VARIATION FOR SINGULAR INTEGRALS ON LIPSCHITZ GRAPHS: L^p AND ENDPOINT ESTIMATES

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Abstract. Let $1 \leq n < d$ be integers and let μ denote the n-dimensional Hausdorff measure restricted to an n-dimensional Lipschitz graph in $\mathbb{R}^d$ with slope strictly less than 1. For $\rho > 2$, we prove that the ρ-variation and oscillation for Calderón-Zygmund singular integrals with odd kernel are bounded operators in $L^p(\mu)$ for $1 < p < \infty$, from $L^1(\mu)$ to $L^{1,\infty}(\mu)$, and from $L^\infty(\mu)$ to $BMO(\mu)$. Concerning the first endpoint estimate, we actually show that such operators are bounded from the space of finite complex Radon measures in $\mathbb{R}^d$ to $L^{1,\infty}(\mu)$.

1. Introduction

Many recent papers on probability, ergodic theory, and harmonic analysis dealt with the topics of ρ-variation and oscillation for martingales and some families of operators (see [Lp], [Bo], [JKRW], [CJRW1], [JSW], [LT], and [OSTTW], for example). In this paper we continue the study developed in [MT1] and [MT2] about the ρ-variation and oscillation for Calderón-Zygmund singular integral operators with odd kernel defined on measures different form the Lebesgue measure. More precisely, we are concerned with variational L^p $(1 < p < \infty)$ and endpoint estimates for such singular integral operators defined on Lipschitz graphs and with respect to the Hausdorff measure.

Throughout the paper $1 \leq n < d$ denote two fixed integers. By an n-dimensional Lipschitz graph $\Gamma \subset \mathbb{R}^d$ we mean any translation and rotation of a set of the type

$$\{x \in \mathbb{R}^d \,:\, x = (y, \mathcal{A}(y)),\, y \in \mathbb{R}^n\},$$

where $\mathcal{A} : \mathbb{R}^n \to \mathbb{R}^{d-n}$ is some Lipschitz function with Lipschitz constant $\mathrm{Lip}(\mathcal{A})$. We say that $\mathrm{Lip}(\mathcal{A})$ is the *slope* of Γ.

Given $1 \leq n < d$ integers, $\epsilon > 0$, and a Radon measure μ in $\mathbb{R}^d$, we consider

$$T_\epsilon \mu(x) := \int_{|x-y|>\epsilon} K(x-y)\, d\mu(y), \qquad \text{for } x \in \mathbb{R}^d, \tag{1}$$

where the kernel $K : \mathbb{R}^d \setminus \{0\} \to \mathbb{C}$ satisfies

$$|K(x)| \leq \frac{C}{|x|^n}, \quad |\partial_{x_i} K(x)| \leq \frac{C}{|x|^{n+1}} \quad \text{and} \quad |\partial_{x_i}\partial_{x_j} K(x)| \leq \frac{C}{|x|^{n+2}}, \tag{2}$$

for all $1 \leq i, j \leq d$ and $x = (x_1, \ldots, x_d) \in \mathbb{R}^d \setminus \{0\}$, $C > 0$ is some constant, and moreover $K(-x) = -K(x)$ for all $x \neq 0$ (i.e. K is odd). We set $\mathcal{T}\mu := \{T_\epsilon \mu\}_{\epsilon>0}$, and given $f \in L^1(\mu)$, we also set $T^\mu_\epsilon f := T_\epsilon(f\mu)$, $T^\mu_* f(x) := \sup_{\epsilon>0} |T^\mu_\epsilon f(x)|$, and $\mathcal{T}^\mu f := \{T^\mu_\epsilon f\}_{\epsilon>0}$. The well-known Cauchy and n-dimensional Riesz transforms are two very important examples of

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such Calderón-Zygmund singular integral operators, and they correspond to the kernels $K(x)=1/x$ for $x\in\mathbb{C}\setminus\{0\}$ and $K(x)=x/|x|^{n+1}$ for $x\in\mathbb{R}^d\setminus\{0\}$ respectively (to be precise, we should consider the scalar components $x_i/|x|^{n+1}$).

Definition 1.1 (ρ-variation)**.** *Let $\mathcal{F}:=\{F_\epsilon\}_{\epsilon>0}$ be a family of functions defined on $\mathbb{R}^d$. Given $\rho>0$, the* ρ-variation *of $\mathcal{F}$ at $x\in\mathbb{R}^d$ is defined by*

$$\mathcal{V}_\rho(\mathcal{F})(x):=\sup_{\{\epsilon_m\}}\left(\sum_{m\in\mathbb{Z}}|F_{\epsilon_{m+1}}(x)-F_{\epsilon_m}(x)|^\rho\right)^{1/\rho},$$

where the pointwise supremum is taken over all decreasing sequences $\{\epsilon_m\}_{m\in\mathbb{Z}}\subset(0,\infty)$.

Given a Radon measure μ in $\mathbb{R}^d$, $f\in L^1(\mu)$, and $x\in\mathbb{R}^d$, we will deal with

$$(\mathcal{V}_\rho\circ\mathcal{T})\mu(x):=\mathcal{V}_\rho(\mathcal{T}\mu)(x),\quad\text{and}\quad(\mathcal{V}_\rho\circ\mathcal{T}^\mu)f(x):=\mathcal{V}_\rho(\mathcal{T}^\mu f)(x).$$

For a Borel set $E\subset\mathbb{R}^d$, we denote by $\mathcal{H}^n_E$ the n-dimensional Hausdorff measure resticted to E. The following result is a direct consequence of [MT2, Theorem 1.3].

Theorem 1.2. *Let $\rho>2$. Let $\Gamma\subset\mathbb{R}^d$ be an n-dimensional Lipschitz graph and set $\mu:=\mathcal{H}^n_\Gamma$. Then, $\mathcal{V}_\rho\circ\mathcal{T}^\mu$ is a bounded operator in $L^2(\mu)$. The norm of this operator is bounded by some constant depending only on n, d, K, ρ, and the slope of Γ.*

In fact [MT2, Theorem 1.3] shows that Theorem 1.2 holds whenever μ is an n-dimensional Ahlfors-David regular uniformly n-rectifiable measure in $\mathbb{R}^d$ (the notions of Ahlfors-David regularity and uniform rectifiability are geometric/measure theoretic conditions about homogeneity and quantitative rectifiability which are trivially satisfied for Lipschitz graphs; see [DS, Part I] for precise definitions). Furthermore, in [MT1] it is also proved that, if $\mu=\mathcal{H}^n_\Gamma$ for some n-dimensional Lipschitz graph $\Gamma\subset\mathbb{R}^d$, $\varphi\in\mathcal{C}^\infty(\mathbb{R})$ is some fixed function such that $\chi_{[2,\infty)}\le\varphi\le\chi_{[1/2,\infty)}$,

$$T^\mu_{\varphi_\epsilon}f(x):=\int\varphi(|x-y|/\epsilon)K(x-y)f(y)\,d\mu(y)\quad\text{for }x\in\mathbb{R}^d\text{ and }f\in L^1(\mu),\tag{3}$$

and $\mathcal{T}^\mu_\varphi:=\{T^\mu_{\varphi_\epsilon}\}_{\epsilon>0}$, then the operator $\mathcal{V}_\rho\circ\mathcal{T}^\mu_\varphi$ is bounded

(a) in $L^p(\mu)$ for all $1<p<\infty$,
(b) from $L^1(\mu)$ to $L^{1,\infty}(\mu)$, and
(c) from $L^\infty(\mu)$ to $BMO(\mu)$ (see Section 4 for the precise definition of $BMO(\mu)$).

Usually, we refer to $\mathcal{T}^\mu$ as *the family of rough truncations of the singular integral operator with kernel K and with respect to μ*, and we refer to $\mathcal{T}^\mu_\varphi$ as *the family of smooth truncations* of the same operator.

The following theorem is one of the main results of this paper. Roughly speaking, under an extra assumption on the slope of the Lipschitz graph, it improves Theorem 1.2 and extends the estimates (a), (b), and (c) above to rough truncations.

Theorem 1.3. *Let $\rho>2$. Let $\Gamma\subset\mathbb{R}^d$ be an n-dimensional Lipschitz graph with slope strictly less than 1 and set $\mu:=\mathcal{H}^n_\Gamma$. Then, $\mathcal{V}_\rho\circ\mathcal{T}^\mu$ is a bounded operator*

(a) *in $L^p(\mu)$ for all $1<p<\infty$,*
(b) *from $L^1(\mu)$ to $L^{1,\infty}(\mu)$, and*
(c) *from $L^\infty(\mu)$ to $BMO(\mu)$,*

The norm of this operator in the cases above is bounded by some constant depending only on n, d, K, the slope of Γ, ρ, and on p in the case of (a).

This theorem generalizes the results in [CJRW2] for the class of kernels given by (2) and, in this sense, it is a natural continuation of the study of variational inequalities for Calderón-Zygmund singular integral operators.

As we pointed out above, Theorem 1.3 was already known for the family $\mathcal{T}_\varphi^\mu$, but the case of rough truncations requires much more work and detail on the estimates due to the lack of regularity on the truncations. Moreover, [MT2, Theorem 1.3] (and so Theorem 1.2) were obtained using the so-called *corona decomposition* (see [DS, Chapter 3 of Part I]), which is a useful tool to deal with L^2 estimates. However, it is very difficult to adapt these techniques to deal with L^p estimates for $p \neq 2$. Thus, Theorem 1.3 does not follow from the variational L^p estimates for $\mathcal{T}_\varphi^\mu$, nor by a simple modification of the proof of Theorem 1.2, it requires a more careful and deeper study.

The other main result of this paper is the following theorem, which strengthens the endpoint estimate (b) of Theorem 1.3. Moreover, in combination with the techniques used in [MT2], we think that the following theorem could be useful to derive L^p $(1 < p < \infty)$ and endpoint estimates for $\mathcal{V}_\rho \circ \mathcal{T}^\mu$ when μ is any n-dimensional AD regular uniformly n-rectifiable measure in $\mathbb{R}^d$, which would enhance [MT2, Theorems 1.3 and 2.3]. We denote by $M(\mathbb{R}^d)$ the space of finite complex Radon measures on $\mathbb{R}^d$ equipped with the norm given by the variation of measures.

Theorem 1.4. *Let $\rho > 2$. Let $\Gamma \subset \mathbb{R}^d$ be an n-dimensional Lipschitz graph with slope strictly less than 1 and set $\mu := \mathcal{H}^n_\Gamma$. Then, $\mathcal{V}_\rho \circ \mathcal{T}$ is a bounded operator from $M(\mathbb{R}^d)$ to $L^{1,\infty}(\mu)$, i.e., there exist a constant $C > 0$ such that, for all $\lambda > 0$ and all $\nu \in M(\mathbb{R}^d)$,*

$$\mu\{x \in \mathbb{R}^d : (\mathcal{V}_\rho \circ \mathcal{T})\nu(x) > \lambda\} \leq \frac{C}{\lambda}\,\|\nu\|.$$

Moreover, the constant C only depends on n, d, K, ρ, and the slope of Γ.

***Remark* 1.5.** We think that the assumption on the smallness of the slope of the Lipschitz graph in Theorems 1.3 and 1.4 is just a technical obstruction due to the arguments we will employ in their proofs. As pointed out in the paragraph above Theorem 1.4, we expect that this assumption will be removed in the future.

The following corollary is a direct consequence of Theorem 1.4.

Corollary 1.6. *Let E be an $\mathcal{H}^n$ measurable n-rectifiable subset of $\mathbb{R}^d$ with $\mathcal{H}^n(E) < \infty$, and let K be an odd kernel satisfying (2). If $\nu \in M(\mathbb{R}^d)$, then the principal values $\lim_{\epsilon \searrow 0} T_\epsilon \nu(x)$ exist for $\mathcal{H}^n$ almost all $x \in E$.*

Given an n-rectifiable set $E \subset \mathbb{R}^d$ with $\mathcal{H}^n(E) < \infty$, as far as the author knows, the existence $\mathcal{H}^n_E$-a.e. of $\lim_{\epsilon \searrow 0} T_\epsilon \nu(x)$ for $\nu \in M(\mathbb{R}^d)$ was already known for odd kernels $K \in \mathcal{C}^\infty(\mathbb{R}^d \setminus \{0\})$ satisfying

$$|\nabla^j K(x)| \leq C_j |x|^{-n-j} \tag{4}$$

for *all* $j = 0, 1, 2, 3, \ldots$, or maybe assuming (4) only for a finite but big number of j's (see [Ma, Theorems 20.15 and 20.27, Remarks 20.16 and 20.19] and the references therein). However, the result is new if one only asks (4) for $j = 0, 1, 2$, and so Corollary 1.6 improves on previous results.

The plan of the paper is the following: In Section 2 we state some preliminary results concerning a Calderón-Zygmund decomposition of general measures and about the Hausdorff measure of a Lipschitz graph on annuli. The proof of Theorem 1.4 is given in Section 3, and in Section 4 we prove Theorem 1.3(c). Finally, in Section 5 we complete the proof of Theorem 1.3 and we also prove Corollary 1.6.

***Remark* 1.7** (oscillation)**.** Let $\mathcal{F} := \{F_\epsilon\}_{\epsilon>0}$ be a family of functions defined on $\mathbb{R}^d$. Fix a decreasing sequence $\{r_m\}_{m\in\mathbb{Z}} \subset (0,\infty)$. The *oscillation* of $\mathcal{F}$ at $x \in \mathbb{R}^d$ is defined by

$$\mathcal{O}(\mathcal{F})(x) := \sup_{\{\epsilon_m\},\{\delta_m\}} \left(\sum_{m\in\mathbb{Z}} |F_{\epsilon_m}(x) - F_{\delta_m}(x)|^2 \right)^{1/2},$$

where the pointwise supremum is taken over all sequences $\{\epsilon_m\}_{m\in\mathbb{Z}}$ and $\{\delta_m\}_{m\in\mathbb{Z}}$ such that $r_{m+1} \leq \epsilon_m \leq \delta_m \leq r_m$ for all $m \in \mathbb{Z}$. We are also interested in the operators $(\mathcal{O}\circ\mathcal{T})\mu(x) := \mathcal{O}(\mathcal{T}\mu)(x)$ and $(\mathcal{O}\circ\mathcal{T}^\mu)f(x) := \mathcal{O}(\mathcal{T}^\mu f)(x)$. Theorems 1.2, 1.3 and 1.4 also hold replacing $\mathcal{V}_\rho$ by $\mathcal{O}$. Moreover, the norm of the corresponding operators is bounded independently of the sequence that defines $\mathcal{O}$. We will only give the proof of Theorems 1.3 and 1.4 for $\mathcal{V}_\rho$, because the case of $\mathcal{O}$ follows by very similar arguments and computations. The details are left for the reader.

As usual, in the paper the letter 'C' stands for some constant which may change its value at different occurrences, and which quite often only depends on n and d. The notation $A \lesssim B$ ($A \gtrsim B$) means that there is some constant C such that $A \leq CB$ ($A \geq CB$), with C as above. Also, $A \approx B$ is equivalent to $A \lesssim B \lesssim A$.

2. Preliminaries

2.1. **Calderón-Zygmund decomposition for general measures.** Given a cube Q in $\mathbb{R}^d$, we denote by $\ell(Q)$ the side length of Q. In this paper, the cubes are assumed to be closed and to have sides parallel to the coordinate axes. Given $\nu \in M(\mathbb{R}^d)$, $a > 1$ and $b > a^n$, we say that a cube Q is (a,b)-$|\nu|$-doubling if $|\nu|(aQ) \leq b|\nu|(Q)$, where aQ is the cube concentric with Q with side length $a\ell(Q)$. For definiteness, if a and b are not specified, by a doubling cube we mean a $(2, 2^{d+1})$-$|\nu|$-doubling cube.

The following two lemmas are already known (see [To2], [To1], or [To3] for example), but since they are essential in this paper, we give their proof for completeness.

Lemma 2.1. *Let $b > a^d$. If ν is a Radon measure in $\mathbb{R}^d$, then for ν-a.e. $x \in \mathbb{R}^d$ there exists a sequence of (a,b)-$|\nu|$-doubling cubes $\{Q_k\}_k$ centered at x with $\ell(Q_k) \to 0$ as $k \to \infty$.*

Proof. Let $Z \subset \mathbb{R}^d$ be the set of points x such that there does not exist a sequence of (a,b)-$|\nu|$-doubling cubes $\{Q_k\}_{k\geq 0}$ centered at x with side length decreasing to 0; and let $Z_j \subset \mathbb{R}^d$ be the set of points x such that there does not exist any (a,b)-$|\nu|$-doubling cube Q centered at x with $\ell(Q) \leq 2^{-j}$. Clearly, $Z = \bigcup_{j\geq 0} Z_j$. Thus, proving the lemma is equivalent to showing that $\nu(Z_j) = 0$ for every $j \geq 0$.

Let Q_0 be a fixed cube with side length 2^{-j} and let $k \geq 1$ be some integer. For each $z \in Q_0 \cap Z_j$, let Q_z be a cube centered at z with side length $a^{-k}\ell(Q_0)$. Since the cubes $a^h Q_z$ are not (a,b)-$|\nu|$-doubling for $h = 0,\ldots,k-1$ and $a^k Q_z \subset 2Q_0$, we have

$$\nu(Q_z) \leq b^{-1}\nu(aQ_z) \leq \cdots \leq b^{-k}\nu(a^k Q_z) \leq b^{-k}\nu(2Q_0). \tag{5}$$

By Besicovitch's theorem, there exists a subfamily $\{z_m\}_m \subset Q_0 \cap Z_j$ such that $Q_0 \cap Z_j \subset \bigcup_m Q_{z_m}$ and moreover $\sum_m \chi_{Q_{z_m}} \leq P_d$. This is a finite family and the number N of points z_m can be easily bounded above as follows: if $\mathcal{L}$ stands for the Lebesgue measure on $\mathbb{R}^d$,

$$N\,(a^{-k}\ell(Q_0))^d = \sum_{m=1}^N \mathcal{L}(Q_{z_m}) \leq P_d\mathcal{L}(2Q_0) = P_d(2\ell(Q_0))^d.$$

Thus, $N \leq P_d 2^d a^{kd}$. As a consequence, since $\{Q_{z_m}\}_{1\leq m\leq N}$ covers $Q_0 \cap Z_j$, by (5),

$$\nu(Q_0 \cap Z_j) \leq \sum_{m=1}^{N} \nu(Q_z) \leq N b^{-k} \nu(2Q_0) \leq P_d 2^d a^{kd} b^{-k} \nu(2Q_0).$$

Since $b > a^d$, the right hand side tends to 0 as $k \to \infty$. Therefore $\nu(Q_0 \cap Z_j) = 0$, and since the cube Q_0 is arbitrary, we are done. □

Lemma 2.2 (Calderón-Zygmund decomposition). *Assume that $\mu := \mathcal{H}^n_{\Gamma\cap B}$, where Γ is an n-dimensional Lipschitz graph and $B \subset \mathbb{R}^d$ is some fixed ball. For every $\nu \in M(\mathbb{R}^d)$ with compact support and every $\lambda > 2^{d+1}\|\nu\|/\|\mu\|$, we have:*

(*a*) *There exists a finite or countable collection of almost disjoint cubes $\{Q_j\}_j$ (that is, $\sum_j \chi_{Q_j} \leq C$) and a function $f \in L^1(\mu)$ such that*

$$|\nu|(Q_j) > 2^{-d-1}\lambda\mu(2Q_j), \tag{6}$$

$$|\nu|(\eta Q_j) \leq 2^{-d-1}\lambda\mu(2\eta Q_j) \quad \text{for } \eta > 2, \tag{7}$$

$$\nu = f\mu \text{ in } \mathbb{R}^d \setminus \Omega \text{ with } |f| \leq \lambda \ \mu\text{-a.e, where } \Omega = \textstyle\bigcup_j Q_j. \tag{8}$$

(*b*) *For each j, let $R_j := 6Q_j$ and denote $w_j := \chi_{Q_j}\big(\sum_k \chi_{Q_k}\big)^{-1}$. Then, there exists a family of functions $\{b_j\}_j$ with $\operatorname{supp} b_j \subset R_j$ and with constant sign satisfying*

$$\int b_j\, d\mu = \int w_j\, d\nu, \tag{9}$$

$$\|b_j\|_{L^\infty(\mu)}\mu(R_j) \leq C|\nu|(Q_j), \text{ and} \tag{10}$$

$$\textstyle\sum_j |b_j| \leq C_0\lambda \quad \text{(where } C_0 \text{ is some absolute constant).} \tag{11}$$

***Proof of* Lemma 2.2**(*a*)**.** Let H be the set of those points from $\operatorname{supp}\mu \cup \operatorname{supp}\nu$ such that there exists some cube Q centered at x satisfying $|\nu|(Q) > 2^{-d-1}\lambda\mu(2Q)$. For each $x \in H$, let Q_x be a cube centered at x such that the preceding inequality holds for Q_x but fails for the cubes Q centered at x with $\ell(Q) > 2\ell(Q_x)$. Notice that the condition $\lambda > 2^{d+1}\|\nu\|/\|\mu\|$ guaranties the existence of Q_x.

Since H is bounded (because μ and ν are compactly supported), we can apply Besicovitch's covering theorem to get a finite or countable almost disjoint subfamily of cubes $\{Q_j\}_j \subset \{Q_x\}_{x\in H}$ which cover H and satisfy (6) and (7) by construction.

To prove (8), denote by Z be the set of points $y \in \operatorname{supp}\nu$ such there does not exist a sequence of $(2, 2^{d+1})$-$|\nu|$-doubling cubes centered at y with side length tending to 0, so that $|\nu|(Z) = 0$, by Lemma 2.1. By the definitions of H and Z, for every $x \in \operatorname{supp}\nu \setminus (H \cup Z)$, there exists a sequence of $(2, 2^{d+1})$-$|\nu|$-doubling cubes P_k centered at x, with $\ell(P_k) \to 0$, such that $|\nu|(P_k) \leq 2^{-d-1}\lambda\mu(2P_k)$, and thus $|\nu|(2P_k) \leq 2^{d+1}|\nu|(P_k) \leq \lambda\mu(2P_k)$. This implies that $\chi_{\mathbb{R}^d\setminus(H\cup Z)}\nu$ is absolutely continuous with respect to μ and that $\chi_{\mathbb{R}^d\setminus H}\nu = \chi_{\mathbb{R}^d\setminus(H\cup Z)}\nu = f\mu$ with $|f| \leq \lambda$ μ-a.e., by the Lebesgue-Radon-Nikodym theorem (see [Ma, pages 36 to 39], for instance). □

***Proof of* Lemma 2.2**(*b*)**.** Assume first that the family of cubes $\{Q_j\}_j$ is finite. Then we may suppose that this family of cubes is ordered in such a way that the sizes of the cubes R_j are non decreasing (i.e. $\ell(R_{j+1}) \geq \ell(R_j)$). The functions b_j that we will construct will be of the form $b_j = c_j\,\chi_{A_j}$, with $c_j \in \mathbb{R}$ and $A_j \subset R_j$. We set $A_1 = R_1$ and $b_1 := c_1\,\chi_{R_1}$, where the constant c_1 is chosen so that $\int_{Q_1} w_1\, d\nu = \int b_1\, d\mu$.

Suppose that $b_1, \ldots, b_{k-1}$ have been constructed, satisfy (9) and $\sum_{j=1}^{k-1} |b_j| \leq C_0\,\lambda$, where C_0 is some constant which will be fixed below. Let $R_{s_1}, \ldots, R_{s_m}$ be the subfamily of

$R_1, \ldots, R_{k-1}$ such that $R_{s_i} \cap R_k \neq \emptyset$. As $\ell(R_{s_i}) \leq \ell(R_k)$ (because of the non decreasing sizes of R_j), we have $R_{s_i} \subset 3R_k$. Taking into account that $\int |b_j|\, d\mu \leq |\nu|(Q_j)$ for $j = 1, \ldots, k-1$ by (9), and using (7) and that $\mu(6R_k) \leq C\mu(R_k)$ (because $\frac{1}{2}R_k = 3Q_k$ intersects suppμ by (7)), we get

$$\sum_i \int |b_{s_i}|\, d\mu \leq \sum_i |\nu|(Q_{s_i}) \leq C|\nu|(3R_k) \leq C\lambda\mu(6R_k) \leq C_2\lambda\mu(R_k).$$

Therefore, $\mu\left\{x \in R_k \,:\, \sum_i |b_{s_i}(x)| > 2C_2\lambda\right\} \leq \mu(R_k)/2$. So, if we set

$$A_k := \left\{x \in R_k \,:\, \textstyle\sum_i |b_{s_i}(x)| \leq 2C_2\lambda\right\},$$

then $\mu(A_k) \geq \mu(R_k)/2$.

The constant c_k is chosen so that for $b_k = c_k\chi_{A_k}$ we have $\int b_k\, d\mu = \int_{Q_k} w_k\, d\nu$. Then we obtain, by (7),

$$|c_k| \leq \frac{|\nu|(Q_k)}{\mu(A_k)} \leq \frac{2|\nu|(\frac{1}{2}R_k)}{\mu(R_k)} \leq C_3\lambda$$

(this calculation also applies to $k = 1$). Thus, $|b_k| + \sum_i |b_{s_i}| \leq (2C_2 + C_3)\,\lambda$. If we choose $C_0 = 2C_2 + C_3$, (11) follows.

Now it is easy to check that (10) also holds. Indeed we have

$$\|b_j\|_{L^\infty(\mu)}\mu(R_j) \leq C|c_j|\mu(A_j) = C\left|\int_{Q_j} w_j\, d\nu\right| \leq C|\nu|(Q_j).$$

Suppose now that the collection of cubes $\{Q_j\}_j$ is not finite. For each fixed N we consider the family of cubes $\{Q_j\}_{1\leq j\leq N}$. Then, as above, we construct functions $b_1^N, \ldots, b_N^N$ with $\operatorname{supp}(b_j^N) \subset R_j$ satisfying $\int b_j^N\, d\mu = \int_{Q_j} w_j\, d\nu$, $\sum_{j=1}^N |b_j^N| \leq C_0\,\lambda$ and $\|b_j^N\|_{L^\infty(\mu)}\mu(R_j) \leq C|\nu|(Q_j)$. Notice that the sign of b_j^N equals the sign of $\int w_j\, d\nu$ and so it does not depend on N.

Then there is a subsequence $\{b_1^k\}_{k\in I_1}$ which is convergent in the weak $*$ topology of $L^\infty(\mu)$ to some function $b_1 \in L^\infty(\mu)$. Now we can consider a subsequence $\{b_2^k\}_{k\in I_2}$ with $I_2 \subset I_1$ which is also convergent in the weak $*$ topology of $L^\infty(\mu)$ to some function $b_2 \in L^\infty(\mu)$. In general, for each j we consider a subsequence $\{b_j^k\}_{k\in I_j}$ with $I_j \subset I_{j-1}$ that converges in the weak $*$ topology of $L^\infty(\mu)$ to some function $b_j \in L^\infty(\mu)$. It is easily checked that the functions b_j satisfy the required properties. □

2.2. **Hausdorff measure of Lipschitz graphs on annuli.** Given $z \in \mathbb{R}^d$ and $0 < a \leq b$, let $A(z, a, b) \subset \mathbb{R}^d$ denote the closed annulus centered at z and with inner radius a and outer radius b. This subsection is devoted to the proof of the following lemma, which yields a key estimate to derive Theorems 1.3 and 1.4.

Lemma 2.3. *Let $\Gamma := \{x \in \mathbb{R}^d \,:\, x = (y, \mathcal{A}(y)),\, y \in \mathbb{R}^n\}$ be the graph of a Lipschitz function $\mathcal{A} : \mathbb{R}^n \to \mathbb{R}^{d-n}$ such that* $\operatorname{Lip}(\mathcal{A}) < 1$. *Then, there exists $C > 0$ depending on n, d, and* $\operatorname{Lip}(\mathcal{A})$, *such that $\mathcal{H}^n_\Gamma(A(z, a, b)) \leq C(b-a)b^{n-1}$ for all $z \in \Gamma$ and all $0 < a \leq b$.*

We need the following auxiliary result.

Lemma 2.4. *Let $1 \leq n < d$. For $x := (x_1, \ldots, x_d) \in \mathbb{R}^d$ we denote*

$$x_H := (x_1, \ldots, x_n, 0, \ldots, 0) \in \mathbb{R}^d \quad \textit{and} \quad x_V := (0, \ldots, 0, x_{n+1}, \ldots, x_d) \in \mathbb{R}^d.$$

Given $x, y \in \mathbb{R}^d \setminus \{0\}$, if there exists $0 < s < 1$ such that $|x_V| \le s|x_H|$, $|y_V| \le s|y_H|$, and $|x_V - y_V| \le s|x_H - y_H|$, then there exists $C > 0$ depending only on s such that

$$|x_V - y_V| \le C\left|\frac{|x|}{|x_H|}\,x_H - \frac{|y|}{|y_H|}\,y_H\right|. \tag{12}$$

Proof. We set $\Phi(x, y) := \big||x||x_H|^{-1}x_H - |y||y_H|^{-1}y_H\big|$. Since Φ is symmetric in x and y, we can assume that $|x_H| \le |y_H|$. If $\langle\cdot,\cdot\rangle$ denotes the scalar product in $\mathbb{R}^d$, using the polarization identity,

$$\begin{aligned}
\Phi(x,y)^2 &= |x|^2 + |y|^2 - 2|x||x_H|^{-1}|y||y_H|^{-1}\langle x_H, y_H\rangle\\
&= |x|^2 + |y|^2 + |x||x_H|^{-1}|y||y_H|^{-1}\big(|x_H - y_H|^2 - |x_H|^2 - |y_H|^2\big)\\
&= |x|^2 + |y|^2 - 2|x||y| + |x||x_H|^{-1}|y||y_H|^{-1}\big(|x_H - y_H|^2 - |x_H|^2 - |y_H|^2 + 2|x_H||y_H|\big)\\
&= \big(|x| - |y|\big)^2 + |x||x_H|^{-1}|y||y_H|^{-1}\big(|x_H - y_H|^2 - (|x_H| - |y_H|)^2\big).
\end{aligned}$$

Since $|x_H - y_H|^2 - (|x_H| - |y_H|)^2 \ge 0$, $|x_H| \le |x|$, and $|y_H| \le |y|$, we have

$$\Phi(x, y)^2 \ge \big(|x| - |y|\big)^2 + |x_H - y_H|^2 - (|x_H| - |y_H|)^2. \tag{13}$$

Assume that $2|x| \le |y|$. Then, using (13),

$$|x_V - y_V| \le |x| + |y| \le \frac{3}{2}|y| = 3\Big(|y| - \frac{1}{2}\,|y|\Big) \le 3(|y| - |x|) \le 3\Phi(x, y),$$

and we obtain (12). By the same arguments, if $2|y| \le |x|$, then $|x_V - y_V| \le 3\Phi(x, y)$ and (12) holds. Thus, from now on we assume $\frac{1}{2}|x| \le |y| \le 2|x|$.

Let $0 < \delta < 1$ be a small number that will be fixed below. Assume that $(1-\delta)|x_H - y_H| \ge \big||y_H| - |x_H|\big|$. Then, by (13),

$$\begin{aligned}
\Phi(x, y)^2 &\ge |x_H - y_H|^2 - (|x_H| - |y_H|)^2 \ge |x_H - y_H|^2 - (1 - \delta)^2|x_H - y_H|^2\\
&= \delta(2 - \delta)|x_H - y_H|^2 \ge \delta(2 - \delta)s^{-2}|x_V - y_V|^2,
\end{aligned}$$

and then (12) holds with $C = s/\sqrt{\delta(2 - \delta)}$.

Therefore, we can suppose that $(1 - \delta)|x_H - y_H| \le \big||y_H| - |x_H|\big| = |y_H| - |x_H|$, since we are also assuming $|x_H| \le |y_H|$. If we set $z = y - x$, we have $(1 - \delta)|z_H| \le |x_H + z_H| - |x_H|$, so $(1 - \delta)|z_H| + |x_H| \le |x_H + z_H|$. Hence,

$$\begin{aligned}
(1 - \delta)^2|z_H|^2 + |x_H|^2 + 2(1 - \delta)|z_H||x_H| &= \big((1 - \delta)|z_H| + |x_H|\big)^2\\
&\le |x_H + z_H|^2 = |x_H|^2 + |z_H|^2 + 2\langle x_H, z_H\rangle
\end{aligned}$$

and we obtain

$$\langle x_H, z_H\rangle \ge -\frac{1}{2}\,\delta(2 - \delta)|z_H|^2 + (1 - \delta)|z_H||x_H|. \tag{14}$$

Using (14), that $\langle x_V, z_V\rangle \ge -|x_V||z_V|$, and that $|x_V| \le s|x_H|$ and $|z_V| \le s|z_H|$, we get

$$\begin{aligned}
\langle x, z\rangle &= \langle x_H + x_V, z_H + z_V\rangle = \langle x_H, z_H\rangle + \langle x_V, z_V\rangle\\
&\ge -\frac{1}{2}\,\delta(2 - \delta)|z_H|^2 + (1 - \delta)|z_H||x_H| - |x_V||z_V|\\
&\ge -\frac{1}{2}\,\delta(2 - \delta)|z_H|^2 + (1 - \delta - s^2)|z_H||x_H|.
\end{aligned} \tag{15}$$

Notice that, if $\delta > 0$ is small enough depending on s, then $-\frac{1}{4}\,(1 - s^2)(1 + s^2)^{-1} < -\frac{3}{2}\,\delta(2 - \delta) < 0$ and $1 - \delta - s^2 > \frac{1}{2}\,(1 - s^2)$. Let $\gamma(x, z)$ be the angle between x and z (by definition,

$0 \leq \gamma(x,z) \leq \pi$). Using that $\langle x,z\rangle = |x||z|\cos(\gamma(x,z))$, that $|x| \leq \sqrt{1+s^2}|x_H|$ and $|z| \leq \sqrt{1+s^2}|z_H|$, and that $|z| \leq |x|+|y| \leq 3|x|$, we finally obtain from (15) that

$$\begin{aligned}\cos(\gamma(x,z)) &\geq -\frac{1}{2}\,\delta(2-\delta)|z_H|^2|x|^{-1}|z|^{-1} + (1-\delta-s^2)|z_H||x_H||x|^{-1}|z|^{-1}\\ &\geq -\frac{3}{2}\,\delta(2-\delta) + (1-\delta-s^2)(1+s^2)^{-1} \geq \frac{1}{4}\,(1-s^2)(1+s^2)^{-1} =: a.\end{aligned}$$

Notice that $a > 0$, because $0 < s < 1$ by hypothesis. Hence, since $\cos(\gamma(-x,y-x)) = \cos(\gamma(-x,z)) = -\cos(\gamma(x,z))$ (because $z = y-x$ and $\langle -x,z\rangle = -\langle x,z\rangle$), we have $c_0 := \cos(\gamma(-x,y-x)) \leq -a < 0$ (notice that $c_0 \leq 0$ implies that $|x| \leq |y|$). By the cosines theorem, $|y|^2 = |x|^2 - |y-x|^2 - 2|x||y-x|c_0$. Since $c_0 < 0$, we solve the second degree equation in $|y-x|$ and we obtain

$$\begin{aligned}|y-x| &= \sqrt{|y|^2-|x|^2(1-c_0^2)} - |x||c_0| = \frac{|y|^2-|x|^2(1-c_0^2)-|x|^2c_0^2}{\sqrt{|y|^2-|x|^2(1-c_0^2)}+|x||c_0|}\\ &= \frac{(|y|-|x|)(|y|+|x|)}{\sqrt{|y|^2-|x|^2(1-c_0^2)}+|x||c_0|} \leq \frac{(|y|-|x|)(|y|+|x|)}{|x||c_0|} \leq (|y|-|x|)\frac{3}{a},\end{aligned}$$

where we also used that $|y| \leq 2|x|$ in the last inequality. Therefore, by (13),

$$|x_V - y_V| \leq |x-y| \leq \frac{3}{a}\,(|y|-|x|) \leq \frac{3}{a}\,\Phi(x,y),$$

and (12) follows with $C = 3/a$, where $a > 0$ only depends on s. This completes the proof of the lemma. $\square$

***Proof* of Lemma 2.3.** We keep the notation introduced in Lemma 2.4. Fix $z \in \Gamma$. We can assume that $z = 0$, by taking a translation of Γ if it is necessary.

For $x \in \mathbb{R}^d$ with $x_H \neq 0$, consider the map

$$\Upsilon(x) := \frac{|x|}{|x_H|}\,x_H + x_V = \sqrt{1+\frac{|x_V|^2}{|x_H|^2}}\,x_H + x_V.$$

It is not difficult to show that Υ is a bilipschitz mapping from (a neighborhood of) the cone

$$L := \{x \in \mathbb{R}^d \setminus \{0\} \,:\, |x_V| \leq \mathrm{Lip}(\mathcal{A})|x_H|\}$$

to (a neighborhood of) the cone

$$L' := \{x \in \mathbb{R}^d \setminus \{0\} \,:\, |x_V| \leq \mathrm{Lip}(\mathcal{A})(1+\mathrm{Lip}(\mathcal{A})^2)^{-1/2}|x_H|\},$$

whose inverse equals

$$\Upsilon^{-1}(x) = \sqrt{1-\frac{|x_V|^2}{|x_H|^2}}\,x_H + x_V.$$

Moreover, when Υ and Υ^{-1} are restricted to L and L' respectively, $\mathrm{Lip}(\Upsilon)$ and $\mathrm{Lip}(\Upsilon^{-1})$ only depend on n, d, and $\mathrm{Lip}(\mathcal{A})$. Hence, since $\Gamma \subset L \cup \{0\}$, for any $0 < a \leq b$ we have

$$\mathcal{H}^n_\Gamma(A(0,a,b)) = \mathcal{H}^n(\Gamma \cap A(0,a,b)) \approx \mathcal{H}^n(\Upsilon(\Gamma \cap A(0,a,b))).$$

Consider the set $\Upsilon(\Gamma)$. Since Γ has slope smaller than 1 (i.e. $\mathrm{Lip}(\mathcal{A}) < 1$), by Lemma 2.4 there exists a constant $C > 0$ depending only on n, d, and $\mathrm{Lip}(\mathcal{A})$ such that for any two points $x, y \in \Upsilon(\Gamma)$ one has $|x_V - y_V| \leq C|x_H - y_H|$. Then, it is known that $\Upsilon(\Gamma)$ is contained in the n-dimensional graph Γ' of some Lipschitz function (see for example the proof of [Ma, Lemma

15.13]). Notice also that, given $0 < a \le b$, $\Upsilon(L \cap A(0,a,b)) \subset \{x \in \mathbb{R}^d : a \le |x_H| \le b\}$. Therefore,

$$\mathcal{H}^n_\Gamma(A(0,a,b)) \approx \mathcal{H}^n(\Upsilon(\Gamma \cap A(0,a,b))) \le \mathcal{H}^n(\Gamma' \cap \{x \in \mathbb{R}^d : a \le |x_H| \le b\}) \lesssim (b-a)b^{n-1},$$

and the lemma is proved. □

***Remark* 2.5.** With a little more of effort, one can show that $\Upsilon(\Gamma)$ is actually a Lipschitz graph. We omit the details.

***Remark* 2.6.** Lemma 2.3 is sharp in the sense that the estimate fails if $\mathrm{Lip}(\mathcal{A}) \ge 1$ (notice that the constant C in Lemma 2.4 for $s = \mathrm{Lip}(\mathcal{A})$ is bigger than $(1+\mathrm{Lip}(\mathcal{A})^2)/(1-\mathrm{Lip}(\mathcal{A})^2)$). Given $\epsilon > 0$, one can easily construct a Lipschitz graph Γ such that $1 < \mathrm{Lip}(\mathcal{A}) < 1+\epsilon$ and such that, for some $z \in \Gamma$ and $r > 0$, Γ contains a set $P \subset \partial B(z,r)$ with $\mathcal{H}^n_\Gamma(P) > 0$. Then, if Lemma 2.3 were true for Γ, we would have $0 < \mathcal{H}^n_\Gamma(P) \le \mathcal{H}^n_\Gamma(A(z, r-\delta, r+\delta)) \lesssim 2\delta(r+\delta)^{n-1}$, and we would have a contradiction by letting $\delta \to 0$. By a similar argument, one can also show that the lemma fails in the limiting case $\mathrm{Lip}(\mathcal{A}) = 1$.

3. $\mathcal{V}_\rho \circ \mathcal{T}$ is a bounded operator from $M(\mathbb{R}^d)$ to $L^{1,\infty}(\mathcal{H}^n_\Gamma)$

This section is devoted to the proof of Theorem 1.4, which is based on a nontrivial modification of the proof of [CJRW2, Theorem B] using the Calderón-Zygmund decomposition developed in Subsection 2.1.

***Proof of* Theorem 1.4.** Set $\mu := \mathcal{H}^n_{\Gamma \cap B}$, where B is some fixed ball in $\mathbb{R}^d$. Let $\nu \in M(\mathbb{R}^d)$ be a finite complex Radon measure with compact support and $\lambda > 2^{d+1}\|\nu\|/\|\mu\|$. We will show that

$$\mu\big(\{x \in \mathbb{R}^d : (\mathcal{V}_\rho \circ \mathcal{T})\nu(x) > \lambda\}\big) \le \frac{C}{\lambda}\,\|\nu\|, \tag{16}$$

where $C > 0$ depends on n, d, K, ρ and Γ, but not on $B \subset \mathbb{R}^d$. Let us check that (16) implies that $\mathcal{V}_\rho \circ \mathcal{T}$ is bounded from $M(\mathbb{R}^d)$ into $L^{1,\infty}(\mathcal{H}^n_\Gamma)$. Suppose that ν is not compactly supported. Set $\nu_N = \chi_{B(0,N)}\,\nu$. Let N_0 be such that $\mathrm{supp}\mu \subset B(0,N_0)$. Then it is not hard to show that, for $x \in \mathrm{supp}\mu$,

$$|(\mathcal{V}_\rho \circ \mathcal{T})\nu(x) - (\mathcal{V}_\rho \circ \mathcal{T})\nu_N(x)| \le C\,\frac{|\nu|(\mathbb{R}^d \setminus B(0,N))}{N - N_0},$$

thus $(\mathcal{V}_\rho \circ \mathcal{T})\nu_N(x) \to (\mathcal{V}_\rho \circ \mathcal{T})\nu(x)$ for all $x \in \mathrm{supp}\mu$ uniformly, and since the estimate (16) holds by assumption for ν_N, letting $N \to \infty$, we deduce that it also holds for ν. Now, by increasing the size of the ball B and monotone convergence, (16) yields

$$\mathcal{H}^n_\Gamma\big(\{x \in \mathbb{R}^d : (\mathcal{V}_\rho \circ \mathcal{T})\nu(x) > \lambda\}\big) \le \frac{C}{\lambda}\,\|\nu\|,$$

as desired. Thus, we only have to verify (16) for all compactly supported ν.

Let $\{Q_j\}_j$ be the almost disjoint family of cubes of Lemma 2.2, and set $\Omega := \bigcup_j Q_j$ and $R_j := 6Q_j$. Then we can write $\nu = g\mu + \nu_b$, with

$$g\mu = \chi_{\mathbb{R}^d \setminus \Omega}\nu + \sum_j b_j\mu \quad \text{and} \quad \nu_b = \sum_j \nu_b^j := \sum_j (w_j\nu - b_j\mu),$$

where the functions b_j satisfy (9), (10), and (11), and $w_j = \chi_{Q_j}\big(\sum_k \chi_{Q_k}\big)^{-1}$.

By the subadditivity of $\mathcal{V}_\rho \circ \mathcal{T}$, we have

$$\begin{aligned}(17)\qquad &\mu\big(\{x \in \mathbb{R}^d : (\mathcal{V}_\rho \circ \mathcal{T})\nu(x) > \lambda\}\big)\\ &\qquad \leq \mu\big(\{x \in \mathbb{R}^d : (\mathcal{V}_\rho \circ \mathcal{T}^\mu)g(x) > \lambda/2\}\big) + \mu\big(\{x \in \mathbb{R}^d : (\mathcal{V}_\rho \circ \mathcal{T})\nu_b(x) > \lambda/2\}\big).\end{aligned}$$

Since $\mathcal{V}_\rho \circ \mathcal{T}^{\mathcal{H}^n_\Gamma}$ is bounded in $L^2(\mathcal{H}^n_\Gamma)$ by Theorem 1.2, it is easy to show that $\mathcal{V}_\rho \circ \mathcal{T}^\mu$ is bounded in $L^2(\mu)$, with a bound independent of B. Notice that $|g| \leq C\lambda$ by (8) and (11). Then, using (10),

$$\begin{aligned}\mu\big(\{x \in \mathbb{R}^d : (\mathcal{V}_\rho \circ \mathcal{T}^\mu)g(x) > \lambda/2\}\big) &\lesssim \frac{1}{\lambda^2}\int |(\mathcal{V}_\rho \circ \mathcal{T}^\mu)g|^2\,d\mu \lesssim \frac{1}{\lambda^2}\int |g|^2\,d\mu\\ &\lesssim \frac{1}{\lambda}\int |g|\,d\mu \leq \frac{1}{\lambda}\Big(|\nu|(\mathbb{R}^d \setminus \Omega) + \sum_j \int_{R_j} |b_j|\,d\mu\Big)\\ &\leq \frac{1}{\lambda}\Big(|\nu|(\mathbb{R}^d \setminus \Omega) + \sum_j |\nu|(Q_j)\Big) \leq \frac{C}{\lambda}\,\|\nu\|.\end{aligned}\tag{18}$$

Set $\widehat{\Omega} := \bigcup_j 2Q_j$. By (6), we have $\mu(\widehat{\Omega}) \leq \sum_j \mu(2Q_j) \lesssim \lambda^{-1}\sum_j |\nu|(Q_j) \lesssim \lambda^{-1}\|\nu\|$. We are going to show that

$$\mu\big(\{x \in \mathbb{R}^d \setminus \widehat{\Omega} : (\mathcal{V}_\rho \circ \mathcal{T})\nu_b(x) > \lambda/2\}\big) \leq \frac{C}{\lambda}\,\|\nu\|,\tag{19}$$

and then (16) is a direct consequence of (17), (18), (19) and the estimate $\mu(\widehat{\Omega}) \lesssim \lambda^{-1}\|\nu\|$.

For simplicity of notation, given $0 < \epsilon \leq \delta$ and $t \in \mathbb{R}^d$, we set $\chi_\epsilon^\delta(t) := \chi_{(\epsilon,\delta]}(|t|)$, so

$$T_\epsilon\nu_b(x) - T_\delta\nu_b(x) = \int \chi_\epsilon^\delta(x-y)K(x-y)\,d\nu_b(y) = (K\chi_\epsilon^\delta * \nu_b)(x).$$

Given $x \in \operatorname{supp}\mu$, let $\{\epsilon_m\}_{m\in\mathbb{Z}}$ be a decreasing sequence of positive numbers (which depends on x, i.e. $\epsilon_m \equiv \epsilon_m(x)$) such that

$$(\mathcal{V}_\rho \circ \mathcal{T})\nu_b(x) \leq 2\Big(\sum_{m\in\mathbb{Z}} |(K\chi_{\epsilon_{m+1}}^{\epsilon_m} * \nu_b)(x)|^\rho\Big)^{1/\rho}.\tag{20}$$

If $R_j \cap A(x,\epsilon_{m+1},\epsilon_m) = \emptyset$ then $(K\chi_{\epsilon_{m+1}}^{\epsilon_m} * \nu_b^j)(x) = 0$, so by (20) and the triangle inequality,

$$\begin{aligned}(\mathcal{V}_\rho \circ \mathcal{T})\nu_b(x) &\leq 2\Big(\sum_{m\in\mathbb{Z}}\Big|\sum_{j:\,R_j\subset A(x,\epsilon_{m+1},\epsilon_m)} (K\chi_{\epsilon_{m+1}}^{\epsilon_m} * \nu_b^j)(x)\Big|^\rho\Big)^{1/\rho}\\ &\quad + 2\Big(\sum_{m\in\mathbb{Z}}\Big|\sum_{j:\,R_j\cap\partial A(x,\epsilon_{m+1},\epsilon_m)\neq\emptyset} (K\chi_{\epsilon_{m+1}}^{\epsilon_m} * \nu_b^j)(x)\Big|^\rho\Big)^{1/\rho}\\ &=: 2\big(IS(x) + BS(x)\big),\end{aligned}$$

and then,

$$\begin{aligned}(21)\qquad &\mu\big(\{x \in \mathbb{R}^d \setminus \widehat{\Omega} : (\mathcal{V}_\rho \circ \mathcal{T})\nu_b(x) > \lambda/2\}\big)\\ &\qquad \leq \mu\big(\{x \in \mathbb{R}^d \setminus \widehat{\Omega} : IS(x) > \lambda/8\}\big) + \mu\big(\{x \in \mathbb{R}^d \setminus \widehat{\Omega} : BS(x) > \lambda/8\}\big).\end{aligned}$$

Let us estimate first $\mu\big(\{x \in \mathbb{R}^d \setminus \widehat{\Omega} : IS(x) > \lambda/8\}\big)$. Since the ℓ^ρ-norm is not bigger than the ℓ^1-norm for $\rho \geq 1$,

$$
\begin{aligned}
IS(x) &\leq \sum_{m\in\mathbb{Z}} \Big| \sum_{j:\, R_j \subset A(x,\epsilon_{m+1},\epsilon_m)} (K\chi_{\epsilon_{m+1}}^{\epsilon_m} * \nu_b^j)(x) \Big| \\
&\leq \sum_{m\in\mathbb{Z}} \sum_{j:\, R_j \subset A(x,\epsilon_{m+1},\epsilon_m)} \Big| \int \chi_{\epsilon_{m+1}}^{\epsilon_m}(x-y) K(x-y)\, d\nu_b^j(y) \Big| \\
&= \sum_j \sum_{m\in\mathbb{Z}:\, A(x,\epsilon_{m+1},\epsilon_m)\supset R_j} \Big| \int \chi_{\epsilon_{m+1}}^{\epsilon_m}(x-y) K(x-y)\, d\nu_b^j(y) \Big| \\
&\leq \sum_j \chi_{\mathbb{R}^d\setminus R_j}(x) \Big| \int K(x-y)\, d\nu_b^j(y) \Big|.
\end{aligned} \tag{22}
$$

Notice that

$$
\begin{aligned}
\int_{\mathbb{R}^d\setminus\widehat{\Omega}} \chi_{\mathbb{R}^d\setminus R_j}(x) \Big| \int K(x-y)\, d\nu_b^j(y) \Big|\, d\mu(x) &\leq \int_{\mathbb{R}^d\setminus R_j} \Big| \int K(x-y)\, d\nu_b^j(y) \Big|\, d\mu(x) \\
\leq \int_{\mathbb{R}^d\setminus 2R_j} \Big| \int K(x-y)\, d\nu_b^j(y) \Big|\, d\mu(x) &+ \int_{2R_j\setminus R_j} \Big| \int K(x-y)\, d\nu_b^j(y) \Big|\, d\mu(x).
\end{aligned} \tag{23}
$$

On one hand, by (10) and using the $L^2(\mu)$ boundedness of the maximal operator T_*^μ (recall that $\mu = \mathcal{H}^n_{\Gamma\cap B}$, where Γ is a Lipschitz graph and B is a ball) and that $\mu(2R_j) \leq C\mu(R_j)$ (because $\frac{1}{2}R_j \cap \operatorname{supp}\mu \neq \emptyset$), we get

$$
\begin{aligned}
\int_{2R_j\setminus R_j} \Big| \int K(x-y) b_j(y)\, d\mu(y) \Big|\, d\mu(x) &\leq \int_{2R_j\setminus R_j} T_*^\mu b_j\, d\mu \\
&\leq \Big(\int_{2R_j} (T_*^\mu b_j)^2\, d\mu \Big)^{1/2} \mu(2R_j)^{1/2} \\
&\lesssim \|b_j\|_{L^2(\mu)} \mu(2R_j)^{1/2} \lesssim \|b_j\|_{L^\infty(\mu)} \mu(R_j) \\
&\lesssim |\nu|(Q_j).
\end{aligned} \tag{24}
$$

On the other hand, since $\operatorname{supp} w_j \subset Q_j = \frac{1}{6}R_j$ and $|w_j| \leq 1$, if $x \in 2R_j \setminus R_j$ we have $\int |K(x-y)w_j(y)|\, d|\nu|(y) \lesssim |\nu|(Q_j)|x - z_j|^{-n}$, where z_j denotes the center of R_j. Hence, using again that $\mu(2R_j) \leq C\mu(R_j) \leq C\ell(R_j)^n$,

$$
\begin{aligned}
\int_{2R_j\setminus R_j} \Big| \int K(x-y) w_j(y)\, d\nu(y) \Big|\, d\mu(x) &\leq \int_{2R_j\setminus R_j} \int |K(x-y) w_j(y)|\, d|\nu|(y)\, d\mu(x) \\
&\lesssim |\nu|(Q_j) \int_{2R_j\setminus R_j} |x - z_j|^{-n}\, d\mu(x) \\
&\lesssim |\nu|(Q_j) \ell(R_j)^{-n} \mu(2R_j) \lesssim |\nu|(Q_j).
\end{aligned} \tag{25}
$$

Since $\nu_b^j(R_j)=0$, $\operatorname{supp}\nu_b^j\subset R_j$, and $\|\nu_b^j\|\lesssim|\nu|(Q_j)$ by (10), we have

$$\begin{aligned}\int_{\mathbb{R}^d\setminus 2R_j}\left|\int K(x-y)\,d\nu_b^j(y)\right|d\mu(x)&\le\int_{\mathbb{R}^d\setminus 2R_j}\int_{R_j}|K(x-y)-K(x-z_j)|\,d|\nu_b^j|(y)\,d\mu(x)\\&\lesssim\int_{\mathbb{R}^d\setminus 2R_j}\int_{R_j}\frac{|y-z_j|}{|x-z_j|^{n+1}}\,d|\nu_b^j|(y)\,d\mu(x)\\&\lesssim\|\nu_b^j\|\int_{\mathbb{R}^d\setminus 2R_j}\frac{\ell(R_j)}{|x-z_j|^{n+1}}\,d\mu(x)\lesssim\|\nu_b^j\|\lesssim|\nu|(Q_j).\end{aligned}$$

Combining this last estimate with (24), (25), and the fact that $\nu_b^j=w_j\nu-b_j\mu$, from (23) we obtain that

$$\int_{\mathbb{R}^d\setminus\widehat{\Omega}}\chi_{\mathbb{R}^d\setminus R_j}(x)\left|\int K(x-y)\,d\nu_b^j(y)\right|d\mu(x)\lesssim|\nu|(Q_j).$$

Finally, using (22) we conclude

$$\begin{aligned}\mu\big(\big\{x\in\mathbb{R}^d\setminus\widehat{\Omega}\,:\,IS(x)>\lambda/8\big\}\big)&\le\frac{8}{\lambda}\int_{\mathbb{R}^d\setminus\widehat{\Omega}}IS(x)\,d\mu(x)\\&\le\frac{8}{\lambda}\sum_j\int_{\mathbb{R}^d\setminus\widehat{\Omega}}\chi_{\mathbb{R}^d\setminus R_j}(x)\left|\int K(x-y)\,d\nu_b^j(y)\right|d\mu(x)\\&\le\frac{C}{\lambda}\sum_j|\nu|(Q_j)\le\frac{C}{\lambda}\|\nu\|.\end{aligned}\tag{26}$$

Let us estimate $\mu\big(\big\{x\in\mathbb{R}^d\setminus\widehat{\Omega}\,:\,BS(x)>\lambda/8\big\}\big)$. Recall that $\epsilon_m\equiv\epsilon_m(x)$. We define

$$\begin{aligned}\psi_m^j(x)&:=\begin{cases}1&\text{if } R_j\cap\partial A(x,\epsilon_{m+1}(x),\epsilon_m(x))\neq\emptyset\\0&\text{if not}\end{cases},\text{ and}\\\theta_k^j(x)&:=\begin{cases}1&\text{if } R_j\cap\partial A(x,2^{-k-1},2^{-k})\neq\emptyset\\0&\text{if not}\end{cases}.\end{aligned}\tag{27}$$

Then, by the triangle inequality, for $x\in\mathbb{R}^d\setminus\widehat{\Omega}$ we have

$$\begin{aligned}BS(x)&=\bigg(\sum_{m\in\mathbb{Z}}\Big|\sum_j\psi_m^j(x)(K\chi_{\epsilon_{m+1}}^{\epsilon_m}*\nu_b^j)(x)\Big|^\rho\bigg)^{1/\rho}\\&\le\bigg(\sum_{m\in\mathbb{Z}}\Big|\sum_j\chi_{\mathbb{R}^d\setminus 2R_j}(x)\psi_m^j(x)(K\chi_{\epsilon_{m+1}}^{\epsilon_m}*\nu_b^j)(x)\Big|^\rho\bigg)^{1/\rho}\\&\quad+\bigg(\sum_{m\in\mathbb{Z}}\Big|\sum_j\chi_{2R_j\setminus 2Q_j}(x)\psi_m^j(x)(K\chi_{\epsilon_{m+1}}^{\epsilon_m}*\nu_b^j)(x)\Big|^\rho\bigg)^{1/\rho}\\&\le\bigg(\sum_{m\in\mathbb{Z}}\Big|\sum_j\chi_{\mathbb{R}^d\setminus 2R_j}(x)\psi_m^j(x)(K\chi_{\epsilon_{m+1}}^{\epsilon_m}*\nu_b^j)(x)\Big|^\rho\bigg)^{1/\rho}\\&\quad+\sum_j\chi_{2R_j\setminus 2Q_j}(x)\bigg(\sum_{m\in\mathbb{Z}}\big|(K\chi_{\epsilon_{m+1}}^{\epsilon_m}*\nu_b^j)(x)\big|^\rho\bigg)^{1/\rho}\\&=:BS_1(x)+BS_2(x).\end{aligned}\tag{28}$$

Notice that $BS_2(x) \leq \sum_j \chi_{2R_j\setminus 2Q_j}(x)(\mathcal{V}_\rho \circ \mathcal{T})\nu_b^j(x)$. Since $\rho \geq 1$, for $x \in 2R_j \setminus 2Q_j$,

$$\begin{aligned}(\mathcal{V}_\rho \circ \mathcal{T})\nu_b^j(x) &\leq (\mathcal{V}_\rho \circ \mathcal{T})(w_j\nu)(x) + (\mathcal{V}_\rho \circ \mathcal{T})(b_j\mu)(x)\\ &\leq (\mathcal{V}_1 \circ \mathcal{T})(w_j\nu)(x) + (\mathcal{V}_\rho \circ \mathcal{T}^\mu)b_j(x)\\ &\lesssim |\nu|(Q_j)|x-z_j|^{-n} + (\mathcal{V}_\rho \circ \mathcal{T}^\mu)b_j(x),\end{aligned}$$

where z_j denotes the center of Q_j (and R_j). Then, similarly to (24) and (25) but using now the $L^2(\mu)$ boundedness of $\mathcal{V}_\rho \circ \mathcal{T}^\mu$ given by Theorem 1.2, we have

$$\begin{aligned}\mu\big(\{x \in \mathbb{R}^d\setminus\widehat{\Omega} \,:\, BS_2(x) > \lambda/16\}\big) &\leq \frac{16}{\lambda}\int_{\mathbb{R}^d\setminus\widehat{\Omega}} BS_2\,d\mu\\ &\leq \frac{16}{\lambda}\int\sum_j \chi_{2R_j\setminus 2Q_j}(\mathcal{V}_\rho \circ \mathcal{T})\nu_b^j\,d\mu = \frac{16}{\lambda}\sum_j\int_{2R_j\setminus 2Q_j}(\mathcal{V}_\rho \circ \mathcal{T})\nu_b^j\,d\mu\\ &\lesssim \frac{1}{\lambda}\sum_j |\nu|(Q_j)\int_{2R_j\setminus 2Q_j}|x-z_j|^{-n}\,d\mu(x) + \frac{1}{\lambda}\sum_j\int_{2R_j\setminus 2Q_j}(\mathcal{V}_\rho \circ \mathcal{T}^\mu)b_j\,d\mu\\ &\lesssim \frac{1}{\lambda}\sum_j |\nu|(Q_j)\ell(Q_j)^{-n}\mu(2R_j) + \frac{1}{\lambda}\sum_j \|(\mathcal{V}_\rho \circ \mathcal{T}^\mu)b_j\|_{L^2(\mu)}\mu(2R_j)^{1/2}\\ &\lesssim \frac{1}{\lambda}\sum_j|\nu|(Q_j) + \frac{1}{\lambda}\sum_j\|b_j\|_{L^\infty(\mu)}\mu(R_j) \lesssim \frac{1}{\lambda}\sum_j|\nu|(Q_j) \leq \frac{C}{\lambda}\|\nu\|.\end{aligned} \tag{29}$$

Therefore, to show that $\mu\big(\{x \in \mathbb{R}^d \setminus \widehat{\Omega} \,:\, BS(x) > \lambda/8\}\big) \leq C\lambda^{-1}\|\nu\|$, by (28) and (29) it is enough to verify that

$$\mu\big(\{x \in \mathbb{R}^d \setminus \widehat{\Omega} \,:\, BS_1(x) > \lambda/16\}\big) \leq \frac{C}{\lambda}\|\nu\|.$$

Without loss of generality, we can assume from the beginning that, for a given $x \in \operatorname{supp}\mu$, either $[\epsilon_{m+1}, \epsilon_m) \subset [2^{-k-1}, 2^{-k})$ for some $k \in \mathbb{Z}$, or $[\epsilon_{m+1}, \epsilon_m) = [2^{-i}, 2^{-k})$ for some $i > k$ (see [CJRW2, page 2130] for a similar argument). Thus, if we set $I_k := [2^{-k-1}, 2^{-k})$, we can decompose $\mathbb{Z} = \mathcal{S} \cup \mathcal{L}$, where

$$\begin{aligned}\mathcal{L} &:= \{m \in \mathbb{Z} \,:\, \epsilon_m = 2^{-k},\, \epsilon_{m+1} = 2^{-i} \text{ for } i > k\},\\ \mathcal{S} &:= \bigcup_{k\in\mathbb{Z}} \mathcal{S}_k, \quad \mathcal{S}_k := \{m \in \mathbb{Z} \,:\, \epsilon_m, \epsilon_{m+1} \in I_k\}.\end{aligned}$$

Then, since $\rho \geq 1$,

$$\begin{aligned}BS_1(x) &\leq \Big(\sum_{m\in\mathcal{L}}\Big|\sum_j \chi_{\mathbb{R}^d\setminus 2R_j}(x)\psi_m^j(x)(K\chi_{\epsilon_{m+1}}^{\epsilon_m} * \nu_b^j)(x)\Big|^\rho\Big)^{1/\rho}\\ &\quad + \Big(\sum_{m\in\mathcal{S}}\Big|\sum_j \chi_{\mathbb{R}^d\setminus 2R_j}(x)\psi_m^j(x)(K\chi_{\epsilon_{m+1}}^{\epsilon_m} * \nu_b^j)(x)\Big|^\rho\Big)^{1/\rho}\\ &=: BS_{\mathcal{L}}(x) + BS_{\mathcal{S}}(x),\end{aligned}$$

and we have

$$\begin{aligned}&\mu\big(\{x \in \mathbb{R}^d \setminus \widehat{\Omega} \,:\, BS_1(x) > \lambda/16\}\big)\\ &\qquad \leq \mu\big(\{x \in \mathbb{R}^d \setminus \widehat{\Omega} \,:\, BS_{\mathcal{L}}(x) > \lambda/32\}\big) + \mu\big(\{x \in \mathbb{R}^d \setminus \widehat{\Omega} \,:\, BS_{\mathcal{S}}(x) > \lambda/32\}\big).\end{aligned} \tag{30}$$

We are going to estimate first $\mu\big(\{x \in \mathbb{R}^d \setminus \widehat{\Omega} \,:\, BS_{\mathcal{L}}(x) > \lambda/32\}\big)$. Given $x \in \mathbb{R}^d \setminus \widehat{\Omega}$ (recall the definitions of $\psi_k^j(x)$ and $\theta_k^j(x)$ in (27)), we have

$$
\begin{aligned}
BS_{\mathcal{L}}(x) &\leq \sum_j \sum_{m\in\mathcal{L}} \chi_{\mathbb{R}^d\setminus 2R_j}(x)\psi_m^j(x)|(K\chi_{\epsilon_{m+1}}^{\epsilon_m} * \nu_b^j)(x)| \\
&\leq \sum_j \sum_{k\in\mathbb{Z}} \chi_{\mathbb{R}^d\setminus 2R_j}(x)\theta_k^j(x)|(K\chi_{2^{-k-1}}^{2^{-k}} * \nu_b^j)(x)| \\
&\leq \sum_j \sum_{k\in\mathbb{Z}\,:\,2^{-k+1}>\ell(R_j)} \chi_{\mathbb{R}^d\setminus 2R_j}(x)\theta_k^j(x)|(K\chi_{2^{-k-1}}^{2^{-k}} * \nu_b^j)(x)|,
\end{aligned}
\tag{31}
$$

where in the second and third inequalities above we used the following facts, respectively:

- assume $m \in \mathcal{L}$, $\epsilon_{m+1} = 2^{-i}$ and $\epsilon_m = 2^{-i+s}$, with $i \in \mathbb{Z}$ and $s \in \mathbb{N}$. Given j such that $R_j \cap \partial A(x, \epsilon_{m+1}, \epsilon_m) \neq \emptyset$, if $R_j \cap A(x, 2^{-k-1}, 2^{-k}) \neq \emptyset$ for some $k \in \mathbb{Z}$, then $R_j \cap \partial A(x, 2^{-k-1}, 2^{-k}) \neq \emptyset$.
- For $x \in \mathbb{R}^d \setminus 2R_j$, if $2^{-k+1} \leq \ell(R_j)$ then we have $\operatorname{supp}\chi_{2^{-k-1}}^{2^{-k}}(x - \cdot) \cap R_j = \emptyset$, so $(K\chi_{2^{-k-1}}^{2^{-k}} * \nu_b^j)(x) = 0$.

Therefore, from (31) and since $|(K\chi_{2^{-k-1}}^{2^{-k}} * \nu_b^j)(x)| \lesssim 2^{(k+1)n}\|\nu_b^j\|$,

$$
\begin{aligned}
\mu\big(\{x \in \mathbb{R}^d \setminus \widehat{\Omega} \,: BS_{\mathcal{L}}(x) > \lambda/32\}\big) &\leq \frac{32}{\lambda}\int_{\mathbb{R}^d\setminus\widehat{\Omega}} BS_{\mathcal{L}}(x)\,d\mu(x) \\
&\leq \frac{32}{\lambda}\sum_j \sum_{k\in\mathbb{Z}\,:\,2^{-k+1}>\ell(R_j)} \int_{\mathbb{R}^d\setminus 2R_j} \theta_k^j(x)|(K\chi_{2^{-k-1}}^{2^{-k}} * \nu_b^j)(x)|\,d\mu(x) \\
&\lesssim \frac{1}{\lambda}\sum_j \sum_{k\in\mathbb{Z}\,:\,2^{-k+1}>\ell(R_j)} 2^{(k+1)n}\|\nu_b^j\| \int \theta_k^j(x)\,d\mu(x).
\end{aligned}
\tag{32}
$$

Let us check that $\int \theta_k^j(x)\,d\mu(x) \lesssim \ell(R_j)2^{-k(n-1)}$. Fix k and j such that $2^{-k+1} > \ell(R_j)$, and take $u \in \frac{9}{10}R_j \cap \operatorname{supp}\mu$ (this u exists because of (7)). There exists $a > 0$ depending only on d such that $\operatorname{supp}\theta_k^j \subset B(u, 2^{-k}a)$; thus, if $\ell(R_j) \geq 2^{-k}b$ for some small constant $b > 0$, $\int \theta_k^j\,d\mu \leq \mu(B(u, 2^{-k}a)) \lesssim 2^{-kn} \leq b^{-1}\ell(R_j)2^{-k(n-1)}$. On the contrary, if $\ell(R_j) < 2^{-k}b$ and b is small enough, then

$$
\operatorname{supp}\theta_k^j \subset A(u, 2^{-k} - b'\ell(R_j), 2^{-k} + b'\ell(R_j)) \cup A(u, 2^{-k-1} - b'\ell(R_j), 2^{-k-1} + b'\ell(R_j))
$$

for some constant $b' > 0$ depending on b and d such that $2^{-k-1} - b'\ell(R_j) > 0$. In that case, since $u \in \operatorname{supp}\mu$, we have $\int \theta_k^j\,d\mu = \mu(\operatorname{supp}\theta_k^j) \lesssim \ell(R_j)2^{-k(n-1)}$ (because $\mu(A(u,r,R)) \lesssim (R-r)R^{n-1}$ for all $0 < r \leq R$ by Lemma 2.3, since Γ has slope smaller than 1), as desired.

Using that $\int \theta_k^j\,d\mu \lesssim \ell(R_j)2^{-k(n-1)}$ and that $\|\nu_b^j\| \lesssim |\nu|(Q_j)$ in (32), we conclude

$$
\begin{aligned}
\mu\big(\{x \in \mathbb{R}^d \setminus \widehat{\Omega} \,:\, BS_{\mathcal{L}}(x) > \lambda/32\}\big) &\lesssim \frac{1}{\lambda}\sum_j \sum_{k\in\mathbb{Z}\,:\,2^{-k+1}>\ell(R_j)} 2^{(k+1)n}\|\nu_b^j\|\ell(R_j)2^{-k(n-1)} \\
&= \frac{1}{\lambda}\sum_j \|\nu_b^j\|\ell(R_j) \sum_{k\in\mathbb{Z}\,:\,2^{-k+1}>\ell(R_j)} 2^{n+k} \\
&\lesssim \frac{1}{\lambda}\sum_j |\nu|(Q_j) \leq \frac{C}{\lambda}\|\nu\|.
\end{aligned}
\tag{33}
$$

It only remains to show $\mu\big(\{x\in\mathbb{R}^d\setminus\widehat\Omega\,:\,BS_{\mathcal S}(x)>\lambda/32\}\big)\le C\lambda^{-1}\|\nu\|$ to finish the proof of the theorem. We set

$$\Phi_m^j(x):=\chi_{\mathbb{R}^d\setminus 2R_j}(x)\psi_m^j(x)(K\chi_{\epsilon_{m+1}(x)}^{\epsilon_m(x)}*\nu_b^j)(x).$$

Recall that $I_r=[2^{-r-1},2^{-r})$. Since the ℓ^ρ-norm is not bigger than the ℓ^2-norm,

$$\begin{aligned}\mu\big(\{x\in\mathbb{R}^d\setminus\widehat\Omega\,:\,BS_{\mathcal S}(x)>\lambda/32\}\big)&\lesssim\frac{1}{\lambda^2}\int_{\mathbb{R}^d\setminus\widehat\Omega}\sum_{m\in\mathcal S}\Big|\sum_j\Phi_m^j(x)\Big|^2d\mu(x)\\&=\frac{1}{\lambda^2}\sum_{k\in\mathbb Z}\int_{\mathbb{R}^d\setminus\widehat\Omega}\sum_{m\in\mathcal S_k}\Big|\sum_{j\,:\,2^{-k+1}>\ell(R_j)}\Phi_m^j(x)\Big|^2d\mu(x)\\&=\frac{1}{\lambda^2}\sum_{k\in\mathbb Z}\int_{\mathbb{R}^d\setminus\widehat\Omega}\sum_{m\in\mathcal S_k}\Big|\sum_{r\in\mathbb Z\,:\,r\ge k-1}\ \sum_{j\,:\,\ell(R_j)\in I_r}\Phi_m^j(x)\Big|^2d\mu(x),\end{aligned}$$

and then by Cauchy-Schwarz inequality,

$$\begin{aligned}&\mu\big(\{x\in\mathbb{R}^d\setminus\widehat\Omega\,:\,BS_{\mathcal S}(x)>\lambda/32\}\big)\\&\quad\lesssim\frac{1}{\lambda^2}\sum_{k\in\mathbb Z}\int_{\mathbb{R}^d\setminus\widehat\Omega}\sum_{m\in\mathcal S_k}\Bigg(\sum_{\substack{r\in\mathbb Z:\\r\ge k-1}}2^{(k-r)/2}\Bigg)\Bigg(\sum_{\substack{r\in\mathbb Z:\\r\ge k-1}}2^{(r-k)/2}\Big|\sum_{j\,:\,\ell(R_j)\in I_r}\Phi_m^j(x)\Big|^2\Bigg)d\mu(x)\\&\quad\lesssim\frac{1}{\lambda^2}\sum_{k\in\mathbb Z}\int_{\mathbb{R}^d\setminus\widehat\Omega}\sum_{m\in\mathcal S_k}\sum_{r\in\mathbb Z\,:\,r\ge k-1}2^{(r-k)/2}\Big|\sum_{j\,:\,\ell(R_j)\in I_r}\Phi_m^j(x)\Big|^2d\mu(x).\end{aligned}$$

Thus, if we set $P_m^r(x):=\sum_{j\,:\,\ell(R_j)\in I_r}\Phi_m^j(x)$, we have seen that

$$\mu\big(\{x\in\mathbb{R}^d\setminus\widehat\Omega\,:\,BS_{\mathcal S}(x)>\lambda/32\}\big)\lesssim\frac{1}{\lambda^2}\sum_{k\in\mathbb Z}\int_{\mathbb{R}^d\setminus\widehat\Omega}\sum_{m\in\mathcal S_k}\sum_{\substack{r\in\mathbb Z:\\r\ge k-1}}2^{(r-k)/2}|P_m^r(x)|^2\,d\mu(x).\tag{34}$$

Let us estimate $P_m^r(x)$ for $m\in\mathcal S_k$ and $r\ge k-1$. Since $\|\nu_b^j\|\lesssim|\nu|(Q_j)\le|\nu|(3Q_j)\lesssim\lambda\mu(6Q_j)$ by (10) and (7), we have

$$\begin{aligned}|P_m^r(x)|&\le\sum_{j\,:\,\ell(R_j)\in I_r}\chi_{\mathbb{R}^d\setminus 2R_j}(x)\psi_m^j(x)|(K\chi_{\epsilon_{m+1}}^{\epsilon_m}*\nu_b^j)(x)|\\&\lesssim\sum_{j\,:\,\ell(R_j)\in I_r}\chi_{\mathbb{R}^d\setminus 2R_j}(x)\psi_m^j(x)2^{kn}\|\nu_b^j\|\lesssim\sum_{\substack{j\,:\,6\ell(Q_j)\in I_r,\\6Q_j\cap\partial A(x,\epsilon_{m+1},\epsilon_m)\neq\emptyset}}2^{kn}\lambda\mu(6Q_j).\end{aligned}\tag{35}$$

It is not difficult to see that, if $\sum_j\chi_{Q_j}\le C$ for some $C>0$, then $\sum_{j\,:\,\ell(6Q_j)\in I_r}\chi_{6Q_j}\le C'$ for all $r\in\mathbb Z$, where $C'>0$ only depends on C (that is, the family of cubes $\mathcal F:=\{6Q_j\}_{j\,:\,\ell(6Q_j)\in I_r}$ has finite overlap uniformly in $r\in\mathbb Z$). We set

$$\Upsilon:=\sum_{\substack{j\,:\,6\ell(Q_j)\in I_r,\\6Q_j\cap\partial A(x,\epsilon_{m+1},\epsilon_m)\neq\emptyset}}\chi_{6Q_j}.$$

If $2^{-k}a\le2^{-r}\le2^{-k+1}$ for some small constant $a>0$ (recall that we are assuming $r\ge k-1$), then there exists a constant $b>0$ depending only on d and a such that $\operatorname{supp}\Upsilon\subset B(x,b2^{-k})$, and then, by the finite overlap of the family $\mathcal{F}$,

$$\sum_{\substack{j\,:\,6\ell(Q_j)\in I_r,\\ 6Q_j\cap\partial A(x,\epsilon_{m+1},\epsilon_m)\neq\emptyset}} \mu(6Q_j) = \int_{B(x,b2^{-k})} \Upsilon\,d\mu \le C'\mu(B(x,b2^{-k})) \lesssim 2^{-kn} \approx 2^{-r}2^{-k(n-1)}.$$

On the contrary, if $2^{-k}a \ge 2^{-r}$ for a small enough (depending on d), then there exists a constant $b>0$ depending on d and a such that $2^{-k-1} > 2^{-r}b$ and $\operatorname{supp}\Upsilon \subset A(x,\epsilon_m - 2^{-r}b,\epsilon_m + 2^{-r}b) \cup A(x,\epsilon_{m+1}-2^{-r}b,\epsilon_{m+1}+2^{-r}b)$, and then, since $m\in\mathcal{S}_k$, $x\in\operatorname{supp}\mu$ and the slope of Γ is smaller than 1, by Lemma 2.3 we have $\mu(\operatorname{supp}\Upsilon) \le \mu(A(x,\epsilon_m-2^{-r}b,\epsilon_m+2^{-r}b)) + \mu(A(x,\epsilon_{m+1}-2^{-r}b,\epsilon_{m+1}+2^{-r}b)) \lesssim 2^{-r}2^{-k(n-1)}$, thus by the finite overlap of the family $\mathcal{F}$,

$$\sum_{\substack{j\,:\,6\ell(Q_j)\in I_r,\\ 6Q_j\cap\partial A(x,\epsilon_{m+1},\epsilon_m)\neq\emptyset}} \mu(6Q_j) = \int_{\operatorname{supp}\Upsilon} \Upsilon\,d\mu \lesssim \mu(\operatorname{supp}\Upsilon) \lesssim 2^{-r}2^{-k(n-1)}.$$

In any case, from (35) we get $|P_m^r(x)| \lesssim 2^{kn}\lambda 2^{-r}2^{-k(n-1)} = 2^{k-r}\lambda$. Therefore, using (34) we obtain that

$$\begin{aligned}
\mu\big(\{x\in\mathbb{R}^d\setminus\widehat{\Omega} \,:\, BS_{\mathcal{S}}(x) > \lambda/32\}\big) &\lesssim \frac{1}{\lambda}\sum_{k\in\mathbb{Z}}\int_{\mathbb{R}^d\setminus\widehat{\Omega}} \sum_{m\in\mathcal{S}_k}\sum_{r\in\mathbb{Z}\,:\,r\ge k-1} 2^{(k-r)/2}|P_m^r(x)|\,d\mu(x)\\
&\le \frac{1}{\lambda}\sum_{k\in\mathbb{Z}}\int_{\mathbb{R}^d\setminus\widehat{\Omega}} \sum_{m\in\mathcal{S}_k}\sum_{\substack{r\in\mathbb{Z}:\\ r\ge k-1}} 2^{(k-r)/2} \sum_{\substack{j\,:\,\ell(R_j)\in I_r,\\ R_j\cap\partial A(x,\epsilon_{m+1},\epsilon_m)\neq\emptyset}} |(K\chi_{\epsilon_{m+1}}^{\epsilon_m} * \nu_b^j)(x)|\,d\mu(x)\\
&\lesssim \frac{1}{\lambda}\sum_{k\in\mathbb{Z}}\int_{\mathbb{R}^d\setminus\widehat{\Omega}} \sum_{m\in\mathcal{S}_k}\sum_{\substack{r\in\mathbb{Z}:\\ r\ge k-1}} 2^{(k-r)/2} \sum_{\substack{j\,:\,\ell(R_j)\in I_r,\\ R_j\cap A(x,2^{-k-1},2^{-k})\neq\emptyset}} 2^{kn}|\nu_b^j|(A(x,\epsilon_{m+1},\epsilon_m))\,d\mu(x)\\
&\le \frac{1}{\lambda}\sum_{k\in\mathbb{Z}}\int_{\mathbb{R}^d\setminus\widehat{\Omega}} \sum_{\substack{r\in\mathbb{Z}:\\ r\ge k-1}} 2^{(k-r)/2+kn} \sum_{\substack{j\,:\,\ell(R_j)\in I_r,\\ R_j\cap A(x,2^{-k-1},2^{-k})\neq\emptyset}} |\nu_b^j|(A(x,2^{-k-1},2^{-k}))\,d\mu(x).
\end{aligned}$$

Hence, if we set

$$\tau_k^j(x) := \begin{cases} 1 & \text{if } R_j\cap A(x,2^{-k-1},2^{-k})\neq\emptyset\\ 0 & \text{if not}\end{cases},$$

we obtain

$$\begin{aligned}
&\mu\big(\{x\in\mathbb{R}^d\setminus\widehat{\Omega} \,:\, BS_{\mathcal{S}}(x) > \lambda/32\}\big)\\
&\qquad\le \frac{1}{\lambda}\sum_{k\in\mathbb{Z}}\int_{\mathbb{R}^d\setminus\widehat{\Omega}} \sum_{\substack{r\in\mathbb{Z}:\\ r\ge k-1}} 2^{(k-r)/2+kn} \sum_{j\,:\,\ell(R_j)\in I_r} \|\nu_b^j\|\tau_k^j(x)\,d\mu(x)\\
&\qquad= \frac{1}{\lambda}\sum_{k\in\mathbb{Z}}\sum_{\substack{r\in\mathbb{Z}:\\ r\ge k-1}} 2^{(k-r)/2+kn} \sum_{j\,:\,\ell(R_j)\in I_r} \|\nu_b^j\|\int_{\mathbb{R}^d\setminus\widehat{\Omega}} \tau_k^j\,d\mu.
\end{aligned}$$

Notice that, if $\ell(R_j)\in I_r$ and $r\ge k-1$, then $\ell(R_j) < 2^{-k+1}$. Hence, there exists a constant $C>0$ such that $\operatorname{supp}\tau_k^j \subset B(z_j, C2^{-k})$ for all $\ell(R_j)\in I_r$ and all $r\ge k-1$ (recall that z_j is the center of R_j), and then $\int_{\mathbb{R}^d\setminus\widehat{\Omega}} \tau_k^j\,d\mu \le \mu(B(z_j,C2^{-k})) \lesssim 2^{-kn}$. Therefore, by exchanging the order of summation and using that $\|\nu_b^j\| \lesssim |\nu|(Q_j)$, we finally obtain

$$\begin{aligned}\mu\big(\{x\in\mathbb{R}^d\backslash\widehat{\Omega}\,:\,BS_{\mathcal{S}}(x)>\lambda/32\}\big) &\lesssim \frac{1}{\lambda}\sum_{k\in\mathbb{Z}}\sum_{r\in\mathbb{Z}\,:\,r\geq k-1} 2^{(k-r)/2}\sum_{j\,:\,\ell(R_j)\in I_r}\|\nu_b^j\|\\ &= \frac{1}{\lambda}\sum_j|\nu|(Q_j)\sum_{r\in\mathbb{Z}\,:\,2^{-r-1}\leq\ell(R_j)<2^{-r}}\;\sum_{k\in\mathbb{Z}\,:\,k\leq r+1}2^{(k-r)/2}\\ &\lesssim \frac{1}{\lambda}\sum_j|\nu|(Q_j)\leq\frac{C}{\lambda}\|\nu\|.\end{aligned} \tag{36}$$

The estimate (19) is a direct consequence of (21), (26), (28), (29), (30), (33), and (36). □

4. $\mathcal{V}_\rho\circ\mathcal{T}^{\mathcal{H}^n_\Gamma}$ is a bounded operator from $L^\infty(\mathcal{H}^n_\Gamma)$ to $BMO(\mathcal{H}^n_\Gamma)$

This section is devoted to the proof of the endpoint estimate (c) of Theorem 1.3. The use of Lemma 2.3 is also essential in this section.

We may assume that $\Gamma=\{(y,\mathcal{A}(y))\,:\,y\in\mathbb{R}^n\}$, where $\mathcal{A}:\mathbb{R}^n\to\mathbb{R}^{d-n}$ is some Lipschitz function with Lipschitz constant $\mathrm{Lip}(\mathcal{A})$. We say that a function $f\in L^1_{loc}(\mathcal{H}^n_\Gamma)$ belongs to $BMO(\mathcal{H}^n_\Gamma)$ if there exists a constant $C>0$ such that

$$\sup_D\inf_{c\in\mathbb{R}}\frac{1}{\mathcal{H}^n_\Gamma(D)}\int_D|f-c|\,d\mathcal{H}^n_\Gamma\leq C,$$

where the supremum is taken over all the sets of the type $D:=\widetilde{D}\times\mathbb{R}^{d-n}$, where $\widetilde{D}$ is a cube in $\mathbb{R}^n$. For convenience of notation, given $a>0$ we define $aD:=a\widetilde{D}\times\mathbb{R}^{d-n}$ and $\ell(aD):=\ell(a\widetilde{D})$. Notice that, since Γ is an n-dimensional Lipschitz graph, we have $\mathcal{H}^n_\Gamma(D)\approx\ell(D)^n$ for all cubes $\widetilde{D}\subset\mathbb{R}^n$. Moreover $(\Gamma,\mathcal{H}^n_\Gamma)$ is a space of homogeneous type, and it is not hard to show that our definition of $BMO(\mathcal{H}^n_\Gamma)$ is equivalent to the classical one for doubling measures (see [To1] for a definition of BMO on doubling measures).

***Proof of* Theorem 1.3(c).** We have to prove that there exists a constant $C>0$ such that, for any $f\in L^\infty(\mathcal{H}^n_\Gamma)$ and any cube $\widetilde{D}\subset\mathbb{R}^n$, there exists some constant c depending on f and $\widetilde{D}$ such that

$$\int_{\widetilde{D}\times\mathbb{R}^{d-n}}\big|(\mathcal{V}_\rho\circ\mathcal{T}^{\mathcal{H}^n_\Gamma})f-c\big|\,d\mathcal{H}^n_\Gamma\leq C\|f\|_{L^\infty(\mathcal{H}^n_\Gamma)}\mathcal{H}^n_\Gamma(\widetilde{D}\times\mathbb{R}^{d-n}). \tag{37}$$

Let f and $\widetilde{D}$ be as above, and set $D:=\widetilde{D}\times\mathbb{R}^{d-n}$, $f_1:=f\chi_{3D}$, and $f_2:=f-f_1$. First of all, by Hölder's inequality, Theorem 1.2, and since $\mathcal{H}^n_\Gamma(3D)\approx\mathcal{H}^n_\Gamma(D)$ because Γ is a Lipschitz graph, we have

$$\begin{aligned}\int_D(\mathcal{V}_\rho\circ\mathcal{T}^{\mathcal{H}^n_\Gamma})f_1\,d\mathcal{H}^n_\Gamma &\leq \mathcal{H}^n_\Gamma(D)^{1/2}\Big(\int\big((\mathcal{V}_\rho\circ\mathcal{T}^{\mathcal{H}^n_\Gamma})f_1\big)^2\,d\mathcal{H}^n_\Gamma\Big)^{1/2}\\ &\lesssim \mathcal{H}^n_\Gamma(D)^{1/2}\Big(\|f_1\|^2_{L^\infty(\mathcal{H}^n_\Gamma)}\mathcal{H}^n_\Gamma(3D)\Big)^{1/2}\lesssim\|f\|_{L^\infty(\mathcal{H}^n_\Gamma)}\mathcal{H}^n_\Gamma(D).\end{aligned} \tag{38}$$

Notice that $|(\mathcal{V}_\rho\circ\mathcal{T}^{\mathcal{H}^n_\Gamma})(f_1+f_2)-(\mathcal{V}_\rho\circ\mathcal{T}^{\mathcal{H}^n_\Gamma})f_2|\leq(\mathcal{V}_\rho\circ\mathcal{T}^{\mathcal{H}^n_\Gamma})f_1$, because $\mathcal{V}_\rho\circ\mathcal{T}^{\mathcal{H}^n_\Gamma}$ is sublinear and positive. Then, for any $c\in\mathbb{R}$,

$$\begin{aligned}|(\mathcal{V}_\rho\circ\mathcal{T}^{\mathcal{H}^n_\Gamma})f-c| &= |(\mathcal{V}_\rho\circ\mathcal{T}^{\mathcal{H}^n_\Gamma})(f_1+f_2)-c|\\ &\leq|(\mathcal{V}_\rho\circ\mathcal{T}^{\mathcal{H}^n_\Gamma})(f_1+f_2)-(\mathcal{V}_\rho\circ\mathcal{T}^{\mathcal{H}^n_\Gamma})f_2|+|(\mathcal{V}_\rho\circ\mathcal{T}^{\mathcal{H}^n_\Gamma})f_2-c|\\ &\leq(\mathcal{V}_\rho\circ\mathcal{T}^{\mathcal{H}^n_\Gamma})f_1+|(\mathcal{V}_\rho\circ\mathcal{T}^{\mathcal{H}^n_\Gamma})f_2-c|,\end{aligned} \tag{39}$$

hence, to prove (37), by (38) and (39) we are reduced to prove that, for some constant $c \in \mathbb{R}$,

$$\int_D \big|(\mathcal{V}_\rho \circ \mathcal{T}^{\mathcal{H}^n_\Gamma})f_2 - c\big|\, d\mathcal{H}^n_\Gamma \leq C\|f\|_{L^\infty(\mathcal{H}^n_\Gamma)}\mathcal{H}^n_\Gamma(D). \tag{40}$$

Set $z_D := (\widetilde{z}_D, \mathcal{A}(\widetilde{z}_D))$, where $\widetilde{z}_D$ is the center of $\widetilde{D} \subset \mathbb{R}^n$, and take $c := (\mathcal{V}_\rho \circ \mathcal{T}^{\mathcal{H}^n_\Gamma})f_2(z_D)$. We may assume that $c < \infty$. By the triangle inequality,

$$\big|(\mathcal{V}_\rho \circ \mathcal{T}^{\mathcal{H}^n_\Gamma})f_2(x) - c\big|^\rho \leq \sup_{\{\epsilon_m \searrow 0\}} \sum_{m\in\mathbb{Z}} |(K\chi^{\epsilon_m}_{\epsilon_{m+1}} * (f_2\mathcal{H}^n_\Gamma))(x) - (K\chi^{\epsilon_m}_{\epsilon_{m+1}} * (f_2\mathcal{H}^n_\Gamma))(z_D)|^\rho.$$

Given $x \in \Gamma \cap D$, let $\{\epsilon_m\}_{m\in\mathbb{Z}}$ be a decreasing sequence of positive numbers (which depends on x) such that

$$\big|(\mathcal{V}_\rho \circ \mathcal{T}^{\mathcal{H}^n_\Gamma})f_2(x) - c\big|^\rho \leq 2\sum_{m\in\mathbb{Z}} |(K\chi^{\epsilon_m}_{\epsilon_{m+1}} * (f_2\mathcal{H}^n_\Gamma))(x) - (K\chi^{\epsilon_m}_{\epsilon_{m+1}} * (f_2\mathcal{H}^n_\Gamma))(z_D)|^\rho.$$

Notice that $|(K\chi^{\epsilon_m}_{\epsilon_{m+1}} * (f_2\mathcal{H}^n_\Gamma))(x) - (K\chi^{\epsilon_m}_{\epsilon_{m+1}} * (f_2\mathcal{H}^n_\Gamma))(z_D)| \leq \|f\|_{L^\infty(\mathcal{H}^n_\Gamma)}(\Theta 1_m + \Theta 2_m)$, where

$$\begin{aligned}
\Theta 1_m &:= \int_{(3D)^c} \chi^{\epsilon_m}_{\epsilon_{m+1}}(x-y)\,|K(x-y) - K(z_D - y)|\, d\mathcal{H}^n_\Gamma(y),\\
\Theta 2_m &:= \int_{(3D)^c} |\chi^{\epsilon_m}_{\epsilon_{m+1}}(x-y) - \chi^{\epsilon_m}_{\epsilon_{m+1}}(z_D-y)||K(z_D-y)|\, d\mathcal{H}^n_\Gamma(y).
\end{aligned}$$

Thus,

$$\big|(\mathcal{V}_\rho \circ \mathcal{T}^{\mathcal{H}^n_\Gamma})f_2(x) - c\,\big| \lesssim \|f\|_{L^\infty(\mathcal{H}^n_\Gamma)}\Big(\sum_{m\in\mathbb{Z}}(\Theta 1_m + \Theta 2_m)^\rho\Big)^{1/\rho}. \tag{41}$$

Since $\rho \geq 1$, we easily have

$$\begin{aligned}
\Big(\sum_{m\in\mathbb{Z}} \Theta 1_m^\rho\Big)^{1/\rho} &\leq \sum_{m\in\mathbb{Z}} \Theta 1_m \lesssim \int_{(3D)^c} \sum_{m\in\mathbb{Z}} \chi^{\epsilon_m}_{\epsilon_{m+1}}(x-y)\frac{|x-z_D|}{|z_D-y|^{n+1}}\, d\mathcal{H}^n_\Gamma(y)\\
&\lesssim \ell(D)\int_{(3D)^c} |z_D - y|^{-n-1}\, d\mathcal{H}^n_\Gamma(y) \lesssim 1.
\end{aligned} \tag{42}$$

The case of $\Theta 2_m$ is more delicate. Since Γ is a Lipschitz graph, there exists an integer $M > 10$ depending only on $\mathrm{Lip}(\mathcal{A})$ such that any $x \in \Gamma \cap D$ satisfies $|x - z_D| < 2^M\ell(D)$. Without loss of generality, we can assume that there exists $m_0 \in \mathbb{Z}$ such that $\epsilon_{m_0} = 2^{M+2}\ell(D)$, just by adding the term $2^{M+2}\ell(D)$ to the fixed sequence $\{\epsilon_m\}_{m\in\mathbb{Z}}$. Obviously, we can also assume that $\epsilon_m > \epsilon_{m+1}$ for all $m \in \mathbb{Z}$.

We set $J_0 := \{m \in \mathbb{Z} : \epsilon_m \leq 2^{M+2}\ell(D)\} = \{m \in \mathbb{Z} : m \geq m_0\}$ and, for $j > M+2$,

$$\begin{aligned}
J^1_j &:= \{m \in \mathbb{Z} : 2^{j-1}\ell(D) \leq \epsilon_{m+1} < \epsilon_m \leq 2^j\ell(D) \text{ and } \epsilon_m - \epsilon_{m+1} \geq 2^M\ell(D)\},\\
J^2_j &:= \{m \in \mathbb{Z} : 2^{j-1}\ell(D) \leq \epsilon_{m+1} < \epsilon_m \leq 2^j\ell(D) \text{ and } \epsilon_m - \epsilon_{m+1} < 2^M\ell(D)\},\\
J^3_j &:= \{m \in \mathbb{Z} : 2^{j-1}\ell(D) \leq \epsilon_{m+1} \leq 2^j\ell(D) < \epsilon_m\}.
\end{aligned}$$

Then $\mathbb{Z} = J_0 \cup \big(\bigcup_{j>M+2}(J^1_j \cup J^2_j \cup J^3_j)\big)$. For the case of $m \in J_0$, we have the easy estimate

$$\begin{aligned}
\Big(\sum_{m\in J_0} \Theta 2_m^\rho\Big)^{1/\rho} &\lesssim \sum_{m\in J_0}\int_{(3D)^c} \big(\chi^{\epsilon_m}_{\epsilon_{m+1}}(x-y) + \chi^{\epsilon_m}_{\epsilon_{m+1}}(z_D-y)\big)\ell(D)^{-n}\, d\mathcal{H}^n_\Gamma(y)\\
&\leq \int_{|x-y|\leq 2^{M+2}\ell(D)} \frac{d\mathcal{H}^n_\Gamma(y)}{\ell(D)^n} + \int_{|z_D-y|\leq 2^{M+2}\ell(D)} \frac{d\mathcal{H}^n_\Gamma(y)}{\ell(D)^n} \lesssim 1.
\end{aligned}$$

Assume that $m \in J_j^1$. Notice that $\operatorname{supp}\big(\chi_{\epsilon_{m+1}}^{\epsilon_m}(x-\cdot)-\chi_{\epsilon_{m+1}}^{\epsilon_m}(z_D-\cdot)\big)\subset A_m(x,z_D)$, where $A_m(x,z_D)$ denotes the symmetric difference between $A(x,\epsilon_{m+1},\epsilon_m)$ and $A(z_D,\epsilon_{m+1},\epsilon_m)$. Notice also that, since $m\in J_j^1$ and $x\in D\cap\Gamma$, the set $A_m(x,z_D)$ is contained in the union of annuli $A_1:=A(x,\epsilon_{m+1}-2^M\ell(D),\epsilon_{m+1}+2^M\ell(D))$ and $A_2:=A(x,\epsilon_m-2^M\ell(D),\epsilon_m+2^M\ell(D))$. Hence, using that $m\in J_j^1$ and Lemma 2.3, we have

$$\begin{aligned}\mathcal{H}^n_\Gamma(\{y\in\mathbb{R}^d\,:\,|\chi_{\epsilon_{m+1}}^{\epsilon_m}(x-y)-\chi_{\epsilon_{m+1}}^{\epsilon_m}(z_D-y)|\neq 0\}) &\le \mathcal{H}^n_\Gamma(A_1\cup A_2)\\ &\lesssim 2^{M+1}\ell(D)\Big(\epsilon_m+2^M\ell(D)\Big)^{n-1}+2^{M+1}\ell(D)\Big(\epsilon_{m+1}+2^M\ell(D)\Big)^{n-1}\\ &\lesssim 2^{j(n-1)}\ell(D)^n.\end{aligned}\tag{43}$$

Using that $|K(z_D-y)|\lesssim(2^j\ell(D))^{-n}$ for all $y\in A_m(x,z_D)\cap(3D)^c$, we get

$$\Theta 2_m\lesssim(2^j\ell(D))^{-n}2^{j(n-1)}\ell(D)^n=2^{-j}$$

and, since $\rho\ge 2$ and J_j^1 contains at most 2^{j-M-1} indices, we have $\sum_{m\in J_j^1}\Theta 2_m^\rho\lesssim 2^{-j}$.

Assume now that $m\in J_j^2$. Then, using Lemma 2.3, we obtain

$$\begin{aligned}&\mathcal{H}^n_\Gamma(\{y\in\mathbb{R}^d\,:\,|\chi_{\epsilon_{m+1}}^{\epsilon_m}(x-y)-\chi_{\epsilon_{m+1}}^{\epsilon_m}(z_D-y)|\neq 0\})\\ &\qquad\le\mathcal{H}^n_\Gamma(\{y\in\mathbb{R}^d\,:\,\chi_{\epsilon_{m+1}}^{\epsilon_m}(x-y)=1\})+\mathcal{H}^n_\Gamma(\{y\in\mathbb{R}^d\,:\,\chi_{\epsilon_{m+1}}^{\epsilon_m}(z_D-y)=1\})\\ &\qquad\lesssim(\epsilon_m-\epsilon_{m+1})\epsilon_m^{n-1},\end{aligned}$$

and, as above, $|K(z_D-y)|\lesssim(2^j\ell(D))^{-n}$ for all $y\in A_m(x,z_D)\cap(3D)^c$. Since $m\in J_j^2$,

$$\begin{aligned}\Theta 2_m^\rho&\lesssim(2^j\ell(D))^{-\rho n}\big((\epsilon_m-\epsilon_{m+1})\epsilon_m^{n-1}\big)^\rho\\ &\lesssim(2^j\ell(D))^{-\rho n}(\epsilon_m-\epsilon_{m+1})(2^M\ell(D))^{\rho-1}(2^j\ell(D))^{(n-1)\rho}\lesssim 2^{-j\rho}\ell(D)^{-1}(\epsilon_m-\epsilon_{m+1})\end{aligned}$$

and then, since $\rho\ge2$ and $j>M+2>12$,

$$\sum_{m\in J_j^2}\Theta 2_m^\rho\lesssim 2^{-j\rho}\sum_{m\in J_j^2}\frac{\epsilon_m-\epsilon_{m+1}}{\ell(D)}\le 2^{-j\rho}2^{j-1}\approx 2^{-j(\rho-1)}\le 2^{-j}.$$

Finally, assume that $m\in J_j^3$. Obviously, the set J_j^3 contains at most one term. If $\epsilon_m-\epsilon_{m+1}<2^M\ell(D)$, arguing as in the case $m\in J_j^2$, we have

$$\begin{aligned}\mathcal{H}^n_\Gamma(\{y\in\mathbb{R}^d\,:\,|\chi_{\epsilon_{m+1}}^{\epsilon_m}(x-y)-\chi_{\epsilon_{m+1}}^{\epsilon_m}(z_D-y)|\neq 0\})&\lesssim(\epsilon_m-\epsilon_{m+1})\epsilon_m^{n-1}\\ &\lesssim 2^M\ell(D)(2^j\ell(D)+2^M\ell(D))^{n-1}\lesssim 2^{j(n-1)}\ell(D)^n,\end{aligned}$$

and then $\Theta 2_m\lesssim 2^{j(n-1)}\ell(D)^n(2^{j-1}\ell(D))^{-n}\lesssim 2^{-j}$. On the contrary, if $\epsilon_m-\epsilon_{m+1}\ge 2^M\ell(D)$, arguing as in the case $m\in J_j^1$, we have $\operatorname{supp}\big(\chi_{\epsilon_{m+1}}^{\epsilon_m}(x-\cdot)-\chi_{\epsilon_{m+1}}^{\epsilon_m}(z_D-\cdot)\big)\subset A_m(x,z_D)\subset A_1\cup A_2$. Similarly to (43), we have

$$\mathcal{H}^n_\Gamma(A_1)\lesssim 2^{M+1}\ell(D)(\epsilon_{m+1}+2^M\ell(D))^{n-1}\lesssim\epsilon_{m+1}^{n-1}\ell(D)\le 2^{j(n-1)}\ell(D)^n,$$

and $|K(z_D-y)|\lesssim(2^j\ell(D))^{-n}$ for all $y\in A_1\cap(3D)^c$. If we denote by $j(\epsilon_m)$ the positive integer such that $2^{j(\epsilon_m)-1}\ell(D)\le\epsilon_m\le 2^{j(\epsilon_m)}\ell(D)$ (obviously, $j(\epsilon_m)>j$), we have $\mathcal{H}^n_\Gamma(A_2)\lesssim\epsilon_m^{n-1}\ell(D)\le 2^{j(\epsilon_m)(n-1)}\ell(D)^n$, and $|K(z_D-y)|\lesssim(2^{j(\epsilon_m)}\ell(D))^{-n}$ for all $y\in A_2\cap(3D)^c$. Hence, $\Theta 2_m\lesssim 2^{j(n-1)}\ell(D)^n(2^j\ell(D))^{-n}+2^{j(\epsilon_m)(n-1)}\ell(D)^n(2^{j(\epsilon_m)}\ell(D))^{-n}\lesssim 2^{-j}+2^{-j(\epsilon_m)}\lesssim 2^{-j}$. Therefore, since J_j^3 contains at most one term, $\sum_{m\in J_j^3}\Theta 2_m^\rho\lesssim 2^{-j\rho}\le 2^{-j}$.

We put all these estimates of $\Theta 2_m$ for m belonging to J_0, J_j^1, J_j^2, and J_j^3 together with (42) in (41) and we conclude that

$$\begin{aligned}\big|(\mathcal{V}_\rho\circ\mathcal{T}^{\mathcal{H}^n_\Gamma})f_2(x)-c\,\big| &\lesssim \|f\|_{L^\infty(\mathcal{H}^n_\Gamma)}\bigg(\sum_{m\in\mathbb{Z}}(\Theta 1_m+\Theta 2_m)^\rho\bigg)^{1/\rho}\\
&\lesssim \|f\|_{L^\infty(\mathcal{H}^n_\Gamma)}\bigg(\sum_{m\in\mathbb{Z}}\Theta 1_m^\rho\bigg)^{1/\rho}+\|f\|_{L^\infty(\mathcal{H}^n_\Gamma)}\bigg(\sum_{m\in J_0}\Theta 2_m^\rho\bigg)^{1/\rho}\\
&\quad+\|f\|_{L^\infty(\mathcal{H}^n_\Gamma)}\bigg(\sum_{j>M+2}\bigg(\sum_{m\in J_j^1}\Theta 2_m^\rho+\sum_{m\in J_j^2}\Theta 2_m^\rho+\sum_{m\in J_j^3}\Theta 2_m^\rho\bigg)\bigg)^{1/\rho}\\
&\lesssim \|f\|_{L^\infty(\mathcal{H}^n_\Gamma)}\bigg(1+1+\bigg(\sum_{j>12}2^{-j}\bigg)^{1/\rho}\bigg)\lesssim \|f\|_{L^\infty(\mathcal{H}^n_\Gamma)}.\end{aligned}$$

Finally, (40) follows by integrating in D this last estimate. This yields the boundedness of $\mathcal{V}_\rho\circ\mathcal{T}^{\mathcal{H}^n_\Gamma}$ from $L^\infty(\mathcal{H}^n_\Gamma)$ to $BMO(\mathcal{H}^n_\Gamma)$. □

5. $\mathcal{V}_\rho\circ\mathcal{T}^{\mathcal{H}^n_\Gamma}$ is a bounded operator in $L^p(\mathcal{H}^n_\Gamma)$ for all $1<p<\infty$

This section is devoted to complete the proof of Theorem 1.3 and Corollary 1.6.

***Proof of* Theorem 1.3(b).** This is a straightforward application of Theorem 1.4. □

***Proof of* Theorem 1.3(a).** Recall from Theorem 1.2 that $\mathcal{V}_\rho\circ\mathcal{T}^{\mathcal{H}^n_\Gamma}$ is bounded in $L^2(\mathcal{H}^n_\Gamma)$. We deduce the L^p boundedness of the positive sublinear operator $\mathcal{V}_\rho\circ\mathcal{T}^{\mathcal{H}^n_\Gamma}$ by interpolation between the pairs $(L^1(\mathcal{H}^n_\Gamma),L^{1,\infty}(\mathcal{H}^n_\Gamma))$ and $(L^2(\mathcal{H}^n_\Gamma),L^2(\mathcal{H}^n_\Gamma))$ for $1<p<2$, and between $(L^2(\mathcal{H}^n_\Gamma),L^2(\mathcal{H}^n_\Gamma))$ and $(L^\infty(\mathcal{H}^n_\Gamma),BMO(\mathcal{H}^n_\Gamma))$ for $2<p<\infty$. Let us remark that, in the latter case, the classical interpolation theorem (see [Du, Theorem 6.8], for instance) would require the operator $\mathcal{V}_\rho\circ\mathcal{T}^{\mathcal{H}^n_\Gamma}$ to be linear. Clearly, this fails in our case. However, an easy modification of the arguments in [Du] using Lemma 5.1 below shows that that interpolation theorem is also valid for positive sublinear operators. Before stating the lemma, let us recall some definitions. Given $f\in L^1_{loc}(\mathcal{H}^n_\Gamma)$, $x\in\mathbb{R}^d$, and a cube $\widetilde{Q}\in\mathbb{R}^n$, set $Q=\widetilde{Q}\times\mathbb{R}^{d-n}$ and define

$$m_Qf:=\frac{1}{\mathcal{H}^n_\Gamma(Q)}\int_Q f\,d\mathcal{H}^n_\Gamma,$$

$Mf(x):=\sup_{Q\ni x}m_Q|f|$, and $M^\sharp f(x):=\sup_{Q\ni x}m_Q|f-m_Qf|$.

Lemma 5.1. *Let $F:L^1_{loc}(\mathcal{H}^n_\Gamma)\to L^1_{loc}(\mathcal{H}^n_\Gamma)$ be a positive and sublinear operator. Then $(M^\sharp\circ F)(f+g)\lesssim(M\circ F)f+(M^\sharp\circ F)g$ for all functions $f,g\in L^1_{loc}(\mathcal{H}^n_\Gamma)$.*

By using Lemma 5.1 and the fact that $\|Mf\|_{L^p(\mathcal{H}^n_\Gamma)}\lesssim\|M^\sharp f\|_{L^p(\mathcal{H}^n_\Gamma)}$ for $f\in L^{p_0}(\mathcal{H}^n_\Gamma)$ and $1\le p_0\le p<\infty$ (see [Du, Lemma 6.9]), one can reprove the interpolation theorem [Du, Theorem 6.8] applied to $\mathcal{V}_\rho\circ\mathcal{T}^{\mathcal{H}^n_\Gamma}$ with minor modifications in the original proof. □

***Proof of* Lemma 5.1.** If F is sublinear and positive, one has that $|F(f)(x)-F(g)(x)|\le F(f-g)(x)$ for all functions $f,g\in L^1_{loc}(\mathcal{H}^n_\Gamma)$. Let $\widetilde{Q}$ be a cube in $\mathbb{R}^n$, and set $Q=\widetilde{Q}\times\mathbb{R}^{d-n}\subset\mathbb{R}^d$. Then, for $x,y\in Q\cap\Gamma$,

$$\begin{aligned}|F(f+g)(y)-m_Q(Fg)|&\le|F(f+g)(y)-Fg(y)|+|Fg(y)-m_Q(Fg)|\\
&\le|Ff(y)|+|Fg(y)-m_Q(Fg)|.\end{aligned}$$

Hence, $m_Q|F(f+g) - m_Q(Fg)| \leq m_Q|Ff| + m_Q|Fg - m_Q(Fg)| \leq (M \circ F)f(x) + (M^\sharp \circ F)g(x)$ and, by taking the supremum over all possible cubes $\widetilde{Q} \subset \mathbb{R}^n$ such that $Q \ni x$, we conclude $(M^\sharp \circ F)(f+g)(x) \lesssim (M \circ F)f(x) + (M^\sharp \circ F)g(x)$ (recall that $(M^\sharp \circ F)h(x) \lesssim \sup_{Q \ni x} \inf_{a \in \mathbb{R}} m_Q|Fh - a|$ for all $h \in L^1_{loc}(\mathcal{H}^n_\Gamma)$). □

***Proof of* Corollary 1.6.** The arguments follow closely the proof of [Ma, Theorem 20.27]. First of all, we may assume that E is a Lipschitz graph with slope smaller than 1, since $\mathcal{H}^n$ almost all E can be covered with countably many $\mathcal{C}^1$ manifolds which in turn can be covered by Lipschitz graphs with small slope. By the Lebesgue decomposition theorem and Radon-Nikodym theorem (see [Ma, Theorem 2.17] for the real case, for example), there exists $f \in L^1(\mathcal{H}^n_E)$ and a finite complex Radon measure ν_s such that $\mathcal{H}^n_E$ and $|\nu_s|$ are mutually singular and $\nu = f\mathcal{H}^n_E + \nu_s$.

Given K satisfying (2), by Theorem 1.3(b) we have $(\mathcal{V}_\rho \circ \mathcal{T}^{\mathcal{H}^n_E})f(x) < \infty$ for $\mathcal{H}^n$ almost all $x \in E$. Therefore, for any decreasing sequence $\{\epsilon_m\}_{m\in\mathbb{Z}}$, $\{T^\mu_{\epsilon_m} f(x)\}_{m\in\mathbb{Z}}$ is a Cauchy sequence, so it is convergent. Thus $\lim_{\epsilon\to 0} T^{\mathcal{H}^n_E}_\epsilon f(x)$ exists for $\mathcal{H}^n$ almost all $x \in E$. Therefore, we may assume that $\nu = \nu_s$. The rest of the proof is almost the same of [Ma, Theorem 20.27] (just replace T^* by $\mathcal{V}_\rho \circ \mathcal{T}$ in the proof in [Ma] and use Theorem 1.4). The details are left for the reader. □

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