Double milling in self-propelled swarms from kinetic theory

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PACS 05.65.+b – Self-organizing systems
PACS 05.20.Dd – Kinetic theory
PACS 02.30.Jr – Partial differential equations

Abstract. - We present a kinetic theory for swarming systems of interacting, self-propelled discrete particles. Starting from the Liouville equation for the many-body problem we derive a kinetic equation for the single particle probability distribution function and the related macroscopic hydrodynamic equations. General solutions include flocks of constant density and fixed velocity and other non-trivial morphologies such as compactly supported mills. Our kinetic theory approach allows us to identify macroscopic structures otherwise not recognized as solutions of the hydrodynamic equations, such as double mills of two superimposed flows. We find the conditions that lead to the existence of such solutions and compare to the case of single rotating mills.

Introduction. – Viewed as a collective, interacting individuals often flow into spectacular coherent patterns [1]. Systems that self-organize can be observed in nature across a wide variety of spatio-temporal scales: schools of fish, flocks of birds and swarms of insects among animals, morphogenetic and bacterial growth at the cellular and subcellular levels. While each of these aggregates follow specific biophysical laws, all of them are able to organize internally, allowing order to arise from an initially disordered state and in the absence of a leader [2].

The ubiquity of the self-organizing phenomenon has lead to the development of several minimal models to describe a collection of interacting agents, both as discrete particles [3–5] or as a continuous density [6,7]. In particular, models of individuals driven by self-propelling forces and pairwise attractive and repulsive interactions have been shown to self-organize in various morphologies. Translationally invariant flocks, rotating mills, rings and clumps have all been observed and classified, so that specific interaction and propulsion values can be associated to specific collective configurations [8–10].

However, as the number of particles grows, it becomes increasingly difficult to follow the dynamics of each individual agent. Indeed, given N individuals, there are about N^2 interactions to compute at each time step, which in the limit of large N may yield to prohibitive calculations. A more compact, continuum approach where particles are represented by a density field, becomes thus desirable. While several continuum models based on heuristic derivations have been presented in the literature, few attempts have been made at trying to derive the hydrodynamic equations starting from discrete models [11–14].

Furthermore, even when parallels between microscopic and macroscopic descriptions exist, there might be cases in which the fullness of a microscopic solution is not necessarily captured by the corresponding macroscopic one. For example, in simulations of rotating mills, discrete particle systems show the possibility of *two* compactly supported structures of roughly the same number of particles circulating in opposite directions. The corresponding macroscopic solution would be a "boring" stationary density since the two mills average out their velocities to zero. Due to the non-linearity of the problem, the trivial superposition of two rotating mills which are solutions to the macroscopic problem, is not necessarily a solution itself.

This paper aims to bridge general microscopic descriptions of self-propelled interacting swarming systems to their macroscopic counterparts, using kinetic theory [15,16] as middle ground. Here, the exact location and velocity of particles are considered irrelevant, but not to the extent that average velocities can be computed *tout court* at every position and every time step. Rather, several velocities may be possible, so that the focus is on determining the probability $f(\mathbf{x}, \mathbf{v}, t)$ that at time t a particle is at position $\mathbf{x}$ with velocity $\mathbf{v}$. Starting from a set of discrete swarming equations we shall thus derive the kinetic equation for $f(\mathbf{x}, \mathbf{v}, t)$ and hence present the corresponding hydrodynamic description. Solutions will be matched to the discrete case and most importantly, our kinetic model will allow us to identify the presence of a new class of solutions, those of double mills, which elude a strictly macroscopic derivation.

Discrete Model. – We consider N interacting, self-propelled particles with Rayleigh friction, governed by the following equations of motion [8,10,12,17]

$$\begin{aligned} \dot{\mathbf{x}}_i &= \mathbf{v}_i, \\ m_i \dot{\mathbf{v}}_i &= m_i(\alpha - \beta\,|\mathbf{v}_i|^2)\mathbf{v}_i - m_i \nabla_{\mathbf{x}_i} \sum_{j \neq i} m_j U(|\mathbf{x}_i - \mathbf{x}_j|). \end{aligned}$$

Here U is a pairwise interaction potential per unit mass and $\alpha, \beta > 0$ are effective values for propulsion and friction forces per unit mass. We will mostly focus our discussion on this "per unit mass" case. On the other hand, the full mass description can be written by simply letting α, β, U refer to global quantities and by setting $m_i = 1$. A common choice for U is the Morse potential composed of attractive and repulsive components

$$U(r) = -C_a e^{-r/\ell_a} + C_r e^{-r/\ell_r},$$

with C_a, C_r attractive and repulsive strengths and ℓ_a, ℓ_r their respective length scales. While the Morse potential is a widely spread choice for interacting swarming systems, in this formulation we keep U general. For simplicity, we assume identical particles and with total mass fixed at $M = N m_i$ such that

$$\dot{\mathbf{x}}_i = \mathbf{v}_i, \tag{1}$$

$$\dot{\mathbf{v}}_i = (\alpha - \beta\,|\mathbf{v}_i|^2)\mathbf{v}_i - \frac{M}{N} \nabla_{\mathbf{x}_i} \sum_{j \neq i} U(|\mathbf{x}_i - \mathbf{x}_j|). \tag{2}$$

The mass normalization can be considered a scaling assumption for the potential amplitude, the so-called "weak coupling limit" [18]. It allows total kinetic and potential energy to bear the same N dependence since, in the unnormalized case, the total kinetic energy is a sum of N terms and the total potential energy scales as $N(N-1)/2$. In the normalized case of Eq. 2 the interaction amplitudes are now N dependent but the fundamental character of the resulting morphologies do not change since we simply introduce a multiplicative factor for U. As an example, in the case of the Morse potential, patterns of aggregation depend on the relative amplitudes $C = C_r/C_a$ and $\ell = \ell_r/\ell_a$ [10]. Both the normalized and unnormalized potentials (where $M = N$ in Eq. 2) thus yield the same type of patterns, whether they be

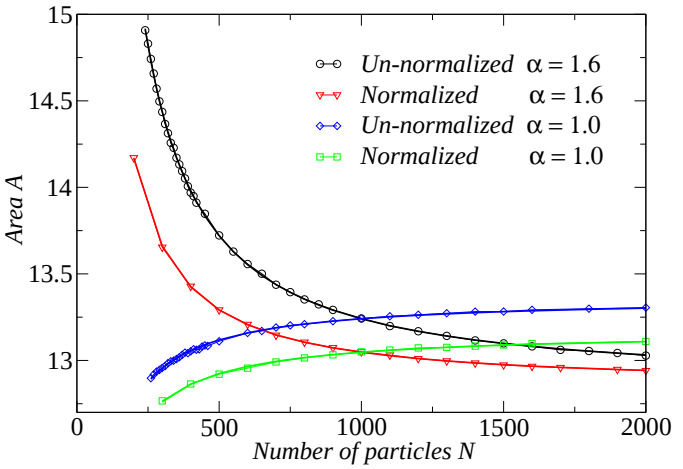

Fig. 1: Area of unidirectional milling structures as a function of N and for $\alpha = 1.6$ and $\alpha = 1.0$. The decreasing curves represent the unnormalized potential and the increasing ones the normalized potential, where $M = 1000$. Milling parameters are chosen as $C_a = 0.5$, $C_r = 1.0$, $\ell_a = 3$, $\ell_r = 0.5$ and $\beta = 0.5$. At fixed α, the normalized and unnormalized curves match at $N = 1000$, corresponding to $m_i = 1$, the reference point for the unnormalized potential. As $N \to \infty$, the asymptotic areas A_∞ differ. The unnormalized curves scale as $|A - A_\infty| \sim N^{-1.2}$ and the normalized ones as $|A - A_\infty| \sim N^{-1}$.

mills or flocks, however the way these structures scale with N is very different. In Fig. 1 we show mill areas of discrete particle systems as a function of N in the so called catastrophic regime, where, in the unnormalized case, rotating mills are expected to collapse to a finite region as the number of agents increases. As can be seen, mills areas scale very differently in the unormalized regime compared to the normalized case.

Collisionless Kinetic Model. – We denote by $f^{(N)}(\{\mathbf{x}_i\}, \{\mathbf{v}_i\}, t)$ the N-particle probability density function of the system, so that the probability of finding each of the i particles at position $\mathbf{x}_i$ and velocity $\mathbf{v}_i$ within a volume $d\mathbf{x}_i\, d\mathbf{v}_i$ in phase space is $f^{(N)}(\{\mathbf{x}_i\}, \{\mathbf{v}_i\}, t) \prod_i d\mathbf{x}_i\, d\mathbf{v}_i$. Conservation of mass allows to write the time evolution of $f^{(N)}$ according to the following Liouville equation

$$\frac{\partial f^{(N)}}{\partial t} + \sum_{i=1}^{N} \left[\mathrm{div}_{\mathbf{x_i}}(\dot{\mathbf{x}}_i f^{(N)}) + \mathrm{div}_{\mathbf{v}_i}(\dot{\mathbf{v}}_i f^{(N)}) \right] = 0, \tag{3}$$

where we have now set $M = 1$ in Eq. 1. The one-particle distribution function $f^{(1)}(\mathbf{x}_1, \mathbf{v}_1, t)$ is defined as

$$f^{(1)}(\mathbf{x}_1, \mathbf{v}_1, t) = \int f^{(N)}\, d\mathbf{x}_2 \ldots d\mathbf{x}_N d\mathbf{v}_2 \ldots d\mathbf{v}_N.$$

Our goal is to find the time evolution of f^1 in the collisionless case using standard tools from kinetic theory [18, 19] and utilizing a mean-field type approach where particles are considered uncorrelated. The kinetic equation for $f^{(1)}$ is in Eq. 5. Here we present its derivation. Integrating the Liouville equation 3 we find that the only term in the sums not to vanish are those for $i = 1$, since the probability density function is assumed to be zero at infinity in phase space. We thus obtain

$$\frac{\partial f^{(1)}}{\partial t} + \int \mathrm{div}_{\mathbf{x}_1}(\mathbf{v}_1 f^{(N)})\, d\mathbf{\Omega}_1 + \int \mathrm{div}_{\mathbf{v}_1}(\dot{\mathbf{v}}_1 f^{(N)})\, d\mathbf{\Omega}_1 = 0,$$

where $d\mathbf{\Omega}_1 = d\mathbf{x}_2 \dots d\mathbf{x}_N d\mathbf{v}_2 \dots d\mathbf{v}_N$ is the volume element. The spatial divergence term reduces to $\mathbf{v}_1 \cdot \nabla_{\mathbf{x}_1} f^{(1)}$ while the momentum term decomposes into

$$\mathrm{div}_{\mathbf{v}_1}[(\alpha - \beta|\mathbf{v}_1|^2)\mathbf{v}_1 f^{(1)}],$$

and

$$\mathrm{div}_{\mathbf{v}_1} \int (\nabla_{\mathbf{x}_1} \sum_{j\neq 1} U_{1,j})\, f^{(N)}\, d\mathbf{\Omega}_1,$$

where $U_{1,j} \equiv U(|\mathbf{x}_1 - \mathbf{x}_j|$. Since particles are indistinguishable the last term can be recast as

$$\int (\nabla_{\mathbf{x}_1} \sum_{j\neq 1} U_{1,j})\, f^{(N)}\, d\mathbf{\Omega}_1 = (N-1) \int \nabla_{\mathbf{x}_1} U_{1,2}\, f^{(N)}\, d\mathbf{\Omega}_1.$$

The last integral can be simplified by letting $d\mathbf{\Omega_1} = d\mathbf{x}_2 d\mathbf{v}_2\, d\mathbf{\Omega}_2$ and performing the integral over $d\mathbf{\Omega}_2$ so that

$$\int \nabla_{\mathbf{x}_1} U_{1,2}\, f^{(N)}\, d\mathbf{x}_2\, d\mathbf{v}_2\, d\mathbf{\Omega}_2 = \int \nabla_{\mathbf{x}_1} U_{1,2}\, f^{(2)}\, d\mathbf{x}_2 d\mathbf{v}_2,$$

where $f^{(2)}$ is the pair correlation function

$$f^{(2)}(\mathbf{x}_1, \mathbf{x}_2, \mathbf{v}_1, \mathbf{v}_2) = \int f^{(N)} d\mathbf{\Omega}_2.$$

We assume $f^{(2)}$ to be factorizable as

$$f^{(2)}(\mathbf{x}_1, \mathbf{x}_2, \mathbf{v}_1, \mathbf{v}_2) \quad = \quad g^{(2)}\, f^{(1)}(\mathbf{x}_1, \mathbf{v}_1)\, f^{(1)}(\mathbf{x}_2, \mathbf{v}_2).$$

The fundamental idea is that the two-body probability density function is the product of single particle probability density functions multiplied by a pair correlation function $g^{(2)}(\mathbf{x}_1, \mathbf{x}_2, \mathbf{v}_1, \mathbf{v}_2)$. If we assume that particles are not strongly correlated, we may fix the correlation coefficient $g^{(2)} = 1$ and further simplify the interaction term as

$$\int \nabla_{\mathbf{x}_1} U_{1,2}\, f^{(2)}\, d\mathbf{x}_2 d\mathbf{v}_2 = f^{(1)} \int \nabla_{\mathbf{x}_1} U_{1,2}\, f^{(1)}\, d\mathbf{x}_2 d\mathbf{v}_2. \tag{4}$$

We simplify the notation by $f^{(1)}(\mathbf{x}_1, \mathbf{v}_1) \equiv f(\mathbf{x}, \mathbf{v})$ such that its integral in the velocity coordinate is defined as the macroscopic density of the system

$$\rho(\mathbf{x}, t) = \int f(\mathbf{x}, \mathbf{v}, t) d\mathbf{v}.$$

Since the interaction term in Eq.4 is independent of $\mathbf{v}$ we may finally write

$$\int \nabla_{\mathbf{x}_1} U_{1,2}\, f^{(1)}\, d\mathbf{x}_2 d\mathbf{v}_2 = (\nabla U \star \rho)\, f,$$

where the $\star$ notation implies convolution between its arguments in position. Eqn. 3 thus reduces to

$$\frac{\partial f}{\partial t} + \mathbf{v} \cdot \nabla_{\mathbf{x}} f + \mathrm{div}_{\mathbf{v}}[(\alpha - \beta|\mathbf{v}|^2)\mathbf{v}\, f] - \frac{N-1}{N}\, \mathrm{div}_{\mathbf{v}}\, [(\nabla U \star \rho) f] = 0,$$

which in limit of very large N can be approximated by

$$\frac{\partial f}{\partial t} + \mathbf{v}\cdot\nabla_{\mathbf{x}} f + \mathrm{div}_{\mathbf{v}}[(\alpha - \beta|\mathbf{v}|^2)\mathbf{v}\, f] - \mathrm{div}_{\mathbf{v}}\left[(\nabla U \star \rho) f\right] = 0. \tag{5}$$

The kinetic Eq. 5 is one of the main results of this paper and determines the spatio-temporal evolution of the one particle probability distribution function. From this derivation we find the total energy E defined as the sum of the potential and kinetic energies, E_U and E_K, respectively

$$\begin{aligned} E(t) &= \frac{1}{2}\int U(|x-y|)\rho(\mathbf{x},t)\rho(\mathbf{y},t)d\mathbf{x}d\mathbf{y} \\ &+ \frac{1}{2}\int |\mathbf{v}|^2 f(\mathbf{x},\mathbf{v},t)d\mathbf{x}d\mathbf{v}. \end{aligned} \tag{6}$$

The total energy can be shown to evolve as

$$\frac{dE}{dt} = \int |\mathbf{v}|^2(\alpha - \beta|\mathbf{v}|^2) f(\mathbf{x},\mathbf{v},t)d\mathbf{x}d\mathbf{v}.$$

The system thus tends towards energy equilibration whenever particle velocities lie on the velocity sphere $\beta|\mathbf{v}|^2 = \alpha$.

Macroscopic-Hydrodynamic equations. – Starting from Eq. 5 we now derive the time evolution for the density ρ by integrating over $d\mathbf{v}$ and, by integrating Eq. 5 over $\mathbf{v}d\mathbf{v}$, for the coarse grained velocity field $\mathbf{u}(\mathbf{x},t)$. The latter is defined through

$$\rho\mathbf{u} = \int \mathbf{v} f(\mathbf{x},\mathbf{v})\, d\mathbf{v}.$$

Integrating Eq. 5 over $d\mathbf{v}$ we obtain the continuity equation

$$\frac{\partial \rho}{\partial t} + \mathrm{div}_{\mathbf{x}}(\rho\mathbf{u}) = 0. \tag{7}$$

Integrating the same equation over $\mathbf{v}d\mathbf{v}$ and using integration by parts, we find

$$\frac{\partial(\rho\mathbf{u})}{\partial t} + \int \left\{ \mathrm{div}_{\mathbf{x}}(\mathbf{v}f)\,\mathbf{v} - [(\alpha - \beta|\mathbf{v}|^2)\mathbf{v}] f \right\} d\mathbf{v} = (\nabla U \star \rho)\rho.$$

To simplify this expression, we introduce the vector $\mathbf{q}_K$, the tensor $\hat{\sigma}_K$ and the scalar δ_K defined as

$$\begin{aligned} \mathbf{q}_K &= \frac{1}{2}\int |\mathbf{v}-\mathbf{u}|^2(\mathbf{v}-\mathbf{u}) f\, d\mathbf{v}, \\ \hat{\sigma}_K &= \int (\mathbf{v}-\mathbf{u})\otimes(\mathbf{v}-\mathbf{u}) f\, d\mathbf{v}, \\ \delta_K &= \int |\mathbf{v}-\mathbf{u}|^2 f\, d\mathbf{v} = d\,\rho\,\theta, \end{aligned}$$

where $\otimes$ represents a tensor product. The above terms represent fluctuations about the coarse grained velocity fields, and are deviations of at least second order. We isolate them so as to neglect them if locally particle velocity fluctuations can be assumed to be small. The temperature of the system, $\theta(\mathbf{x},t)$ is defined by the normalized-mass variance of the distribution in velocity, where d is the dimension. In this case $d = 2$. Using this notation, we can compute all terms in the velocity equation

$$\int \mathrm{div}_{\mathbf{x}}(\mathbf{v}f)\mathbf{v}d\mathbf{v} = \mathrm{div}_{\mathbf{x}}\,\hat{\sigma}_K + \mathrm{div}_{\mathbf{x}}(\rho\mathbf{u}\otimes\mathbf{u}),$$

$$\alpha \int \mathbf{v} f d\mathbf{v} = \alpha \rho \mathbf{u},$$

and

$$\beta \int |\mathbf{v}|^2 \mathbf{v} f d\mathbf{v} = 2\mathbf{q}_K + 2\mathbf{u}\,\hat{\sigma}_K + |\mathbf{u}|^2 \rho \mathbf{u} + \delta_K \mathbf{u}.$$

Collecting all three we can finally write the momentum equation

$$\begin{aligned} \frac{\partial(\rho \mathbf{u})}{\partial t} + \mathrm{div}_{\mathbf{x}}(\rho \mathbf{u} \otimes \mathbf{u}) = (\alpha - \beta|\mathbf{u}|^2)\rho\,\mathbf{u} - (\nabla U \star \rho)\,\rho \\ -\mathrm{div}_{\mathbf{x}}\,\hat{\sigma}_K - 2\,\beta\,\mathbf{q}_K - 2\,\beta\,\mathbf{u}\,\hat{\sigma}_K \\ -\beta\,\delta_K\,\mathbf{u}. \end{aligned} \tag{8}$$

In order to close the moment system we may assume that fluctuations are small and that the distribution is mono-kinetic, i.e., $f(\mathbf{x}, \mathbf{v}, t) = \rho(\mathbf{x}, t)\,\delta(\mathbf{v} - \mathbf{u}(\mathbf{x}, t))$, where δ stands for the Dirac delta. In this case, the macroscopic system reduces to the continuity equation Eq. 7 coupled to

$$\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla_{\mathbf{x}} \mathbf{u} = (\alpha - \beta|\mathbf{u}|^2)\mathbf{u} - \nabla U \star \rho. \tag{9}$$

Eqs. 7-9 were already proposed in [12] based on computations of the empirical measure associated to N particles [20]. Here, they are rediscovered and understood as a first-order closure hydrodynamic system to the kinetic Eq. 5. We can also write the equations for energy transport. The kinetic energy density $\mathcal{E}_K$ and potential energy density $\mathcal{E}_U$ are defined, respectively, as

$$\mathcal{E}_K = \frac{1}{2}\int |\mathbf{v}|^2 f d\mathbf{v} = \rho \frac{|\mathbf{u}|^2}{2} + \frac{\delta_K}{2} = \rho \frac{|\mathbf{u}|^2}{2} + \frac{d}{2}\rho\theta,$$

and

$$\mathcal{E}_U = \frac{1}{2}(U \star \rho)\,\rho.$$

In order to determine the evolution of the kinetic energy density we multiply Eq. 5 by $|\mathbf{v}|^2/2$, integrate over $d\mathbf{v}$ and integrate by parts to obtain

$$\begin{aligned} \frac{\partial \mathcal{E}_K}{\partial t} + \int \frac{|\mathbf{v}|^2}{2} \mathbf{v} \cdot \nabla_{\mathbf{x}} f d\mathbf{v} - \int |\mathbf{v}|^2 (\alpha - \beta|\mathbf{v}|^2)\, f\, d\mathbf{v} \\ + \int \mathbf{v}(\nabla U \star \rho) f\, d\mathbf{v} = 0. \end{aligned}$$

As before, we can rewrite all integral terms as

$$\int \frac{|\mathbf{v}|^2}{2} \mathbf{v} \cdot \nabla_{\mathbf{x}} f d\mathbf{v} = \mathrm{div}_{\mathbf{x}}\,\mathcal{E}_K \mathbf{u} + \mathrm{div}_{\mathbf{x}}\,\mathbf{q}_K + \mathrm{div}_{\mathbf{x}}\,\mathbf{u}\,\hat{\sigma}_K,$$

$$\int (\nabla_{\mathbf{x}} U \star \rho)\, f\, \mathbf{v}\, d\mathbf{v} = (\nabla_{\mathbf{x}} U \star \rho)\,\rho \mathbf{u},$$

and

$$\int (\alpha - \beta|\mathbf{v}|^2)|\mathbf{v}|^2\, f d\mathbf{v} = 2\alpha \mathcal{E}_K + \beta \int |\mathbf{v}|^4\, f d\mathbf{v}.$$

Using the chain rule and the continuity equation for ρ we find

$$\frac{\partial \mathcal{E}_U}{\partial t} = -\mathrm{div}_{\mathbf{x}}\, \mathcal{E}_U \mathbf{u} + \frac{1}{2}\left[(\nabla_{\mathbf{x}} U \star \rho\mathbf{u}) + (\nabla_{\mathbf{x}} U \star \rho)\,\mathbf{u}\right]\rho,$$

so that upon defining $\mathcal{E} = \mathcal{E}_K + \mathcal{E}_U$ we can write

$$\begin{aligned}\frac{\partial \mathcal{E}}{\partial t} + \mathrm{div}_{\mathbf{x}}\mathcal{E}\,\mathbf{u} + \mathrm{div}_{\mathbf{x}}(\mathbf{q}_K + \mathbf{u}\,\hat{\sigma}_K) = -\beta \int |\mathbf{v}|^4 f d\mathbf{v} \\ + \frac{1}{2}\left[(\nabla_{\mathbf{x}} U \star \rho\mathbf{u}) - (\nabla_{\mathbf{x}} U \star \rho)\mathbf{u}\right]\rho + 2\alpha\mathcal{E}_K. \end{aligned} \tag{10}$$

Finally, the latter term can be expressed as

$$\int |\mathbf{v}|^4 f d\mathbf{v} = |\mathbf{u}|^2(2\mathcal{E}_K + \delta_K) + 8\mathbf{u}\cdot\mathbf{q}_K + \tau_K + 4g_K,$$

where τ_K and g_K are defined as

$$\tau_K = \int |\mathbf{v} - \mathbf{u}|^4 f d\mathbf{v}$$

and

$$g_K = \int \left[\mathbf{u}\cdot(\mathbf{v}-\mathbf{u})\right]^2 d\mathbf{v}.$$

Again, in the limit of small fluctuations $\theta \simeq 0$, the above reduces to $2\mathcal{E}_K = |\mathbf{u}|^2 + \delta_K \sim |\mathbf{u}|^2$, and the energy density transport Eq. 10 can be written as

$$\begin{aligned}\frac{\partial \mathcal{E}}{\partial t} + \mathrm{div}_{\mathbf{x}}\mathcal{E}\,\mathbf{u} = 2\alpha|\mathbf{u}|^2 - \beta|\mathbf{u}|^4 \\ + \frac{1}{2}\left[(\nabla_{\mathbf{x}} U \star \rho\mathbf{u}) - (\nabla_{\mathbf{x}} U \star \rho)\mathbf{u}\right]\rho.\end{aligned}$$

Eqs.7, 8 and 10 constitute the hydrodynamic description of our general swarming system.

Single-Milling and Flocking Patterns: Mono-Kinetic solutions. – We now try to find weak solutions to the kinetic Eq. 5 of the mono-kinetic form

$$f(\mathbf{x},\mathbf{v},t) = \rho(\mathbf{x},t)\,\delta(\mathbf{v} - \mathbf{u}(\mathbf{x},t)), \tag{11}$$

where the constraints on ρ and $\mathbf{u}$ will be imposed by the weak formulation analysis. Note that ρ is the macroscopic density obtained by integrating f over $d\mathbf{v}$, and that integrating Eq. 11 over $\mathbf{v}d\mathbf{v}$ leads to the macroscopic velocity $\mathbf{u}$. Furthermore, in this ansatz all fluctuating terms are strictly zero, since all microscopic velocities are identically set to $\mathbf{u}$. Upon substituting Eq. 11 into Eq. 5, we multiply by a test function $\psi(\mathbf{x},\mathbf{v},t)$, in order to find weak form solutions. Integrating over $d\mathbf{v}$, we find that all $\mathbf{v}$ values turn into $\mathbf{u}$. The other spatial and temporal integrations can be resolved via integration by parts so that

$$\int \frac{\partial f}{\partial t}\,\psi\, dt d\mathbf{x}\, d\mathbf{v} = -\int \frac{\partial[\psi]}{\partial t}\,\rho\, dt\, d\mathbf{x}, \tag{12}$$

where the $[\cdot]$ notation is meant as the partial derivative of ψ evaluated at its arguments $(\mathbf{x},\mathbf{u}(\mathbf{x},t),t)$. This quantity can be written through the chain rule as follows

$$\frac{\partial \psi}{\partial t} = \frac{\partial[\psi]}{\partial t} + \frac{\partial \mathbf{u}}{\partial t}\cdot\nabla_{\mathbf{v}}\psi,$$

where $\frac{\partial \psi}{\partial t}$ refers to the derivative in time of the function $\psi(\mathbf{x}, \mathbf{u}(\mathbf{x},t),t)$. Following this notation for all terms in the kinetic equation, we find time derivative to be

$$\int \frac{\partial f}{\partial t}\,\psi\, dt d\mathbf{x}\, d\mathbf{v} = -\int \frac{\partial \psi}{\partial t}\,\rho\, dt\, d\mathbf{x} + \int \frac{\partial \mathbf{u}}{\partial t}\cdot \nabla_{\mathbf{v}}\psi\,\rho\, dt\, d\mathbf{x}.$$

Similarly for the transport term we find

$$\begin{aligned}\int \mathrm{div}_{\mathbf{x}}(\mathbf{v}f)\psi\, dt\, d\mathbf{x}\, d\mathbf{v} = & -\int \rho\,\mathbf{u}\cdot\nabla_{\mathbf{x}}\psi\, dt\, d\mathbf{x} \\ & + \int \rho\,\mathbf{u}\cdot(\nabla_{\mathbf{x}}\mathbf{u}\cdot\nabla_{\mathbf{v}}\psi)\, dt\, d\mathbf{x}.\end{aligned}$$

We need not perform the same operations of completing the derivative in the interaction term, $U\star\rho$ since there is no implicit dependence of ψ on $\mathbf{u}$. If we now impose that $\beta|\mathbf{u}(\mathbf{x},t)|^2 = \alpha$ the frictional terms balance each other, so that the weak formulation of the kinetic equation for this particular mono-kinetic distribution reads

$$\begin{aligned}\int \left[\rho\frac{\partial \mathbf{u}}{\partial t} + \rho\,\mathbf{u}\cdot\nabla_{\mathbf{x}}\mathbf{u} + (\nabla_{\mathbf{x}}U\star\rho)\,\rho\right]\cdot\nabla_{\mathbf{v}}\psi\, dt\, d\mathbf{x} \\ + \int\left(\frac{\partial \rho}{\partial t} + \mathrm{div}_{\mathbf{x}}(\mathbf{u}\rho)\right)\psi\, dt\, d\mathbf{x} = 0.\end{aligned}$$

By choosing ψ to be velocity independent, $\psi = \psi(\mathbf{x},t)$ or $\psi = \psi(\mathbf{x},t)\mathbf{v}$ the continuity and momentum balance equations, Eqs. 7 and 8, are recovered with the additional constraint $\beta|\mathbf{u}(\mathbf{x},t)|^2 = \alpha$. Moreover, it follows from the procedure above that the ansatz in Eq. 11 is a weak solution to Eq. 5 if and only if Eqs. 7 and 8 are satisfied for the density and mean velocity. In the particular case of steady solutions, we find the following equations for the density and mean velocity of the solutions

$$\mathrm{div}_{\mathbf{x}}(\rho\mathbf{u}) = 0, \tag{13}$$

$$\mathbf{u}\cdot\nabla_{\mathbf{x}}\mathbf{u} = -(\nabla_{\mathbf{x}}U\star\rho) \tag{14}$$

$$\beta|\mathbf{u}(\mathbf{x})|^2 = \alpha. \tag{15}$$

One family of solutions for these equations of the form in Eq. 11 is given by constant ρ and $\mathbf{u}$. This corresponds to the translationally invariant, flocking solution. We can obtain another family of mono-kinetic solutions by assuming that particles undergo a circular motion with constant speed $\sqrt{\alpha/\beta}$. These single milling solutions were proposed in [8] and found numerically. We will come back to this particular set of solutions below.

Double Milling Patterns: Hydrodynamic Superpositions at Kinetic Level. – In the previous section, we showed that a possible solution to the kinetic equation is found by imposing that all particles travel at a fixed speed given by $\beta|\mathbf{u}(\mathbf{x},t)|^2 = \alpha$ describing a macroscopic density ρ that satisfies the proper continuity and momentum equations in Eqs. 14 and 15. Here, we look for the conditions that must be met in order to ensure that a superposition of such solutions exist. For concreteness, we consider the case of the superposition of two densities ρ_1 and ρ_2, with velocity $\mathbf{u}_1$ and $\mathbf{u}_2$ respectively

$$f = \rho_1\delta(\mathbf{v}-\mathbf{u}_1(\mathbf{x},t)) + \rho_2\delta(\mathbf{v}-\mathbf{u}_2(\mathbf{x},t)), \tag{16}$$

satisfying $\beta|\mathbf{u}_i(\mathbf{x},t)|^2 = \alpha$, $i = 1,2$. With this definition we find

$$\rho = \rho_1 + \rho_2, \tag{17}$$

$$\rho\mathbf{u} = \rho_1\mathbf{u}_1 + \rho_2\mathbf{u}_2. \tag{18}$$

Upon inserting Eq. 16 into Eq. 5, multiplying by the test function ψ and completing the temporal and spatial integrals we find

$$\begin{aligned}
&\int \left[\rho_1 \frac{\partial \mathbf{u}_1}{\partial t} + \rho_1\, \mathbf{u}_1 \cdot \nabla_{\mathbf{x}} \mathbf{u_1} + (\nabla_{\mathbf{x}} U \star \rho)\, \rho_1\right] \cdot \nabla_{\mathbf{v}} \psi_1 \, dt\, d\mathbf{x} \\
+ &\int \left[\rho_2 \frac{\partial \mathbf{u}_2}{\partial t} + \rho_2\, \mathbf{u}_2 \cdot \nabla_{\mathbf{x}} \mathbf{u_2} + (\nabla_{\mathbf{x}} U \star \rho)\, \rho_2\right] \cdot \nabla_{\mathbf{v}} \psi_2 \, dt\, d\mathbf{x} \\
&+ \int \left(\frac{\partial \rho_1}{\partial t} + \mathrm{div}_{\mathbf{x}}(\mathbf{u}_1 \rho_1)\right) \psi_1 \, dt\, d\mathbf{x} \\
&+ \int \left(\frac{\partial \rho_2}{\partial t} + \mathrm{div}_{\mathbf{x}}(\mathbf{u}_2 \rho_2)\right) \psi_2 \, dt\, d\mathbf{x} = 0,
\end{aligned} \tag{19}$$

where the subscripts $\psi_{1,2}$ signify that the quantities of interest are evaluated at $\mathbf{v} = \mathbf{u}_1$ and $\mathbf{v} = \mathbf{u}_2$ respectively. As before, choosing ψ to be a function only of $\mathbf{x}$ and t leads to the following conservation equation

$$\frac{\partial(\rho_1 + \rho_2)}{\partial t} + \mathrm{div}_{\mathbf{x}}(\rho_1 u_1 + \rho_2 u_2) = 0,$$

which in light of Eqs. 17 and 18 is the continuity equation 7. We can also choose $\psi = \mathbf{v}$ so that the only non zero terms of Eq. 19 give

$$\begin{aligned}
\frac{\partial \mathbf{u}_1}{\partial t}\rho_1 + \frac{\partial \mathbf{u}_2}{\partial t}\rho_2 + (\nabla_{\mathbf{x}} U \star \rho)(\rho_1 + \rho_2) \\
+ \rho_1 \mathbf{u}_1 \cdot \nabla_{\mathbf{x}} \mathbf{u}_1 + \rho_2 \mathbf{u}_2 \cdot \nabla_{\mathbf{x}} \mathbf{u}_2 = 0.
\end{aligned} \tag{20}$$

This is the general condition for the two subpopulations to satisfy. The steady state assumption with time independent $\mathbf{u}_1$ and $\mathbf{u}_2$ leads to

$$\rho(\nabla_{\mathbf{x}} U \star \rho) + \rho_1 \mathbf{u}_1 \cdot \nabla_{\mathbf{x}} \mathbf{u}_1 + \rho_2 \mathbf{u}_2 \cdot \nabla_{\mathbf{x}} \mathbf{u}_2 = 0.$$

The above formulation differs from the mono-kinetic case of Eq. 11 in a non-trivial way. We can nonetheless recast the above result by assuming $2\rho_1 = 2\rho_2 = \rho$ and by imposing $\mathbf{u}_1 = -\mathbf{u}_2$ so that

$$\rho(\nabla_{\mathbf{x}} U \star \rho) + \rho \mathbf{u}_1 \cdot \nabla_{\mathbf{x}} \mathbf{u}_1 = 0.$$

Dividing by ρ we find, that similarly to the mono-velocity case, the superposition in Eq. 16 is a solution to the kinetic equation if and only if

$$\mathbf{u}_1 \cdot \nabla_{\mathbf{x}} \mathbf{u}_1 + (\nabla_{\mathbf{x}} U \star \rho) = 0. \tag{21}$$

We may thus conclude that double milling solutions, where half the particles travel at the same speed $\mathbf{u}$ and the rest at its exact opposite $-\mathbf{u}$, exist provided the continuity equation 7, the modified momentum equation 21 and the speed constraints are met. Furthermore, this result allows us to conclude that finding steady solutions $\rho, \mathbf{u}$ for the single mill case, automatically yields double milling solutions with dual velocities since Eqs. 7 and 21 and the speed constraints are the same. Note that in the dual-velocity case the average macroscopic velocity is zero and therefore, double milling solutions cannot be explained at the macroscopic hydrodynamic level. The true nature of these solutions arise solely from a kinetic theory approach.

Density profiles for Single and Double Milling Patterns. – We can now find steady state solutions for Eqs. 13-15. Milling solutions can be found by setting $\mathbf{u}$ in a rotatory state:

$$\mathbf{u} = \sqrt{\frac{\alpha}{\beta}} \frac{\mathbf{x}^{\perp}}{|\mathbf{x}|}.$$

In this case the transport term can be rewritten as

$$\mathbf{u} \cdot \nabla_{\mathbf{x}} \mathbf{u} = \frac{\alpha}{\beta} \frac{\mathbf{x}}{|\mathbf{x}|^{\mathbf{2}}}$$

so that, as shown in [8]

$$U \star \rho = D - \frac{\alpha}{\beta} \ln |\mathbf{x}|, \tag{22}$$

where D is a constant. Solutions to Eq. 22 were numerically found in [8] and matched to single mill particle patterns in [12]. From our previous discussion, these solutions also apply to superimposed mills of velocities $\mathbf{u}_1, -\mathbf{u}_1$. While the total velocity is zero, the two populations coexist rotating in opposite directions such that the total density satisfies Eq. 22.

Conclusions. – We have developed a kinetic theory aimed at describing self-propelling swarming systems driven by general pairwise interactions. This description allows us to find a new class of macroscopic solutions to swarming systems corresponding to double, superimposed mills of interacting particles. These are indeed observed in discrete simulations but they cannot be identified from the corresponding hydrodynamic equations since the inherent dual velocity distributions yield a macroscopic average of zero. We find the conditions under which double mills can coexist. Due to the non-linearity of the problem these solutions are not trivial since the superposition of two existing solutions does not necessarily satisfy the governing equations of motion. Future development includes a full numerical solution of the kinetic equations, both in one and two dimensions and the introduction of noise effects into the dynamics.

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This research was carried out during the thematic program "Optimal Transport" at the Institute for Pure and Applied Mathematics at UCLA. We are grateful to the institute for providing an excellent atmosphere for research and for some financial support. MRD acknowledges support from the NSF (DMS-0719462) and T. Chou for careful reading of the manuscript. JAC acknowledges partial support from Spanish-MCI project MTM2008-06349-C03-03.

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