$W^{1,q}$ estimates for the extremal solution of reaction-diffusion problems

Manel Sanchón

Departament de Matemàtica Aplicada i Anàlisi, Universitat de Barcelona, Gran Via 585, 08007 Barcelona, Spain

Abstract

We establish a new $W^{1,2n-1,n-2}$ estimate for the extremal solution of $-\Delta u = \lambda f(u)$ in a smooth bounded domain $\Omega$ of $\mathbb{R}^n$, which is convex, for arbitrary positive and increasing nonlinearities $f \in C^1(\mathbb{R})$ satisfying $\lim_{t \to +\infty} f(t)/t = +\infty$.

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1. Introduction

Let $\Omega$ be a smooth bounded domain of $\mathbb{R}^n$ and consider the reaction-diffusion problem

$$\begin{cases} 
-\Delta u = \lambda f(u) & \text{in } \Omega, \\
 u = 0 & \text{on } \partial \Omega,
\end{cases}$$

(1.1)$_\lambda$

where $\lambda$ is a positive parameter and $f$ is a positive and increasing $C^1$ function satisfying

$$\lim_{t \to +\infty} \frac{f(t)}{t} = +\infty.$$  

(1.2)

Crandall and Rabinowitz [7] proved, using the Implicit Function Theorem, the existence of an extremal parameter $\lambda^* \in (0, +\infty)$ such that problem (1.1)$_\lambda$ admits a classical minimal solution $u_\lambda$ for all $\lambda \in (0, \lambda^*)$. Here, minimal means that it is smaller than any other nonnegative solution. Moreover,
the least eigenvalue of the linearized operator at $u_\lambda$, $-\Delta - \lambda f'(u_\lambda)$, is positive for all $\lambda \in (0, \lambda^*)$. Alternatively, this can be reached by using an iteration argument to obtain that $u_\lambda$ is an absolute minimizer of the associated energy functional

$$J(u_\lambda) := \int_\Omega |\nabla u_\lambda|^2 - \lambda F(u_\lambda) \, dx,$$

in the convex set $\{ w \in H^1_0(\Omega) : 0 \leq w \leq u_\lambda \}$, where $F'' = f$. In particular, $u_\lambda$ will be semi-stable in the sense that the second variation of energy at $u_\lambda$ is nonnegative definite:

$$Q_{u_\lambda}(\varphi) := \int_\Omega |\nabla \varphi|^2 - \lambda f'(u_\lambda)\varphi^2 \, dx \geq 0 \quad \text{for all } \varphi \in C^1_0(\Omega).$$

Brezis et al. [1] proved that there is no weak solution for $\lambda > \lambda^*$, while the increasing limit

$$u^* := \lim_{\lambda \uparrow \lambda^*} u_\lambda$$

is a weak solution of the extremal problem (1.1)$_{\lambda^*}$, i.e., $u^* \in L^1(\Omega)$, $f(u^*) \dist(\cdot, \partial \Omega) \in L^1(\Omega)$, and

$$\int_\Omega u^*(-\Delta \varphi) \, dx = \lambda \int_\Omega f(u^*)\varphi \, dx \quad \text{for all } \varphi \in C^2_0(\overline{\Omega}).$$

This solution is known as extremal solution of the extremal problem (1.1)$_{\lambda^*}$.

The study of the regularity of the extremal solution started to growth after Brezis and Vázquez raised some open problems in [2]. In this direction, Nedev [10] proved, in an unpublished preprint, that $u^* \in H^1_0(\Omega)$ for every positive and increasing nonlinearity $f$ satisfying (1.2) when the domain is convex (see also Theorem 2.9 in [3]). The proof uses the Pohožaev identity and the fact that $u_\lambda$ is an absolute minimizer of the functional $J$, defined in (1.3), on the compact set $\{ w \in H^1_0(\Omega) : 0 \leq w \leq u_\lambda \}$, and hence, $J(u_\lambda) \leq J(0) = 0$.

Our main result establishes that $u^* \in W^{1,\frac{2n}{n-2}}_0(\Omega)$ for any convex domain $\Omega$ and any nonlinearity $f$ satisfying the above assumptions. In particular, $u^* \in H^1_0(\Omega)$. We prove it using a geometric Sobolev inequality on the graph of minimal solutions $u_\lambda$.

**Theorem 1.1.** Let $\Omega$ be a smooth bounded domain of $\mathbb{R}^n$ with $n \geq 3$ and $f$ a positive and increasing $C^1$ function satisfying (1.2). Let $u_\lambda \in C^2_0(\overline{\Omega})$ be the
minimal solution of $(1.1)_\lambda$ for $\lambda \in (0, \lambda^*)$ and

$$I(t) := \int_{\{u_\lambda \geq t\}} (1 + |\nabla u_\lambda|^2)^{\frac{n-2}{2}} \, dx, \quad t \in (0, \|u_\lambda\|_{L^\infty(\Omega)}).$$

There exists a positive constant $C$ depending only on $n$ such that the following inequality holds

$$C I(t)^{\frac{2n-4}{n}} \leq \frac{1}{t^2} \left( \int_{\{u_\lambda \leq t\}} (1 + |\nabla u_\lambda|^2)^{\frac{n-4}{n}} \, dx \right) I(t) + \left( \int_{\{u_\lambda = t\}} (1 + |\nabla u_\lambda|^2)^{\frac{n-4}{n}} \, dS \right)^2$$

for a.e. $t \in (0, \|u_\lambda\|_{L^\infty(\Omega)})$.

If in addition $\Omega$ is convex then the extremal solution $u^* \in W_0^{1,\frac{n-4}{n-2}}(\Omega)$.

In the last decade several authors studied the regularity of the extremal solution (see the monograph by Dupaigne and references therein). However, there are few results for general reaction terms $f$ (i.e., positive and increasing nonlinearities satisfying $(1.2)$). Cabrè established that $u^* \in L^\infty(\Omega)$ when $n \leq 4$ and the domain is convex. More recently, Cabrè and the author proved for $n \geq 5$ that there exists a constant $C$ depending only on $n$ such that

$$\left( \int_{\{u_\lambda > t\}} (u_\lambda - t)^{\frac{2n-4}{n}} \, dx \right)^{\frac{n-4}{n}} \leq C \left( \int_{\{u_\lambda \leq t\}} |\nabla u_\lambda|^4 \, dx \right)^{1/2}$$

for all $t \in (0, \|u_\lambda\|_{L^\infty(\Omega)})$. As a consequence, it is proved that the extremal solution $u^*$ belongs to $L^{2n/(n-2)}(\Omega)$ when the domain is convex and the dimension $n \geq 5$. The first step in the proof of both results is to take $\varphi = |\nabla u_\lambda|\eta$ as a test function in the semistability condition $(1.4)$ and use the following geometric identity

$$(\nabla |\nabla u_\lambda| + \lambda f'(u_\lambda)|\nabla u_\lambda|) |\nabla u_\lambda| = \tilde{A}^2 |\nabla u_\lambda|^2 + |\nabla_T |\nabla u_\lambda||^2$$

in $\{x \in \Omega : |\nabla u_\lambda| > 0\}$, where $\tilde{A}^2(x)$ denotes the second fundamental form at $x$ of the $(n - 1)$-dimensional hypersurface $\{y \in \Omega : |u_\lambda(y)| = |u_\lambda(x)|\}$ and $\nabla_T$ is the tangential gradient with respect to this level set. Sternberg and Zumbrun made this choice to obtain

$$Q_{u_\lambda}(|\nabla u_\lambda|\eta) = \int_{\Omega \cap \{|\nabla u_\lambda| > 0\}} |\nabla u_\lambda|^2 |\nabla \eta|^2 - \left(\tilde{A}^2 |\nabla u_\lambda|^2 + |\nabla_T |\nabla u_\lambda||^2\right) \eta^2 \, dx$$
for every Lipschitz function \( \eta \) in \( \overline{\Omega} \) such that \( \eta|_{\partial \Omega} \equiv 0 \), where \( Q_{u_{\lambda}} \) is the quadratic form defined in (1.4). The second step in the proof is to choose an appropriate function \( \eta = \eta(u) \) and use the coarea formula and a Sobolev inequality on the \( (n - 1) \)-dimensional hypersurface \( \{ y \in \Omega : u_{\lambda}(y) = u_{\lambda}(x) \} \).

The first ingredient in the proof of Theorem 1.1 is the following identity, analogue to (1.6), involving the second fundamental form of Graph \( (u_{\lambda}) \).

**Proposition 1.2.** Let \( u \in C^3_0(\Omega) \) be a positive function and \( v(x, x_{n+1}) := u(x) - x_{n+1} \) for all \( (x, x_{n+1}) \in \Omega \times \mathbb{R} \). Let \( \nu = -\frac{\nabla u}{|\nabla u|} \in \mathbb{R}^{n+1} \) be the unit normal vector to Graph \( (u_{\lambda}) \), \( A^2 \) the second fundamental form of Graph \( (u_{\lambda}) \), and \( \nabla T \varphi := \nabla \varphi - (\nu \cdot \nabla \varphi) \nu \) for every \( \varphi \in C^1(\mathbb{R}^{n+1}) \). The following identity holds

\[
(\Delta |\nabla v| + \nu \cdot \nabla \Delta v) |\nabla v| = A^2 |\nabla v|^2 + |\nabla T |\nabla v||^2 \quad \text{in } \Omega. \tag{1.7}
\]

In particular, if \( u \in C^2(\Omega) \) is a solution of (1.1) and \( f \in C^1(\mathbb{R}) \) then

\[
(\Delta |\nabla v| + \lambda f'(u) |\nabla v|) |\nabla v| = \lambda f'(u) + A^2 |\nabla v|^2 + |\nabla T |\nabla v||^2 \quad \text{in } \Omega. \tag{1.8}
\]

**Remark 1.3.** (i) Let \( u \in C^2(\Omega) \) be a solution of (1.1). Note that

\[
\Delta v = \sum_{i=1}^{n+1} v_{ii} = \sum_{i=1}^{n} u_{ii} = \Delta u
\]

and

\[
\nabla \Delta v = (\nabla \Delta u, 0) = (-(\lambda f'(u)) \nabla u, 0) \in \mathbb{R}^{n+1}.
\]

(ii) Farina, Sciunzi, and Valdinoci [9] and Cesaroni, Novaga, and Valdinoci [6] recently used identity (1.6) to obtain one-dimensional symmetry of solutions to some reaction-diffusion equations. In this sense identity (1.8) could be useful by itself.

The main novelty in the proof of Theorem 1.1 is that we use a Sobolev inequality on the \( n \)-dimensional hypersurface

\[
\text{Graph}(u_{\lambda}) = \{ (x, x_{n+1}) \in \Omega \times \mathbb{R} : x_{n+1} = u_{\lambda}(x) \} \subset \mathbb{R}^{n+1},
\]

instead on the level sets \( \{ y \in \Omega : u_{\lambda}(y) = u_{\lambda}(x) \} \) of \( u_{\lambda} \) as in [4, 5], and the geometric identity (1.8). More precisely, define \( v_{\lambda}(x, x_{n+1}) := u_{\lambda}(x) - x_{n+1} \) for every \( \lambda \in (0, \lambda^*) \). Taking \( \varphi = |\nabla v_{\lambda}| \eta \) in the semistability condition (1.4) and using identity (1.8), we obtain

\[
\int_{\Omega} \left( \lambda f'(u_{\lambda}) + A^2 |\nabla v_{\lambda}|^2 + |\nabla T |\nabla v_{\lambda}|^2 \right) \eta^2 \, dx \leq \int_{\Omega} |\nabla \eta|^2 |\nabla v_{\lambda}|^2 \, dx \quad (1.9)
\]
for every Lipschitz function $\eta$ in $\overline{\Omega}$ such that $\eta|_{\partial \Omega} \equiv 0$. Choosing $\eta = \min\{u_\lambda, t\}$ as a test function in (1.9) and using a geometric Sobolev inequality on the $n$-dimensional hypersurface $\{(x, x_{n+1}) \in \text{Graph}(u_\lambda) : x_{n+1} \geq t\}$ (see Theorem 2.1 below) we prove inequality (1.5) in Theorem 1.1. The $W^{1,2}_{n-1,n-2}$ estimate for the extremal solution follows from (1.5) and the convexity of the domain.

The paper is organized as follows. In section 2 we recall a Sobolev inequality on $n$-dimensional hypersurfaces with boundary and we prove the geometric identities established in Proposition 1.2. In section 3 we prove Theorem 1.1.

## 2. Geometric identities and inequalities. Proof of Proposition 1.2

The first ingredient in the proof of Theorem 1.1 is the following Sobolev inequality on $n$-dimensional hypersurfaces (see section 28.5.3 in [3]): Let $M \subset \mathbb{R}^{n+1}$ be a $C^2$ immersed $n$-dimensional compact hypersurface with $n \geq 2$. There exists a constant $C(n)$ depending only on the dimension $n$ such that, for every $\phi \in C^1(M)$ it holds

\[
C(n) \left( \int_M |\phi|^{\frac{2n}{n-1}} \, dV \right)^{\frac{n-1}{n}} \leq \int_M \left( |H\phi| + |\nabla \phi| \right) \, dV + \int_{\partial M} |\phi| \, dS,
\]

where $H$ is the mean curvature of $M$.

Let $p^* := np/(n-p)$ be the critical Sobolev exponent. Replacing $\phi$ by $\phi^{\alpha}$ in (2.1), with $\alpha = 2^*/1^* = 2(n-1)/(n-2)$, and using Hölder and Minkowski inequalities it is standard to obtain the following result.

**Theorem 2.1 ([3])**. Let $M \subset \mathbb{R}^{n+1}$ be a $C^2$ immersed $n$-dimensional compact hypersurface with $n \geq 3$. There exists a constant $C = C(n)$ depending only on the dimension $n$ such that, for every $\phi \in C^1(M)$ it holds

\[
C \left( \int_M |\phi|^{2^*} \, dV \right)^{\frac{2n-1}{n}} \leq \left( \int_M |\phi|^{2^*} \, dV \right) \left( \int_M (|H\phi|^2 + |\nabla \phi|^2) \, dV \right) + \left( \int_{\partial M} |\phi|^{2^*} \, dS \right)^2,
\]

where $H$ is the mean curvature of $M$ and $2^* = 2n/(n-2)$.
The second ingredient is identity \((1.8)\) in Proposition \([1.2]\). Before to prove it let us introduce some notation. Let \(\Omega\) be a smooth bounded domain of \(\mathbb{R}^n\), \(v \in C^2(\Omega \times \mathbb{R})\), and

\[
\nu(x, x_{n+1}) = -\frac{\nabla v}{|\nabla v|}(x, x_{n+1})
\]

the unit normal vector to the level set of \(v\) passing throughout \((x, x_{n+1}) \in \{|\nabla v| \neq 0\}\). Recall that the eigenvalues of \(\nu\) are the \(n\) principal curvatures \(\kappa_1, \ldots, \kappa_n\) of the level sets of \(v\) and zero. In particular, the second fundamental form \(A^2 := \kappa_1^2 + \cdots + \kappa_n^2\) of the level sets of \(v\) is given by \(A^2 = \nu^i_j \nu^j_i\), where as usual Einstein summation convention is used. We denote the gradient along the level sets of \(v\) by \(\nabla_T\), i.e.,

\[
\nabla_T \phi = \nabla \phi - (\nabla \phi \cdot \nu)\nu \quad \text{for any} \quad \phi \in C^1(\mathbb{R}^{n+1}).
\]

Let us prove the identities established in Proposition \([1.2]\).

**Proof of Proposition \([1.2]\).** Let \(u \in C^3_0(\overline{\Omega})\) be a positive function and define \(v(x, x_{n+1}) = u(x) - x_{n+1}\) for all \(x \in \Omega\).

We claim that

\[
-\frac{v_{ij}}{|\nabla v|} = \frac{(\nu^i|\nabla v|)_j}{|\nabla v|} = \nu^i \nabla^j \log|\nabla v| + \nu^j \nabla^i \log|\nabla v| + \nu^i \nabla^j \nu^j_i - \nu^j \nabla^i \nu^i_j
\]

and \(v_{ij} = v_{ji}\) for all \(i, j = 1, \ldots, n + 1\), we obtain

\[
\nu^i_j = \nu^i_j + \nu^j_i \nabla_T \log|\nabla v| - \nu^i \nabla_T \log|\nabla v| \quad \text{for all} \quad i, j = 1, \ldots, n + 1.
\]

We prove the claim multiplying the previous equality by \(\nu^j\) and noting that \(\nu^j_i \nu^i_j = 0\) for every \(j = 1, \ldots, n + 1\) and \(\nabla_T \log|\nabla v| \cdot \nu = 0\).

Now, using \(\nu^j_i \nu^i_j = A^2\) and \(\nabla_T \log|\nabla v| = \nu^j_i \nu^i_j\), we compute

\[
\Delta|\nabla v| = -(v_{ij} \nu^j)i = -\nu \cdot \nabla \Delta v - v_{ij} \nu^j_i
= -\nu \cdot \nabla \Delta v + (|\nabla v| \nu^j) \nu^j_i
= -\nu \cdot \nabla \Delta v + |\nabla v| \nu^j_i \nu^j_i + |\nabla v| \nabla_T \log|\nabla v|
= -\nu \cdot \nabla \Delta v + (A^2 + |\nabla_T \log|\nabla v|)^2) |\nabla v|
\]

to obtain identity \((1.7)\).
If \( u \in C^2(\Omega) \) is a solution of (1.1) and \( f \in C^1(\mathbb{R}) \), then by standard regularity results for uniformly elliptic equations one has \( u \in C^3(\overline{\Omega}) \). From (1.7) and noting that

\[
\nabla \Delta v = (-\lambda f'(u)\nabla u, 0) \quad \text{and} \quad \nu = \frac{1}{|\nabla v|}(-\nabla u, 1),
\]

we obtain

\[
\Delta |\nabla v| = -\lambda f'(u)\frac{|\nabla u|^2}{|\nabla v|} + (A^2 + |\nabla \log |\nabla v||^2)|\nabla v|
\]

proving the proposition.

3. Proof of Theorem 1.1

Let \( u_\lambda \) be the minimal solution of (1.1) for \( \lambda \in (0, \lambda^*) \). Choosing \( \varphi = \sqrt{1 + |\nabla u_\lambda|^2}\eta \) as a test function in the semistability condition (1.4) and using Proposition 1.2, we first obtain (1.9).

**Lemma 3.1.** Assume that \( \Omega \) is a smooth bounded domain of \( \mathbb{R}^n \) and \( f \) a positive and increasing \( C^1 \) function satisfying (1.2). Let \( u_\lambda \) be the minimal solution of (1.1) and \( v_\lambda(x, x_{n+1}) := u_\lambda(x) - x_{n+1} \) for \( \lambda \in (0, \lambda^*) \). The following inequality holds

\[
\int_{\Omega} (\lambda f'(u_\lambda) + A^2|\nabla v_\lambda|^2 + |\nabla T|\nabla v_\lambda|^2) \eta^2 \, dx \leq \int_{\Omega} |\nabla v_\lambda|^2 |\nabla \eta|^2 \, dx \tag{3.1}
\]

for every Lipschitz function \( \eta \) in \( \overline{\Omega} \) with \( \eta|_{\partial \Omega} \equiv 0 \), where \( A^2 \) and \( \nabla T \) are as in Proposition 1.2.

**Proof.** In order to improve the notation, let us denote \( u_\lambda = u \) and \( v_\lambda = v \) for \( \lambda \in (0, \lambda^*) \). Choosing \( \varphi = |\nabla v|\eta \) as a test function in (1.4) and integrating by parts we get

\[
0 \leq Q_u(|\nabla v|\eta)
\]

\[
= \int_{\Omega} |\nabla v|^2 |\nabla \eta|^2 + |\nabla v| |\nabla v| \cdot \nabla \eta^2 + |\nabla v|^2 \eta^2 - \lambda f'(u)|\nabla v|^2 \eta^2 \, dx
\]

\[
= \int_{\Omega} |\nabla v|^2 |\nabla \eta|^2 - (\text{div}(|\nabla v| |\nabla v|)) - |\nabla v|^2 + \lambda f'(u)|\nabla v|^2) \eta^2 \, dx
\]

\[
= \int_{\Omega} |\nabla v|^2 |\nabla \eta|^2 - (|\nabla v| \Delta |\nabla v| + \lambda f'(u)|\nabla v|^2) \eta^2 \, dx.
\]

Inequality (3.1) follows directly from identity (1.8). \( \square \)
Finally, using Lemma 3.1 and the geometric Sobolev inequality established in Theorem 2.1 we prove Theorem 1.1.

Proof of Theorem 1.1. Let $u_{\lambda} \in C^2_0(\Omega)$ be the minimal solution of (1.1)$_\lambda$ for $\lambda \in (0, \lambda^*)$ and $t \in (0, \|u_{\lambda}\|_{L^\infty(\Omega)})$. Define $v_{\lambda}(x, x_{n+1}) = u_{\lambda}(x) - x_{n+1}$. Let $M_t := \{(x, x_{n+1}) \in \text{Graph}(u_{\lambda}) : x_{n+1} \geq t\}$ and $dV = \sqrt{1 + |\nabla u_{\lambda}|^2} \, dx$ its element of volume.

We start by proving inequality (1.5). On the one hand, taking $\eta = \min\{u_{\lambda}, t\}$ as a test function in (3.1), using that $f$ is an increasing function, and $H^2 = (\kappa_1 + \cdots + \kappa_n)^2 \leq nA^2 = n(\kappa_1^2 + \cdots + \kappa_n^2)$, we obtain

$$\int_{M_t} \left( H^2 |\nabla v_{\lambda}| + |\nabla T| |\nabla v_{\lambda}|^2 \right) dV \leq \int_{\{u_{\lambda} \geq t\}} \left( nA^2 |\nabla v_{\lambda}|^2 + \frac{1}{4} |\nabla T| |\nabla v_{\lambda}|^2 \right) dx$$

$$\leq \frac{n}{t^2} \int_{\{u_{\lambda} \leq t\}} |\nabla v_{\lambda}|^2 |\nabla u_{\lambda}|^2 dx \quad (3.2)$$

for all $t \in (0, \|u_{\lambda}\|_{L^\infty(\Omega)})$.

Therefore, applying Theorem 2.1 with $M = M_t$ and $\phi = |\nabla v_{\lambda}|^{1/2}$, we obtain

$$C \left( \int_{M_t} |\nabla v_{\lambda}|^{\frac{n}{n-2}} dV \right)^{\frac{2n-1}{n}} \leq \frac{n}{t^2} \left( \int_{\{u_{\lambda} \leq t\}} |\nabla v_{\lambda}|^2 |\nabla u_{\lambda}|^2 dx \right) \left( \int_{M_t} |\nabla v_{\lambda}|^{\frac{n}{n-2}} dV \right)^{\frac{2n-1}{n}}$$

$$+ \left( \int_{\partial M_t} |\nabla v_{\lambda}|^{\frac{n}{n-2}} dS \right)^2, \quad (3.3)$$

where $C$ is a constant depending only on $n$. This is inequality (1.5).

Assume in addition that $\Omega$ is convex. To prove that the extremal solution $u^*$ belongs to $W^{1,2\frac{n-1}{n-2}}_0(\Omega)$ we only need to bound the integrals on $\{u_{\lambda} \leq t\}$ and $\partial M_t$, for some $t$, by a constant independent of $\lambda$ and then let $\lambda$ tend to $\lambda^*$. The same argument was done in the proof of Theorem 2.7 [5]. However, for convinience to the reader we sketch the proof.

Since $\Omega$ is convex, there exist positive constants $\varepsilon$ and $\gamma$ independent of $\lambda$ such that

$$\|u_{\lambda}\|_{L^\infty(\Omega_\varepsilon)} \leq \frac{1}{\gamma} \|u^*\|_{L^1(\Omega)} \quad \text{for all } \lambda < \lambda^*, \quad (3.4)$$

where $\Omega_\varepsilon := \{x \in \Omega : \text{dist}(x, \partial \Omega) < \varepsilon\}$ (see Proposition 4.3 [5] and references therein). Moreover, if $\lambda^*/2 < \lambda < \lambda^*$, then $u_{\lambda} \geq u_{\lambda^*/2} > c \text{dist}(\cdot, \partial \Omega)$ for
some positive constant $c$ independent of $\lambda \in (\lambda^*/2, \lambda^*)$. Therefore, letting $t := c\varepsilon/2$, we have $\{x \in \Omega : u_\lambda(x) \leq t\} \subset \Omega_{\varepsilon/2} \subset \Omega_{\varepsilon}$.

Note that $u_\lambda$ is a solution of the linear equation $-\Delta u_\lambda = h(x) := \lambda f(u_\lambda(x))$ in $\Omega_{\varepsilon}$ and that, by (3.4), $u_\lambda$ and the right hand side $h$ are bounded in $L^\infty(\Omega_{\varepsilon})$ by a constant independent of $\lambda$. Hence, using interior and boundary estimates for the linear Poisson equation and (3.3), we deduce that

$$\left( \int_{M_t} |\nabla v_\lambda|^{\frac{n}{n-2}} dV \right)^{\frac{2(n-1)}{n}} \leq C_1 \int_{M_t} |\nabla v_\lambda|^{\frac{n}{n-2}} dV + C_2$$

for some constants $C_1$ and $C_2$ independent of $\lambda$.

Finally, noting that $2(n-1)/n > 1$ (since $n \geq 3$) and $|\nabla u_\lambda| \leq |\nabla v_\lambda|$ we obtain

$$\int_{\{u_\lambda \geq t\}} |\nabla u_\lambda|^{\frac{n}{n-2}+1} dx \leq \int_{\{u_\lambda \geq t\}} |\nabla v_\lambda|^{\frac{n}{n-2}+1} dx = \int_{M_t} |\nabla v_\lambda|^{\frac{n}{n-2}} dV \leq C,$$

for some constant $C$ independent of $\lambda$. Letting $\lambda$ tend to $\lambda^*$ in the previous inequality we conclude that $u^* \in W^{1,2}_{0,\varepsilon} = W^{1,2}_{0,\varepsilon}(\Omega)$ proving the theorem. \qed

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References


