey products of $H^*(BK;\mathbb{F}_p)$ can be obtained from the ring structure of $\mathbb{F}_p[G]$ by means of the Yoneda cocomplex (e.g. see [5, Theorem 2.3 and Lemma 2.4]). Hence, if $\mathbb{F}_2[G] \cong \mathbb{F}_2[H]$ as rings, then $H^*(BH;\mathbb{F}_2) \cong H^*(BG;\mathbb{F}_2)$ as algebras with iterated Massey products. Then, according to Theorem 1.1, $BG \simeq BH$ (recall that the classifying space of a finite p-group is a p-complete space), and $G \cong H$ as groups. $\square$

We finish this introduction recalling that iterated Massey products of degree one classes have been previously used by I.C. Borge in [5] and [4] to provide a cohomological classification of finite p-groups. Unfortunately, her results cannot be applied in our topological framework: in Remark 5.5, we give an example showing that iterated Massey products of degree one classes cannot isolate the homotopy type of the classifying space of quaternion groups.

Organization of the paper: In Section 2, we briefly review the definition, main properties and tools needed in the computation of iterated Massey products. Section 3 is devoted to give representations of maximal nilpotency class 2-groups in $\mathrm{GL}(\mathbb{F}_2)$. In Section 4, we compute iterated Massey products in the cohomology of these groups and finally, in Section 5, we use this structure to prove Theorem 1.1 case by case.

Notation: In general the following notations will be used in the rest of the paper: group elements are denoted with lower case letters (x,y, ...), while roman capital letters (X, Y, ...) and calligraphic capital letters ($\mathcal{X}$ and $\mathcal{Y}$) are used to denote cohomology generators, and topological spaces respectively.

Unless otherwise stated, cohomology means cohomology with trivial coefficients over the field $\mathbb{F}_2$, so $H^*(\mathcal{X}) = H^*(\mathcal{X};\mathbb{F}_2)$.

2. Iterated Massey products: definition and properties

In this section we quickly review the theory of iterated Massey products. A more detailed description can be found in [12], or [13].

Let R be a ring with unit. Consider $C^*(G,R)$ the cochain algebra of the group G with coefficients in R, and d the coboundary operator in $C^*(G,R)$.

Definition 2.1. Let $\{X_i\}_{1\leq i\leq n}$ be homogeneous elements in $H^*(G,R)$. A *defining system for the nth order iterated Massey product* $\langle X_1,\ldots,X_n\rangle$, is an upper triangular matrix

$$M=\{m_{i,j}\mid 1\leq i\leq n+1, i<j\leq n+1, (i,j)\neq(1,n+1)\}$$

with coefficients in $C^*(G,R)$ such that:

(i) $m_{i,i+1}$ is a representative for X_i and

(ii) $dm_{i,j}=\displaystyle\sum_{k=i+1}^{j-1} m_{i,k}\cup m_{k,j}$ $(j\neq i+1)$.

Definition 2.2. Given M a defining system for $\langle X_1,\ldots,X_n\rangle$, the *value of the product relative to this defining system*, denoted $\langle X_1,\ldots,X_n\rangle_M$, is the element in $H^*(G;R)$ represented by the cocycle:

$$\sum_{k=2}^{n} m_{1,k}\cup m_{k,n+1}\,.$$

The *nth order iterated Massey product* $\langle X_1,\ldots,X_n\rangle$ is defined as the set of elements which can be written as $\langle X_1,\ldots,X_n\rangle_M$ for some defining system M. The *indeterminacy in the iterated Massey product* $\langle X_1,\ldots,X_n\rangle$ is defined as the set of elements Z which can be expressed as $Z=Y_1-Y_2$ for Y_1 and Y_2 in $\langle X_1,\ldots,X_n\rangle$.

We now enumerate some of the properties which are used later:

(i) The iterated Massey product $\langle X_1,\ldots,X_n\rangle$ is not defined for all $X_1,\ldots,X_n$ in $H^*(G;R)$. For example, $\langle X_1,X_2,X_3\rangle$ is defined if and only if $X_1\cup X_2=X_2\cup X_3=0$.

(ii) The degree of an element in $\langle X_1,\ldots,X_n\rangle$ is $\sum\deg(X_i)-n+2$.

(iii) If $f\colon\mathcal{X}\to\mathcal{Y}$ is a continuous map of topological spaces, and $Y_1,\ldots,Y_r$ are cohomology classes in $H^*(\mathcal{Y};R)$ such that $\langle Y_1,\ldots,Y_r\rangle$ is defined, then so is $\langle f^*(Y_1),\ldots,f^*(Y_r)\rangle$ and

$$f^*(\langle Y_1,\ldots,Y_r\rangle)\subset\langle f^*(Y_1),\ldots,f^*(Y_r)\rangle.$$

Moreover, if f^* is an isomorphism, equality holds.

The next result follows from May's proof of [13, Theorem 1.5]:

Lemma 2.3. *Let $f\colon\mathcal{X}\to\mathcal{Y}$ be a continuous map such that $f^k\colon H^k(\mathcal{Y};R)\to H^k(\mathcal{X};R)$ is an isomorphism for $k\leq n$. Let $Y_1,\ldots,Y_r$ be elements in $H^*(\mathcal{Y};R)$ such that $\sum\deg(Y_i)-r+2\leq n$. Then:*

(a) *$\langle Y_1,\ldots,Y_r\rangle$ is defined if and only if $\langle f^*(Y_1),\ldots,f^*(Y_r)\rangle$ is so.*

(b) *$f^*(\langle Y_1,\ldots,Y_r\rangle)=\langle f^*(Y_1),\ldots,f^*(Y_r)\rangle$.*

Let p be a fixed prime. In this work, we compare the mod-p cohomology algebras (with iterated Massey products) of spaces. This is done by considering a special kind of morphisms:

Definition 2.4. Let $\varphi\colon H^*(\mathcal{Y};\mathbb{F}_p)\to H^*(\mathcal{X};\mathbb{F}_p)$ be a morphism (not necessarily induced by a continuous map of topological spaces). We say that φ is an *$\mathcal{M}$-isomorphism* if

(1) φ is an $\mathbb{F}_p$-algebras isomorphism and

(2) for all $r \geq 1$ and $Y_1, \ldots, Y_r$ elements in $H^*(\mathcal{Y};\mathbb{F}_p)$ such that $\langle Y_1,\ldots,Y_r\rangle$ is defined, then $\langle \varphi(Y_1),\ldots \varphi(Y_r)\rangle$ is defined and

$$\varphi(\langle Y_1,\ldots,Y_r\rangle) = \langle \varphi(Y_1),\ldots \varphi(Y_r)\rangle$$

Definition 2.5. Let $\mathcal{X}$ and $\mathcal{Y}$ be topological spaces. We say that $\mathcal{X}$ and $\mathcal{Y}$ are $\mathcal{M}$-*comparable* if there exists an $\mathcal{M}$-isomorphism $\varphi\colon H^*(\mathcal{Y};\mathbb{F}_p) \to H^*(\mathcal{X};\mathbb{F}_p)$. We say that $\mathcal{X}$ *is determined by its* $\mathcal{M}$-*cohomology* if for any p-complete space $\mathcal{Y}$ having the homotopy type of a CW-complex and such that it is $\mathcal{M}$-comparable to $\mathcal{X}$ we have that $\mathcal{X} \simeq \mathcal{Y}$.

The easiest examples of spaces determined by its $\mathcal{M}$-cohomology are provided by classifying spaces of some particular families of p-groups:

Proposition 2.6. *Let E be an elementary abelian p-group, then BE is determined by its $\mathcal{M}$-cohomology.*

Proof. Let E be a rank r elementary abelian p-group. First we prove the algebra structure of $H^*(BE;\mathbb{F}_p)$ with iterated Massey products determines the Steenrod operations and the Bockstein spectral sequence. We consider two cases:

- If $p=2$, then $H^*(BE;\mathbb{F}_2) = \mathbb{F}_2[x_1,\ldots,x_r]$ where every x_i has degree 1. Therefore, the unstability axiom forces $\operatorname{Sq}^1 x_i = x_i^2$, and this determines completely the Steenrod operations and the Bockstein spectral sequence.
- If $p>2$, then $H^*(BE;\mathbb{F}_2) = \Lambda(u_1,\ldots,u_r)\otimes\mathbb{F}_2[v_1,\ldots,v_r]$ where every u_i has degree 1, and every v_i has degree 2. Again, the unstability axiom forces $\mathcal{P}^1 u_i = 0$ and $\mathcal{P}^1 v_i = v_i^2$, and this determines completely the Steenrod operations. Moreover, according to [12, Theorem 19], $\beta_1(u_i) = \langle u_i, \overset{p)}{\ldots}, u_i\rangle = v_i$, what completely determines the Bockstein spectral sequence.

Finally, BE is determined by its cohomology with Steenrod operations and Bockstein spectral sequence [7, Proposition 1.5], thus the result follows. □

2.1. **Iterated Massey products of degree one elements.** In this subsection we recall the work of W.G. Dwyer [11, Section 2], that relates iterated Massey products of degree one elements in the cohomology of a group and representations of this group in the upper triangular matrices.

Let $U(R,n)$ be the multiplicative group of all upper triangular $n\times n$ matrices over R which agree with the identity matrix along the diagonal. The subgroup $Z(R,n)$ of $U(R,n)$ consists of matrices which are identically zero except along the diagonal and at position $(1,n)$. We get that $Z(R,n)\cong R$ and it is contained in the center of $U(R,n)$, so it gives rise to the central extension:

$$Z(R,n) \to U(R,n) \to \overline{U}(R,n) \overset{\text{def}}{=} U(R,n)/Z(R,n)\,. \tag{1}$$

Given a group homomorphism $\phi\colon G \to U(R,n)$, the image of $g\in G$ is a matrix with coefficients $\phi_{i,j}(g)\in R$. Remark that the elements $\phi_{i,i+1}(g)$ satisfy the equation:

$$\phi_{i,i+1}(g_1g_2) = \phi_{i,i+1}(g_1) + \phi_{i,i+1}(g_2)\,.$$

So $\phi_{i,i+1}$ are group morphisms from G to R, and thus represent cohomology classes in $H^1(G;R)$. These elements are called *near diagonal elements* of ϕ.

Theorem 2.7 ([11]). *Let $X_1, \dots, X_n$ be elements in $H^1(G;R)$. There is a one-one correspondence $M \leftrightarrow \phi_M$ between defining systems M for $\langle X_1, \dots, X_n\rangle$ and group homomorphisms $\phi_M \colon G \to \overline{U}(R, n+1)$ which have $-X_1, \dots, -X_n$ as near-diagonal components. Moreover, given ϕ_M, we can pull back the extension in Equation (1), getting an extension of G by R. Then $\langle X_1, \dots, X_n\rangle_M$ is exactly the characteristic class of this extension.*

2.2. **Iterated Massey products in higher degrees.** In order to compute iterated Massey products of higher degree elements we use the Yoneda cocomplex [4, Section 2.5]:

Fix G a p-group and $P_\bullet \to \mathbb{F}_p$ a $\mathbb{F}_p[G]$-free resolution with differentials $\partial_i \colon P_{i+1} \to P_i$.

Definition 2.8. The *Yoneda cocomplex* $\mathrm{Hom}^{(\bullet)}_{\mathbb{F}_p[G]}(P_\bullet, P_\bullet)$ is defined as:

- In degree i we have the $\mathbb{F}_p[G]$-module:
$$\mathrm{Hom}^{i}_{\mathbb{F}_p[G]}(P_\bullet, P_\bullet) = \prod_{n\in\mathbb{Z}} \mathrm{Hom}_{\mathbb{F}_p[G]}(P_{n+i}, P_n), \quad i \geq 0$$
and 0 otherwise.
- The differential of $\phi^i = \{\phi^i_n \colon P_{n+i} \to P_n\}_{n\in\mathbb{Z}} \in \mathrm{Hom}^{i}_{\mathbb{F}_p[G]}(P_\bullet, P_\bullet)$, is defined as:
$$\delta^i(\phi^i_n) = \partial_{n-1}\phi^i_n - (-1)^i \phi^i_{n-1}\partial_{n+i-1}\,.$$
- The algebra structure in the cohomology of $\mathrm{Hom}^{(\bullet)}_{\mathbb{F}_p[G]}(P_\bullet, P_\bullet)$ is induced by the composition of elements as a cochain morphisms.

The following result tells us that we can use this tool to work with the cohomology of a group:

Theorem 2.9 ([4]). *$H^i(\mathrm{Hom}^{(\bullet)}_{\mathbb{F}_p[G]}(P_\bullet, P_\bullet), \delta^\bullet) \simeq \mathrm{Ext}^i_{\mathbb{F}_p[G]}(\mathbb{F}_p, \mathbb{F}_p)$ for all $i \geq 0$.*

3. Representations of maximal nilpotency class 2-groups

The maximal nilpotency class finite 2-groups are precisely the dihedral, quaternion and semidihedral groups. In this section we deal with these three families of 2-groups, obtaining a representation of these groups on $\overline{U}(\mathbb{F}_2, n)$ which will allow us to compute iterated Massey products of some degree one elements.

3.1. **Notation.** Consider the following finite presentations of the dihedral, quaternion and semidihedral groups:

$$\begin{aligned} D_{2^n} &= \langle x, y \mid x^{2^{n-1}} = 1, y^2 = 1, yxy^{-1} = x^{-1}\rangle, \\ Q_{2^n} &= \langle x, z \mid x^{2^{n-1}} = 1, z^2 = x^{2^{n-2}}, zxz^{-1} = x^{-1}\rangle \text{ and} \\ SD_{2^n} &= \langle x, t \mid x^{2^{n-1}} = 1, t^2 = 1, txt^{-1} = x^{2^{n-2}-1}\rangle\,. \end{aligned} \tag{2}$$

And also the following cohomology rings with coefficients in $\mathbb{F}_2$:

$$\begin{aligned} &H^*(BD_4) \cong \mathbb{F}_2[X, Y], \\ &H^*(BD_{2^n}) \cong \mathbb{F}_2[X, Y, W]/(X^2 + XY) \text{ for } n \geq 3, \\ &H^*(BQ_8) \cong \mathbb{F}_2[X, Y, V]/(X^2 + XY + Y^2, X^2Y + XY^2), \\ &H^*(BQ_{2^n}) \cong \mathbb{F}_2[X, Y, V]/(X^2 + XY, Y^3) \text{ for } n \geq 4 \text{ and} \\ &H^*(BSD_{2^n}) \cong \mathbb{F}_2[X, Y, U, V]/(X^2 + XY, XU, X^3, U^2 + (X^2 + Y^2)V) \\ &\qquad \text{for } n \geq 4. \end{aligned} \tag{3}$$

where $\deg(X) = \deg(Y) = 1$, $\deg(W) = 2$, $\deg(U) = 3$ and $\deg(V) = 4$ in all the cases.

The results in this paper will not require to keep the structure of algebra over the Steenrod Algebra, but in some proofs we will use that such a structure exists. More precisely we will use that it is unstable and that it is known that $\mathrm{Sq}^1(W) = WY$ in $H^*(BD_{2^n})$ $(n \geq 3)$.

There is also a tower of principal fibrations:

$$\cdots \xrightarrow{\pi_n} BD_{2^n} \xrightarrow{\pi_{n-1}} BD_{2^{n-1}} \xrightarrow{\pi_{n-2}} \cdots \xrightarrow{\pi_3} BD_8 \xrightarrow{\pi_2} BD_4 \tag{4}$$

where π_2 is classified by the class $X^2 + XY \in H^2(BD_4)$ and π_n by $W \in H^2(BD_n)$ for $n \geq 3$.

The quaternion and semidihedral groups fit in the following central extensions:

$$\mathbb{Z}/2 \to Q_{2^n} \to D_{2^{n-1}} \text{ and } \mathbb{Z}/2 \to SD_{2^n} \to D_{2^{n-1}} \tag{5}$$

classified by the following classes in $H^2(BD_{2^{n-1}})$: $X^2 + XY + Y^2$ in the case Q_8, $W + Y^2$ in the case Q_{2^n} $(n \geq 4)$ and $W + X^2$ in the case SD_{2^n} $(n \geq 4)$.

3.2. **Representations of maximal nilpotency class 2-groups.** In this subsection we will give a explicit minimal degree faithful representation over $\mathbb{F}_2$ of the maximal nilpotency class 2-groups. The definition of the images of the generators of each group is defined using the matrices described below:

Consider A_n the $2^n \times 2^n$ matrix defined inductively:

$$A_0 = (1) \quad , \quad A_n = \begin{pmatrix} A_{n-1} & A_{n-1} \\ 0 & A_{n-1} \end{pmatrix} , \tag{6}$$

where 0 means a matrix with all the entries equal zero.

Consider also the $2^n \times 2^n$-matrix B_n with entries $b_{i,j}$:

$$b_{i,j} \stackrel{\text{def}}{=} \begin{cases} 1 & \text{if } i = j \text{ or } j = i + 1, \\ 0 & \text{otherwise.} \end{cases} \tag{7}$$

Finally consider the $2^n \times 2^n$-matrix C_n with entries $c_{i,j}$:

$$c_{i,j} \stackrel{\text{def}}{=} \begin{cases} 1 & \text{if } i = j, \\ 1 & \text{if } i = 1 \text{ and } j = 2^n, \\ 0 & \text{otherwise.} \end{cases} \tag{8}$$

The following calculation gives the property needed to compute the iterated Massey products:

Lemma 3.1. *Let B be a matrix in $\mathrm{GL}_m(\mathbb{F}_2)$ as defined in Equation (7) (now m is not necessarily of the form 2^n). Then:*

(a) *The order of B is 2^n, where $2^{n-1} < m \leq 2^n$.*
(b) *If l is a positive integer such that $2^l < \mathrm{Order}(B)$, then the coefficients in positions $(i, i+2^l)$ in B^{2^l} are equal to 1 for i from 1 to $m - 2^l$.*

Proof. B is the sum of the identity Id and a nilpotent matrix N. Now we can compute $(\mathrm{Id} + N)^l$ using the Newton coefficients formula and both (a) and (b) follow. □

Lemma 3.2. *Fix $n \geq 1$. The matrices A_n, B_n and C_n defined in Equations (6), (7) and (8) have the following properties:*

(a) $A_n^2 = \mathrm{Id}$,

(b) *C_n is in the center of the invertible upper triangular matrices, so, in particular, commute with A_n and B_n, and*
(c) *$A_nB_nA_n$ has all the entries over the diagonal equal* 1.

Proof. (a) Use induction on n and the fact that all coefficient are taken in $\mathbb{F}_2$.
(b) If U is an invertible upper triangular $2^n\times 2^n$-matrix, the matrix UC_n has the same entries as U but the last element of the first row, which changes by $+1$. Multiplying C_nU then we have to add 1 to the first element of the last column, getting the same result.
(c) This is a direct computation using induction on n.

□

Now we will use the matrices A_n, B_n and C_n to construct the following $(2^n+1)\times(2^n+1)$ matrices:

$$x_n \stackrel{\text{def}}{=} \left(\begin{array}{c|ccccc} 1 & 1 & 1 & 0 & \cdots & 0 \\ \hline 0 & & & & & \\ \vdots & & & B_n & & \\ 0 & & & & & \end{array}\right), \; y_n \stackrel{\text{def}}{=} \left(\begin{array}{c|ccc} 1 & 0 & \cdots & 0 \\ \hline 0 & & & \\ \vdots & & A_n & \\ 0 & & & \end{array}\right),$$

$$z_n \stackrel{\text{def}}{=} \left(\begin{array}{c|c|c} 1 & 0 \;\cdots\; 0 & 0 \;\cdots\; 0 \;\; 1 \;\; 0 \\ \hline 0 & & \\ \vdots & A_{n-1} & A_{n-1}C_{n-1} \\ 0 & & \\ \hline 0 & & \\ \vdots & 0 & A_{n-1} \\ 0 & & \end{array}\right) \quad \text{and}$$

$$t_n \stackrel{\text{def}}{=} \left(\begin{array}{c|c|c} 1 & 0 \;\cdots\; 0 & 0 \;\cdots\; 0 \\ \hline 0 & & \\ \vdots & A_{n-1} & A_{n-1}C_{n-1} \\ 0 & & \\ \hline 0 & & \\ \vdots & 0 & A_{n-1} \\ 0 & & \end{array}\right).$$

Lemma 3.3. *The $(2^n+1)\times(2^n+1)$ matrices x_n, y_n, z_n and t_n defined above have the following properties:*
(a) *x_n has order 2^{n+1},*
(b) *y_n has order 2,*
(c) *$y_nx_ny_n = x_n^{-1}$,*
(d) *$z_n^2 = x_n^{2^n}$, in particular z_n has order 4,*
(e) *$z_nx_nz_n^{-1} = x_n^{-1}$,*
(f) *t_n has order 2 and*
(g) *$t_nx_nt_n^{-1} = x^{2^n-1}$.*

So x_n and y_n generate a subgroup isomorphic to $D_{2^{n+2}}$; x_n and z_n a subgroup isomorphic to $Q_{2^{n+2}}$, and x_n and t_n a subgroup isomorphic to $SD_{2^{n+2}}$.

Proof. All can be checked again by induction and using Lemma 3.2. □

Now we want to check that the previous representations of the dihedral, quaternion and semidihedral groups are the smaller rank ones. To check this we will compute the exponent of a Sylow 2-subgroup of a general linear group over $\mathbb{F}_2$.

Lemma 3.4. *Let M be an element of order 2^n in $\mathrm{GL}_m(\mathbb{F}_2)$. Then $2^{n-1} < m$. Moreover, there exists an element of order 2^n in the linear group $\mathrm{GL}_{2^{n-1}+1}(\mathbb{F}_2)$.*

Proof. Let M be an element of order 2^n, then M satisfies the polynomial $X^{2^n} - 1$. As we are in characteristic 2 we get $X^{2^n} - 1 = (X-1)^{2^n}$. The minimal polynomial of M must be of the form $(X-1)^r$, and the characteristic polynomial must be $(X-1)^m$, with $r \leq m$. If $2^{n-1} \geq r$, then M would satisfy the polynomial $(X-1)^{2^{n-1}} = (X^{2^{n-1}} - 1)$, so the order of M would be at most 2^{n-1}. So we get $2^{n-1} < r \leq m$.

Finally, the element x_{n-1} in Lemma 3.3 is of order 2^n in $\mathrm{GL}_{2^{n-1}+1}(\mathbb{F}_2)$. □

4. Iterated Massey products in the cohomology of maximal nilpotency class 2-groups

This section is devoted to the computation of the iterated Massey products in the cohomology of dihedral, quaternion and semidihedral groups. The results presented here as Lemmas 4.2, 4.3, 4.4 and 4.5 can be summarized in the following theorem:

Theorem 4.1. *Consider D_{2^n} $(n \geq 2)$, Q_{2^n} $(n \geq 3)$ and SD_{2^n} $(n \geq 4)$ the dihedral, quaternion and semidihedral groups of order 2^n, and the generators of their cohomology as denoted in Equation* (3)*. Then:*

(a) *Neither W, $W + Y^2$ nor $W + X^2$ is contained in an iterated Massey product of degree one elements in the cohomology of D_{2^n} of order less than 2^{n-1}.*
(b) *The 2^{n-1}th order iterated Massey product in the cohomology of D_{2^n} defined as $\langle X, X+Y, \ldots, X, X+Y\rangle$ contains W, $W+Y^2$ and $W+X^2$.*
(c) *The 2^{n-1}th order iterated Massey product in the cohomologies of D_{2^n}, Q_{2^n} and SD_{2^n} defined as $\langle X, X+Y, \ldots, X, X+Y\rangle$ does not contain the zero element.*
(d) *The mth order iterated Massey product $\langle X, X+Y, X, X+Y, \ldots\rangle$ is not defined for $m > 2^{n-1}$ in any of the cohomologies of D_{2^n}, Q_{2^n} and SD_{2^n}.*
(e) *$\langle Y, Y^2, Y, Y^2\rangle = \{V\}$ in the cohomology of Q_{2^n}.*
(f) *$\langle X, X^2, Y\rangle = \{U, U+Y^3\}$ and $\langle X, X^2, X, X^2\rangle = \{V, V+YU\}$ in the cohomology of SD_{2^n}.*

The proof of this theorem will be done in the next two subsections: the first one is devoted to the computations of iterated Massey products of degree one elements while the second will use the Yoneda cocomplex to prove last two statements in the theorem.

4.1. Iterated Massey products of degree one elements.

The representations described in Section 3 allows us to compute some iterated Massey products in the cohomology of the dihedral, quaternion and semidihedral groups:

Lemma 4.2. *Consider the cohomology of D_{2^n}, $n \geq 3$ as denoted in Equation* (3)*. Then:*

(a) *Neither W, $W+Y^2$ nor $W+X^2$ is contained in an iterated Massey product of degree one elements of order less than 2^{n-1}.*

(b) *The* 2^{n-1}*th order iterated Massey product* $\langle X, X+Y, \dots, X, X+Y\rangle$ *contains* W, $W+Y^2$ *and* $W+X^2$.

Proof. Assume first that an iterated Massey product of length m contains either W, $W+Y^2$ or $W+X^2$. Then we would have the following diagram:

$$\begin{tikzcd}
 & \mathbb{Z}/2 \arrow[r, equal] \arrow[d] & \mathbb{Z}/2 \arrow[d] \\
K \arrow[r] \arrow[d, equal] & G \arrow[r] \arrow[d] & U(2, m+1) \arrow[d] \\
K \arrow[r] & D_{2^n} \arrow[r] & \overline{U}(2, m+1)
\end{tikzcd}$$

where the bottom right square is a pull-back, K is defined as the kernel of the horizontal arrows and the vertical lines are central extensions.

As either W, $W+Y^2$ or $W+X^2$ classifies the extension, then G is isomorphic to either $D_{2^{n+1}}$, $Q_{2^{n+1}}$ or $SD_{2^{n+1}}$. The center of G is exactly $\mathbb{Z}/2$. If K is non trivial, as it is a normal subgroup in G, then K intersects non-trivially the center of G, so it contains the center of G. But this implies that the center of G maps injectively to D_{2^n}, and it contradicts the exactness of the vertical line. This implies that K is trivial, so, there is an injection of G in $U(2, m+1)$, so by Lemma 3.4, $m \geq 2^{n-1}$.

The representations in Lemma 3.3 and Theorem 2.7 tell us that the 2^{n-1}th order iterated Massey product $\langle X, X+Y, \dots, X, X+Y\rangle$ contains W, $W+Y^2$ and $W+X^2$. □

The following Lemma will use the same notation for the generators of $H^1(G)$ for different G, as noted in Equation (3) and the result applies to all of them:

Lemma 4.3. *Consider the cohomology of the dihedral, quaternion and semidihedral groups of order* 2^n *as denoted in Equation* (3)*. Then, for all these groups:*

(a) *The* 2^{n-1}*th order iterated Massey product defined as* $\langle X, X+Y, \dots, X, X+Y\rangle$ *does not contain the zero element.*
(b) *The* m*th order iterated Massey product* $\langle X, X+Y, X, X+Y, \dots\rangle$ *is not defined for* $m > 2^{n-1}$.

Proof. Assume that the zero element is in a 2^{n-1}th order iterated Massey product of type $\langle X, X+Y, \dots, X, X+Y\rangle$ of the cohomology of G, where G is either D_{2^n}, Q_{2^n} or SD_{2^n}. Then there would be a group morphism from G to $\overline{U}(\mathbb{F}_2, 2^{n-1}+1)$ which lifts to a group morphism from G to $U(\mathbb{F}_2, 2^{n-1}+1)$ such that the image of x is a matrix with all the entries in position $(i, i+1)$ equals 1. The order of such an element is 2^n, bigger than the order of x, getting a contradiction.

(b) can be deduced from (a): if a mth order iterated Massey product $\langle X, X+Y, X, X+Y, \dots\rangle$ with $m > 2^{n-1}$ is defined, then the zero element must be in all the strictly shorter subproducts, in particular in the 2^{n-1}th order product of this type. □

4.2. **Iterated Massey products in higher degrees.** This subsection is devoted to the computation of some iterated Massey products involving powers of degree one cohomology generators and having as a result the higher degree generators.

We start with the quaternion groups.

Lemma 4.4. *Consider the notation in Equation* (3) *for the cohomology of* Q_{2^n}*, the quaternion group of order* 2^n *with* $n \geq 3$*. Then*

$$\langle Y, Y^2, Y, Y^2 \rangle = \{V\}\,.$$

Proof. To compute this iterated Massey product we must consider a projective resolution of these groups and find a representative of each element in the Yoneda cocomplex for the generator Y.

We can find a projective resolution of Q_{2^n} in [10, pp. 253], but before giving it we need some notation: consider x and z the generators of Q_{2^n} as given in Equation (2), and consider the following elements in $\mathbb{F}_2[Q_{2^n}]$: $I = 1+x$, $J = 1+z$, $K = 1+xz$, $L = 1+x+x^2+\cdots+x^{2^{n-2}-1}$, $N_x = 1+x+x^2+\cdots+x^{2^{n-1}-1}$ and $N = \sum_{g\in Q_{2^n}} g$.

These elements satisfy the following relations: $L = I^{2^{n-2}-1}$, $N_x = I^{2^{n-1}-1}$, $I^{2^{n-2}} = J^2 = K^2$, $I^{2^{n-1}} = J^4 = K^4 = 0$, $KI = IJ$, $K = I+J+IJ$ and $N = JN_x = N_xJ = KN_x = N_xK$.

A projective resolution of $\mathbb{F}_2$ as $\mathbb{F}_2[Q_{2^n}]$ module is given by the following periodic data:

$$\mathbb{F}_2 \xleftarrow{\varepsilon} P_0 \xleftarrow{\partial_1} P_1 \xleftarrow{\partial_2} P_2 \xleftarrow{\partial_3} P_3 \xleftarrow{\partial_4} P_4 \xleftarrow{\partial_5} P_5 \cdots$$

where $P_{4i} \cong P_{4i+3} \cong \mathbb{F}_2[Q_{2^n}]$ and $P_{4i+1} \cong P_{4i+2} \cong \mathbb{F}_2[Q_{2^n}]^2$ and the differentials: $\partial_{4i+1} = (I \quad J)$, $\partial_{4i+2} = \left(\begin{smallmatrix} L & K \\ J & I \end{smallmatrix}\right)$, $\partial_{4i+3} = \left(\begin{smallmatrix} I \\ K \end{smallmatrix}\right)$ and $\partial_{4i} = (N)$.

The element X in $H^1(BQ_{2^n})$ is represented in the Yoneda cocomplex by a cochain map $X_i\colon P_{i+1} \to P_i$ defined as follows (to cover the case Q_8 here we are using the convention $I^0 = 1$):

$$X_{4i} = (1\,0),\ X_{4i+1} = \begin{pmatrix} I^{2^{n-2}-2} & 1 \\ 0 & 1+I \end{pmatrix}, X_{4i+2} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \text{ and } X_{4i+3} = (I^{2^{n-1}-2}J)\,.$$

The element Y in $H^1(BQ_{2^n})$ is represented in the Yoneda cocomplex by a cochain map $Y_i\colon P_{i+1} \to P_i$ defined as follows: $Y_{4i} = (0 \quad 1)$, $Y_{4i+1} = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$, $Y_{4i+2} = \left(\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right)$ and $Y_{4i+3} = (N_x)$.

Finally the element V in $H^4(BQ_{2^n})$ is represented by a cochain map $V_i\colon P_{i+4} \to P_i$ which is the identity.

With all this data, compute products in these generators corresponds to compose these cochain maps, so we can easily find cochain maps representing Y^2 and Y^3.

Now we sketch here how to compute these iterated Massey product $\langle Y, Y^2, Y, Y^2\rangle$:

(i) We must find the following coefficients in a defining system:

$$\begin{pmatrix} 1 & Y & \alpha & \beta & \\ 0 & 1 & Y^2 & \alpha & \gamma \\ 0 & 0 & 1 & Y & \alpha \\ 0 & 0 & 0 & 1 & Y^2 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

such that $\delta\alpha = Y^3$, $\delta\beta = Y\alpha + \alpha Y$, $\delta\gamma = Y^2\alpha + \alpha Y^2$: remark that we just need to compute the maps in low degrees, just enough to compose them and get the product. $\alpha_i\colon P_{i+2} \to P_i$ can be taken as $\alpha_0 = (1 \quad 0)$, $\alpha_1 = \left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right)$, $\alpha_2 = \left(\begin{smallmatrix} J \\ L \end{smallmatrix}\right)$ and $\alpha_3 = (K \quad L + N_x)$. $\beta_i\colon P_{i+2} \to P_i$ can be taken as: $\beta_0 = (0 \quad 0)$, $\beta_1 = \left(\begin{smallmatrix} 0 \\ 0 \end{smallmatrix}\right)$, $\beta_2 = \left(\begin{smallmatrix} 1+J+J^2 \\ L(1+J) \end{smallmatrix}\right)$ and $\beta_3 = (K \quad L + N_x)$. Finally, $\gamma_i\colon P_{i+3} \to P_i$ can be taken as: $\gamma_0 = (0)$, $\gamma_1 = \left(\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right)$, $\gamma_2 = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$ and $\gamma_3 = (0 \quad 1)$.

(ii) Consider now the cochain $Y\gamma + \alpha^2 + \beta Y$ composing the maps described above and we obtain that it is equivalent to V.

This procedure gives us an element in $\langle Y, Y^2, Y, Y^2\rangle$, but we have done several choices. Let us see now that if we choose other elements we get again V: by [12, Theorem 3] we can fix the representatives of Y and Y^2 to construct any defining system. Assume we change all coefficients α by (possibly) α', α'', α''', β by β', and finally γ by γ' getting a new defining system. Then we get, by direct computation, that the result of this defining system differs from V by an element which can be written as $\alpha(a''X^2+b''Y^2)+(aX^2+bY^2)\alpha$, for $a, a'', b, b'' \in \mathbb{F}_2$. We use again the Yoneda cocomplex for all this generators to compute it, and this always give the zero element in cohomology, so

$$\langle Y, Y^2, Y, Y^2\rangle = \{V\}.$$

□

Lemma 4.5. *Let SD_{2^n} be a semidihedral group of order 2^n, with $n \geq 4$. Fix X, Y, U and V the generators in $H^*(SD_{2^n})$ as in Equation (3). Then*

$$\langle X, X^2, Y\rangle = \{U, U + Y^3\} \text{ and } \langle X, X^2, X, X^2\rangle = \{V, V + YU\}.$$

Proof. We begin fixing a basis of the cohomology as graded vector space in low degrees: $\{X, Y\}$ in degree one, $\{X^2, Y^2\}$ in degree 2, $\{U, Y^3\}$ in degree 3 and $\{Y^4, YU, V\}$ in degree 4.

We also consider the projective resolution of $\mathbb{F}_2$ as $\mathbb{F}_2[SD_{2^n}]$ module given in [14], using the generators x and t of SD_{2^n} as in Equation (2): define $I = 1 + x$, $J = 1 + t$, $L = 1 + x + x^2 + \cdots + x^{2^{n-2}-2}$ and $N_x = 1 + x + x^2 + \cdots + x^{2^{n-1}-1}$.

These elements satisfy the following relations, which are useful to do all the computations: $L = I^{2^{n-2}-1} + x^{2^{n-2}-1}$, $I^{2^{n-1}} = J^2 = 0$, $IJ = (1+tL)I$, $(1+tL^n)I = I(1+tL^{n-1})$, $tN_x = N_x t$, $tI^{2^{n-1}-2} = I^{2^{n-1}-2}t$, $I^{2^{n-2}-2}t = tI^{2^{n-2}-2}x^{2^{n-2}-2}$, $(1 + tL^{2i})^2 = 0$, $(1 + tL^{2i+1})^2 = N_x$ and $L^{2^{n-1}} = 1$.

Then a projective resolution is given by:

$$\mathbb{F}_2 \xleftarrow{\varepsilon} P_0 \xleftarrow{\partial_1} P_1 \xleftarrow{\partial_2} P_2 \xleftarrow{\partial_3} P_3 \xleftarrow{\partial_4} P_4 \xleftarrow{\partial_5} P_5 \cdots$$

where $P_i \cong \mathbb{F}_2[SD_{2^n}]^{i+1}$ and the differentials defined inductively: $\partial_1 = (I \quad J)$,

$$\partial_{2i} = \left(\begin{array}{c|cccc} N_x & 1 + tL^i & 0 & \cdots & 0 \\ \hline 0 & & & & \\ \vdots & & \partial_{2i-1} & & \\ 0 & & & & \end{array}\right) \quad \text{and}$$

$$\partial_{2i+1} = \left(\begin{array}{c|ccccc} I & 1+tL^i & i & 0 & \cdots & 0 \\ \hline 0 & & & & & \\ \vdots & & & \partial_{2i} & & \\ 0 & & & & & \end{array}\right) .$$

We proceed now giving cochain maps representing the generators. This free resolutions does not allow us to give inductive formulas, but as we are just interested in 4th order iterated Massey products of elements in degrees 1 and 2 we just need to give the first 4 steps of the maps.

The element $X \in H^1(BSD_{2^n})$ is represented in the Yoneda cocomplex by a cochain map $X_i\colon P_{i+1} \to P_i$ with

$$X_0 = (1 \quad 0), \quad X_1 = \left(\begin{array}{c|cc} I^{2^{n-1}-2} & t(L+I^{2^{n-2}-2}) & 0 \\ \hline 0 & X_0 & \end{array}\right),$$

$$X_2 = \left(\begin{array}{c|ccc} 1 & t(L+I^{2^{n-2}-2}) & 1 & 0 \\ \hline 0 & & & \\ 0 & & X_1 & \end{array}\right) \quad \text{and}$$

$$X_3 = \left(\begin{array}{c|cccc} I^{2^{n-1}-2} & 0 & 0 & 0 & 0 \\ \hline 0 & & & & \\ 0 & & X_2 & & \\ 0 & & & & \end{array}\right) .$$

The element $Y \in H^1(BSD_{2^n})$ is represented in the Yoneda cocomplex by a cochain map $Y_i\colon P_{i+1} \to P_i$ with $Y_i = (0|\,\mathrm{Id}_{i+1})$ where Id_{i+1} is the $(i+1)\times(i+1)$ identity matrix.

The elements in $H^3(BSD_{2^n})$ are determined by maps $P_3 \to \mathbb{F}_2$ which can be lifted to cochain maps, and we can write them as matrices 1×4. Using the previous representatives Y^3 is determined by $(0\,0\,0\,1)$, $X^3 = X^2Y = XY^2$ (which is a coboundary) is represented by $(0\,0\,1\,0)$ and $(0\,1\,0\,0)$ cannot be lifted. As $(1\,0\,0\,0)$ can be lifted and, taking into account the dimension of the cohomology in degree 3, it can be considered as a representative for U or $U+Y^3$. The important fact is that checking that U or $U+Y^2$ are in $\langle X, X^2, Y\rangle$ will reduce to see that the first coordinate in the result of a defining system of this iterated Massey product is non zero.

The same arguments work for detecting which can be V as a map from $P_4 \to \mathbb{F}_2$: $(0\,0\,0\,0\,1)$ determines Y^4, $(0\,0\,0\,1\,0)$ is a coboundary, $(0\,0\,1\,0\,0)$ cannot be lifted, $(0\,1\,0\,0\,0)$ determines YU, so V (or $V+YU$) can be taken as $(1\,0\,0\,0\,0)$. Again, the computations will focus on the information which tell us that either V or $V+YU$ belongs to this iterated Massey product: that is, there is an element of the form $(1 * * * 0)$ in $\langle X, X^2, X, X^2\rangle$, seen as a map from P_4 to $\mathbb{F}_2$.

With all this data we can sketch the computations which tell us that either U or $U+Y^3$ belongs to $\langle X, X^2, Y\rangle$:

(i) We must find α and β in a defining system, so that $\delta\alpha = X^3$ and $\delta\beta = X^2Y$. By the previous identification of the cohomology classes in the Yoneda cocomplex, $X^3 = X^2Y$, so we can take $\alpha = \beta$. A direct computation gives us that we can take $\alpha_i\colon P_{i+2} \to P_i$

as:

$$\alpha_0 = (1\,0\,0) \text{ and } \alpha_1 = \begin{pmatrix} 1 + I^{2^{n-1}-3} & t(L + I^{2^{n-2}-2}) & x^2 I^{2^{n-1}-5} & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}.$$

(ii) Consider now the cochain $X\alpha + \alpha Y = X\alpha + \beta Y$. We just need to compute the evaluation of the morphism at level 0:

$$\epsilon(\alpha_0 X_2 + Y_0 \alpha_1) = (1\,0\,1\,0)\,,$$

as a map from P_3 to $\mathbb{F}_2$. This element corresponds either to $U + Y^3$ or U.

To see that $V \in \langle X, X^2, X, X^2 \rangle$ we can use the same procedure as in the proof of Lemma 4.4, using the matrices X_i described above. We need α, β and γ in a defining system such that $\delta\alpha = X^3$, $\delta\beta = X\alpha + \alpha X$ and $\delta\gamma = X^2\alpha + \alpha X^2$.

By definition, α may be taken the same α considered in the computation of $\langle X, X^2, Y \rangle$. Moreover, as $X\alpha + \alpha X$ gives the same representative as X^3 in the Yoneda cocomplex, we can take also $\beta = \alpha$. Finally, this election of α makes $X^2\alpha + \alpha X^2$ to be the zero element, so we can take $\gamma = 0$.

Now we can proceed in the computation of the first coordinate of $X\gamma + \alpha^2 + \beta X^2$:

$$\epsilon(0 + \alpha_0\alpha_2 + X_0 X_1 \alpha_2)\,.$$

We just need the first and last columns of α_2, which we can see that have $(1, 0, 0)$ and $(0, 0, 0)$ as coefficients respectively. As the result of the computations is the sum of the first two rows, we get that the result is of the form $(1 * * * 0)$ as a map from P_4 to $\mathbb{F}_2$, obtaining either V or $V + YU$.

Finally we must deal with the indeterminacy in these iterated Massey products to get the final result.

Applying [13, Proposition 2.3], in a triple Massey product, two elements in $\langle X, X^2, Y \rangle$ differ by an element of the form $\langle Z, Y \rangle + \langle X, Z' \rangle$, for Z, Z' elements in $H^2(BSD_{2^n})$. As Y^3 is the only element which can be constructed in this way we get

$$\langle X, X^2, Y \rangle = \{U, U + Y^3\}.$$

As in the proof of Lemma 4.4, two elements in $\langle X, X^2, X, X^2 \rangle$ differ by an element which can be expressed as $\alpha(a''X^2 + b''Y^2) + (aX^2 + bY^2)\alpha$, with α a cochain such that $\delta\alpha = X^3$ and $a, a'', b, b'' \in \mathbb{F}_2$. Any element of this form gives a map $\mathbb{F}_2[G]^5 \to \mathbb{F}_2[G] \to \mathbb{F}_2$ with zeros in the first and last coordinates and the coefficient a in the second. So just the element YU is in the indeterminacy, getting that:

$$\langle X, X^2, X, X^2 \rangle = \{V, V + YU\}.$$

□

5. Cohomological uniqueness

In this section we will work with topological spaces of the homotopy type of CW-complexes. We will use the cohomology over $\mathbb{F}_2$, so we have to consider 2-complete spaces in the sense of A.K. Bousfield and D. Kan [6].

5.1. **Dihedral groups.** Consider D_{2^n} the dihedral group of order 2^n, and its cohomology for $n \geq 3$ as defined in Equations (2) and (3).

The following lemma uses the fact that, for any topological space $\mathcal{X}$, $H^*(\mathcal{X})$ is an unstable algebra over the Steenrod algebra, but it does not require that this structure must be the same as the one in $H^*(BD_{2^n})$:

Lemma 5.1. *Let $\mathcal{X}$ be an space such that $H^*(\mathcal{X}) \cong H^*(BD_{2^n})$ as algebras. Let $\phi\colon \mathcal{X} \to BD_{2^n}$ be a map inducing the identity in degree one cohomology. Then either ϕ^* is an isomorphism or $\phi^*(W) = 0$.*

Proof. According to the hypothesis, ϕ^* is an isomorphism if and only if W is in the image of ϕ^*. In other words ϕ^* is not an isomorphism if and only if $\phi^*(W) = aX^2 + bY^2$ for $a, b \in \mathbb{F}_2$. Assume then that $\phi^*(W) = aX^2 + bY^2$. Applying Sq^1 in both sides we get $\mathrm{Sq}^1(W) = WY$ and $\mathrm{Sq}^1(aX^2 + bY^2) = 0$. So $0 = \mathrm{Sq}^1(\phi^*(W)) = \phi^*(\mathrm{Sq}^1(W)) = (aX^2 + bY^2)Y$ which implies $a = b = 0$. $\square$

Theorem 5.2. *BD_{2^n} is determined by its $\mathcal{M}$-cohomology.*

Proof. Fix $\mathcal{X}$ a 2-complete topological space having the homotopy type of a CW-complex and $\mathcal{M}$-comparable to BD_{2^n}.

For $n = 2$, $D_4 \cong \mathbb{Z}/2 \times B\mathbb{Z}/2$ and the result follows from Proposition 2.6.

Assume that $n \geq 3$. Then we should give a map $\phi_n\colon \mathcal{X} \to BD_{2^n}$ inducing an isomorphism in cohomology up to degree 2. Consider the tower of principal fibrations in Equation (4), where each π_k corresponds to the central extension:

$$1 \to \mathbb{Z}/2 \to D_{2^{k+1}} \to D_{2^k} \to 1$$

classified either by $X^2 + XY$ if $k = 2$ or by W when $k > 2$.

Consider $\phi_2\colon \mathcal{X} \to BD_4$ a map classifying the classes X and Y. The composite

$$\mathcal{X} \xrightarrow{\phi_2} BD_4 \xrightarrow{X^2+XY} K(\mathbb{Z}/2, 2)$$

is nullhomotopic, so ϕ_2 factors as a composition $\pi_2 \circ \phi_3$, with $\phi_3\colon \mathcal{X} \to BD_8$.

Now, if we assume that $\phi_k\colon \mathcal{X} \to BD_{2^k}$ inducing the identity in H^1 is defined, this map will extend to a map $\phi_{k+1}\colon \mathcal{X} \to BD_{2^{k+1}}$ if and only if $\phi_k^*(W) = 0$. Using Lemmas 3.4, 5.1 and the fact that

$$\phi_k^*(\langle X, X+Y, X, X+Y, \overset{2^k-2}{\cdots} \rangle_{H^*(BD_{2^k})}) \subset \langle X, X+Y, X, X+Y, \overset{2^k-2}{\cdots} \rangle_{H^*(\mathcal{X})}$$

we get that $\phi_k(W) = 0$ for all $k < n$ (the subscripts in the formula indicate the algebra where the iterated Massey products are considered). So, the map ϕ_k extends to $\phi_{k+1}\colon \mathcal{X} \to BD_{2^{k+1}}$ for $k < n$.

It remains to check the last step: if $\phi_n(W) = 0$, then it extends to ϕ_{n+1}. Such a map $\phi_{n+1}\colon \mathcal{X} \to BD_{2^{n+1}}$ inducing the identity in H^1 cannot exist because $\langle X, X+Y, X, X+Y, \overset{2^{n+1}}{\cdots} \rangle$ is defined in $H^*(BD_{2^{n+1}})$ and it is not defined in $H^*(\mathcal{X})$ by Lemma 3.4.

So $\phi_n(W) \neq 0$, and by Lemma 5.1, ϕ_n is an isomorphism in cohomology and $\mathcal{X} \simeq BD_{2^n}$. $\square$

5.2. **Quaternion groups.**

Lemma 5.3. *Consider the notation in Equation* (3) *for the cohomology of the dihedral and quaternion groups. Let $\mathcal{X}$ be an space $\mathcal{M}$-comparable to BQ_{2^n}, and $\phi\colon \mathcal{X} \to BD_{2^r}$ be a map such that $\phi^*(X) = X$ and $\phi^*(Y) = Y$. Then $\phi^*(W) = 0$ or $\phi^*(W) = Y^2$.*

Proof. Recall $\{X^2, Y^2\}$ is a basis of $H^2(\mathcal{X})$. Then there are $a, b \in \mathbb{F}_2$ such that $\phi^*(W) = aX^2 + bY^2$. If we apply Sq^1 to both sides of the identity we get $aX^2Y = 0$ (notice $Y^3 = 0$ in $H^*(\mathcal{X})$), getting the desired result. □

Theorem 5.4. *BQ_{2^n} is determined by its $\mathcal{M}$-cohomology.*

Proof. Let $\mathcal{X}$ be a 2-complete space having the homotopy type of a CW-complex and $\mathcal{M}$-comparable to BQ_{2^n}.

We must consider the cases $n = 3$ and $n \neq 3$ separately.

Consider Q_8 and its cohomology as in Equations (2) and (3). Let $\phi_2\colon \mathcal{X} \to B(\mathbb{Z}/2 \times \mathbb{Z}/2)$ be the map classifying the elements X and Y in cohomology. Such a map factorizes through $\phi_3\colon \mathcal{X} \to Q_8$ because $\phi_2^*(X^2 + XY + Y^2) = 0$, so we have a map ϕ_3 inducing the identity in H^1. Use now Lemma 4.4 to get that it must be an isomorphism in cohomology, so a homotopy equivalence.

Assume now $n \geq 4$ and that $\mathcal{X}$ is a $\mathbb{F}_2$-complete space $\mathcal{M}$-comparable to BQ_{2^n}. Let $\phi_2\colon \mathcal{X} \to B(\mathbb{Z}/2 \times \mathbb{Z}/2)$ be the map classifying the elements X and Y in cohomology. Consider the tower of principal fibrations

$$\cdots \to BD_{2^k} \overset{\pi_{k-1}}{\to} BD_{2^{k-1}} \overset{\pi_{n-2}}{\to} \cdots \overset{\pi_3}{\to} BD_8 \overset{\pi_2}{\to} B(\mathbb{Z}/2 \times \mathbb{Z}/2).$$

Recall that the map π_2 is classified by the class $X^2 + XY$ and each π_i for $i \geq 3$ is classified by W.

Since $\phi_2^*(X^2 + XY) = 0$, ϕ_2 lifts to $\phi_3\colon \mathcal{X} \to BD_8$, a map such that $\phi_3^*(X) = X$ and $\phi_3^*(Y) = Y$.

Assume now that $k \geq 3$ and $\phi_k\colon \mathcal{X} \to BD_{2^k}$ such that $\phi_k^*(X) = X$, $\phi_k^*(Y) = Y$. We see that $\phi_k^*(W) = 0$ when $k < n-1$, which implies that ϕ_k lifts to $\phi_{n-1}\colon \mathcal{X} \to BD_{2^{n-1}}$ such that $\phi_{n-1}^*(X) = X$, $\phi_{n-1}^*(Y) = Y$.

If $\phi_k^*(W) \neq 0$, then $\phi_k^*(W) = Y^2$ Lemma 5.3. This implies $\phi_k^*(Y^2 + W) = 0$ and there would be a map $\tilde{\phi}_{k+1}\colon \mathcal{X} \to BQ_{2^{k+1}}$ which is an isomorphism in cohomology till degree 3. This implies, by Lemma 2.3, that both $\mathcal{X}$ and $BQ_{2^{k+1}}$ have the same iterated Massey products involving degree one elements. But by Lemma 4.3, as $k+1 < n$ there are iterated Massey products in $H^*(\mathcal{X})$ which are not defined in $H^*(BQ_{2^{k+1}})$, getting a contradiction from the assumption $\phi_k^*(W) \neq 0$.

It remains to see that ϕ_{n-1} lifts to $\phi_n\colon \mathcal{X} \to BQ_{2^n}$, i.e. that $\phi_{n-1}(W) = Y^2$: if $\phi_{n-1}(W) \neq Y^2$ then $\phi_{n-1}(W) = 0$ by Lemma 2.3, and we would get a map $\tilde{\phi}_n\colon \mathcal{X} \to D_{2^n}$. If such a $\tilde{\phi}_n$ exists, using again by Lemma 2.3, there would exist either a map $\tilde{\phi}_{n+1}\colon \mathcal{X} \to D_{2^{n+1}}$ or a map $\tilde{\phi}'_{n+1}\colon \mathcal{X} \to Q_{2^{n+1}}$, but neither $\tilde{\phi}_{n+1}$ nor $\tilde{\phi}'_{n+1}$ could exist because in the cohomology of both targets there are products of type $\langle X, X+Y, X, \ldots, X+Y \rangle$ of length 2^n which are not defined in $H^*(\mathcal{X})$ by Lemma 4.3, getting a contradiction.

This implies that we have a map $\phi_n\colon \mathcal{X} \to BQ_{2^n}$ which is the identity in cohomology till degree 3, and by Lemma 4.4, $\phi_n^*(V) = V$. Then ϕ_n^* is an isomorphism and ϕ_n is a homotopy equivalence. □

Remark 5.5. Observe that iterated Massey products involving elements in degree greater than one cannot be avoided. Consider, as in [3], $\mathcal{X} = S^3/Q_{2^n} \times BS^3$ where the action of Q_{2^n} is by left multiplication on S^3 when one considers Q_{2^n} as a discrete subgroup of the Lie group S^3.

$\mathcal{X}$ is a 2-good space, and using [3], we have that $H^*(\mathcal{X}) \cong H^*(BQ_{2^n})$ as $\mathbb{F}_2$-algebras. Moreover, the iterated products of degree one elements are the same: all the information about the iterated Massey products of elements in degree one can be read in the three first steps of a minimal projective resolution of $\mathbb{F}_2[Q_{2^n}]$, and the three steps agree with the minimal projective resolution of $C^*(\mathcal{X}, \mathbb{F}_2)$. Finally, these are not homotopy equivalent spaces up to 2 completion because, for example, they have different homotopy groups.

5.3. **Semidihedral groups.** Consider SD_{2^n} a semidihedral group of order 2^n and its cohomology with the notation in Equations (2) and (3).

Lemma 5.6. *Let $\mathcal{X}$ be a topological space $\mathcal{M}$-comparable to BSD_{2^n}, and $\phi\colon \mathcal{X} \to BD_{2^r}$ ($r \geq 3$) be a map such that $\phi^*(X) = X$ and $\phi^*(Y) = Y$. Then $\phi^*(W) = 0$ or $\phi^*(W) = X^2$.*

Proof. Consider $\{X^2, Y^2\}$ as basis of $H^2(\mathcal{X})$ as $\mathbb{F}_2$ vector space. Then there are a and b in $\mathbb{F}_2$ such that $\phi^*(W) = aX^2 + bY^2$. If we apply Sq^1 to both sides of the identity we get $bY^3 = 0$, getting the desired result. □

Theorem 5.7. *BSD_{2^n} is determined by its $\mathcal{M}$-cohomology.*

Proof. Fix $\mathcal{X}$ a 2-complete space having the homotopy type of a CW-complex and $\mathcal{M}$-comparable to BSD_{2^n} ($n \geq 4$).

Consider $\phi_2\colon \mathcal{X} \to BD_4$ a map classifying X and Y. As $\phi_2^*(X^2 + XY) = 0$ this map will factorize with a map $\phi_3\colon \mathcal{X} \to BD_8$ such that $\phi_3^1\colon H^1(BD_8) \to H^1(\mathcal{X})$ is the identity.

Assume now that we have a map $\phi_k\colon \mathcal{X} \to BD_{2^k}$ which induces the identity in H^1 and with $k < n-1$. Then, by Lemma 5.6, $\phi_k^*(W) = X^2$ or $\phi_k^*(W) = 0$.

In the first case, $\phi_k^*(W + X^2) = 0$, so there would be a map $\tilde{\phi}_{k+1}\colon \mathcal{X} \to BSD_{2^{k+1}}$ which is the identity in cohomology in degrees one and two. So, by Lemma 2.3, they must have the same iterated Massey products of degree one elements. But Lemma 4.3 tells that if $k+1 < n$ there are 2^{n-1}th order iterated Massey products which are defined in the cohomology of $\mathcal{X}$, but not in $H^*(BSD_{2^{b+1}})$, and this contradicts Lemma 2.3.

So we are in the second case and we have a map $\phi_{k+1}\colon \mathcal{X} \to BD_{2^{k+1}}$ inducing the identity in H^1. This procedure can be done till $\phi_{n-1}\colon \mathcal{X} \to BD_{2^{n-1}}$ inducing the identity in H^1. Again, by Lemma 5.6, $\phi_{n-1}^*(W) = 0$ or $\phi_{n-1}^*(W) = X^2$.

In the first case, we would obtain a map $\tilde{\phi}_n\colon \mathcal{X} \to BD_{2^n}$ inducing the identity in H^1. By the previous arguments such a map would induce a map either $\tilde{\phi}_{n+1}\colon \mathcal{X} \to BD_{2^{n+1}}$ or $\tilde{\phi}_{n+1}\colon \mathcal{X} \to BSD_{2^{n+1}}$ inducing the identity in H^1, and this cannot happen because Lemmas 4.2 and 4.3 imply that there are iterated Massey products of degree one elements defined in $H^*(BD_{2^{n+1}})$ and $H^*(BSD_{2^{n+1}})$ which are not defined in $H^*(\mathcal{X})$.

So $\phi^*_{n-1}(W) = X^2$ and we have a map $\phi_n \colon \mathcal{X} \to BSD_{2^n}$ inducing the identity in H^1. Now we use now Lemma 4.5 to see that it must be an isomorphism in cohomology, and therefore a homotopy equivalence: as $\phi^*_n(\{U, U+Y^3\}) = \phi^*_n(\langle X, X^2, Y\rangle) \subset \langle X, X^2, Y\rangle = \{U, U+Y^3\}$ we get that either $\phi^*_n(U) = U$ or $\phi^*_n(U+Y^3) = U$, so U is in the image of ϕ^*_n. Using the same argument applied to $\langle X, X^2, X, X^2\rangle = \{V, V+YU\}$ we get that V is in the image of ϕ^*_n. This implies that all generators are in the image, and, up to degree 4 all are finite dimensional vector spaces, so an epimorphism is an isomorphism. □

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