ERRATA TO "A DUAL CHARACTERIZATION OF THE $\mathcal{C}^1$ HARMONIC CAPACITY AND APPLICATIONS", DUKE MATH. J. 153 (1), 1–22

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ABSTRACT. In our previous paper [MMT] we studied some questions related to the $\mathcal{C}^1$ and Lipschitz harmonic capacities. A serious error was found in the arguments. In this note we explain how this error, that we have not been able to fix, affects to the results claimed in that paper.

In the paper [MMT] we studied some topics in connection with the $\mathcal{C}^1$ and Lipschitz harmonic capacities. Given a compact set $F \subset \mathbb{R}^n$, its $\mathcal{C}^1$ harmonic capacity is

$$\kappa_c(F) = \sup\langle 1, \Delta\varphi\rangle,$$

where the sup is taken over all functions $\varphi \in \mathcal{C}^1(\mathbb{R}^n)$ which are harmonic in $\mathbb{R}^n \setminus F$, such that $\|\nabla\varphi\|_\infty \leq 1$, with $\nabla\varphi$ vanishing at ∞. If one asks φ to be locally Lipschitz in $\mathbb{R}^n$ instead of $\mathcal{C}^1$, then one gets the Lipschitz harmonic capacity $\kappa(F)$.

Shortly after the publication of [MMT] we discovered an error in the proof of Proposition 3.2, where we claimed that if F is a compact smooth subset of $\mathbb{R}^n$ and μ is a vectorial measure which is orthogonal to

$$B(F) = \big\{f \in \mathcal{C}(F)^n : \ f = \nabla\varphi, \, \varphi \in \mathcal{C}^1(\mathbb{R}^n), \nabla\varphi(\infty) = 0, \, \mathrm{supp}\Delta\varphi \subset F\big\},$$

then $\mu|_{\partial F}$ is also orthogonal to $B(F)$. Unfortunately, this statement turns out to be false. Indeed, consider $F = \overline{B}(0,1)$ and let Γ be a $\mathcal{C}^1$ closed curve contained in F such that both lengths $\mathcal{H}^1(\Gamma \cap \partial F)$ and $\mathcal{H}^1(\Gamma \cap \overset{\circ}{F})$ are positive. Take the vectorial measure $\mu = t(x)\, d\mathcal{H}^1|_\Gamma(x)$, where $t(x)$ stands for a continuous unitary vector field tangent to Γ at x. Then μ is orthogonal to all $\mathcal{C}^1$ gradients in $\mathbb{R}^n$, and so in particular to $B(F)$. However, it is easy to check that, in general, $\mu|_{\partial F}$ is not orthogonal to $B(F)$.

The failure of Proposition 3.2 has serious consequences for some of the results in [MMT]. In particular, Theorem 3.3 should be rewritten as follows:

The three authors are partially supported by grants SGR2009-00420 (Generalitat de Catalunya) and MTM2007-62817 (MEC, Spain). Albert Mas is also supported by grant AP2006-02416 (Programa FPU del MEC, Spain).

Key words and phrases. Lipschitz and $\mathcal{C}^1$ harmonic capacities, Dirichlet and Neumann problems, orthogonal measures.

2000 Mathematics Subject Classification. Primary 31B05, Secondary 31B15.

Theorem 3.3'. *We have*

$$\kappa_c(F) = \min\left\{\|\eta d\sigma_o + \mu\|_F : \mu \in B(F)^\perp\right\}, \tag{0.1}$$

where $d\sigma_o$ is the surface measure of $\partial_o F$, i.e., the restriction of $d\sigma$ to the outer boundary $\partial_o F$, and η stands for the outward normal unit vector.

Recall that in Theorem 3.3 in [MMT] the measures μ in the minimum in (0.1) were assumed to be supported on $\partial_o F$. On the other hand, in Theorem 3.1 from [MMT] the measures $\mu \in B(F)^\perp$ supported on $\partial_o F$ were characterized. This result is still correct. However, to be able to exploit the characterization of κ_c given by Theorem 3.3' one needs a description of *all* the measures $\mu \in B(F)^\perp$. By arguments more or less analogous to ones in Theorem 3.1 of [MMT], one gets the following:

Theorem 3.1'. *Given a $\mathcal{C}^1$ vector field g on ∂F, let g_τ be the tangential component of g and $g_\eta \cdot \eta$ the normal one. Denote by u_g the unique harmonic extension of g_η to $F^c \cup \{\infty\}$ and let $A_0(F)$ be the class of the $\mathcal{C}^1$ vector fields g such that $u_g(\infty) = 0$.*

Let $G(F)$ be the vector space of linear combinations of measures of the form $t(x)\, d\mathcal{H}^1|_\Gamma(x)$, where Γ is a $\mathcal{C}^1$ closed curve contained F and $t(\cdot)$ is a continuous tangential vector field on Γ with constant modulus.

The set of measures $gd\sigma + \nu$, with $g \in A_0(F)$, $\mathrm{div} g_\tau = \nabla u_g \cdot \eta$ on ∂F, and $\nu \in G(F)$, is a weak dense subset of $B(F)^\perp$. Therefore, any $\mu \in B(F)^\perp$ splits into a measure in $B(F)^\perp$ supported on ∂F plus a measure which annihilates all the gradients of functions in $\mathcal{C}^1(F)$.*

Using the characterization of κ_c by duality from [MMT, Theorem 3.3], in [MMT, Theorem 4.1] we deduced that, for any compact set $E \subset \mathbb{R}^n$, the Lipschitz harmonic capacity $\kappa(E)$ coincides with $\kappa(\partial_o E)$. For the proof it was essential that we could assume that the measures appearing in (0.1) were supported on $\partial_0 F$. Unfortunately, from Theorem 3.3' and the characterization of the measures $\mu \in B(F)^\perp$ given by Theorem 3.1', we have not been able to show that $\kappa(E) = \kappa(\partial_o E)$. To prove or disprove the latter statement (or even that $\kappa(E)$ is comparable to $\kappa(\partial_o(E))$) is an open problem.

Finally, in [MMT, Theorem 4.2] we claimed that, given a continuous function $f : [0,d]^{n-1} \to \mathbb{R}$, for the graph $\Gamma = \{(x, f(x)) \in \mathbb{R}^n : x \in [0,d]^{n-1}\}$, one has

$$\kappa(\Gamma) \geq C\, d^{n-1}, \tag{0.2}$$

where C is some positive absolute constant. This would solve an open question raised by Volberg. As shown in [MMT], this follows from the semiadditivity of κ and from the fact $\kappa(E) = \kappa(\partial_o E)$ for all compact sets E. However, since the latter assertion is now open, (0.2) remains a conjecture.

References

[MMT] A. Mas, M. Melnikov and X. Tolsa, *A dual characterization of the $\mathcal{C}^1$ harmonic capacity and applications,* Duke Math. J. 153 (1), 1–22.

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