ASYMPTOTICS OF STEADY STATES OF A SELECTION MUTATION EQUATION FOR SMALL MUTATION RATE

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Abstract. We consider a selection mutation equation for the density of individuals with respect to a continuous phenotypic evolutionary trait in which the competition term for an individual of a given trait depends on the traits of all the other individuals giving then an infinite dimensional nonlinearity. Mutation is modelled by means of an integral operator. We prove existence of steady states and we show that their asymptotic profile, when the mutation rate goes to zero, is a Cauchy distribution.

1. **Introduction.** In this paper we consider a selection mutation equation for the density of individuals of a biological population with respect to a continuous phenotypic evolutionary trait. Selection mutation equations in the continuous framework were introduced in [10] and [18] in order to explain the maintenance of variability in a continuum of alleles due to the balance effect of selection and mutation. A continuation of this work can be found in [3] and more extensively in [4].

With an ecological point of view this type of equations has been used to model the evolution of phenotypical traits (see for instance [5, 6, 11]).
One of the main goals in this topic is the study of the existence and the properties of the stationary solutions of these equations. An important feature of these previous works is that the feedback variable (also called environment, see e.g [21]) is finite-dimensional and therefore the equations reduce to linear when a finite number of quantities is considered fixed. This is what happens when individuals compete

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for a given number of different nutrients (see e.g. [11], [5]).
More precisely, this means that these equations can be written in the form $f_t = A(E(f))f$ where f denotes the density of individuals with respect to the evolutionary trait, f_t is time derivative, E is a (usually linear) function from the state space to a finite dimensional space and $A(E)$ is a linear operator on the state space. Then the problem of looking for steady states is equivalent to finding positive eigenfunctions corresponding to the zero eigenvalue of the linear operator plus solving a fixed point problem in a finite dimensional space (nonlinear due to the nonlinear dependence of the operator $A(E)$ with respect to E, see [7, 8, 9]).

However, in many applications, the environment is not finite dimensional (see [26, 27, 22], for instance), we thus consider (like in [12] and [16]) in the present paper the more general case where the competitive stress that an individual of a given trait feels (undergoes) turns out to be the sum of the individual competitions caused by all the other individuals, typically in such a way that they are stronger when individuals have closer traits.

In this article, we model mutations by means of an integral operator like in [27, 25, 11, 23, 2]. Notice that mutations have also been modelled using a diffusion operator in e.g. [22, 26, 6, 19].

Typically, only a small proportion of offspring are affected by mutations having an influence on the considered phenotype. Notice that this assumption of rare mutations is at the basis of the so-called Adaptive Dynamics Theory (see [20, 14]), where the ecological time scale (that is, the time scale of the selection phenomena) is considered as infinitely faster than the evolutionary time scale (that is the time scale of mutations). On the other hand pure selection models for continuous phenotypic traits are often used in theoretical works on biological evolution (see e.g. [15]). For mathematical studies on pure selection models see for instance [1, 12, 24].

We are thus interested in an asymptotic study of the steady states of the population when the proportion of mutants goes to zero. In [7, 8, 9], it has been shown (in the case of a finite-dimensional environment) that the steady states tend to concentrate around a Dirac mass at an ESS value of the trait (that is, a value of the trait such that if there is no mutation and all individuals share this trait, they are uninvadable by small populations with a different trait value, see [20]). In the case of a small mutation modelled by a diffusion, formal asymptotics (see [18, 17]) suggest that the asymptotic profile of the population should be a Gaussian distribution. In this paper we show rigorously that in the case of mutation modelled by an integral operator, the asymptotic profile of a steady state is a Cauchy distribution (see (1)).

In Section 2, we introduce the model and some notation that will be useful throughout the article.

In Section 3, using Schauder's fixed point theorem, we show (under reasonable technical assumptions on the coefficients) that the model admits a non-trivial steady-state if (and only if) the per capita growth rate is positive for some value of the trait when the population is small.

Section 4 is devoted to an asymptotic analysis of the shape of steady populations, when the mutation rate tends to zero. As in [7, 8, 9], we consider cases where the monomorphic population $f = \delta_0$ would be globally stable in the corresponding pure selection model, and we study what happens when the mutation rate is small, but not zero. In order to do it, we perform a rescaling on both dependent and independent variables and we obtain that the steady states tend, when the mutation rate goes to zero, to a Cauchy distribution:

$$f^\varepsilon(x) \sim \frac{1/\varepsilon}{C_1 + C_2(x/\varepsilon)^2}, \tag{1}$$

where ε is the mutation rate, and C_1, C_2 are constants that can be computed explicitly.

2. **The model.** Let us consider the following selection mutation model

$$\begin{cases} \partial_t f(t,x) &= \Big((1-\mu)b(x) - d_0(x) - \int_X d(x,y)f(t,y)\mathrm{d}y\Big) f(t,x) \\ & + \quad \mu \int_X b(y)\beta(x,y)f(t,y)\mathrm{d}y, \\ f(0,x) &= f_0(x), \end{cases} \tag{2}$$

for the density of individuals $f(t,x)$ with respect to an (abstract) evolutionary trait $x \in X$, a bounded interval of $\mathbb{R}$ such that $0 \in \mathrm{Int}\ X$.
The parameter $\mu \in [0,1]$ stands for the probability of mutation in every reproduction, $b(x)$ is the trait specific birth rate and $\beta(x,y)$ is the density of probability that the offspring of an individual with trait y has trait x.
The quantity $(1-\mu)b(x)f(t,x)$ gives then the density of newborns per unit of time due to faithful reproduction and $\mu \int_X b(y)\beta(x,y)f(t,y)\mathrm{d}y$ the density of newborns per unit of time due to mutant reproduction. The trait specific per capita death rate is given by the terms $d_0(x)$ and $\int_X d(x,y)f(t,y)\mathrm{d}y$. The last one is the only nonlinear term which models the interaction between individuals through competition for resources.
Typically we think of a function $d(x,y) = \tilde{d}(|x-y|)$ such that the function $\tilde{d}$ is bounded below by a strictly positive constant (meaning that any individual competes with all individuals having any value of the trait) and having a maximum at zero (meaning that the maximum competition occurs between identical individuals). Obviously this means that the function d has critical points on the diagonal. However we will assume a weaker hypothesis on d (see Assumption 3 in section 4).
To simplify notations in the proofs, we will denote from now on by $\mu m(x,y)$ the rate of mutants of trait x produced by a parent of trait y:

$$m(x,y) := b(y)\beta(x,y),$$

and by a_μ the intrinsic growth rate if we forget the mutant newborns:

$$a_\mu(x) := (1-\mu)b(x) - d_0(x).$$

We will also define the fitness of an individual of trait x as

$$s[f(t,\cdot)](x) = a_\mu(x) - \int_X d(x,y)f(t,y)\,dy.$$

3. **Existence of steady-states.** This section is devoted to prove existence of steady states of (2). We introduce the notation

$$d \tilde{*} f(x) := \int_{y\in X} d(x,y)f(y)\,dy,$$

and the same for the mutation kernel m. To avoid cases where the population concentrates on the boundary, we will make the following assumption:

Assumption 1: For any $f \in L^1_+(X)$ such that $\max s[f] = \min(d \tilde{*} f - (1-\mu)b + d_0) \leq 1$, the maximum of $s[f] = (1-\mu)b - d_0 - d \tilde{*} f$ is reached in the interior of X.

This assumption seems difficult to check. However it is possible to give easier conditions on the parameters of the model that imply it and that we state in the following remark.

Remark 1. Notice that $\max s[f] = \min(d \tilde{*} f - (1-\mu)b + d_0) \leq 1$ implies that $\|f\|_{L^1} \leq \frac{\max((1-\mu)b - d_0) + 1}{\min d}$.
Assumption 1 is then satisfied if one of the following two conditions is satisfied:

- There exists $\bar{x} \in \mathrm{Int}\, X$ such that
$$(1-\mu)b(\bar{x}) - d_0(\bar{x}) = \max((1-\mu)b - d_0)$$
and for any $x \in \partial X$,
$$\begin{aligned}(1-\mu)b(x) - d_0(x) \quad &\leq \quad \max((1-\mu)b - d_0) \\ &\qquad -(\|d\|_\infty - \min d)\frac{\max((1-\mu)b - d_0) + 1}{\min d}.\end{aligned}$$
Indeed, this implies that for $x \in \partial X$
$$\begin{aligned} s[f](x) - s[f](\bar{x}) \quad &\leq \quad \|d\|_\infty \|f\|_{L^1} - \max((1-\mu)b - d_0) \\ &\qquad - \min d \|f\|_{L^1} + ((1-\mu)b - d_0)(x) \\ &\leq \quad 0. \end{aligned}$$
Then, $s[f](\bar{x}) \geq s[f](x)$, and the maximum of $s[f](x)$ is necessarily reached in the interior of X.
- Let $X = [x_1, x_2]$,
$$\begin{array}{lcl} (1-\mu)b'(x_1) - d_0'(x_1) - \|\partial_1 d\|_\infty \frac{\max((1-\mu)b - d_0)+1}{\min d} & \geq & 0 \\ (1-\mu)b'(x_2) - d_0'(x_2) + \|\partial_1 d\|_\infty \frac{\max((1-\mu)b - d_0)+1}{\min d} & \leq & 0. \end{array}$$
This implies that $\frac{d}{dx} s[f](x_1) > 0 > \frac{d}{dx} s[f](x_2)$, and the maximum of $s[f]$ is then reached in the interior of X.

In order to prove the theorem of existence of steady states we first prove the following technical lemma.

Lemma 3.1. *Let C_1, C_2, C_3, C_4, C_5, $C_6 > 0$. If $\delta > 0$ is small enough, then there exists $\hat{\alpha} > 0$ such that:*
$$\hat{\alpha} \leq \frac{(1-\delta)C_1}{\left(C_2 + C_3\left(C_4 + \frac{C_5}{\hat{\alpha}}\delta\right)\right)} - C_6\delta. \tag{3}$$

Proof. $\hat{\alpha}$ satisfies (3) if and only if
$$0 \geq \hat{\alpha}\left((C_2 + C_3C_4)\hat{\alpha} - C_1\right) + \delta((C_1 + C_3C_5)\hat{\alpha} + C_6(C_2\hat{\alpha} + C_3(C_4\hat{\alpha} + C_5\delta)))$$
which is satisfied (for instance) by $\hat{\alpha} = \frac{C_1}{2(C_2 + C_3C_4)} > 0$, if $\delta > 0$ is small enough. $\square$

Theorem 3.2. *Let $\mu \in [0,1]$, b, $d_0 \in W^{2,\infty}(X)$, $d \in W^{2,\infty}(X \times X)$, $\beta \in W^{1,\infty}(X \times X)$ such that $\min d > 0$, $\min \beta > 0$, $\min b > 0$ and such that $\max((1-\mu)b - d_0) > 0$. Under Assumption 1 there exists a non trivial (i.e. non 0 everywhere) steady state $\bar{f} \in W^{1,\infty}(X)$ of (2). Moreover, if (for some $k \in \mathbb{N}$) b, $d_0 \in W^{k,\infty}(X)$ and d, $\beta \in W^{k,\infty}(X \times X)$, then $\bar{f} \in W^{k,\infty}(X)$.*

Proof. Let $\delta > 0$. We define,

$$F(f) := (1-\delta)f + \delta\frac{\mu m\tilde{*}f}{-s[f]},$$

and (for $\bar{\alpha} > 0$, $\bar{\Lambda} > 0$, $\gamma > 0$ to be chosen later) the sets

$$\mathcal{F} := \left\{f \in L^1_+;\ \alpha(f) \geq \bar{\alpha},\ \|f\|_{L^1} \leq \bar{\Lambda}\right\},$$

where $\alpha(f) := \min\{-s[f]\}$, and

$$\mathcal{G} := \mathcal{F} \cap \left\{g \in W^{1,\infty}(X);\ \|g'\|_\infty \leq \gamma\right\}.$$

We notice that $\mathcal{G}$ is a convex, bounded and closed set in $(C(X), \|\cdot\|_\infty)$. Then, thanks to Ascoli Theorem, it is compact in $(C(X), \|\cdot\|_\infty)$. We will show that it is not empty (see (11) below), and that, for δ small enough, one can find $\bar{\alpha}$, $\bar{\Lambda}$ such that $F(\mathcal{G}) \subset \mathcal{G}$. We can then apply Schauder's fixed point theorem to the set $\mathcal{G}$ and obtain the existence of $\bar{f} \in \mathcal{G}$ such that

$$\bar{f} = F(\bar{f}) = (1-\delta)\bar{f} + \delta\frac{\mu m\tilde{*}\bar{f}}{-s[\bar{f}]},$$

and therefore

$$0 = (s[\bar{f}])\bar{f} + \mu m\tilde{*}\bar{f},$$

which proves the existence of a steady-state $\bar{f} \in W^{1,\infty}(X)$ of (2). Notice that $\bar{f}$ is non trivial because $0 \notin \mathcal{F}$ since $\alpha(0) = \min(-a_\mu) = -\max((1-\mu)b - d_0)$.
Let us prove that, for δ small enough, $F(\mathcal{G}) \subset \mathcal{G}$. We will do it in four steps.

Step 1: We bound $\{\alpha(F(f));\ f \in \mathcal{F}\}$ from below.

$$\begin{aligned}
\alpha(F(f)) &= \min\{-s[F(f)]\}\\
&= \min\left\{d\tilde{*}\left[(1-\delta)f + \delta\frac{\mu m\tilde{*}f}{-s[f]}\right] - a_\mu\right\}\\
&\geq \min\{-s[f]\} + \delta\min\left\{d\tilde{*}\left(\frac{\mu m\tilde{*}f}{-s[f]} - f\right)\right\}\\
&\geq \alpha(f) + \delta\left(\mu(\min d)\,(\min m)\,\|f\|_{L^1}\int_X\frac{dx}{-s[f]} - \|d\|_\infty\|f\|_{L^1}\right).
\end{aligned}$$

Thanks to Assumption 1, if $\alpha(f) \leq 1$, there exists $x_0 \in \mathrm{Int}X$ such that $(-s[f])(x_0) = \min(-s[f]) = \alpha(f)$. Then, $(-s[f])'(x_0) = 0$, and

$$(-s[f])(x) \leq \alpha(f) + \frac{1}{2}\|(-s[f])''\|_\infty(x-x_0)^2,$$

which gives the following estimate on $\int_X\frac{dx}{-s[f]}$, provided that $\alpha(f) \leq \min\left(\frac{\|a''_\mu\|_\infty}{8}|X|^2, 1\right)$,

$$\begin{aligned}
\int_X\frac{dx}{-s[f]} &\geq \int_0^{|X|/2}\frac{dy}{\alpha(f) + \frac{1}{2}(\|a''_\mu\|_\infty + \|\partial^2_{11}d\|_\infty\|f\|_{L^1})y^2}\\
&= \sqrt{\frac{2}{\alpha(f)\left(\|a''_\mu\|_\infty + \|\partial^2_{11}d\|_\infty\|f\|_{L^1}\right)}}\arctan\left(\sqrt{\frac{\|a''_\mu\|_\infty + \|\partial^2_{11}d\|_\infty\|f\|_{L^1}}{2\alpha(f)}}\frac{|X|}{2}\right)\\
&\geq \frac{\pi}{2\sqrt{2\alpha(f)\left(\|a''_\mu\|_\infty + \|\partial^2_{11}d\|_\infty\|f\|_{L^1}\right)}}.
\end{aligned}$$

Then,

$$\alpha(F(f))-\alpha(f) \geq \delta\|f\|_{L^1}\left\{\mu(\min d)\,(\min m)\,\frac{\frac{\pi}{2\sqrt{2}}}{\sqrt{\alpha(f)\,(\|a_\mu''\|_\infty+\|\partial_{11}^2 d\|_\infty\bar\Lambda)}}-\|d\|_\infty\right\}$$
$$\geq 0,$$

if $\alpha(f) \leq \min\left\{\frac{\frac{\pi^2}{8}\,\mu^2\,(\min d)^2\,(\min m)^2}{\|d\|_\infty^2\,(\|a_\mu''\|_\infty+\|\partial_{11}^2 d\|_\infty\bar\Lambda)},\ \frac{\|a_\mu''\|_\infty}{8}|X|^2,\ 1\right\}$.

We define a constant $\tilde\Lambda$ that will be used in Step 2 by:

$$\tilde\Lambda := \frac{1}{\min d}\left(\mu\|m\|_\infty|X|+\max a_\mu\right). \tag{4}$$

We also define the constant C_α by

$$\begin{aligned} C_\alpha \quad := \quad & (\|a_\mu''\|_\infty+\|\partial_{11}^2 d\|_\infty\tilde\Lambda)\min\left\{\frac{\frac{\pi^2}{8}\,\mu^2\,(\min d)^2\,(\min m)^2}{\|d\|_\infty^2\,(\|a_\mu''\|_\infty+\|\partial_{11}^2 d\|_\infty\tilde\Lambda)},\right. \\ & \left.\frac{\|a_\mu''\|_\infty}{8}|X|^2,\ 1,\ \mu\|m\|_\infty|X|\right\}, \end{aligned} \tag{5}$$

(the last term $\mu\|m\|_\infty|X|$ will be useful at the end of Step 3) and $\tilde\alpha_{\bar\Lambda}$ by:

$$\tilde\alpha_{\bar\Lambda} := \frac{C_\alpha}{(\|a_\mu''\|_\infty+\|\partial_{11}^2 d\|_\infty\bar\Lambda)}. \tag{6}$$

From now on we assume $\bar\Lambda \geq \tilde\Lambda$ and $\bar\alpha \leq \tilde\alpha_{\bar\Lambda}$.
Let us take $f \in \mathcal{F}$. If $\alpha(f) \leq \tilde\alpha_{\bar\Lambda}$ then we have just proved that $\alpha(F(f)) \geq \alpha(f) \geq \bar\alpha$.
On the other hand if $\alpha(f) \geq \tilde\alpha_{\bar\Lambda}$ then

$$\begin{aligned} \alpha(F(f)) \quad &= \quad \min\left\{(1-\delta)(-s[f])+\delta\left[d\tilde{*}\left(\frac{\mu m\tilde{*}f}{-s[f]}\right)-a_\mu\right]\right\} \\ &\geq \quad (1-\delta)\tilde\alpha_{\bar\Lambda}+\delta\min\left\{d\tilde{*}\left(\frac{\mu m\tilde{*}f}{-s[f]}\right)-a_\mu\right\} \\ &\geq \quad (1-\delta)\tilde\alpha_{\bar\Lambda}-\delta\max a_\mu \\ &= \quad \frac{(1-\delta)C_\alpha}{(\|a_\mu''\|_\infty+\|\partial_{11}^2 d\|_\infty\bar\Lambda)}-\delta\max a_\mu. \end{aligned}$$

That is, we have shown that for any $f \in \mathcal{F}$,

$$\alpha(F(f)) \geq \frac{(1-\delta)C_\alpha}{(\|a_\mu''\|_\infty+\|\partial_{11}^2 d\|_\infty\bar\Lambda)}-\delta\max a_\mu \ \text{ or } \ \alpha(F(f)) \geq \bar\alpha. \tag{7}$$

Step 2: We bound $\{\|F(f)\|_{L^1};\, f \in \mathcal{F}\}$ from above.

$$\begin{aligned} \|F(f)\|_{L^1} \quad &= \quad \int\left[f+\delta\left(\frac{\mu m\tilde{*}f}{-s[f]}-f\right)\right]dx \\ &= \quad \|f\|_{L^1}+\delta\left[\int\frac{\mu m\tilde{*}f}{-s[f]}\,dx-\|f\|_{L^1}\right] \\ &\leq \quad \|f\|_{L^1}+\delta\left[\frac{\mu\|m\|_\infty|X|}{\min(-s[f])}-1\right]\|f\|_{L^1} \\ &\leq \quad \|f\|_{L^1}+\delta\left[\frac{\mu\|m\|_\infty|X|}{\|f\|_{L^1}\min d-\max a_\mu}-1\right]\|f\|_{L^1}. \end{aligned}$$

So, if $\|f\|_{L^1} \geq \frac{1}{\min d}\left(\mu\|m\|_\infty|X| + \max a_\mu\right) = \tilde{\Lambda}$ then $\|F(f)\|_{L^1} \leq \|f\|_{L^1} \leq \bar{\Lambda}$. We consider next $f \in \mathcal{F}$ such that $\|f\|_{L^1} \leq \tilde{\Lambda}$. Then

$$\begin{aligned}
\|F(f)\|_{L^1} &\leq (1-\delta)\|f\|_{L^1} + \delta\mu\|m\|_\infty\|f\|_{L^1}\frac{|X|}{\alpha(f)} \\
&\leq (1-\delta)\tilde{\Lambda} + \delta\mu\|m\|_\infty\tilde{\Lambda}\frac{|X|}{\bar{\alpha}} \\
&\leq \tilde{\Lambda}(1+\delta\mu\|m\|_\infty\frac{|X|}{\bar{\alpha}}).
\end{aligned}$$

That is, we have shown that for any $f \in \mathcal{F}$,

$$\|F(f)\|_{L^1} \leq \tilde{\Lambda}(1+\delta\mu\|m\|_\infty\frac{|X|}{\bar{\alpha}}) \text{ or } \|F(f)\|_{L^1} \leq \bar{\Lambda}. \tag{8}$$

Step 3: We show that if $\delta > 0$ is small enough, there exist $0 < \bar{\alpha}$, $\bar{\Lambda} < \infty$ such that $F(\mathcal{F}) \subset \mathcal{F}$, and $\mathcal{F} \neq \emptyset$.

Thanks to steps 1 and 2, in order to show that $F(\mathcal{F}) \subset \mathcal{F}$, we need to show that for $\delta > 0$ small enough, $\bar{\alpha} > 0$ and $\bar{\Lambda} < \infty$ can be chosen such that:

$$\left\{\begin{aligned}
\bar{\alpha} &\leq \frac{(1-\delta)C_\alpha}{(\|a_\mu''\|_\infty + \|\partial_{11}^2 d\|_\infty\bar{\Lambda})} - \delta\max a_\mu, \\
\bar{\Lambda} &\geq \tilde{\Lambda}(1+\delta\mu\|m\|_\infty\tfrac{|X|}{\bar{\alpha}}).
\end{aligned}\right. \tag{9}$$

In order to show that such a choice of $\bar{\alpha}$, $\bar{\Lambda}$ is posssible, we apply Lemma 3.1 and get that for $\delta > 0$ small enough, there exists $\hat{\alpha} > 0$ satisfying

$$\hat{\alpha} \leq \frac{(1-\delta)C_\alpha}{\|a_\mu''\|_\infty + \|\partial_{11}^2 d\|_\infty\left(\tilde{\Lambda}(1+\delta\mu\|m\|_\infty\frac{|X|}{\hat{\alpha}})\right)} - \delta\max a_\mu. \tag{10}$$

We define then $\bar{\alpha} := \hat{\alpha}$, and $\bar{\Lambda} := \tilde{\Lambda}(1+\delta\mu\|m\|_\infty\frac{|X|}{\hat{\alpha}})$. The second equation of (9) is satisfied thanks to the definition of $\bar{\Lambda}$, and the first equation of (9) is satisfied thanks to (10). Finally, $\bar{\alpha} \leq \tilde{\alpha}_{\bar{\Lambda}}$ and $\bar{\Lambda} \geq \tilde{\Lambda}$ thanks to (10). It follows that $F(\mathcal{F}) \subset \mathcal{F}$.

Finally, in order to show that $\mathcal{F}$ is not empty, we consider the constant function $g \in L^1(X)$:

$$g(x) := \frac{\tilde{\Lambda}}{|X|}. \tag{11}$$

Then, $\|g\|_{L^1} = \tilde{\Lambda} \leq \bar{\Lambda}$, and

$$\begin{aligned}
\alpha(g) &= \min\{-s[g]\} \\
&\geq \frac{\tilde{\Lambda}}{|X|}|X|\min d - \max a_\mu \\
&= \mu\|m\|_\infty|X| \\
&\geq \tilde{\alpha}_{\bar{\Lambda}},
\end{aligned}$$

thanks to the definitions of (4)-(6). Then, $\alpha(g) \geq \bar{\alpha}$, and $g \in \mathcal{F}$ which cannot be empty. Notice that $g \in \mathcal{G}$.

Step 4: We conclude.

We choose δ, $\bar\alpha$, $\bar\Lambda > 0$ such that $F(\mathcal{F}) \subset \mathcal{F}$, which is possible thanks to Step 3. We compute the first derivative of $F(f)$ as follows:

$$F(f)' = (1-\delta)f' + \delta\left[\frac{\mu\partial_1 m\tilde{*}f}{-s[f]} + (\mu m\tilde{*}f)\frac{a'_\mu - \partial_1 d\tilde{*}f}{(-s[f])^2}\right],$$

and then, if $f \in \mathcal{F} \cap W^{1,\infty}(X)$,

$$\|F(f)'\|_\infty \leq (1-\delta)\|f'\|_\infty + \gamma\delta, \tag{12}$$

where $\gamma := \frac{\mu\|\partial_1 m\|_\infty\bar\Lambda}{\bar\alpha} + \mu\|m\|_\infty\bar\Lambda\frac{\|a'_\mu\|_\infty + \|\partial_1 d\|_\infty\bar\Lambda}{\bar\alpha^2}$.
Then, $F(\mathcal{G}) \subset \mathcal{G}$ thanks to Step 3 and (12). Thus, we can apply Schauder's fixed point theorem, which proves the existence of a steady-state $\bar f \in W^{1,\infty}(X)$ of (2), and concludes the proof. □

Remark 2. If b, $d_0 \in W^{k,\infty}(X)$, d, $\beta \in W^{k,\infty}(X\times X)$, the same argument can be used to build a set $\mathcal{G}^k \subset W^{k,\infty}(X)$ such that $F(\mathcal{G}^k) \subset \mathcal{G}^k$. It follows then that the steady-state $\bar f$ given by Theorem 3.2 satisfies $\bar f \in W^{k,\infty}(X)$.

4. Asymptotics.

In this section we perform an asymptotical analysis of the steady states for small mutation rate. Since the mutation rate is supposed to be very small, we denote it in this section by ε. The model (2) then reads (recall the notation at the end of Section 2)

$$\partial_t f^\varepsilon(t,x) = s[f^\varepsilon(t,\cdot)](x)f^\varepsilon(t,x) + \varepsilon\int_X m(x,y)f^\varepsilon(t,y)\,dy \text{ for } t\geq 0,\ x\in X. \tag{13}$$

Assumption 2: We assume that $b'(0) = d_0'(0) = 0$. We also assume that $d \in W^{3,\infty}(X\times X)$, b, $d_0 \in W^{3,\infty}(X)$, and for some $1 > \bar\varepsilon > 0$,

$$\begin{aligned}\max_{x\in X}\max\left(b''(x),(1-\bar\varepsilon)b''(x)\right) - \min_{x\in X} d_0''(x)\\ + \frac{\max(b-d_0) + \bar\varepsilon\max_{y\in X}\|m(\cdot,y)\|_{L^1(X)}}{\min d}\|\partial_{11}^2 d\|_\infty \leq -\delta < 0.\end{aligned} \tag{14}$$

Remark 3. Notice that Assumption 2 implies that $b-d_0$ admits a unique maximum at $x=0$ and that for any $\varepsilon\in(0,\bar\varepsilon)$, $a_\varepsilon = (1-\varepsilon)b - d_0$ has also a unique maximum at $x = 0$.

Assumption 3: The following "symmetry" condition holds:

$$\forall x\in X,\quad \partial_1 d(x,x) = 0.$$

Assumption 4: The mutation kernel satisfies:

$$m\geq 0,\ \min_{X\times X} m > 0,\ m\in C^1(X\times X)\cap L^\infty(X\times X).$$

The next theorem gives the result on the asymptotics of the steady states.

Theorem 4.1. *Assume Assumptions 2, 3 and 4. For $\varepsilon \in (0, \bar{\varepsilon})$ (where $\bar{\varepsilon}$ is defined as in Remark 3), let f^ε be a steady-state of (13). Then, there exists $\bar{x}^\varepsilon = O(\varepsilon^{-1/3})$, such that*

$$\varepsilon f^\varepsilon(\varepsilon(\bar{x}^\varepsilon + x)) = \frac{m(0,0)\frac{a_0(0)}{d(0,0)} + O(\sqrt{\varepsilon}) + O(\varepsilon x)}{\frac{2(m(0,0)\pi)^2}{-[a_0''(0) - \frac{a_0(0)}{d(0,0)}\partial_{11}^2 d(0,0)]} + O(\sqrt{\varepsilon}) + \frac{1}{2}\left(-[a_0''(0) - \frac{a_0(0)}{d(0,0)}\partial_{11}^2 d(0,0)] + O(\sqrt{\varepsilon}) + O(\varepsilon x)\right)x^2}.$$

Remark 4. Note that this theorem shows that asymptotically, f^ε is of the form of a Cauchy distribution centered in $\varepsilon\bar{x}^\varepsilon = O(\varepsilon^{\frac{2}{3}})$.
If mutations are modelled by a diffusion (instead of an integral operator as in this article), formal asymptotics (see [18, 17]) suggest that the asymptotic profile of the population should be a Gaussian distribution.

Proof. We introduce the change of variable $f^\varepsilon(t,x) = \frac{1}{\varepsilon} g^\varepsilon(t, \frac{x}{\varepsilon})$, and consider the nontrivial steady-states $u^\varepsilon \geq 0$ for the equation on g^ε

$$\forall x \in \varepsilon^{-1}X, \quad 0 = s^\varepsilon[u^\varepsilon](x)u^\varepsilon(x) + \varepsilon^2 \int_{\varepsilon^{-1}X} m(\varepsilon x, \varepsilon y)u^\varepsilon(y)\,dy, \tag{15}$$

where $s^\varepsilon[u^\varepsilon](x) := a_\varepsilon(\varepsilon x) - \int_{\varepsilon^{-1}X} d(\varepsilon x, \varepsilon y)u^\varepsilon(y)\,dy$. Then,

$$u^\varepsilon(x) \quad = \quad \frac{\int_{\varepsilon^{-1}X} m(\varepsilon x, \varepsilon y)u^\varepsilon(y)\,dy}{-\frac{1}{\varepsilon^2}s^\varepsilon[u^\varepsilon](x)}. \tag{16}$$

Let $\bar{x}^\varepsilon \in \varepsilon^{-1}X$ be such that

$$s^\varepsilon[u^\varepsilon](\bar{x}^\varepsilon) := \max_{x \in \varepsilon^{-1}X} s^\varepsilon[u^\varepsilon](x). \tag{17}$$

Remark 5. Notice that, since $u^\varepsilon(x) \geq 0$, the second term on the right hand side in (15) is strictly positive and therefore $u^\varepsilon(x) > 0$ and $s^\varepsilon[u^\varepsilon](x) < 0$ for $x \in \varepsilon^{-1}X$.

Step 1: We show that $\frac{1}{\varepsilon^2}\partial_{xx}^2 s^\varepsilon[u^\varepsilon] \leq -\delta$.

Since u^ε satisfies (15),

$$\begin{aligned} 0 \quad &= \quad \int_{\varepsilon^{-1}X}\left[\left(a_\varepsilon(\varepsilon x) - \int_{\varepsilon^{-1}X} d(\varepsilon x, \varepsilon y)u^\varepsilon(y)\,dy\right)u^\varepsilon(x)\right]dx \\ &\quad + \int_{\varepsilon^{-1}X}\left[\varepsilon^2 \int_{\varepsilon^{-1}X} m(\varepsilon x, \varepsilon y)u^\varepsilon(y)\,dy\right]dx \\ &\leq \quad \left(\max a_\varepsilon - (\min d)\,\|u^\varepsilon\|_{L^1(\varepsilon^{-1}X)}\right)\|u^\varepsilon\|_{L^1(\varepsilon^{-1}X)} \\ &\quad + \varepsilon\left(\max_{y\in X}\|m(\cdot,y)\|_{L^1(X)}\right)\|u^\varepsilon\|_{L^1(\varepsilon^{-1}X)}, \end{aligned}$$

we can thus bound $\|u^\varepsilon\|_{L^1(\varepsilon^{-1}X)}$ from above as follows

$$\|u^\varepsilon\|_{L^1(\varepsilon^{-1}X)} \leq \frac{\max a_\varepsilon + \varepsilon \max_{y\in X}\|m(\cdot,y)\|_{L^1(X)}}{\min d}. \tag{18}$$

Moreover $\partial_{xx}^2 s^\varepsilon[u^\varepsilon]$ satisfies

$$\begin{aligned} \partial_{xx}^2 s^\varepsilon[u^\varepsilon](x) \quad &= \quad \varepsilon^2 a_\varepsilon''(\varepsilon x) - \varepsilon^2 \int_{\varepsilon^{-1}X} \partial_{11}^2 d(\varepsilon x, \varepsilon y)u^\varepsilon(y)\,dy \\ &\leq \quad \varepsilon^2\left[\max_X a_\varepsilon'' + \|u^\varepsilon\|_{L^1(\varepsilon^{-1}X)}\|\partial_{11}^2 d\|_{L^\infty(X\times X)}\right]. \end{aligned}$$

Thanks to (18) and Assumption 2,

$$\begin{aligned}\frac{1}{\varepsilon^2}\partial^2_{xx}s^\varepsilon[u^\varepsilon](x) &\leq \max_X a''_\varepsilon \\ &+ \frac{\max_X a_\varepsilon + \varepsilon \max_{y\in X}\|m(\cdot,y)\|_{L^1(X)}}{\min d}\|\partial^2_{11}d\|_{L^\infty(X\times X)} \\ &\leq -\delta < 0,\end{aligned}$$

which proves Step 1.

Remark 6. Step 1 proves that $s^\varepsilon[u^\varepsilon]$ is concave. Then, $\bar{x}^\varepsilon$ is well defined (unique) by (17).

Step 2: We show that $|\varepsilon\bar{x}^\varepsilon| = O(\sqrt{\varepsilon})$.

We show that if $\varepsilon\bar{x}^\varepsilon > 0$ then $\varepsilon\bar{x}^\varepsilon \leq O(\sqrt{\varepsilon})$. The case $\varepsilon\bar{x}^\varepsilon < 0$ can be dealt with in the same way.
Since $\varepsilon\bar{x}^\varepsilon > 0$, $[0,\varepsilon\bar{x}^\varepsilon) \subset X$, and thanks to the definition (17) of $\bar{x}^\varepsilon$,

$$0 \leq \partial_x s^\varepsilon[u^\varepsilon](\bar{x}^\varepsilon). \tag{19}$$

Then,

$$\begin{aligned}a'_\varepsilon(\varepsilon\bar{x}^\varepsilon) &\geq \int_{\varepsilon^{-1}X}\partial_1 d(\varepsilon\bar{x}^\varepsilon,\varepsilon y)u^\varepsilon(y)\,dy \\ &= \int_{|y-\bar{x}^\varepsilon|\leq 1/\sqrt{\varepsilon}}\partial_1 d(\varepsilon\bar{x}^\varepsilon,\varepsilon y)u^\varepsilon(y)\,dy \\ &\quad+\int_{|y-\bar{x}^\varepsilon|\geq 1/\sqrt{\varepsilon}}\partial_1 d(\varepsilon\bar{x}^\varepsilon,\varepsilon y)u^\varepsilon(y)\,dy.\end{aligned} \tag{20}$$

We perform a Taylor expansion to estimate the first term on the right hand side of the equality in (20), using Assumption 3 and (18)

$$\begin{aligned}\int_{|y-\bar{x}^\varepsilon|\leq 1/\sqrt{\varepsilon}}\partial_1 d(\varepsilon\bar{x}^\varepsilon,\varepsilon y)u^\varepsilon(y)\,dy &\geq \int_{|y-\bar{x}^\varepsilon|\leq 1/\sqrt{\varepsilon}}\partial_1 d(\varepsilon\bar{x}^\varepsilon,\varepsilon\bar{x}^\varepsilon)u^\varepsilon(y)\,dy \\ &\quad+\varepsilon\int_{|y-\bar{x}^\varepsilon|\leq 1/\sqrt{\varepsilon}}\partial^2_{12}d(\varepsilon\bar{x}^\varepsilon,\varepsilon\theta(y))(y-\bar{x}^\varepsilon)u^\varepsilon(y)\,dy \\ &\geq 0-\varepsilon\|\partial^2_{12}d\|_\infty\frac{1}{\sqrt{\varepsilon}}\|u^\varepsilon\|_{L^1} \\ &\geq -C_1\sqrt{\varepsilon},\end{aligned}$$

for some $C_1 > 0$. To estimate the second term of (20), we first estimate $s^\varepsilon[u^\varepsilon]$, thanks to a Taylor expansion: for $\varepsilon x \in X$,

$$\begin{aligned}s^\varepsilon[u^\varepsilon](x) &= s^\varepsilon[u^\varepsilon](\bar{x}^\varepsilon)+(x-\bar{x}^\varepsilon)\partial_x s^\varepsilon[u^\varepsilon](\bar{x}^\varepsilon)+\frac{1}{2}(x-\bar{x}^\varepsilon)^2\partial^2_{xx}s^\varepsilon[u^\varepsilon](\theta) \\ &\leq s^\varepsilon[u^\varepsilon](\bar{x}^\varepsilon)-\frac{\varepsilon^2}{2}\delta(x-\bar{x}^\varepsilon)^2,\end{aligned} \tag{21}$$

where we used (19) to estimate $\partial_x s^\varepsilon[u^\varepsilon](x)$, and Step 1 to estimate $\partial^2_{xx} s^\varepsilon[u^\varepsilon](\theta)$. Then, using (16), we get (thanks to Remark 5):

$$\begin{aligned}&\int_{|y-\bar{x}^\varepsilon|\geq 1/\sqrt{\varepsilon}} \partial_1 d(\varepsilon\bar{x}^\varepsilon,\varepsilon y)u^\varepsilon(y)\,dy\\ &\qquad \geq -\|\partial_1 d\|_\infty \int_{|y-\bar{x}^\varepsilon|>1/\sqrt{\varepsilon}} \frac{||m||_\infty \|u^\varepsilon\|_{L^1}}{-\frac{1}{\varepsilon^2}s^\varepsilon[u^\varepsilon](\bar{x}^\varepsilon)+\frac{\delta}{2}(y-\bar{x}^\varepsilon)^2}\,dy,\end{aligned}$$

and then (thanks to Remark 5 and (18)),

$$\begin{aligned}&\int_{|y-\bar{x}^\varepsilon|\geq 1/\sqrt{\varepsilon}} \partial_1 d(\varepsilon\bar{x}^\varepsilon,\varepsilon y)u^\varepsilon(y)\,dy\\ &\qquad \geq -\|\partial_1 d\|_\infty \|u^\varepsilon\|_{L^1} \int_{|y-\bar{x}^\varepsilon|>1/\sqrt{\varepsilon}} \frac{||m||_\infty}{0+\frac{1}{2}\delta(y-\bar{x}^\varepsilon)^2}\,dy\\ &\qquad \geq -C_2\sqrt{\varepsilon},\end{aligned} \tag{22}$$

for some $C_2>0$. Finally, thanks to estimates (21) and (22), (20) becomes

$$a'_\varepsilon(\varepsilon\bar{x}^\varepsilon)\geq -C\sqrt{\varepsilon},$$

where $C=C_1+C_2$. We assumed that $\varepsilon\bar{x}^\varepsilon>0$. Then, thanks to Assumption 2, $a'_\varepsilon(\varepsilon\bar{x}^\varepsilon)\leq -\varepsilon\bar{x}^\varepsilon\delta$, and thus,

$$\varepsilon\bar{x}^\varepsilon \leq \frac{-a'_\varepsilon(\varepsilon\bar{x}^\varepsilon)}{\delta}\leq \frac{C}{\delta}\sqrt{\varepsilon}.$$

Remark 7. Step 2 implies in particular that for $\varepsilon>0$ small enough, $\varepsilon\bar{x}^\varepsilon\in \operatorname{Int} X$, and then,

$$0=\partial_x s^\varepsilon[u^\varepsilon](\bar{x}^\varepsilon).$$

Thanks to this equality and a Taylor expansion, (16) becomes:

$$u^\varepsilon(x) = \frac{\int_{\varepsilon^{-1}X} m(\varepsilon x,\varepsilon y)u^\varepsilon(y)\,dy}{-\frac{1}{\varepsilon^2}s^\varepsilon[u^\varepsilon](\bar{x}^\varepsilon)+\frac{1}{2\varepsilon^2}(-\partial^2_{xx}(s^\varepsilon[u^\varepsilon])(\theta))(x-\bar{x}^\varepsilon)^2} \tag{23}$$

where $\theta\in[\bar{x}^\varepsilon,x]$ (or $\theta\in[x,\bar{x}^\varepsilon]$, if $x<\bar{x}^\varepsilon$).

Step 3: We bound $Q^\varepsilon := -\frac{1}{\varepsilon^2}s^\varepsilon[u^\varepsilon](\bar{x}^\varepsilon)$ from above and from below.

Let us remind what we showed in Step 1, that is,

$$\forall x\in\varepsilon^{-1}X,\quad -\partial^2_{xx}(s^\varepsilon[u^\varepsilon])(x)\geq \delta\varepsilon^2>0. \tag{24}$$

Then, thanks to (23), ($Q^\varepsilon>0$ thanks to Rem 5):

$$\begin{aligned}u^\varepsilon(\bar{x}^\varepsilon+x) &\leq \frac{\int_{\varepsilon^{-1}X} m(\varepsilon(\bar{x}^\varepsilon+x),\varepsilon y)u^\varepsilon(y)\,dy}{Q^\varepsilon+\frac{1}{2}\delta\,x^2}\\ &\leq \|m\|_\infty \frac{\int_{\varepsilon^{-1}X}u^\varepsilon(y)\,dy}{Q^\varepsilon+\frac{1}{2}\delta\,x^2},\end{aligned} \tag{25}$$

and then,

$$\begin{aligned}\int_{\varepsilon^{-1}X}u^\varepsilon &\leq \|m\|_\infty\left(\int_{\varepsilon^{-1}X}u^\varepsilon\right)\int_{\varepsilon^{-1}X-\bar{x}^\varepsilon}\frac{dx}{Q^\varepsilon+\frac{1}{2}\delta\,x^2}\\ &\leq \|m\|_\infty\left(\int_{\varepsilon^{-1}X}u^\varepsilon\right)\int_{\mathbb{R}}\frac{dx}{Q^\varepsilon+\frac{1}{2}\delta\,x^2}\\ &= \|m\|_\infty\left(\int_{\varepsilon^{-1}X}u^\varepsilon\right)\frac{\sqrt{2}\pi}{\sqrt{Q^\varepsilon\delta}},\end{aligned}$$

which yields the following upper bound on Q^ε (uniform in ε):

$$Q^\varepsilon \leq \frac{2\pi^2 \|m\|_\infty^2}{\delta}.$$

On the other hand, since

$$\begin{aligned}
\frac{1}{\varepsilon^2}\partial^2_{xx}s^\varepsilon[u^\varepsilon](x) &= a''_\varepsilon(\varepsilon x) - \int_{\varepsilon^{-1}X} \partial^2_{11} d(\varepsilon x, \varepsilon y) u^\varepsilon(y)\, dy \\
&\geq \left[\min a''_\varepsilon - \|u^\varepsilon\|_{L^1(\varepsilon^{-1}X)} \|\partial^2_{11} d\|_{L^\infty(X\times X)}\right] \\
&\geq \left[\min a''_\varepsilon - \frac{\max a_{\bar\varepsilon} + \bar\varepsilon \max_{y\in X} \|m(\cdot,y)\|_{L^1(X)}}{\min d} \|\partial^2_{11} d\|_\infty\right] \\
&=: -\tilde\delta,
\end{aligned}$$

(where we have used (18)) we can obtain, using (23) the following estimate:

$$\begin{aligned}
\int_{\varepsilon^{-1}X} u^\varepsilon &\geq (\min m)\ \|u^\varepsilon\|_{L^1(\varepsilon^{-1}X)} \int_{\varepsilon^{-1}X-\bar x^\varepsilon} \frac{dx}{Q^\varepsilon - \frac{1}{2\varepsilon^2}\partial^2_{xx}s^\varepsilon[u^\varepsilon](\theta)\, x^2} \\
&\geq (\min m)\ \|u^\varepsilon\|_{L^1(\varepsilon^{-1}X)} \int_{\varepsilon^{-1}X-\bar x^\varepsilon} \frac{dx}{Q^\varepsilon + \frac{1}{2}\tilde\delta\, x^2}.
\end{aligned}$$

Since X is an interval, $\varepsilon^{-1}X - \bar x^\varepsilon$ contains the interval $(\frac{-1}{2\varepsilon}|X|, 0]$, or $[0, \frac{1}{2\varepsilon}|X|)$. Then,

$$\begin{aligned}
\frac{1}{\min m} &\geq \int_0^{\frac{1}{2\varepsilon}|X|} \frac{dx}{Q^\varepsilon + \frac{1}{2}\tilde\delta\, x^2} \\
&= \frac{\pi}{\sqrt{2Q^\varepsilon\tilde\delta}} - \int_{\frac{1}{2\varepsilon}|X|}^\infty \frac{dx}{Q^\varepsilon + \frac{1}{2}\tilde\delta\, x^2} \\
&\geq \frac{\pi}{\sqrt{2Q^\varepsilon\tilde\delta}} - O(\varepsilon).
\end{aligned}$$

For $\varepsilon > 0$ small enough, we thus get a (uniform in ε) lower bound on Q^ε:

$$Q^\varepsilon \geq \frac{\pi^2 (\min m)^2}{2\tilde\delta} > 0.$$

Step 4: We estimate $\int_{\varepsilon^{-1}X} u^\varepsilon$, and Q^ε.

We first estimate $\int_{\varepsilon^{-1}X} u^\varepsilon$. Thanks to Step 3, $Q^\varepsilon = O(1)$. Then,

$$
\begin{aligned}
O(1) &= \frac{1}{\varepsilon^2} s^\varepsilon[u^\varepsilon](\bar{x}^\varepsilon) \\
&= \frac{1}{\varepsilon^2}\left(a_\varepsilon(\varepsilon\bar{x}^\varepsilon) - \int_{\varepsilon^{-1}X} d(\varepsilon\bar{x}^\varepsilon, \varepsilon y) u^\varepsilon(y)\, dy\right) \\
&= \frac{1}{\varepsilon^2}\left(a_\varepsilon(\varepsilon\bar{x}^\varepsilon) - \int_{|y-\bar{x}^\varepsilon|\le 1/\sqrt{\varepsilon}} d(\varepsilon\bar{x}^\varepsilon, \varepsilon y) u^\varepsilon(y)\, dy\right) \\
&\quad -\frac{1}{\varepsilon^2}\left(\int_{|y-\bar{x}^\varepsilon|>1/\sqrt{\varepsilon}, y\in\varepsilon^{-1}X} d(\varepsilon\bar{x}^\varepsilon, \varepsilon y) u^\varepsilon(y)\, dy\right) \\
&\ge \frac{1}{\varepsilon^2}\left(a_\varepsilon(\varepsilon\bar{x}^\varepsilon) - \int_{|y-\bar{x}^\varepsilon|\le 1/\sqrt{\varepsilon}} \left(d(\varepsilon\bar{x}^\varepsilon, 0) + O(\varepsilon y)\right) u^\varepsilon(y)\, dy\right) \\
&\quad -\frac{1}{\varepsilon^2}\left(\|d\|_\infty \int_{|y-\bar{x}^\varepsilon|>1/\sqrt{\varepsilon}, y\in\varepsilon^{-1}X} u^\varepsilon(y)\, dy\right).
\end{aligned}
$$

We estimate (using (18), Step 2 and (25) for the second estimate),

$$
\begin{aligned}
\left|\int_{|y-\bar{x}^\varepsilon|\le 1/\sqrt{\varepsilon}} O(\varepsilon y) u^\varepsilon(y)\, dy\right| &\le \int_{|y-\bar{x}^\varepsilon|\le 1/\sqrt{\varepsilon}} Cst\, \|\varepsilon y\|_\infty u^\varepsilon(y)\, dy \\
&\le Cst\sqrt{\varepsilon}\|u^\varepsilon\|_{L^1(\varepsilon^{-1}X)} \\
&= O(\sqrt{\varepsilon}),
\end{aligned} \tag{26}
$$

$$
\begin{aligned}
\left|\int_{|y-\bar{x}^\varepsilon|>1/\sqrt{\varepsilon}, y\in\varepsilon^{-1}X} u^\varepsilon(y)\, dy\right| &\le \|m\|_{L^\infty(X\times X)}\|u^\varepsilon\|_{L^1(\varepsilon^{-1}X)} \int_{|y|>1/\sqrt{\varepsilon}} \frac{dy}{\frac{1}{2}\delta y^2} \\
&= O(\sqrt{\varepsilon}).
\end{aligned} \tag{27}
$$

Then,

$$
\begin{aligned}
O(\varepsilon^2) &\ge a_\varepsilon(\varepsilon\bar{x}^\varepsilon) - d(\varepsilon\bar{x}^\varepsilon, 0)\int_{|y-\bar{x}^\varepsilon|\le 1/\sqrt{\varepsilon}} u^\varepsilon(y)\, dy - C\sqrt{\varepsilon} \\
&= a_\varepsilon(\varepsilon\bar{x}^\varepsilon) - d(\varepsilon\bar{x}^\varepsilon, 0)\int_{\varepsilon^{-1}X} u^\varepsilon(y)\, dy - O(\sqrt{\varepsilon}).
\end{aligned}
$$

We thus get

$$
d(\varepsilon\bar{x}^\varepsilon, 0)\int_{\varepsilon^{-1}X} u^\varepsilon(y)\, dy \ge a_\varepsilon(\varepsilon\bar{x}^\varepsilon) - O(\varepsilon^2) - O(\sqrt{\varepsilon}). \tag{28}
$$

On the other hand, since

$$
\begin{aligned}
O(1) &= \frac{1}{\varepsilon^2} s^\varepsilon[u^\varepsilon](\bar{x}^\varepsilon) \\
&\le \frac{1}{\varepsilon^2}\left(a_\varepsilon(\varepsilon\bar{x}^\varepsilon) - \int_{|y-\bar{x}^\varepsilon|\le 1/\sqrt{\varepsilon}} \left(d(\varepsilon\bar{x}^\varepsilon, 0) + O(\varepsilon y)\right) u^\varepsilon(y)\, dy\right) \\
&\quad +\frac{1}{\varepsilon^2}\left(\|d\|_\infty \int_{|y-\bar{x}^\varepsilon|>1/\sqrt{\varepsilon}, y\in\varepsilon^{-1}X} u^\varepsilon(y)\, dy\right),
\end{aligned}
$$

we obtain

$$
d(\varepsilon\bar{x}^\varepsilon, 0)\int_{\varepsilon^{-1}X} u^\varepsilon(y)\, dy \le a_\varepsilon(\varepsilon\bar{x}^\varepsilon) - O(\varepsilon^2) + O(\sqrt{\varepsilon}). \tag{29}
$$

From (28) and (29) we obtain

$$\begin{aligned}\int_{\varepsilon^{-1}X} u^\varepsilon(y)\,dy &= \frac{a_\varepsilon(\varepsilon\bar{x}^\varepsilon)}{d(\varepsilon\bar{x}^\varepsilon,0)} + O(\sqrt{\varepsilon}) \\ &= \frac{a_0(0)}{d(0,0)} + O(\sqrt{\varepsilon}). \end{aligned} \tag{30}$$

Next, using (23) and (27), we estimate Q^ε as follows,

$$\begin{aligned}\int_{\varepsilon^{-1}X} u^\varepsilon(x)\,dx &= \int_{|x-\bar{x}^\varepsilon|\le 1/\sqrt{\varepsilon}} u^\varepsilon(x)\,dx + \int_{|x-\bar{x}^\varepsilon|\ge 1/\sqrt{\varepsilon},\, x\in\varepsilon^{-1}X} u^\varepsilon(x)\,dx \\ &= \int_{|x-\bar{x}^\varepsilon|\le 1/\sqrt{\varepsilon}} \frac{\int_{\varepsilon^{-1}X} m(\varepsilon x,\varepsilon y)u^\varepsilon(y)\,dy}{Q^\varepsilon + \frac{1}{2\varepsilon^2}(-\partial^2_{xx}s^\varepsilon[u^\varepsilon](\theta))(x-\bar{x}^\varepsilon)^2}\,dx \\ &\quad + \int_{|x-\bar{x}^\varepsilon|\ge 1/\sqrt{\varepsilon},\, x\in\varepsilon^{-1}X} u^\varepsilon(x)\,dx \\ &= \int_{|x-\bar{x}^\varepsilon|\le 1/\sqrt{\varepsilon}} \frac{\int_{\varepsilon^{-1}X}\left(m(0,\varepsilon y) + O(\varepsilon\frac{1}{\sqrt{\varepsilon}})\right)u^\varepsilon(y)\,dy}{Q^\varepsilon + \frac{1}{2\varepsilon^2}(-\partial^2_{xx}s^\varepsilon[u^\varepsilon](\bar{x}^\varepsilon) + \varepsilon^2 O(\varepsilon\,1/\sqrt{\varepsilon}))(x-\bar{x}^\varepsilon)^2}\,dx \\ &\quad + O(\sqrt{\varepsilon}). \end{aligned} \tag{31}$$

We estimate, using (26) and (27),

$$\begin{aligned}\int_{\varepsilon^{-1}X} m(0,\varepsilon y)u^\varepsilon(y)\,dy &= \int_{|y-\bar{x}^\varepsilon|\le 1/\sqrt{\varepsilon}} m(0,\varepsilon y)u^\varepsilon(y)\,dy \\ &\quad + \int_{|y-\bar{x}^\varepsilon|\ge 1/\sqrt{\varepsilon}} m(0,\varepsilon y)u^\varepsilon(y)\,dy \\ &= \int_{|y-\bar{x}^\varepsilon|\le 1/\sqrt{\varepsilon}} \left(m(0,0) + O(\varepsilon\,1/\sqrt{\varepsilon})\right)u^\varepsilon(y)\,dy \\ &\quad + \int_{|y-\bar{x}^\varepsilon|\ge 1/\sqrt{\varepsilon}} m(0,\varepsilon y)u^\varepsilon(y)\,dy \\ &= m(0,0)\int_{\varepsilon^{-1}X} u^\varepsilon(x)\,dx + O(\sqrt{\varepsilon}), \end{aligned}$$

and we estimate:

$$\begin{aligned}&\int_{|x-\bar{x}^\varepsilon|\le 1/\sqrt{\varepsilon}} \frac{dx}{Q^\varepsilon + \frac{1}{2\varepsilon^2}(-\partial^2_{xx}s^\varepsilon[u^\varepsilon](\bar{x}^\varepsilon) + \varepsilon^2 O(\sqrt{\varepsilon}))(x-\bar{x}^\varepsilon)^2} \\ &= \int_{\mathbb{R}} \frac{dy}{Q^\varepsilon + \frac{1}{2\varepsilon^2}(-\partial^2_{xx}s^\varepsilon[u^\varepsilon](\bar{x}^\varepsilon) + \varepsilon^2 O(\sqrt{\varepsilon}))y^2} \\ &- \int_{|y|\ge 1/\sqrt{\varepsilon}} \frac{dy}{Q^\varepsilon + \frac{1}{2\varepsilon^2}(-\partial^2_{xx}s^\varepsilon[u^\varepsilon](\bar{x}^\varepsilon) + \varepsilon^2 O(\sqrt{\varepsilon}))y^2} \\ &= \frac{\sqrt{2}\pi}{\sqrt{Q^\varepsilon\left(-\frac{1}{\varepsilon^2}\partial^2_{xx}s^\varepsilon[u^\varepsilon](\bar{x}^\varepsilon) + O(\sqrt{\varepsilon})\right)}} + O(\sqrt{\varepsilon}), \end{aligned}$$

because, thanks to Step 1, we know that

$$0 \le \int_{|y|\ge 1/\sqrt{\varepsilon}} \frac{dy}{Q^\varepsilon + \frac{1}{2\varepsilon^2}(-\partial^2_{xx}s^\varepsilon[u^\varepsilon](\bar{x}^\varepsilon) + \varepsilon^2 O(\sqrt{\varepsilon}))y^2} \le \int_{|y|\ge 1/\sqrt{\varepsilon}} \frac{dy}{\frac{\delta}{2}y^2} = \frac{4\sqrt{\varepsilon}}{\delta}.$$

Then, (31) becomes:

$$\begin{aligned}\int_{\varepsilon^{-1}X} u^\varepsilon(x)\,dx &= \left(m(0,0)\int_{\varepsilon^{-1}X} u^\varepsilon(x)\,dx + O(\sqrt{\varepsilon})\right)\\ &\quad \left(\frac{\sqrt{2}\pi}{\sqrt{Q^\varepsilon\left(-\frac{1}{\varepsilon^2}\partial^2_{xx}s^\varepsilon[u^\varepsilon](\bar{x}^\varepsilon)+O(\sqrt{\varepsilon})\right)}} + O(\sqrt{\varepsilon})\right),\end{aligned}$$

so that

$$\begin{aligned}Q^\varepsilon &= \frac{2(m(0,0)\pi)^2}{(-\varepsilon^{-2}\partial^2_{xx}s^\varepsilon[u^\varepsilon](\bar{x}^\varepsilon)+O(\sqrt{\varepsilon}))} + O(\sqrt{\varepsilon})\\ &= \frac{2(m(0,0)\pi)^2}{\frac{-1}{\varepsilon^2}\partial^2_{xx}s^\varepsilon[u^\varepsilon](\bar{x}^\varepsilon)} + O(\sqrt{\varepsilon}).\end{aligned}$$

To estimate Q^ε further, we estimate $-\frac{1}{\varepsilon^2}\partial^2_{xx}s^\varepsilon[u^\varepsilon](\bar{x}^\varepsilon)$ using (30),

$$\begin{aligned}\partial^2_{xx}s^\varepsilon[u^\varepsilon](\bar{x}^\varepsilon) &= \varepsilon^2 a''_\varepsilon(\varepsilon\bar{x}^\varepsilon) - \int_{\varepsilon^{-1}X}\varepsilon^2\partial^2_{11}d(\varepsilon\bar{x}^\varepsilon,\varepsilon y)u^\varepsilon(y)\,dy\\ &= \varepsilon^2\Big[a''_\varepsilon(\varepsilon\bar{x}^\varepsilon) - \int_{|y-\bar{x}^\varepsilon|\le 1/\sqrt{\varepsilon}}\partial^2_{11}d(\varepsilon\bar{x}^\varepsilon,\varepsilon y)u^\varepsilon(y)\,dy\\ &\quad - \int_{|y-\bar{x}^\varepsilon|\ge 1/\sqrt{\varepsilon}}\partial^2_{11}d(\varepsilon\bar{x}^\varepsilon,\varepsilon y)u^\varepsilon(y)\,dy\Big]\\ &= \varepsilon^2\Big[a''_\varepsilon(0)+O(\varepsilon\bar{x}^\varepsilon)\\ &\quad -\int_{|y-\bar{x}^\varepsilon|\le 1/\sqrt{\varepsilon}}\left(\partial^2_{11}d(0,0)+O(\varepsilon\bar{x}^\varepsilon)+O(\varepsilon y)\right)u^\varepsilon(y)\,dy + O(\sqrt{\varepsilon})\Big]\\ &= \varepsilon^2\Big[a''_0(0)-\partial^2_{11}d(0,0)\int_{\varepsilon^{-1}X}u^\varepsilon + O(\sqrt{\varepsilon})\Big]\\ &= \varepsilon^2\Big[a''_0(0)-\frac{a(0)}{d(0,0)}\partial^2_{11}d(0,0)+O(\sqrt{\varepsilon})\Big].\end{aligned}$$

Finally, we obtain the following estimate on Q^ε:

$$Q^\varepsilon = \frac{2(m(0,0)\pi)^2}{-[a''_0(0)-\frac{a_0(0)}{d(0,0)}\partial^2_{11}d(0,0)]} + O(\sqrt{\varepsilon}).$$

Step 5: We show that f^ε asymptotically has a Cauchy-like profile.

Thanks to (23), (27) and the estimates obtained in Step 4, we get (with $\theta\in[\bar{x}^\varepsilon,x]$, or $\theta\in[x,\bar{x}^\varepsilon]$ if $x<\bar{x}^\varepsilon$),

$$
\begin{aligned}
u^\varepsilon(\bar{x}^\varepsilon+x) &= \frac{\int_{\varepsilon^{-1}X} m(\varepsilon(\bar{x}^\varepsilon+x),\varepsilon y)u^\varepsilon(y)\,dy}{Q^\varepsilon+\frac{1}{2\varepsilon^2}(-\partial_{xx}^2(s^\varepsilon[u^\varepsilon])(\theta))x^2} \\
&= \frac{\int_{|y-\bar{x}^\varepsilon|\le 1/\sqrt{\varepsilon}} \left(m(\varepsilon(\bar{x}^\varepsilon+x),\varepsilon\bar{x}^\varepsilon)+O(\varepsilon(y-\bar{x}^\varepsilon))\right)u^\varepsilon(y)\,dy}{Q^\varepsilon+\frac{1}{2\varepsilon^2}\left(-\partial_{xx}^2(s^\varepsilon[u^\varepsilon])(\bar{x}^\varepsilon)+\varepsilon^2 O(\varepsilon(\theta-\bar{x}^\varepsilon))\right)x^2} \\
&+ \frac{\int_{|y-\bar{x}^\varepsilon|\ge 1/\sqrt{\varepsilon}} m(\varepsilon(\bar{x}^\varepsilon+x),\varepsilon y)u^\varepsilon(y)\,dy}{Q^\varepsilon+\frac{1}{2\varepsilon^2}\left(-\partial_{xx}^2(s^\varepsilon[u^\varepsilon])(\bar{x}^\varepsilon)+\varepsilon^2 O(\varepsilon(\theta-\bar{x}^\varepsilon))\right)x^2} \\
&= \frac{m(\varepsilon(\bar{x}^\varepsilon+x),\varepsilon\bar{x}^\varepsilon)\int_{\varepsilon^{-1}X}u^\varepsilon(y)\,dy+O(\sqrt{\varepsilon})}{Q^\varepsilon+\frac{1}{2\varepsilon^2}\left(-\partial_{xx}^2(s^\varepsilon[u^\varepsilon])(\bar{x}^\varepsilon)+\varepsilon^2 O(\varepsilon x)+\varepsilon^2 O(\varepsilon\bar{x}^\varepsilon)\right)x^2} \\
&= \frac{m(0,0)\frac{a(0)}{d(0,0)}+O(\sqrt{\varepsilon})+O(\varepsilon x)}{\frac{2(m(0,0)\pi)^2}{-[a_0''(0)-\frac{a_0(0)}{d(0,0)}\partial_{11}^2 d(0,0)]}+O(\sqrt{\varepsilon})+\frac{1}{2}(-[a_0''(0)-\frac{a_0(0)}{d(0,0)}\partial_{11}^2 d(0,0)]+O(\sqrt{\varepsilon})+O(\varepsilon x))x^2}.
\end{aligned}
$$

Step 6: We improve our estimate on $\bar{x}^\varepsilon$.

Thanks to Remark 7,

$$
\begin{aligned}
0 &= \partial_x s^\varepsilon[u^\varepsilon](\bar{x}^\varepsilon) \\
&= \varepsilon a_\varepsilon'(\varepsilon\bar{x}^\varepsilon)-\varepsilon\int_{\varepsilon^{-1}X}\partial_1 d(\varepsilon\bar{x}^\varepsilon,\varepsilon y)u^\varepsilon(y)\,dy \\
&= \varepsilon^2 a_\varepsilon''(\varepsilon\theta)\bar{x}^\varepsilon-\varepsilon\int_{\varepsilon^{-1}X}\partial_1 d(\varepsilon\bar{x}^\varepsilon,\varepsilon y)u^\varepsilon(y)\,dy,
\end{aligned}
$$

where $\theta\in[0,\bar{x}^\varepsilon]$ (or $\theta\in[\bar{x}^\varepsilon,0]$, if $\bar{x}^\varepsilon<0$). Then,

$$
\bar{x}^\varepsilon=\frac{1}{\varepsilon a_\varepsilon''(\varepsilon\theta)}\int_{\varepsilon^{-1}X}\partial_1 d(\varepsilon\bar{x}^\varepsilon,\varepsilon y)u^\varepsilon(y)\,dy.
$$

We consider $C>0$ small enough so that $[\bar{x}^\varepsilon-C,\bar{x}^\varepsilon+C]\subset\varepsilon^{-1}X$ (C will be precisely chosen later). Then,

$$
\begin{aligned}
&\left|\int_{\varepsilon^{-1}X}\partial_1 d(\varepsilon\bar{x}^\varepsilon,\varepsilon y)u^\varepsilon(y)\,dy\right| \\
&\quad\le\left|\int_{|y-\bar{x}^\varepsilon|\le C}\partial_1 d(\varepsilon\bar{x}^\varepsilon,\varepsilon y)u^\varepsilon(y)\,dy\right| \\
&\qquad+\left|\int_{|y-\bar{x}^\varepsilon|>C,\,y\in\varepsilon^{-1}X}\partial_1 d(\varepsilon\bar{x}^\varepsilon,\varepsilon y)u^\varepsilon(y)\,dy\right|.
\end{aligned}
$$

We estimate the first term using a Taylor expansion of $y\mapsto\partial_1 d(\varepsilon\bar{x}^\varepsilon,y)$ around $y=\varepsilon\bar{x}^\varepsilon$ (we assumed that $d\in W^{2,\infty}$), and we estimate the second term using (25).

Then, using Assumption 3,

$$\begin{aligned}
&\left|\int_{\varepsilon^{-1}X} \partial_1 d(\varepsilon\bar{x}^\varepsilon, \varepsilon y) u^\varepsilon(y)\, dy\right| \\
&\quad \le \left|\int_{|y-\bar{x}^\varepsilon|\le C} \left[\partial_1 d(\varepsilon\bar{x}^\varepsilon, \varepsilon\bar{x}^\varepsilon) + \partial^2_{12} d(\varepsilon\bar{x}^\varepsilon, \varepsilon\bar{x}^\varepsilon)\varepsilon(y-\bar{x}^\varepsilon) + O(\varepsilon^2|\bar{x}^\varepsilon - y|^2)\right] u^\varepsilon(y)\, dy\right| \\
&\qquad + \|\partial_1 d\|_\infty \int_{|z|>C,\, z\in\mathbb{R}} \frac{\|m\|_\infty \|u^\varepsilon\|_{L^1(\varepsilon^{-1}X)}}{\frac{1}{2}\delta\, z^2}\, dz \\
&\quad \le \left|\varepsilon\partial^2_{12} d(\varepsilon\bar{x}^\varepsilon, \varepsilon\bar{x}^\varepsilon) \int_{|y-\bar{x}^\varepsilon|\le C} (\bar{x}^\varepsilon - y) u^\varepsilon(y)\, dy\right| + O(\varepsilon^2 C^2)\|u^\varepsilon\|_{L^1(\varepsilon^{-1}X)} \\
&\qquad + O(C^{-1}).
\end{aligned} \tag{32}$$

To estimate $\int_{|y-\bar{x}^\varepsilon|\le C}(\bar{x}^\varepsilon - y)u^\varepsilon(y)\, dy$, we use (23) as follows (with $\theta \in (y, \bar{x}^\varepsilon)$ if $y \le \bar{x}^\varepsilon$, or $\theta \in (\bar{x}^\varepsilon, y)$ otherwise, and in particular $|\theta - \bar{x}^\varepsilon| \le C$),

$$\begin{aligned}
&\int_{|y-\bar{x}^\varepsilon|\le C} (\bar{x}^\varepsilon - y) u^\varepsilon(y)\, dy \\
&\quad = \int_{|y-\bar{x}^\varepsilon|\le C} \frac{(\bar{x}^\varepsilon - y)\left(\int_{\varepsilon^{-1}X} m(\varepsilon y, \varepsilon z) u^\varepsilon(z)\, dz\right)}{Q^\varepsilon + \frac{1}{2\varepsilon^2}(-\partial^2_{xx} s^\varepsilon[u^\varepsilon](\theta))(y-\bar{x}^\varepsilon)^2}\, dy \\
&\quad = \int_{|y-\bar{x}^\varepsilon|\le C} \frac{(\bar{x}^\varepsilon - y)\left(\int_{\varepsilon^{-1}X} m(\varepsilon y, \varepsilon z) u^\varepsilon(z)\, dz\right)}{Q^\varepsilon + \frac{1}{2\varepsilon^2}(-\partial^2_{xx} s^\varepsilon[u^\varepsilon](\bar{x}^\varepsilon))(y-\bar{x}^\varepsilon)^2}\, dy \\
&\qquad - \int_{|y-\bar{x}^\varepsilon|\le C} \frac{\frac{1}{2\varepsilon^2}(\bar{x}^\varepsilon - y)\left[\partial^2_{xx} s^\varepsilon[u^\varepsilon](\bar{x}^\varepsilon) - \partial^2_{xx} s^\varepsilon[u^\varepsilon](\theta)\right]}{\left[Q^\varepsilon + \frac{1}{2\varepsilon^2}(-\partial^2_{xx} s^\varepsilon[u^\varepsilon](\theta))(y-\bar{x}^\varepsilon)^2\right]} \times \\
&\qquad\qquad \frac{(y-\bar{x}^\varepsilon)^2\left(\int_{\varepsilon^{-1}X} m(\varepsilon y, \varepsilon z) u^\varepsilon(z)\, dz\right)}{\left[Q^\varepsilon + \frac{1}{2\varepsilon^2}(-\partial^2_{xx} s^\varepsilon[u^\varepsilon](\bar{x}^\varepsilon))(y-\bar{x}^\varepsilon)^2\right]}\, dy \\
&\quad =: I_1 + I_2.
\end{aligned} \tag{33}$$

We use symmetry arguments to estimate I_1:

$$\begin{aligned}
I_1 &= \int_{|y-\bar{x}^\varepsilon|\le C} \frac{(\bar{x}^\varepsilon - y)\left[\int_{\varepsilon^{-1}X} \left(m(\varepsilon\bar{x}^\varepsilon, \varepsilon z) + O(\varepsilon(\bar{x}^\varepsilon - y))\right) u^\varepsilon(z)\, dz\right]}{Q^\varepsilon + \frac{1}{2\varepsilon^2}(-\partial^2_{xx} s^\varepsilon[u^\varepsilon](\bar{x}^\varepsilon))(y-\bar{x}^\varepsilon)^2}\, dy \\
&= 0 + \int_{|y-\bar{x}^\varepsilon|\le C} \frac{(\bar{x}^\varepsilon - y) O(\varepsilon(\bar{x}^\varepsilon - y))}{Q^\varepsilon + \frac{1}{2\varepsilon^2}(-\partial^2_{xx} s^\varepsilon[u^\varepsilon](\bar{x}^\varepsilon))(y-\bar{x}^\varepsilon)^2}\, dy \int_{\varepsilon^{-1}X} u_\varepsilon(z)\mathrm{d}z \\
&= O(\varepsilon C^2)\left\|\frac{1}{Q^\varepsilon + \frac{1}{2\varepsilon^2}(-\partial^2_{xx} s^\varepsilon[u^\varepsilon](\bar{x}^\varepsilon))(y-\bar{x}^\varepsilon)^2}\right\|_{L^1(\varepsilon^{-1}X)} \\
&= O(\varepsilon C^2).
\end{aligned}$$

To estimate the term I_2 of (33), we notice that (we recall that $|\theta - \bar{x}^\varepsilon| \le C$),

$$\begin{aligned}
\left|\partial^2_{xx} s^\varepsilon[u^\varepsilon](\theta) - \partial^2_{xx} s^\varepsilon[u^\varepsilon](\bar{x}^\varepsilon)\right| &\le \|\partial^3_{xxx} s^\varepsilon[u^\varepsilon]\|_\infty(\theta - \bar{x}^\varepsilon) \\
&\le \varepsilon^3\left(\|a_\varepsilon\|_{W^{3,\infty}} + Cst\,\|d\|_{W^{3,\infty}}\right)(\theta - \bar{x}^\varepsilon) \\
&\le Cst\,\varepsilon^3 C,
\end{aligned}$$

and that, thanks to Step 1 (see (24)),

$$\frac{(y-\bar{x}^\varepsilon)^2}{Q_\varepsilon + \frac{1}{2\varepsilon^2}(-\partial^2_{xx} s^\varepsilon[u^\varepsilon](\theta))(y-\bar{x}^\varepsilon)^2} < \frac{1}{\frac{\delta}{2}}. \tag{34}$$

We can thus estimate the second term of (33) using Step 1,

$$\begin{aligned} |I_2| &\leq \int_{|y-\bar{x}^\varepsilon|\leq C} \frac{\frac{1}{2\varepsilon^2}C\,\left[Cst\,\varepsilon^3 C\right]}{\frac{\delta}{2}} \frac{\|m\|_\infty\, Cst}{\left[Q^\varepsilon + \frac{1}{2\varepsilon^2}(-\partial^2_{xx}s^\varepsilon[u^\varepsilon](\bar{x}^\varepsilon))(y-\bar{x}^\varepsilon)^2\right]}\,dy \\ &\leq Cst\,\varepsilon\, C^2 \int_{\mathbb{R}} \frac{dy}{\left[Q^\varepsilon + \frac{\delta}{2}y^2\right]} \\ &\leq O(\varepsilon C^2). \end{aligned}$$

Using these estimates in (32), we get

$$\left| \int_{|y-\bar{x}^\varepsilon|\leq C} (\bar{x}^\varepsilon - y)u^\varepsilon(y)\,dy \right| \leq O(\varepsilon C^2),$$

and then, $\bar{x}^\varepsilon$ can be estimated as follows, thanks to (32):

$$\begin{aligned} |\bar{x}^\varepsilon| &\leq \frac{1}{\varepsilon|a''_\varepsilon(\varepsilon\theta)|}\left|\int_{\varepsilon^{-1}X} \partial_1 d(\varepsilon\bar{x}^\varepsilon, \varepsilon y)u^\varepsilon(y)\,dy\right| \\ &= \frac{1}{\varepsilon|a''_\varepsilon(\varepsilon\theta)|}\left[\left|\varepsilon\partial^2_{12}d(\varepsilon\bar{x}^\varepsilon,\varepsilon\bar{x}^\varepsilon)\int_{|y-\bar{x}^\varepsilon|\leq C}(\bar{x}^\varepsilon - y)u^\varepsilon(y)\,dy\right| + O(\varepsilon^2C^2) + O(C^{-1})\right] \\ &\leq \frac{1}{\varepsilon}\left[\varepsilon\, O(\varepsilon C^2) + O(\varepsilon^2 C^2) + O(C^{-1})\right] \\ &\leq O(\varepsilon C^2) + O(\varepsilon^{-1}C^{-1}). \end{aligned}$$

If we choose $C := \varepsilon^{\frac{-2}{3}}$, then, for $\varepsilon > 0$ small enough, the assumption $[\bar{x}^\varepsilon - C, \bar{x}^\varepsilon + C] \subset \varepsilon^{-1}X$ (that we assumed on C at the beginning of this proof) is satisfied, since, by Step 2,

$$\begin{aligned} [\bar{x}^\varepsilon - C, \bar{x}^\varepsilon + C] &\subset B\left(0, |\bar{x}^\varepsilon| + \varepsilon^{\frac{-2}{3}}\right) \\ &\subset B\left(0, O(\varepsilon^{\frac{-1}{2}}) + \varepsilon^{\frac{-2}{3}}\right) \\ &\subset \varepsilon^{-1}X. \end{aligned}$$

Our estimates then apply to this specific choice of C, and yield the following estimate on $\bar{x}^\varepsilon$:

$$\bar{x}^\varepsilon = O(\varepsilon^{\frac{-1}{3}}).$$

This estimate ends the proof of the theorem. □

REFERENCES

[1] A. S. Ackleh, B. G. Fitzpatrick and H. Thieme, *Rate distributions and survival of the fittest: A formulation on the space of measures*, Discrete Contin. Dynam. Systems. Ser. B., **5** (2005), 917–928.

[2] J. Cleveland and A. S. Ackleh. *Evolutionary Game Theory on Measure Spaces*, Manuscript.

[3] R. Bürger and I. M. Bomze, *Stationary distributions under mutation-selection balance: Structure and properties*, Adv. in Appl. Probab., **28** (1996), 227–251.

[4] R. Bürger, "The mathematical theory of selection, recombination, and mutation". Wiley Series in Mathematical and Computational Biology. John Wiley & Sons, Ltd., Chichester, 2000.

[5] À. Calsina, C. Perelló and J. Saldaña, *Non-local reaction-diffusion equations modelling predator-prey coevolution*, Publ. Mat., **38** (1994), 315–325.

[6] À. Calsina and C. Perelló, *Equations for biological evolution*, Proc. Roy. Soc. Edinburgh Sect. A, **125** (1995), 939–958.

[7] À. Calsina and S. Cuadrado, *Small mutation rate and evolutionarily stable strategies in infinite dimensional adaptive dynamics*, J. Math. Biol., **48** (2004), 135–159.

[8] À. Calsina and S. Cuadrado, *Stationary solutions of a selection mutation model: The pure mutation case*, Math. Models Meth. Appl. Sci., **15** (2005), 1091–1117.

[9] S. Cuadrado, *Equilibria of a predator-prey model of phenotype evolution*, J. Math. Anal. Appl., **354** (2009), 286–294.

[10] J. F. Crow and M. Kimura, *The theory of genetic loads*, in Proc XIth Int. Congr. Genetics, (1964) 495–505.

[11] O. Diekmann, P. E. Jabin, S. Mischler and B. Perthame, *The dynamics of adaptation: an illuminating example and a Hamilton-Jacobi approach*, Theor. Popul. Biol., **67** (2005), 257–271.

[12] L. Desvillettes, P.E. Jabin, S. Mischler and G. Raoul, *On selection dynamics for continuous structured population models*, Commun. Math. Sci., **6** (2008), no 3 , 729–747.

[13] M. Doebeli, H.J. Blok, O. Leimar, U. Dieckmann, *Multimodal pattern formation in phenotype distributions of sexual populations*, Proc. Roy. Soc. B, **274** (2007), 347–357.

[14] U. Dieckmann and R. Law, *The dynamical theory of coevolution: A derivation from stochastic ecological processes*, J. Math. Biol., **34**(1996), 579–612.

[15] M. Doebeli, U. Diekmann, *Evolutionary Branching and Sympatric Speciation Caused by Different Types of Ecological Interactions*, Am. Nat., **156** (2000), 77-101.

[16] S. Genieys, V. Volpert and P.Auger. *Pattern and waves for a model in population dynamics with nonlocal consumption of resources*, Math. Model. Nat. Phenom., **1** (2006), no. 1, 65–82.

[17] I. Gudelj, C. D. Coman and R. E. Beardmore, *Classifying the role of trade-offs in the evolutionary diversity of pathogens*, Proc. R. Soc. Lond. Ser. A Math. Phys. Eng. Sci., **462** (2006), 97–116.

[18] M. Kimura, *A stochastic model concerning the maintenance of genetic variability in quantitative characters*, Proc. Natl. Acad. Sci. U.S.A, **54** (1965), 731–736.

[19] P. Magal and G. F. Webb, *Mutation, selection and recombination in a model of phenotype evolution*, Discrete Contin. Dynam. Systems, **6** (2000), 221–236.

[20] J. A. J. Metz, S. A. H. Geritz, G. Meszéna, F. A. J. Jacobs, and J. S. van Heerwaasden, *Adaptive dynamics, a geometrical study of the consequences of nearly faithful reproduction*, In S. J. van Strien and S. M. Verduyn Lunel, editors, Stochastic and Spatial Structures of Dynamical Systems, pages 183–231, Amsterdam, 1996.

[21] S.D. Mylius and O. Diekmann, *On evolutionary unbeatable life histories, optimization and the need to be specific about density dependence*, Oikos, **74** (1995), 218–224.

[22] J. Polechová, N.H. Barton, *Speciation through competition: a critical review*, Evolution **59**(6) (2005), 1194–210.

[23] G. Raoul, *Long time evolution of populations under selection and vanishing mutations*, preprint CMLA-ENS Cachan **32** (2009), submitted.

[24] G. Raoul, *Stability of Evolutionary attractors for continuous structured populations*, preprint CMLA-ENS Cachan **1** (2009), accepted in Monatsh. Math.

[25] J. Saldaña, S. F. Elena and R. V. Solé, *Coinfection and superinfection in RNA populations: A selection-mutation model*, Math. Biosci., **183** (2003), 135–160.

[26] A. Sasaki, *Clumped Distribution by Neighborhood Competition*, J. Theor. Biol., **186**(1997), 415-430.

[27] A. Sasaki, H.C.J. Godfray, *A model for the coevolution of resistance and virulence in coupled host-parasitoid interactions*, Proc. R. Soc. Lond. B **266** no 1418 (1999), 455–463.

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