CHOOSING BY MEANS OF APPROVAL-PREFERENTIAL VOTING. THE REVISED APPROVAL CHOICE

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Abstract

We look at procedures for making a collective choice through approval-preferential voting. A new procedure of this kind is proposed that is in the spirit of Condorcet's last ideas on elections, namely making sure that a good choice is made rather than aiming at the best choice but not being so sure about it.

Le mieux est le mortal ennemi du bien
(Montesquieu, 1720/1755)

1 The problem: How to make a collective choice by means of approval-preferential voting?

Approval-preferential ballots give two kinds of information. On the one hand, the voter can approve or disapprove options by themselves; on the other hand, he can also express preferences between pairs of options. Naturally, both kinds of information are required to be consistent with each other.

1.1 The Swiss procedure

An interesting example of approval-preferential voting is provided by multiple-choice referenda as they are conducted in the Swiss Confederation and its cantons. More specifically, approval-preferential voting arises in two cases: (a) popular initiatives, in which case the government can put forward a counter-proposal; and (b) the so-called 'constructive' referenda —in use in only a few cantons— where a proposal from the government can be followed by one or more counter-proposals from groups of voters. In these cases, the voters are asked two sets of questions. The main set is about every proposal

by itself to see whether the voter approves it or not. For a proposal to be adopted it must be approved by a majority of voters. In the event of more than one proposal being approved by a majority, a choice is made on the basis of the answers to the second set of questions, where the voter is asked to express his preferences between the different proposals.

In the more frequent case of two proposals, the second part reduces to a single question, namely which of the two proposals is preferred to the other. This information determines which option is chosen when both proposals satisfy the condition of being approved by a majority. This procedure was put forward in 1976 by Christoph Haab [20], and was adopted at the federal level in 1987 (see [24, p. 165]).

This procedure is quite reasonable. However, it may well happen that only one proposal is approved by a majority but at the same time a majority of the voters prefer the collectively disapproved one (sic). Imagine, for instance, that the votes are as follows:

$$25 : \ a \,|\, b \,, \quad 35 : \ b \,{>}\, a \,|\,, \quad 40 : \ |\, b \,{>}\, a \,, \tag{1}$$

where the numbers mean quantities of voters, $x\,{>}\,y$ means that x is preferred to y, and a bar indicates that the options at its left are approved and those at its right are disapproved. One easily checks that proposal a is approved by a majority of 60%, whereas b is disapproved by a majority of 65%. Therefore, the specified procedure results in proposal a being carried through.

However, one can also check that a majority of 75% expressed that they prefer b to a, which conflicts with the decision that has been adopted. The situation is very much that of a Condorcet cycle [31, ch. 9]. In fact, in the present context, approving a proposal amounts to preferring it to the *status quo*, i.e. leaving things as they are. From now on, we will denote such a default option by 0. So in the preceding example there are three options —$a, b, 0$— and the collective preferences form a cycle, namely $a\,{>}\,0\,{>}\,b\,{>}\,a$. Quite interestingly, such cycles are not just an academic possibility, but they have occurred in practice, as in the referendum that was held on the 28th November 2004 in the canton of Bern [3].

In the following we will refer to this procedure as the *Swiss Procedure*.

In the constructive referenda one can have more than two proposals besides the *status quo*. In this case, it could happen that three (or more) of the proposals were approved by a majority but the collective preferences about them formed a Condorcet cycle. In the cantons of Bern and Nidwalden such situations are regulated by applying first the Copeland rule [31, p. 206–209] restricted to the set of approved options, and then, if necessary, some tie-breaking rule ([2, Art. 139.7], [27, Art. 44.3]).

1.2   The ranking approach: rank all options, including the default one, then choose

Another interesting real case of approval-preferential voting is the voting procedure used by the Debian Project [16] (see also [32]). Since 1998, the votes of this organization systematically include a default option usually described as "further discussion" or "none of the above". In this connection, it is explicitly stated that "Options which the voters rank above the default option are options they find acceptable. Options ranked below the default options are options they find unacceptable" [16, v 1.1, §A.6].

The procedure for making a choice is described in [16, Appendix A] and is sometimes called *Schwartz Sequential Dropping*. According to [29], it is closely related to making use of another procedure that actually ranks all the options and then choosing the top-ranked one.

This ranking procedure, that we call the method of *Path Scores*, will be dealt with in detail in Section 2. For the moment, it will suffice to say that it complies with the Condorcet principle, i.e. it ranks first the Condorcet winner whenever it exists. Recall that a Condorcet winner means an option that beats every other in the sense that a majority of voters prefers the former to the latter [26].

Instead of the method of path scores, one can consider any other ranking method, such as the *Borda* count, the method of *Condorcet-Kemény-Young*, or the method of *Ranked Pairs*. As a general reference for these and other methods, we refer the reader to [31].

The last two mentioned methods comply also with the Condorcet principle, as the method of path scores. For three options, all of them amount to resolving any Condorcet cycle by dropping the weakest, i.e. less supported, of the three majoritarian views in conflict. In the case of example (1), this means dropping the view of approving a, which leads to adopting the default option 0.

So these methods allow strongly supported preferences to overturn the approval information, which seems reasonable enough.

1.3   Should a small preference differential prevail over a large approval differential?

However, we might be giving too much importance to preferences. Consider, for instance, the following example (from [9, eq. (109)]):

$$1/2 + \epsilon : \ a > b \,|\,, \qquad 1/2 - \epsilon : \ b \,|\, a \,, \tag{2}$$

where the numbers of voters are normalized to add up to one and ϵ is a

small positive quantity (for instance, 2ϵ could correspond to a single voter and the total number of voters could be one million and one). As one can see, both a and b are approved —i.e. preferred to 0— by a majority of voters; besides, a is preferred to b also by a majority. So a is a Condorcet winner, and therefore it will be chosen by the above Condorcet-compliant methods. However, the majorities in favour of a are quite slight, whereas b is approved by a whole unanimity. So, we are allowing a tiny preference differential to overcome a huge approval differential.

As one can easily check, the Swiss procedure also chooses a.

Examples like this suggest that one should perhaps completely forget about preferences and take into account only the approval information. However, it still seems that there should be a reasonable way to take into account the preferential information in order to make a better choice.

For later reference, the *Approval Choice* procedure will mean simply throwing away the preferential information and choosing the most approved option. More precisely, since later on we will allow for votes where some options are neither approved nor disapproved, we understand that the approval choice procedure chooses the option that maximizes the number of approvals minus the number of disapprovals, as it is advocated in [19].

1.4   Condorcet's last ideas

The point that we are leading to was formulated by Condorcet in the following way ([11, §XIII, p. 307], [25, p. 177–178], emphasis is ours):

> It is generally more important to *be sure* of electing men who are *worthy* of holding office than to have *a small probability* of electing *the worthiest man.*

The latest works of Condorcet on voting and elections, from 1788 to his death in 1794, are indeed dominated by this idea and by the aim of being able to deal with a large number of candidates, in which case paired comparisons become rather cumbersome [14, 25].

Concerning the meaning of 'being sure' and 'probability' in the above quotation, in another place Condorcet says simply the following ([10, §XIII, p. 193], [25, p. 139]):

> We consider a proposition asserted by 15 people, say, more probable than its contradictory asserted by only 10.

In the same spirit as example (2), he gives the following one ([14, p. 34–35], [25, p. 241]):

$$5 : a > c > \ldots \,|\, \ldots, \qquad 4 : b > c > \ldots \,|\, \ldots, \tag{3}$$

where he assumes a large number of candidates and, although he does not use approval bars, he explicitly says that all voters consider c worthy of the place. So the Condorcet winner a is considered the worthiest candidate by a slight majority, but c is considered worthy by unanimity. By the way, this and other examples show that Condorcet was accepting the possibility of making a choice different from the Condorcet winner.

1.5   Condorcet's practical methods

The methods that Condorcet proposed in connection with the preceding ideas are often referred to as Condorcet's "practical" methods. In general terms, his proposals are essentially two. In both of them, the voter is required to produce an ordered list of approved candidates. Unlike proper approval voting, however, the length of this list is fixed: "It should not be too short, to give a good chance that one of the candidates will obtain a majority, [...] nor should it be too long, [so that] the voters can still complete the list without having to nominate candidates they consider unworthy" ([10, p. 203], [25, p. 143]).

In his first practical proposal, formulated in 1788 [10, Article 5, p. 193–211] (translated in [25, p. 141–147]), Condorcet chooses the most approved candidate, conditioned to having obtained a majority, and the preferential information is used only in the event of ties. If no candidate has a majority of approvals, then he simply proposes to run a secod round after having asked the voters to extend their lists with a certain number of additional candidates. So this proposal was very much in the spirit of approval voting.

In his second and final practical proposal, formulated in 1789 [11] and yet in 1793 [12], Condorcet makes a more substantial use of the preferential information ([14, p. 41–42], [25, p. 249–250]): "If one candidate has the absolute majority of first votes, he will be elected. If one candidate has the absolute majority of first votes and second votes together, he will be elected. If several candidates obtain this majority, the one with the most votes will be preferred. If one candidate has the absolute majority of the three votes together, he will be elected, and if several candidates obtain this majority, the one with the most votes will be preferred."

Except for superficial variations, this idea spread and/or was rediscovered several times. Shortly after Condorcet's proposal, it was adopted in Geneva, where it was analyzed in 1794 by Simon Lhuilier [23]. Later on, in the beginning of the twentieth century it was adopted by several American institutions, starting from the city of Grand Junction (Colorado, USA), where this method was introduced by James W. Bucklin (see [21, §278] and [31,

p. 203–206]). Another example of its use are ballroom dancing competitions, where this idea is used since 1947/48 under the name of *Skating System* in order to combine the rankings given by the adjudicators [15]. More recently, it was proposed again by Murat R. Sertel in 1986 under the name of *Majoritarian Compromise* (see [28]). On account of its origins, we will refer to this procedure as the *Condorcet-Bucklin* method (credit to Condorcet is already acknowledged in [21, p. 490]).

The last two of the implementations that we mentioned in the preceding paragraph are not, properly speaking, about approval-preferential voting, since they assume that every voter ranks all the options and no default option is considered. By the way, in this case the Condorcet-Bucklin procedure amounts to use as main comparison criterion the median rank of each candidate, i.e. the median value of the ranks assigned to him by the different voters, and to choose the candidate that has the lowest median rank and that is ranked in this position or better by the largest number of voters.

The Condorcet-Bucklin procedure is clearly aimed at making sure that the chosen option is approved by a majority. To this effect, it is essential that every vote be confined to options that the voter really approves of. Therefore, the voter should be allowed to rank as few options as he wishes, which will easily come up in practice anyway. Of course, it may happen that no option is approved by a majority, in which case it would be appropriate to choose the default option or to declare a void choice.

If the votes contain any preferential information below approval, then the Condorcet-Bucklin procedure is bound to leave this information out of consideration.

As one can easily check, in the case of example (1), this procedure chooses the most approved option, namely a (which is the only one that is approved by a majority). In contrast, in example (2) it does not choose the unanimously approved option b, but option a, which is approved only by a slight majority.

1.6   Approval-preferential procedures in parliamentary elections

A real example where truncated rankings are used and where they can be interpreted as ordered lists of approved options are the elections to the Legislative Assemblies of Queensland and New South Wales (Australia), where every constituency elects a single representative on the basis of the (possibly) truncated rankings that are expressed by the electors [18]. Long ago, starting from 1892 in Queensland, the choice was made according to the so-called *Contingent Vote* system, that amounts to a instant runoff between

the two candidates that obtained the most first-choice votes. At present, starting from 1980 in New South Wales, the choice is made by means of the *Alternative Vote* system [31, p.193–195].

As it is well-known, a major flaw of these systems is their *lack of monotonicity* [31, p.194].

1.7 Taking into account the preferences between non-approved options

The methods of the preceding sections 1.5 and 1.6 do not take into account the preferences that a voter could have between his non-approved options. This is unfair towards the voters who do not approve at all the chosen option and would rather prefer some other non-approved option.

In order to take into account all preferences one could certainly use the methods of Section 1.2 after having introduced a default option. By the way, one could include among them the Condorcet-Bucklin method for complete rankings (which in the case of (1) chooses neither a nor 0, but b!). However, as we argued in Sections 1.3 and 1.4, this approach is too preference-oriented. Instead, one should give some sort of priority to the approval information.

The existing proposals in this direction are essentially some more elaborated versions of the Swiss procedure that we presented in Section 1.1.

One of them is the *Preference Approval Voting* procedure that was put forward in 2009 by Steven Brams and Remzi Sanver [4]. When more than two options are approved by a majority, this procedure restricts the attention to these options and the preferential information about them is used to single out, if possible, their Condorcet winner; if this is not possible, then other rules are applied that make further use of the approval scores.

A simpler possibility is the *Approval Voting with a Runoff*, considered in 2010 by Remzi Sanver [28]. Here, the preferences are used only to compare between the two most approved options.

Anyway, in the case of (2) both these procedures keep choosing a, like the Swiss procedure, against the point made in Section 1.5 that an overwhelming approval for an option should prevail over a slightly majoritarian preference for another.

1.8 Upper semicontinuity

In this section we consider the effect of small variations in the profile, i.e. the amount of voters who sustain each opinion. In real votes and elections the amounts of voters are certainly discrete quantities. However, when the

total number of voters is very large, the corresponding relative amounts admit of really small variations, so that it becomes reasonable to consider that the profile can change in a continuous way.

Let us consider such a continuous change and let us assume that this leads to our choice switching suddenly from a certain option x to another one y. We postulate that this should entail that at the boundary one has a tie between x and y (and possibly some other option). More generally, the dependence of the choice set on the profile should have the following property: for every fixed profile $\bar{u}$, if u is another profile close enough to $\bar{u}$ then the choice set for u is contained in that for $\bar{u}$. In the established terminology about set functions [22], we are requiring the choice set to be an upper semicontinuous function of the profile.

This property is easily violated by the methods that successively apply different criteria. For instance, in the case of example (2) both the Swiss procedure and the Condorcet-Bucklin one choose a for $\epsilon > 0$ and b for $\epsilon \leq 0$. In order to get $\{a, b\}$ for $\epsilon = 0$ one can consider modifying these rules by replacing proper majority requirements by weak majority ones (greater than or equal to 50%). This achieves the desired result in the particular case of (2), but the problem persists in other examples. Consider, for instance, the following one:

$$
\begin{array}{ccc}
2+\epsilon: a>e>b>c>d, & 2-\epsilon: b>c>a>d>e, & \\
1: a>b>c>d>e, & 1: b>d>c>a>e, & 2: c>d>a>b>e.
\end{array}
\tag{4}
$$

One can check that the Condorcet-Bucklin procedure chooses a for small positive ϵ and b for negative ϵ; at the boundary $\epsilon = 0$ it chooses a or b depending on whether proper majority or weak majority is considered, but neither of both variants chooses $\{a, b\}$.

1.9 Dealing with ties and with incomplete information

In practice, voters often do not have an opinion on some options. Besides, they may also rank equally some that they know. In order to properly deal with such possibilities, one must begin by distinguishing between them when interpreting the ballots.

For instance, the Debian voting rules state that "Unranked options are considered to be ranked equally with one another" [16, §A.6.1]. However, this is really questionable. In Condorcet's words, "When someone votes for one particular candidate, he simply asserts that he considers that candidate better than the others, and makes no assertion whatsoever about the respective merits of these other candidates. His judgement is therefore incomplete" ([10, p. 194], [25, p. 139]).

A vote where two options x and y are really ranked equally with each other can be assimilated to half a vote where x is preferred to y together with half another vote with the reverse preference. In contrast, a vote that expresses neither a preference nor a tie between x and y should contribute neither to the number of voters who prefer x to y nor to the number of those who prefer y to x.

In this connection, it is quite standard to take the view that "Ranked options are considered preferred to all unranked options" [16, §A.6.1]. However, in some contexts it could be more appropriate to interpret that no comparison is made between a ranked option and an unranked one.

One should also be aware that not approving an option is not the same as really disapproving it.

On account of all these considerations, it is certainly desirable that the ballots be designed so as to make as clear as possible what does the voter really mean. Besides, it is most important to clearly specify how will the ballots be interpreted.

Once the information has been properly interpreted and collected, the problem remains of how should one deal with it. In fact, many existing methods assume that complete information is given, and quite often it is not clear at all how should they be extended to the general incomplete case.

1.10 A new proposal

As we will see, the *Revised Approval Choice* procedure that is proposed in this paper satisfactorily solves most of the preceding issues. More specifically, it has the following properties:

- It deals with the general case of possibly incomplete information.

- It takes into account the preferences between non-approved options.

- It is monotonic with respect to an option being raised in the votes.

- Its set of choices is an upper semicontinuous function of the profile.

And, most outstandingly:

- It fulfils in a precise sense the idea of making sure that a good choice is made rather than aiming at the best choice but not being so sure about it.

The revised approval choice procedure is based upon the path scores, a collection of numbers that can also be used for collectively ranking all the options. Even so, however, the revised approval choice need not be the same as the top-ranked option.

2 Ranking all the options

2.1 We are dealing with a finite set of options A. For the moment, our goal will be to rank all of these options, without any special consideration for a possible default option.

We will be based upon the numbers of voters who expressed a preference for x over y, where x and y vary over all ordered pairs of *different* options. These numbers will be denoted by V_{xy}. Instead of them, most of the time we will be dealing with the fractions $v_{xy} = V_{xy}/V$, where V denotes the total number of votes. We will refer to V_{xy} and v_{xy} respectively as the absolute and relative **preference scores** associated with the ordered pair xy, and the whole collection of these scores will be called the (absolute or relative) **Llull matrix** of the vote.

Notice that v is a continuous function of the profile u that we were considering in Section 1.8. Therefore, the upper semicontinuity of a set of choices as a function of u reduces to the analogous property as a function of v.

The preference scores are bound to satisfy the inequality $v_{xy} + v_{yx} \leq 1$. The equality sign corresponds to the case where the preferential information about x and y is complete.

2.2 We think of the relative preference score v_{xy} as a degree of collective belief in the proposition

$$p_{xy} : \quad x \text{ is preferable to } y. \tag{5}$$

We take the view that $\overline{p}_{xy}$ can be identified with p_{yx} (see the discussion in [7, §3]). We assume also that preferences are transitive, that is

$$p_{xy} \wedge p_{yz} \rightarrow p_{xz}, \quad \text{for any pairwise different } x, y, z \in A. \tag{6}$$

Such an implication provides a flow of belief from the premises to the conclusion. More specifically, we adhere to the classical principle that the conclusion can be believed at least in the same degree as the minimum of the premises, i.e. in the degree $\min(v_{xy}, v_{yz})$. If v_{xz} is larger than this number, nothing changes. But if v_{xz} is smaller than it, then we can take that minimum as a revised degree of belief in the proposition p_{xz}.

By proceeding in this way along all implications of the form (6), and by iterating this procedure, one eventually reaches —in a finite number of steps— an invariant system of values (v_{xy}^*) that can be viewed as revised

degrees of belief in p_{xy}, or revised preference scores. We call them **path scores** since they are given by the formula

$$v^*_{xy} \; = \; \text{Max min} \left(v_{x_0 x_1}, v_{x_1 x_2}, \ldots, v_{x_{m-1} x_m} \right), \tag{7}$$

where the Max operator considers all paths $x_0 x_1 \ldots x_m$ of length $m \geq 1$ from $x_0 = x$ to $x_m = y$ with all x_i pairwise different. There is a standard way to compute them, namely the Floyd-Warshall algorithm [13, §25.2], whose computing time grows only as N^3, where N stands for the number of options.

2.3   Being a revised degree of belief, the path score v^*_{xy} measures how strong is the evidence, either direct or indirect, in favour of p_{xy}. Analogously, v^*_{yx} measures the evidence in favour of p_{yx}, i.e. against p_{xy}. In contrast to the original preference scores, here one can have $v^*_{xy} + v^*_{yx} > 1$, which corresponds to the case where there is evidence both in favour of p_{xy} and against it. Anyway, it makes sense to accept p_{xy} whenever the balance is in its favour, i.e. whenever $v^*_{xy} > v^*_{yx}$. Besides, the larger the margin $v^*_{xy} - v^*_{yx}$, the more convinced can be this decision of accepting p_{xy}. The main idea can be compared to that of the adversarial system of justice.

   As a result of this methodology, it turns out that these decisions are guaranteed to be always in agreement with the transitivity implications (6) (unlike the analogous decisions associated with the original scores):

Theorem 2.1 (Schulze, 1998). *If* $v^*_{xy} > v^*_{yx}$ *and* $v^*_{yz} > v^*_{zy}$ *then* $v^*_{xz} > v^*_{zx}$.

This fundamental fact was pointed out in 1998 by Markus Schulze, who gave a proof of it in a mailing list about election methods. The proof can be found also in [29, 30, 6].

   If one has $v^*_{xy} \neq v^*_{yx}$ for any x and y, this procedure determines a total ordering of all the options. Generally speaking, however, the binary relation formed by the pairs xy such that $v^*_{xy} > v^*_{yx}$ is only a partial order. If two options x and y satisfy $v^*_{xy} = v^*_{yx}$, then both xy and yx are excluded from this relation, and x and y should be ranked equally. Therefore, in order to rank all the options one is led to consider the non-strict inequality $v^*_{xy} \geq v^*_{yx}$ and the binary relation formed by the pairs xy that satisfy it. However, this latter relation need not be transitive, which calls for considering its transitive closure in order to properly rank all the options. From now on we will refer to this transitive closure as the **ranking relation,** and we will denote it by the symbol $\succeq$. So, $x \succeq y$ if and only if there exists a path $x_0 x_1 \ldots x_m$ from $x_0 = x$ to $x_m = y$ such that $v^*_{x_i x_{i+1}} \geq v^*_{x_{i+1} x_i}$ for all $i < m$.

Having obtained a whole ranking, if a choice is required it is quite natural to go for the top-ranked options, that is, those options x that satisfy $x \succeq y$ for any $y \neq x$. In the following we will refer to them as **top-ranked choices,** and the set that contains all of them will be denoted by T.

The next statement characterizes the set of top-ranked choices directly in terms of the relation defined by the inequality $v^*_{xy} > v^*_{yx}$. In this connection, we will use the following terminology: A set X of options is a **dominant set** for v^* if and only if it is not empty and it satisfies

$$v^*_{xy} > v^*_{yx}, \qquad \text{for all } x \in X \text{ and } y \notin X. \tag{8}$$

Besides, X is a **minimal dominant set** for v^* if and only if it is a dominant set for v^* and no proper subset of X has this property.

Theorem 2.2. *The set T of top-ranked choices is the only minimal dominant set.*

Proof. Let us begin by checking that property (8) holds for $X = T$. To this effect, it suffices to show that having $x \in T$ and the opposite inequality $v^*_{yx} \geq v^*_{xy}$ for some y implies $y \in T$. In fact, this opposite inequality entails that $y \succeq x$, whereas $x \in T$ means that $x \succeq z$ for all $z \neq x$. By transitivity, it follows that $y \succeq z$ for all $z \neq y$, which means that $y \in T$.

In order to complete the proof, it suffices to show that any dominant set X contains the whole of T. In other words, the existence of $x \in X$ and $y \in T \setminus X$ implies some violation of (8). In fact, the assumption that $y \in T$ entails $y \succeq x$. This means that there exists a path $y_0 y_1 \ldots y_m$ from $y_0 = y$ to $y_m = x$ such that $v^*_{y_i y_{i+1}} \geq v^*_{y_{i+1} y_i}$ for all $i < m$. Since $y \notin X$ and $x \in X$, there exists some i such that $y_i \notin X$ and $y_{i+1} \in X$. For this i the preceding inequality is a violation of (8). $\square$

Corollary 2.3. *An option x is the only top-ranked choice if and only if it satisfies $v^*_{xy} > v^*_{yx}$ for all $y \neq x$.*

If a Condorcet winner exists, then it is the only top-ranked choice. For this and other results we refer to [29, 30, 31, 6, 8]. In particular, in [6, 8] it was shown that this ranking method can be extended to a rating procedure that allows to sense how close are options to each other.

2.4 The top-ranked choices are well behaved in connection with upper semi-continuity:

Proposition 2.4. *The set of top-ranked choices is an upper semicontinuous function of the preference scores.*

Proof. Let $T(v)$ denote the set of top-ranked choices as a function of the Llull matrix v. For every $x \in A$ we will denote by $\Sigma(x)$ the set of Llull matrices for which x is a top-ranked choice. In order to show that T is upper semicontinuous it suffices to show that $\Sigma(x)$ is closed for any $x \in A$ [22, §7.1.4]. Let us assume that $\Sigma(x) \ni v_n \to v$. We want to see that $v \in \Sigma(x)$, that is, $x \succeq_v y$ for any $y \neq x$ (where the subindex indicates which Llull matrix are we talking about). We know that $x \succeq_{v_n} y$. This means that for every n there exists a path $x_0 x_1 \dots x_m$ from $x_0 = x$ to $x_m = y$ such that $(v_n)^*_{x_i x_{i+1}} \geq (v_n)^*_{x_{i+1} x_i}$ for all i. The path in question may depend on n. However, since the possible paths from x to y are finite in number, we can assume —by extracting a subsequence— that we are dealing with the same path for all n. Now, since the path scores are continuous functions of the original preference scores, the preceding non-strict inequalities remain true in the limit $n \to \infty$, which ensures that $x \succeq_v y$ as we wanted to show. $\square$

2.5 Let us consider now the issue of monotonicity. More specifically, we assume that a particular option a is raised to a more preferred status in the ballots *without any change in the preferences about the other options*. This results in the preference scores v_{xy} being modified into new values $\tilde{v}_{xy}$ such that

$$\tilde{v}_{ay} \geq v_{ay}, \quad \tilde{v}_{xa} \leq v_{xa}, \quad \tilde{v}_{xy} = v_{xy}, \qquad \forall x, y \neq a. \tag{9}$$

One would expect that if a is a top-ranked choice for v then it is also a top-ranked choice for $\tilde{v}$. This is true when a is the only top-ranked choice. In fact, according to Proposition 2.3, this amounts to say that $v^*_{ay} > v^*_{ya}$ for any $y \neq a$, and [6, Theorem 12.1] ensures that these inequalities continue to hold for $\tilde{v}$. However, if a is not the only top-ranked choice for v then it may cease to be top-ranked for $\tilde{v}$. An example of this phenomenon is given in §4.4.

2.6 *Remark.* Instead of the top-ranked choices, i.e. the minimal dominant set of v^*, one could consider its so-called minimal undominated set, also known as Schwartz set, or other choice sets that are introduced in the literature (see for instance [17, 1]). In particular, it would be interesting to know whether any of them enjoys both properties of upper semicontinuity and monotonicity.

2.7   In the next section we will make use of the two following facts about the path scores. The first one follows easily from formula (7) and the second one is an immediate consequence of the first.

Lemma 2.5. $v_{xz}^* \geq \min(v_{xy}^*, v_{yz}^*)$ *for any pairwise different* x, y, z.

Lemma 2.6.   *If* $v_{yz}^* > v_{xz}^*$, *then* $v_{xz}^* \geq v_{xy}^*$. *If* $v_{xy}^* > v_{xz}^*$, *then* $v_{xz}^* \geq v_{yz}^*$.

3   Making a good choice. The revised approval choice

3.1   In this section our goal is to make sure that the option that we choose is preferable to another one that has been previously fixed as default option.

One can certainly think of the default option as the one that will be adopted if none of the others is considered preferable to it. Alternatively, one can simply identify it with the bar that we have been using in the Introduction to mark the boundary between approved and disapproved options in the ballots. In the following we will denote the default option by 0.

Anyway, we will adopt the view that an option x being preferred to the default option is equivalent to x being approved; and similarly, the default option being preferred to x is equivalent to x being disapproved. Consequently, the preference scores of the form v_{x0} and v_{0x} can be identified respectively with the approval and disapproval scores.

Now, in the case of approval-preferential voting one can improve upon this information by means of the path scores v_{x0}^* and v_{0x}^*. In fact, these numbers can be viewed as revised approval and disapproval scores that take into account the preference scores between non-default options.

As we said in Section 2.3, it makes sense to accept p_{x0}, i.e. to approve x, whenever the difference $v_{x0}^* - v_{0x}^*$ is positive. Besides, the larger this margin, the more convinced will be this decision. Therefore, it makes sense to choose an option x that maximizes this margin, i.e. that satisfies

$$v_{x0}^* - v_{0x}^* \geq v_{y0}^* - v_{0y}^*, \qquad \text{for any } y \in A \setminus \{x\}. \tag{10}$$

In the following we will refer to such an option as a **revised approval choice,** and the set formed by all of them will be denoted by $C(v)$.

Although v_{xy}^* is defined only for $x \neq y$, in the expression $v_{x0}^* - v_{0x}^*$ we will allow for $x = 0$, in which case we define this expression to be equal to zero. So 0 will be included in $C(v)$ if and only if $v_{x0}^* - v_{0x}^* \leq 0$ for all $x \in A$. In this case, the options contained in $C(v)$ are not properly approved, but they belong to the boundary between approval and disapproval.

3.2 In [9, §7] we considered a slightly different approach, where the transitivity clauses (6) were cut down to only those with $y = 0$. Besides that, we were using also a different notation: instead of p_{x0} and p_{0x} we were writing respectively g_x and $\overline{g}_x$, and instead of v_{xy} we were writing $v(p_{xy})$, $v(g_x)$ or $v(\overline{g}_y)$ depending on the case. Anyway, it turns out that the values that are obtained for v_{x0}^* and v_{0x}^* are exactly the same in both approaches:

Proposition 3.1. *The values of v_{x0}^* and v_{0x}^* do not change when the transitivity implications* (6) *are reduced to only those that involve* 0.

Proof. It suffices to check that formulas (106) and (107) of [9] coincide with the present formula (7) when these changes of notation are taken into account. □

This fact allows to transfer here the following monotonicity result:

Theorem 3.2 (Camps, Mora and Saumell, 2014 [9, Thm. 7.1])**.** *If x is a revised approval choice and some votes are modified by raising x to a better position (with no other change) then x remains a revised approval choice.*

The next result is concerned with the dependence of the revised approval choices on the preference scores.

Theorem 3.3. *The set of revised approval choices is an upper semicontinuous function of the preference scores.*

Proof. Let $\Omega(x)$ denote the set of Llull matrices for which x is a revised approval choice. In order to show that C is upper semicontinuous, it suffices to show that $\Omega(x)$ is closed for any $x \in A$ [22, §7.1.4]. This is easily checked by considering the inequalities $(v_n)_{x0}^* - (v_n)_{0x}^* \geq (v_n)_{y0}^* - (v_n)_{0y}^*$ and using the fact that the path scores are continuous functions of the original preference scores. □

Remark. It is not difficult to show also that for any $x \in C(v)$ there exists a sequence $v_n \to v$ with $v_n \neq v$ such that $x \in C(v_n)$ for all n. In other words, the set $\Omega(x)$ contains no isolated points. Together with the preceding theorem, this allows to write the following equality, where $B(v, r)$ stands for the ball or radius r centered at v:

$$C(v) = \bigcap_{r>0} \bigcup_{w \in B(v,r) \setminus \{v\}} C(w). \tag{11}$$

Like in standard approval voting, and in contrast to plurality voting, the revised approval choice has a good behaviour when adding or deleting similar options.

The concept of similar options is suitably modelled by considering a set $C \subset A \setminus \{0\}$ of non-default options such that, for every voter and each $y \notin C$, therefore including $y = 0$, the preference of that voter between an $x \in C$ and that y is the same for all $x \in C$. Such a set C is often called a set of **clones**.

In plurality voting, replacing a single option x by a set of clones C that includes x, may easily change the choice from x to some option outside C. In contrast, in approval voting all members of C will get exactly the same approval and disapproval scores, so $x \in C$ being an approval choice implies that any other element of C is also an approval choice. The following result establishes exactly the same property for the revised approval choice.

Theorem 3.4. *If $C \subset A \setminus \{0\}$ is a set of clones and $x \in C$ is a revised approval choice, then any other element of C is also a revised approval choice.*

Proof. It suffices to check that the values of v_{x0}^* and v_{0x}^* are the same for all $x \in C$. This is guaranteed by the proofs of [6, Lem. 11.2] and [6, Prop. 11.3], which remain valid in the general, possibly incomplete, case. □

Remark. Notice that the revised approval choice is insensitive to the preferences of the voters between the members of C. Even if one of them is unanimously preferred to any other, all of them get exactly the same revised scores v_{x0}^* and v_{0x}^*.

By the way, this suggests that the case of multiple revised approval choices could be dealt with by means of some tie-breaking rule that takes into account the existing preferences between these options. However, it is not clear whether this can be done without losing the properties of monotonicity or upper semicontinuity (see §1.8, 2.5, 2.6).

3.3 Finally, we will see that the revised approval choice fulfils, in a certain sense, Condorcet's idea of making sure that a good choice is made rather than aiming at the best choice but not being so sure about it.

Let us assume that x is a revised approval choice and y is not. This does not exclude the possibility of having $v_{yx}^* > v_{xy}^*$, which corresponds to considering y preferable to x. In particular, this will be the case whenever x is not a top-ranked choice. However, for proper revised approval choices, i.e. satisfying the strict inequality $v_{x0}^* - v_{0x}^* > 0$, the next theorem ensures

that the confidence that x is preferable to the default option is greater than the confidence that y is preferable to x.

Theorem 3.5. *If $v_{x0}^* - v_{0x}^* > 0$ and $v_{x0}^* - v_{0x}^* > v_{y0}^* - v_{0y}^*$, then $v_{x0}^* - v_{0x}^* > v_{yx}^* - v_{xy}^*$.*

Proof. We will argue by contradiction. More specifically, we will arrive at contradiction from the following inequalities

$$v_{x0}^* > v_{0x}^*, \tag{12}$$

$$v_{x0}^* + v_{0y}^* > v_{0x}^* + v_{y0}^*, \tag{13}$$

$$v_{0x}^* + v_{yx}^* \geq v_{x0}^* + v_{xy}^*, \tag{14}$$

where (13) and (14) are equivalent respectively to the first inequality of the hypothesis and to the negation of the conclusion. We will distinguish two cases: (a) $v_{x0}^* \geq v_{yx}^*$; (b) $v_{yx}^* > v_{x0}^*$.

Case (a). Here we are assuming that

$$v_{x0}^* \geq v_{yx}^*. \tag{15}$$

This implies that $\min(v_{yx}^*, v_{x0}^*) = v_{yx}^*$. Therefore, by Lemma 2.5,

$$v_{y0}^* \geq v_{yx}^*. \tag{16}$$

Now, by concatenating (13), (16) and (14), we get

$$v_{x0}^* + v_{0y}^* > v_{0x}^* + v_{y0}^* \geq v_{0x}^* + v_{yx}^* \geq v_{x0}^* + v_{xy}^*. \tag{17}$$

Therefore,

$$v_{0y}^* > v_{xy}^*, \tag{18}$$

and, by Lemma 2.6,

$$v_{xy}^* \geq v_{x0}^*. \tag{19}$$

Finally, by concatenating (15), (14) and (19), we get

$$v_{0x}^* + v_{x0}^* \geq v_{0x}^* + v_{yx}^* \geq v_{x0}^* + v_{xy}^* \geq 2v_{x0}^*, \tag{20}$$

that is $v_{0x}^* \geq v_{x0}^*$, in contradiction with (12).

Case (b). Here we are assuming that

$$v_{yx}^* > v_{x0}^*. \tag{21}$$

This implies that $\min(v_{yx}^*, v_{x0}^*) = v_{x0}^*$. Therefore, by Lemma 2.5,

$$v_{y0}^* \geq v_{x0}^*. \tag{22}$$

Now, by concatenating (13) and (22), we get

$$v^*_{x0} + v^*_{0y} \; > \; v^*_{0x} + v^*_{y0} \; \geq \; v^*_{0x} + v^*_{x0}. \tag{23}$$

Therefore,

$$v^*_{0y} \; > \; v^*_{0x}, \tag{24}$$

and, by Lemma 2.6,

$$v^*_{0x} \; \geq \; v^*_{yx}. \tag{25}$$

However, by combining this inequality with (21) we get a contradiction with (12).   □

4   Examples

Let us finish by looking at some illustrative examples.

4.1   Let us start by example (1). As we have been saying, a proposal x being approved amounts to its being preferred to the default option 0. By taking this into account, the ballots (1) result in the absolute preference scores and margins that are shown below in (26). Since we are interested only in pairs xy with $x \neq y$, we use the diagonal cells for specifying the simultaneous labelling of rows and columns by the existing options; so the cell located in row x and column y gives information about the preference of x over y.

$$(V_{xy}) = \begin{array}{|c|c|c|} \hline a & 25 & 60 \\ \hline 75 & b & 35 \\ \hline 40 & 65 & 0 \\ \hline \end{array}, \qquad (V_{xy}-V_{yx}) = \begin{array}{|c|c|c|} \hline a & -50 & 20 \\ \hline 50 & b & -30 \\ \hline -20 & 30 & 0 \\ \hline \end{array}. \tag{26}$$

The numbers V_{x0} and $V_{x0} - V_{0x}$ that appear in the last columns of these matrices are nothing else than the approval scores and the corresponding margins. According to their values, the approval choice is a.

In order to work out the revised approval choice as well as the top-ranked one, we must compute the path scores and their corresponding margins. Having only three options, this is easily done by hand. For instance, $V^*_{ab} = \max\left(V_{ab}, \min(V_{a0}, V_{0b})\right) = \max\left(25, \min(60, 65)\right) = 60$. By proceeding in this way for all pairs of options, we get

$$(V^*_{xy}) = \begin{array}{|c|c|c|} \hline a & 60 & 60 \\ \hline 75 & b & 60 \\ \hline 65 & 65 & 0 \\ \hline \end{array}, \qquad (V^*_{xy}-V^*_{yx}) = \begin{array}{|c|c|c|} \hline a & -15 & -5 \\ \hline 15 & b & -5 \\ \hline 5 & 5 & 0 \\ \hline \end{array}. \tag{27}$$

As one can see, these path scores result in their margins being positive only in the lower triangle. So, we get the ranking is $0 \succ b \succ a$, with the default option 0 at the top. In accordance with the convention that was made at the end of §3.1, the revised approval choice is also 0.

4.2 Let us look now at the Bern 2004 referendum mentioned in §1.1. The data, taken from [3, Table 1], are as follows, where $a, b, 0$ stand respectively for the amendment of the parliament, the people's amendment, and the status quo:

$$(V_{xy}) = \begin{array}{|c|c|c|} \hline a & 101\,586 & 109\,812 \\ \hline 106\,863 & b & 104\,144 \\ \hline 102\,796 & 106\,832 & 0 \\ \hline \end{array}, \quad (V_{xy} - V_{yx}) = \begin{array}{|c|c|c|} \hline a & \text{-}5\,277 & 7\,016 \\ \hline 5\,277 & b & \text{-}2\,688 \\ \hline \text{-}7\,016 & 2\,688 & 0 \\ \hline \end{array}. \quad (28)$$

The total number of voters was $225\,758$, which is larger than any of the numbers $V_{xy} + V_{yx}$. This is due to the fact that some voters did not answer all the questions. Anyway, the approval choice is again a. Besides, it is the only proposal that is approved by a majority, which entails that it is also the choice of the Swiss procedure.

In this case, the path scores and their margins are as follows:

$$(V^*_{xy}) = \begin{array}{|c|c|c|} \hline a & 106\,832 & 109\,812 \\ \hline 106\,863 & b & 106\,863 \\ \hline 106\,832 & 106\,832 & 0 \\ \hline \end{array}, \quad (V^*_{xy} - V^*_{yx}) = \begin{array}{|c|c|c|} \hline a & \text{-}31 & 2\,980 \\ \hline 31 & b & 31 \\ \hline \text{-}2\,980 & \text{-}31 & 0 \\ \hline \end{array}. \quad (29)$$

According to the signs of these margins, we get the ranking $b \succ a \succ 0$. The top-ranked option is therefore b. However, the revised approval choice is a, which realises the maximum value for the revised margin over 0, namely $2\,980$. In agreement with Theorem 3.5, this revised margin for a being preferred to the default option is greater than the one for b being preferred to a, which is equal to 31.

4.3 Next we give an example where the revised approval choice is neither the approval choice nor the top-ranked one. In this example we assume that the votes have the form of truncated rankings. Specifically, they are as follows:

$$\begin{array}{ll} 9:\ a>b>c>d>0\,, & 1:\ b>a>c>d>0\,, \\ 1:\ d>0\,, \quad 5:\ a>d>0>b>c\,, & 9:\ c\,. \end{array} \quad (30)$$

Like in Debian elections (see §1.9), we will interpret that any unranked option is less preferred than any ranked one. However, we will not infer

anything about the preference between two unranked options. By applying these rules, we get the absolute preference scores and margins that are shown next.

$$
(V_{xy}) = \begin{array}{|c|c|c|c|c|}
\hline
a & 14 & 15 & 15 & 15 \\
\hline
1 & b & 15 & 10 & 10 \\
\hline
9 & 9 & c & 19 & 19 \\
\hline
1 & 6 & 6 & d & 16 \\
\hline
1 & 6 & 6 & 0 & 0 \\
\hline
\end{array} , \quad
(V_{xy} - V_{yx}) = \begin{array}{|c|c|c|c|c|}
\hline
a & 13 & 6 & 14 & 14 \\
\hline
-13 & b & 6 & 4 & 4 \\
\hline
-6 & -6 & c & 13 & 13 \\
\hline
-14 & -4 & -13 & d & 16 \\
\hline
-14 & -4 & -13 & -16 & 0 \\
\hline
\end{array} . \quad (31)
$$

By looking at the preference scores, we see that $V_{ax} > V/2 = 25/2$ for every $x \neq a$. So a is a Condorcet winner. However, by inspecting the margins we see that the approval choice is d, since this option realises the maximum value of the margin over 0, namely 16.

In order to work out the revised approval choice(s) and the top-ranked one(s), we compute the path scores (using the Floyd-Warshall algorithm mentioned in §2.2) and their corresponding margins. The resulting values are as follows:

$$
(V_{xy}^*) = \begin{array}{|c|c|c|c|c|}
\hline
a & 14 & 15 & 15 & 15 \\
\hline
9 & b & 15 & 15 & 15 \\
\hline
9 & 9 & c & 19 & 19 \\
\hline
6 & 6 & 6 & d & 16 \\
\hline
6 & 6 & 6 & 6 & 0 \\
\hline
\end{array} , \quad
(V_{xy}^* - V_{yx}^*) = \begin{array}{|c|c|c|c|c|}
\hline
a & 5 & 6 & 9 & 9 \\
\hline
-5 & b & 6 & 9 & 9 \\
\hline
-6 & -6 & c & 13 & 13 \\
\hline
-9 & -9 & -13 & d & 10 \\
\hline
-9 & -9 & -13 & -10 & 0 \\
\hline
\end{array} . \quad (32)
$$

The only top-ranked choice is a (by Corollary 2.3), which we already knew since the method of path scores is guaranteed to choose the Condorcet winner whenever it exists. However, the revised approval choice is c, which realises the maximum value for the revised margin over 0, namely 13.

Again these results agree with Theorem 3.5 in that the revised margin for c being preferred to the default option is greater than the one for a being preferred to c, which is equal to 6 (and similarly for b or d instead of a).

4.4   Finally, we give an example (already referred to in [6, §12]) that exhibits a lack of monotonicity of the property of being a top-ranked option. Consider, for instance, the following votes:

$$
\begin{array}{lll}
1: a > d > b > e > c, & 1: b > a > c > e > d, & 1: b > c > a > d > e, \\
1: b > c > d > e > a, & 1: b > e > c > a > d, & 1: d > a > b > c > e, \\
2: e > a > c > d > b, & 1: e > c > a > d > b, & 1: b > d > c > a > e,
\end{array} \quad (33)
$$

The resulting preference scores and path scores together and their corresponding margins are as follows:

$$(V_{xy})=\begin{array}{|c|c|c|c|c|}\hline a & 5 & 5 & 7 & 5 \\\hline 5 & b & 7 & 5 & 7 \\\hline 5 & 3 & c & 7 & 5 \\\hline 3 & 5 & 3 & d & 5 \\\hline 5 & 3 & 5 & 5 & e \\\hline\end{array}, \quad (V_{xy}-V_{yx})=\begin{array}{|c|c|c|c|c|}\hline a & 0 & 0 & 4 & 0 \\\hline 0 & b & 4 & 0 & 4 \\\hline 0 & -4 & c & 4 & 0 \\\hline -4 & 0 & -4 & d & 0 \\\hline 0 & -4 & 0 & 0 & e \\\hline\end{array}, \quad (34)$$

$$(V^*_{xy})=\begin{array}{|c|c|c|c|c|}\hline a & 5 & 5 & 7 & 5 \\\hline 5 & b & 7 & 7 & 7 \\\hline 5 & 5 & c & 7 & 5 \\\hline 5 & 5 & 5 & d & 5 \\\hline 5 & 5 & 5 & 5 & e \\\hline\end{array}, \quad (V^*_{xy}-V^*_{yx})=\begin{array}{|c|c|c|c|c|}\hline a & 0 & 0 & 2 & 0 \\\hline 0 & b & 2 & 2 & 2 \\\hline 0 & -2 & c & 2 & 0 \\\hline -2 & -2 & -2 & d & 0 \\\hline 0 & -2 & 0 & 0 & e \\\hline\end{array}. \quad (35)$$

As one can see, there are quite a few pairs xy such that $v^*_{xy} = v^*_{yx}$. As a consequence, it turns out that the ranking relation $\succeq$ that we defined in §2.3 is a whole tie. The set of top-ranked choices is therefore the whole of A.

Assume now that the last ballot in (33) is modified by replacing $b > d$ by $d > b$. The monotonicity property would require d to continue being a top-ranked option. However, it is not so. In fact, one gets the ranking $a > b > c > d > e$, where the only top-ranked choice is a.

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