Rectifiable sets and the Traveling Salesman Problem

Martí Prats

Universitat Autònoma de Barcelona

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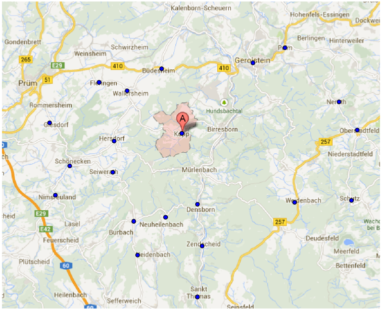
First session

The Traveling Salesman Problem



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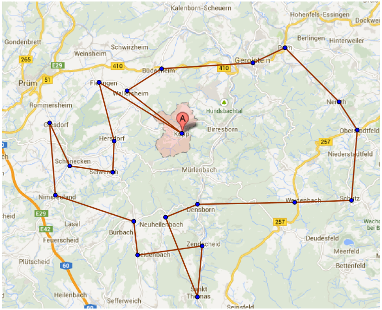
The Traveling Salesman Problem



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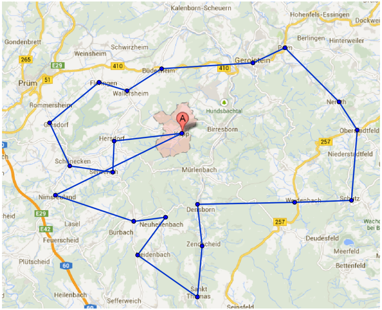

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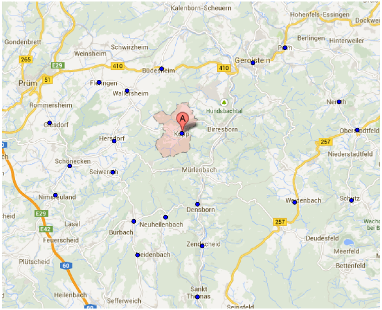

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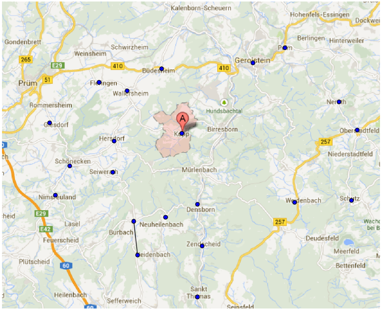

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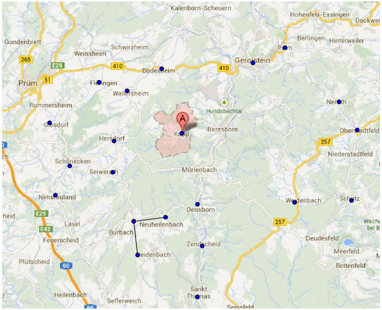

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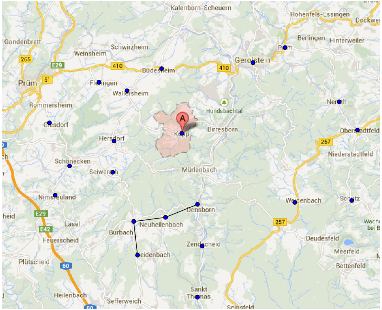

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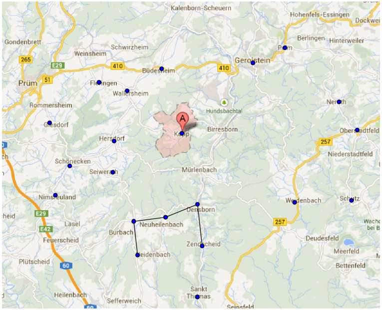

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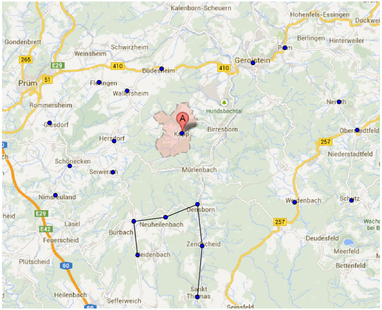

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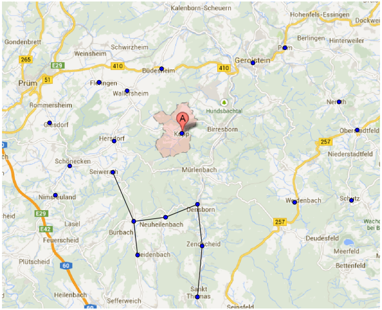

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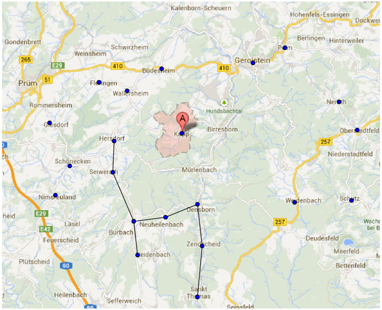

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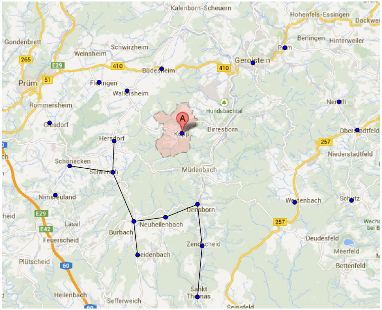

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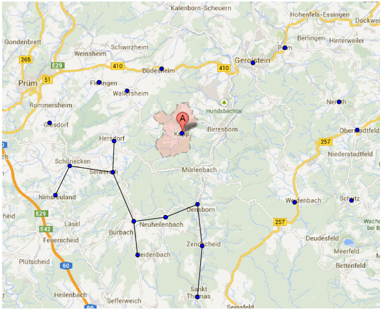

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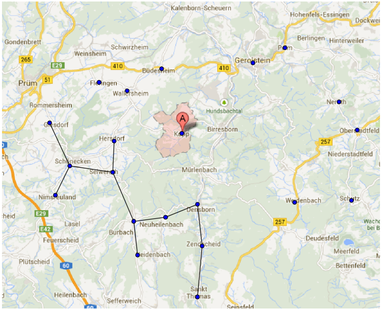

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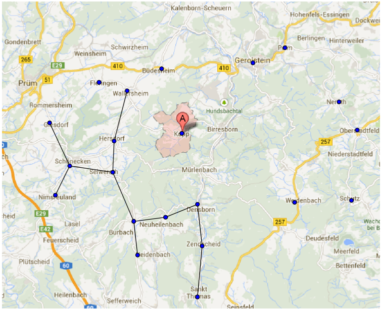

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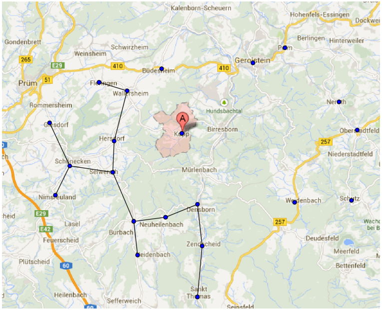

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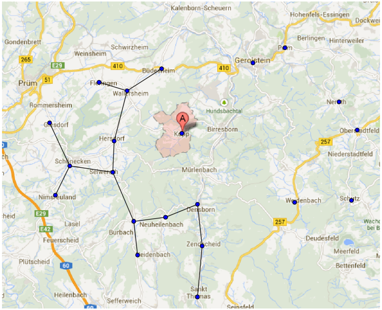

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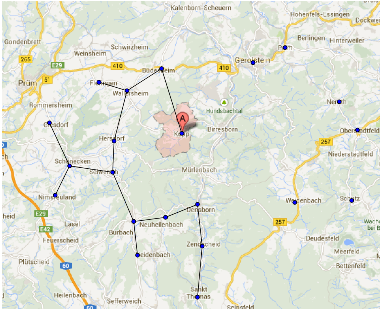

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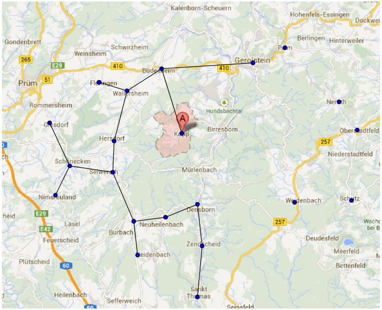

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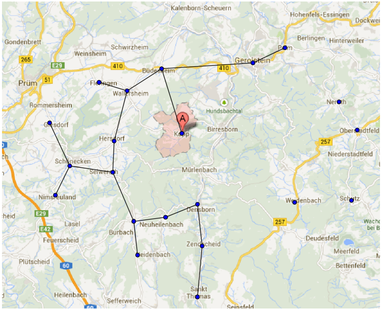

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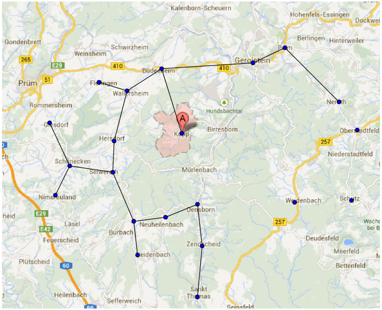

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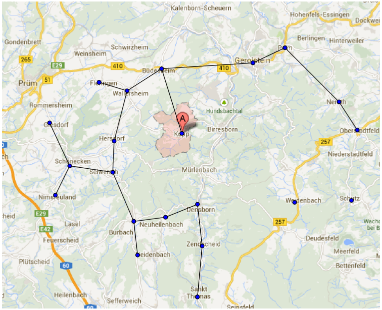

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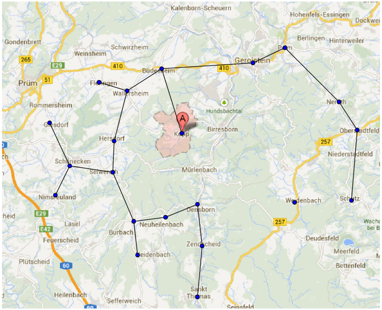

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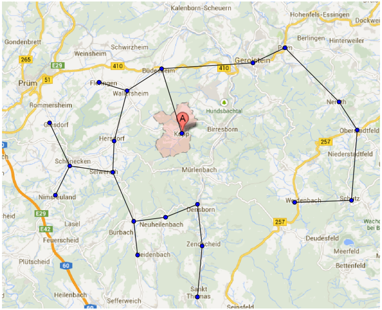
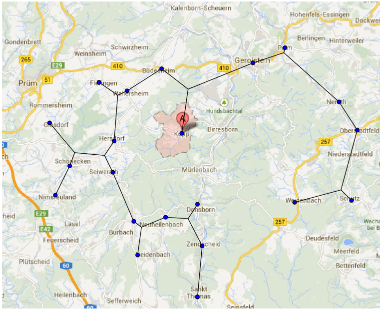


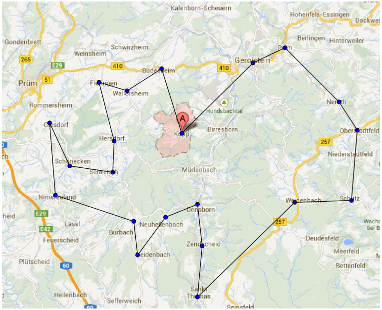
The greedy algorithm gives us the minimal spanning tree: choose a vertex.
Find the closer one to it.
Find the shortest segment with endpoint in one of the previous, and keep doing it until you have united all of them.
The resulting spanning tree is minimal in length. Call it G.

The Traveling Salesman Problem



Suppose K connects all these points with minimal length. You can do a tour with double length...

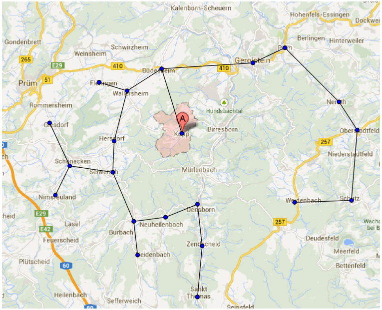
The Traveling Salesman Problem



$2\ell(K) \geq \ell(T)$.

Suppose K connects all these points with minimal length.
You can do a tour with double length...
But you can shorten it by taking straight lines instead of repeating vertices. Call the minimal tour T.

The Traveling Salesman Problem



$2\ell(K) \geq \ell(T) > \ell(G)$.

Suppose K connects all these points with minimal length.
You can do a tour with double length...
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Obviously, T contains a spanning tree and G is minimal among them.

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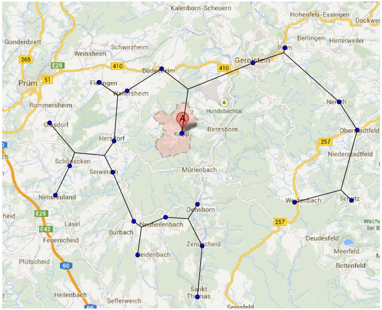

$2\ell(K) \geq \ell(T) > \ell(G) \geq \ell(K)$:

Suppose K connects all these points with minimal length.
You can do a tour with double length...
But you can shorten it by taking straight lines instead of repeating vertices. Call the minimal tour T.

Obviously, T contains a spanning tree and G is minimal among them.
Furthermore, K is shorter than any spanning tree.
The greedy algorithm provides us with the best route up to constant 2.

Non-finite sets

- When it comes to a non-finite set E, the Traveling Salesman Problem consists in finding a minimal rectifiable curve $\Gamma \supset E$. This would give us also a minimal tour up to a constant.

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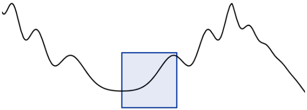
Non-finite sets

- When it comes to a non-finite set E, the Traveling Salesman Problem consists in finding a minimal rectifiable curve $\Gamma \supset E$. This would give us also a minimal tour up to a constant.
- We say that a set is rectifiable when the set is contained in the image of a finite interval by a Lipschitz function.
- One necessary condition for E to be rectifiable is that the Hausdorff one-dimensional (outer) measure of the set, $\mathcal{H}^1(E)$, is finite, but it is not sufficient unless E is connected [Falconer].

The Peter Jones' Betas (1)

Definition

Let Q be a (dyadic) square of side $\ell(Q)$.



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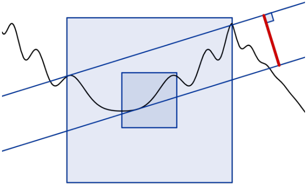
We write $3Q$ for the concentric square with triple side-length,



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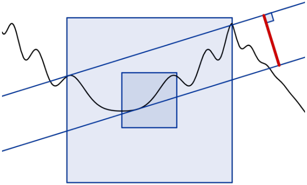
Let Q be a (dyadic) square of side $\ell(Q)$. We write $3Q$ for the concentric square with triple side-length, and call $\beta_E(Q)$ to the width of the narrowest strip containing $E \cap 3Q$ divided by $\ell(3Q)$.



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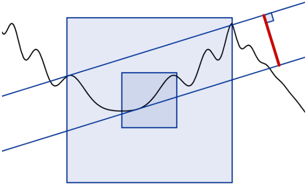
Notice that $\beta_E(Q) \leq 1$. We also have that

$$\beta_E(Q) = \frac{2}{\ell(3Q)} \inf_L \{ \sup_{E \cap 3Q} \mathrm{dist}(z, L) \}.$$



The Peter Jones' Betas (2)

Definition

Given a set E, we associate to it the coefficient

$$\beta^2(E) = \mathrm{diam}(E) + \sum_{\substack{Q \in \Delta \\ \ell(Q) \leq \mathrm{diam}(E)}} \beta_E^2(Q)\ell(Q).$$

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Notice that $\beta_E(Q)$ is adimensional, and we can see that β^2 has a linear behavior up to a constant.

Main Result

Theorem

Suppose $E \subset \mathbb{C}$ is a bounded set. Then E is contained in a rectifiable curve if and only if $\beta^2(E)$ is finite. Moreover, there are constants c_1, c_2 such that

$$c_1\beta^2(E) \leq \inf_{\Gamma \supset E} \mathcal{H}^1(\Gamma) \leq c_2\beta^2(E) \tag{1}$$

where the infimum is taken over all rectifiable curves containing E.

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where the infimum is taken over all rectifiable curves containing E.

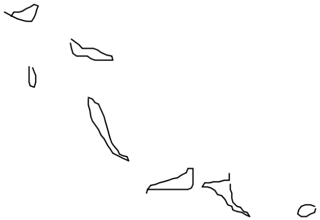
Notice that, even though we do not find the best path for the salesman, we bound the distance the salesman must travel if he designs his route wisely.

Finding a good route

We are about to prove that $\beta^2(E) < \infty$ implies that E is rectifiable, with

$$\inf_{\Gamma \supset E} \mathcal{H}^1(\Gamma) \leq c_2 \beta^2(E).$$

Breaking a rectangle into two



- Consider a given bounded set E_0.

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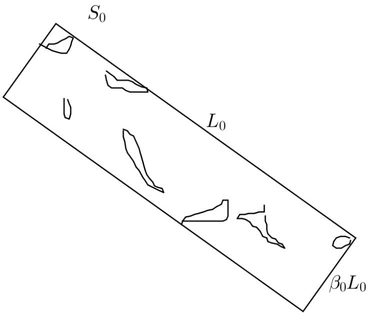
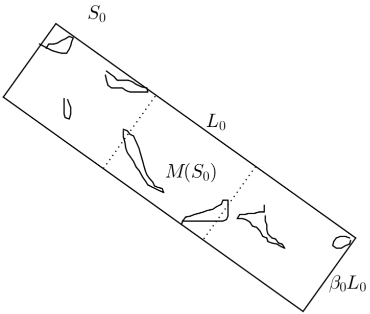


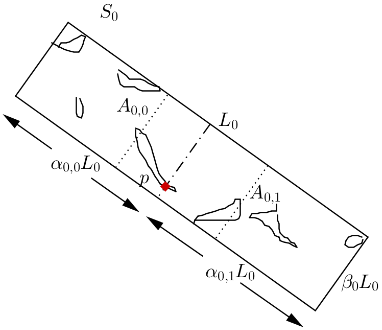
- Consider a given bounded set E_0.
- Cover it by a strip of minimal width and shrink to a rectangle S_0.

Breaking a rectangle into two



- Consider a given bounded set E_0.
- Cover it by a strip of minimal width and shrink to a rectangle S_0.
- Divide in three equal parts.

Breaking a rectangle into two



Case 1: The middle third has a point $p \in E_0$.

- Divide S_0 in two rectangles $A_{0,j}$ by p.
- $\alpha_{0,0} + \alpha_{0,1} = 1.$

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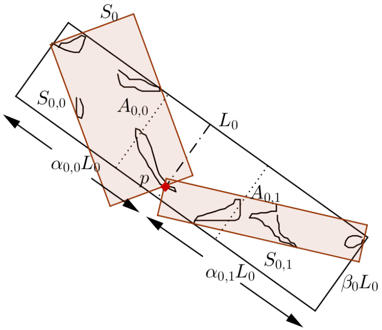

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- Divide S_0 in two rectangles $A_{0,j}$ by p.
- $\alpha_{0,0} + \alpha_{0,1} = 1$.
- Call $E_{0,j} = E_0 \cap A_{0,j}$.
- Cover each part $E_{0,j}$ by a rectangle as before.

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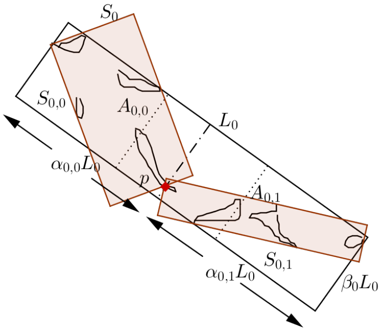

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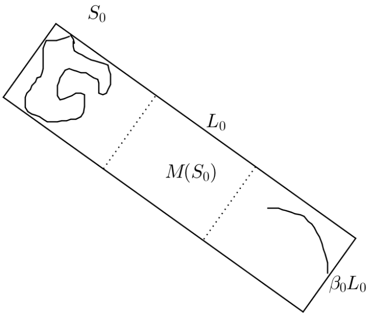
- Cover each part $E_{0,j}$ by a rectangle as before.
- $L_{0,j} \leq \sqrt{\alpha_{0,j}^2 + \beta_0^2} L_0 \leq (1 + 5\beta_0^2)\alpha_{0,j} L_0$.

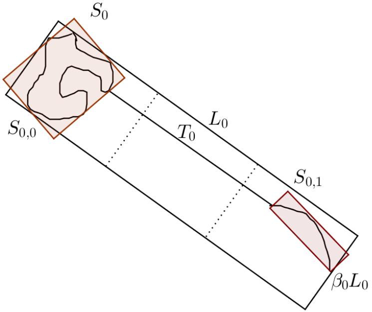
Breaking a rectangle into two



Case 2: The middle third has a no points in E_0.

- Fix $\alpha_{0,0} = \alpha_{0,1} = 1/2$.

Breaking a rectangle into two



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- Call $E_{0,j} = E_0 \cap S_0 \cap S_{0,j}$.

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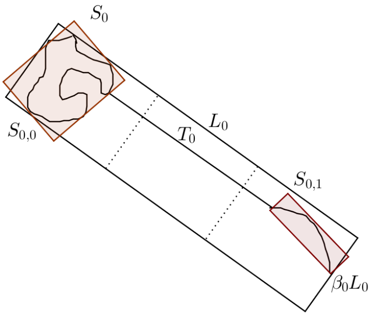

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- Call $E_{0,j} = E_0 \cap S_0 \cap S_{0,j}$.
- Still $L_{0,j} \leq (1+5\beta_0^2)\alpha_{0,j} L_0$.
- The minimal segment joining them $|T_0| \leq (1+\beta_0^2)L_0$.

Iteration

- Cover E by a rectangle S_0 as before. Break the rectangle as shown to get $S_{0,0}$ and $S_{0,1}$. In case 2 you get also a segment T_0.

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- Iterate the process as usual. After n steps, you have 2^n rectangles S_I covering E_I, $I = (0, i_1, \ldots, i_n)$, with long sides L_I, weights $\alpha_I \in [1/3, 2/3]$ and factors β_I.

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- If case 2 is applied to S_I, we get also a segment T_I connecting the sets $E_{I,0}$ and $E_{I,1}$ which are contained into its sons $S_{I,j}$.

Iteration

- Cover E by a rectangle S_0 as before. Break the rectangle as shown to get $S_{0,0}$ and $S_{0,1}$. In case 2 you get also a segment T_0.
- Iterate the process as usual. After n steps, you have 2^n rectangles S_I covering E_I, $I = (0, i_1, \ldots, i_n)$, with long sides L_I, weights $\alpha_I \in [1/3, 2/3]$ and factors β_I.
- If case 2 is applied to S_I, we get also a segment T_I connecting the sets $E_{I,0}$ and $E_{I,1}$ which are contained into its sons $S_{I,j}$.
- After $N = 25$ steps the diameter of a rectangle drops by at least $1/2$.

Bounds for the length (1): the rectangles

Let R_n be the sum of the diameters of the rectangles at stage n.

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Arguing by induction, we will have the uniform bound

$$R_n \leq R_0 + \sum_{|I| \leq n} 5\beta_I^2 L_I \lesssim \mathrm{diam}(E) + \sum \beta_I^2 L_I.$$

Bounds for the length (2): the segments in wide rectangles

If case 2 is applied and $L_{I,0} + L_{I,1} \geq 0.9 L_I$, then $\beta_I \geq \widetilde{\beta}$ for a fixed constant $\widetilde{\beta}$.

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Thus, the sum of the lengths of the middle segments created from applications of case 2 with $L_{I,0} + L_{I,1} \geq 0.9L_I$ is at most $\sum \beta_I^2 L_I$.

Bounds for the length (3): the segments in narrow rectangles

Now write

$$R_n = I_n + II_n$$

where II_n is the sum of the lengths of the rectangles at stage n to which case 2 will be applied and for which $L_{I,0} + L_{I,1} < 0.9L_I$. Let T_{n+1} denote the sum of the lengths of the segments created when $L_{I,0} + L_{I,1} < 0.9L_I$ at stage n.

Bounds for the length (3): the segments in narrow rectangles

Now write

$$R_n = I_n + II_n$$

Then

$$R_{n+1} \leq \left(I_n + C \sum_{|I|=n+1} \beta_I^2 L_I \right) + 0.9 II_n.$$

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Moreover, as $|T_I| \leq (1+\beta_I^2)L_I$, we have

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Summing both inequalities,

$$R_{n+1} + 0.1 T_{n+1} \leq R_n + C \sum_{|I|=n+1} \beta_I^2 L_I.$$

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Equivalently,

$$0.1 T_{n+1} \le R_n - R_{n+1} + C \sum_{|I|=n+1} \beta_I^2 L_I.$$

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As R_n is uniformly bounded by $C\mathrm{diam}(E) + C\sum \beta_I^2 L_I$, also

$$\sum_n T_n \lesssim \mathrm{diam}(E) + \sum \beta_I^2 L_I.$$

Summing cubes

It only remains to bound

$$\sum \beta_I^2 L_I \leq \sum_Q \beta_E^2(Q)\ell(Q).$$

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Given I chose a dyadic cube Q_I such that $d_I := \mathrm{diam}(S_I) \leq \ell(Q_I) < 2d_I$ and with $Q_I \cap S_I \neq \emptyset$. As $3Q_I \supset S_I$, we will have the bound

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It will suffice to prove that

$$\sum_{I:Q_I=Q} \beta_I^2 L_I \leq C\beta_E^2(Q)\ell(Q).$$

Summing cubes

Indeed, write $\mathcal{F}(I)$ for the father index of I, $(0, i_1, ..., i_{n-1})$, and call $J \geq I$ if $I = \mathcal{F}^j(J)$ for some $j \geq 0$.

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$$|J| - |I| \leq 24.$$

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We can classify the independent branches where Q occurs as follows:

$$\{I : Q_I = Q\} = \bigcup_{I : d_{\mathcal{F}(I)} > \ell(Q) \geq d_I} \bigcup_{j=0}^{24} \{J \geq I : |J| = |I| + j \text{ and } Q_J = Q\}$$

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Then, using the previous bound, given a dyadic cube Q we have

$$\sum_{I: Q_I = Q} \beta_I^2 L_I \lesssim \beta_E(Q)\ell(Q) \sum_{j=0}^{24} \sum_{\substack{J: Q_J = Q \\ d_{\mathcal{F}^{j+1}(I)} > \ell(Q) \geq d_{\mathcal{F}^j(I)}}} \beta_J$$

Summing cubes

Now, for $j \leq 24$, the domains E_J appearing in the sum

$$\sum_{\substack{J: Q_J = Q \\ d_{\mathcal{F}^{j+1}(I)} > \ell(Q) \geq d_{\mathcal{F}^{j}(I)}}} \beta_J$$

are contained in disjoint convex polygons $\widetilde{S}_J$ of width $\beta_J L_J$ and diameter comparable to L_J.

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are contained in disjoint convex polygons $\widetilde{S}_J$ of width $\beta_J L_J$ and diameter comparable to L_J. One can see that

$$\beta_J L_J^2 \approx \mathrm{Area}(\widetilde{S}_J).$$

Summing cubes

At the same time, all of them are in a strip of width $\beta_E(Q)\ell(Q)$ and contained in $3Q$. The areas of the polygons are bounded in consequence by the area of this strip intersected with $3Q$. Thus,

$$\sum_{\substack{J:Q_J=Q \\ d_{\mathcal{F}^j(I)}>\ell(Q)\geq d_{\mathcal{F}^{j-1}(I)}}} \beta_J L_J^2 \leq C\beta_E(Q)\ell(Q)^2$$

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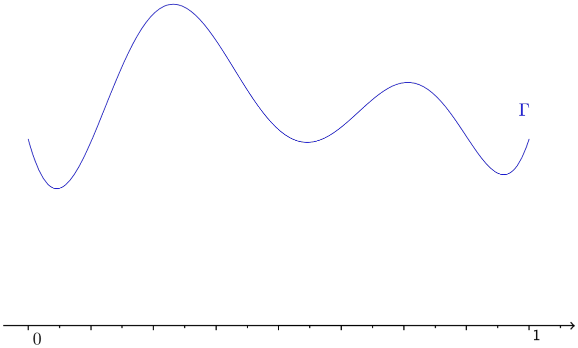
Taking into account that $L_J \approx \ell(Q)$ and summing in j we obtain

$$\sum_{I:Q_I=Q} \beta_I^2 L_I \leq C\beta_E^2(Q)\ell(Q).$$

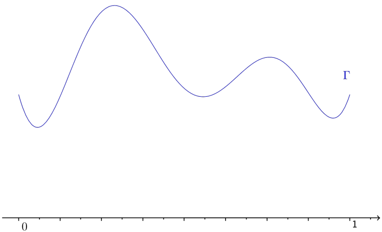
Lipschitz graphs

Lemma

Let Γ be the graph of a Lipschitz function. For $E \subset \Gamma$, $\beta^2(E) \leq C\mathcal{H}^1(\Gamma)$.



Lipschitz graphs



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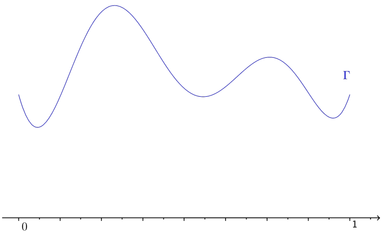
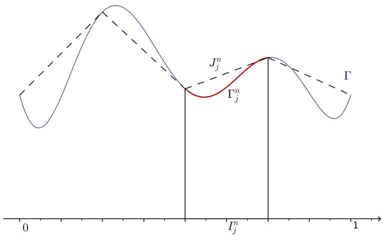


Let $\Gamma = \{0 \le x \le 1, y = f(x)\}$, where f is Lipschitz with constant M. It is enough to show the case $E = \Gamma$ and $f(0) = f(1)$.

Lipschitz graphs



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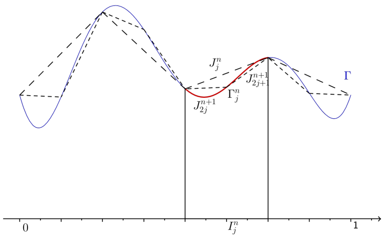
Let I_j^n be the jth dyadic interval of length 2^{-n}, call its image graph Γ_j^n and let J_j^n be the segment uniting the endpoints of Γ_j^n.

Lipschitz graphs



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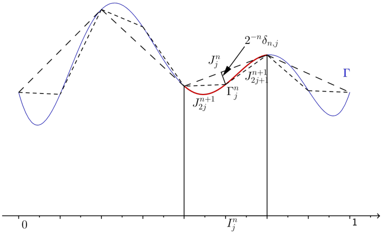
Lipschitz graphs



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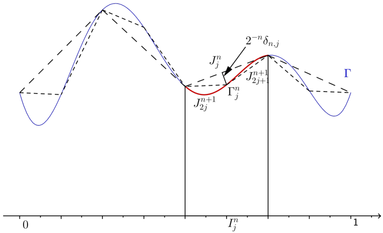
$$2^{-n}\delta_{n,j}^2 \lesssim \ell(J_{2j}^{n+1}) + \ell(J_{2j+1}^{n+1}) - \ell(J_j^n)$$

with constant depending on M.

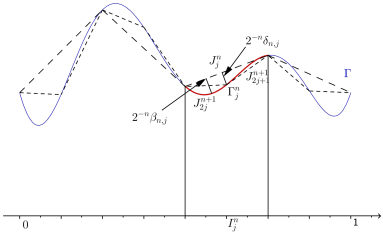
Lipschitz graphs



This implies

$$\sum_{m,k} c2^{-m}\delta_{m,k}^2 \leq 2\ell(\Gamma).$$

Lipschitz graphs



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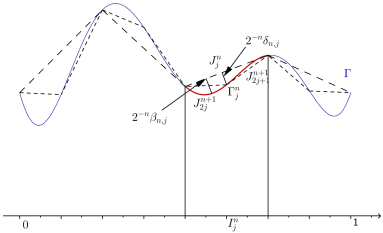
Now, by the triangular inequality,

$$\beta_{n,j} := 2^n \sup\{\mathrm{dist}(z, J_j^n) : z \in \Gamma_j^n\},$$

$$\beta_{n,j} \leq \sum_{m=n}^{\infty} 2^{n-m} \sup\{\delta_{m,k} : I_k^m \subset I_j^n\}.$$

Lipschitz graphs



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Using Hölder inequalities and other standard arguments for series, one gets

$$\sum_{n,j} 2^{-n}\beta_{n,j}^2 \lesssim \sum_{m,k} 2^{-m}\delta_{m,k}^2 \lesssim 2\ell(\Gamma).$$

Hölderizing

Indeed,

$$\sum_{n,j} 2^{-n} \beta_{n,j}^2 \leq \sum_{n,j} 2^{-n} \left(\sum_{m=n}^{\infty} 2^{n-m} \sup_{I_k^m \subset I_j^n} \delta_{m,k} \right)^2$$

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$$\sum_{n,j} 2^{-n}\beta_{n,j}^2 \leq \sum_{n,j} 2^{-n}\left(\sum_{m=n}^{\infty} 2^{n-m} \sup_{I_k^m \subset I_j^n} \delta_{m,k}\right)^2$$

$$\leq \sum_{n,j} 2^{-n} \sum_{m=n}^{\infty} 2^{3\frac{n-m}{2}} \sup_{I_k^m \subset I_j^n} \delta_{m,k}^2 \sum_{m=n}^{\infty} 2^{\frac{n-m}{2}}$$

Hölderizing

Indeed,

$$\begin{aligned}
\sum_{n,j} 2^{-n}\beta_{n,j}^2 &\leq \sum_{n,j} 2^{-n}\left(\sum_{m=n}^{\infty} 2^{n-m} \sup_{I_k^m\subset I_j^n} \delta_{m,k}\right)^2 \\
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&\leq C\sum_{n,j} \sum_{\substack{m\geq n \\ I_k^m\subset I_j^n}} 2^{\frac{n}{2}} 2^{-\frac{3m}{2}} \delta_{m,k}^2
\end{aligned}$$

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$$\leq C\sum_{m,k}\left(\sum_{n=0}^{m}\sum_{I_j^n\supset I_k^m} 2^{\frac{n}{2}}\right)2^{-\frac{3m}{2}}\delta_{m,k}^2$$

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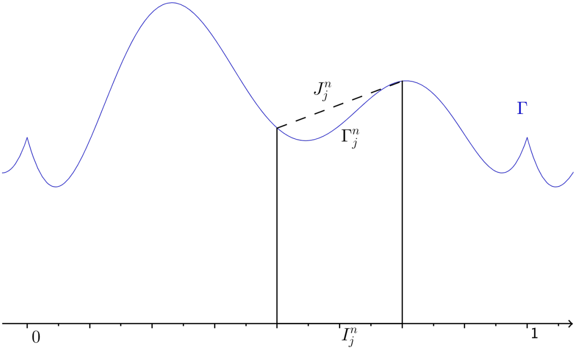
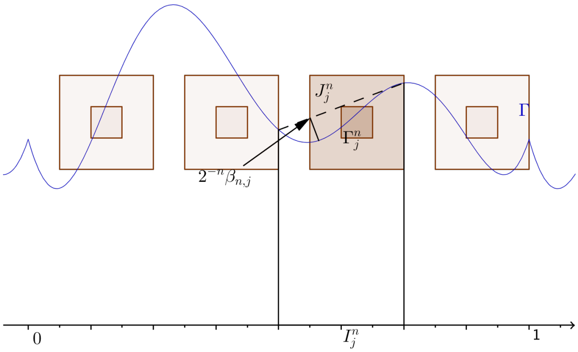
$$\leq C \sum_{m,k} 2^{\frac{m}{2}} 2^{-\frac{3m}{2}} \delta_{m,k}^2 \lesssim \ell(\Gamma)$$

Final step

Finally we extend the function periodically and obtain some translated coefficients $\beta_{n,j}(t)$ related to $\Gamma(t) = (Id \times f)([t, 1+t]) \subset \mathbb{C}$ verifying the last inequality as well.



Final step



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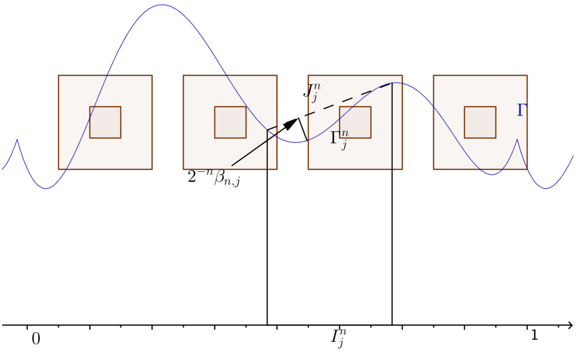

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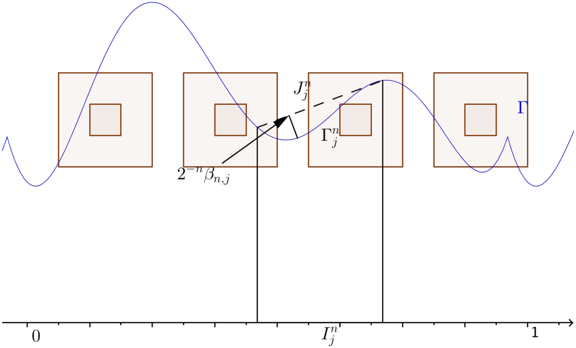

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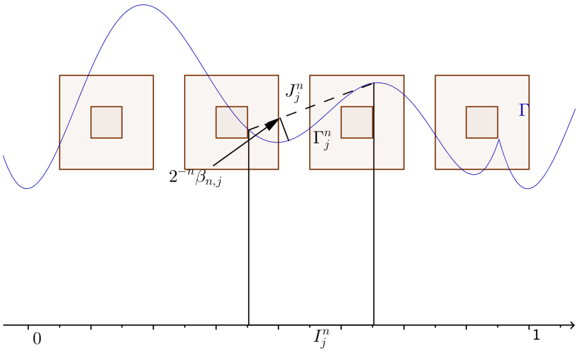

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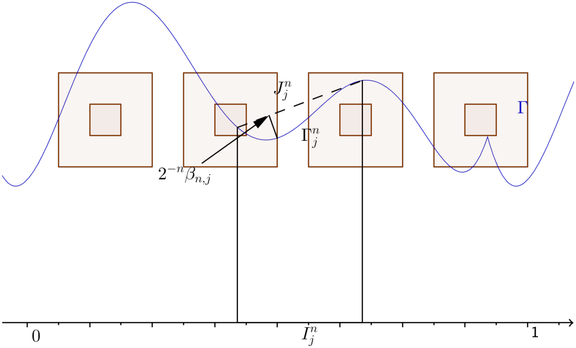

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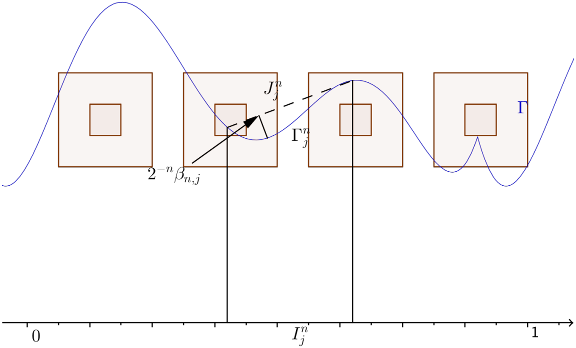

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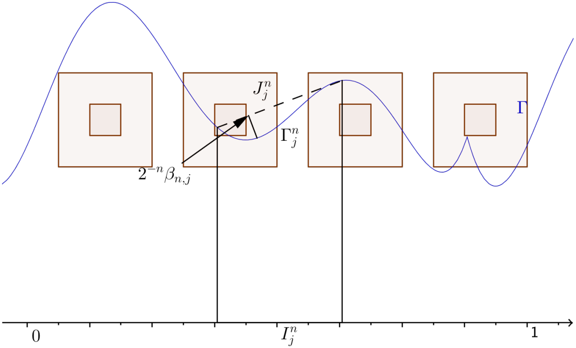

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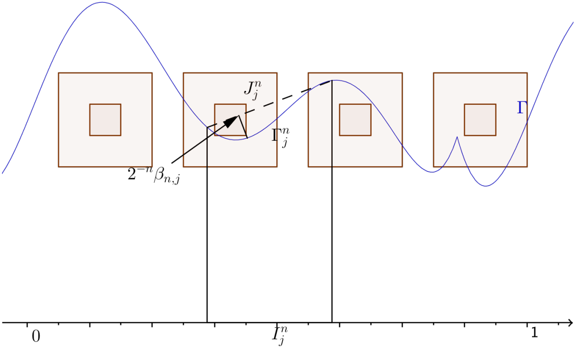

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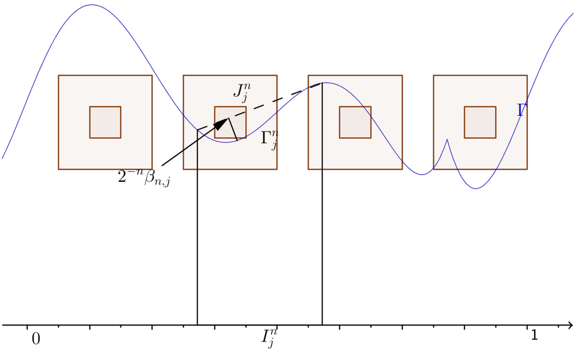

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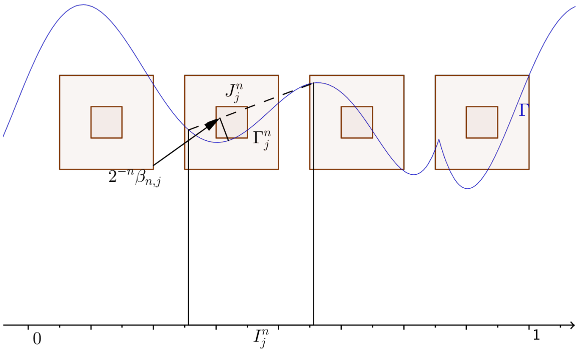

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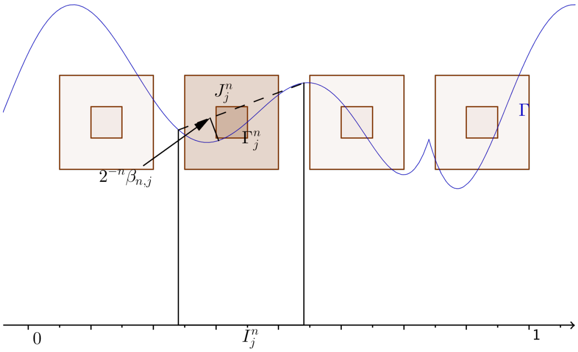

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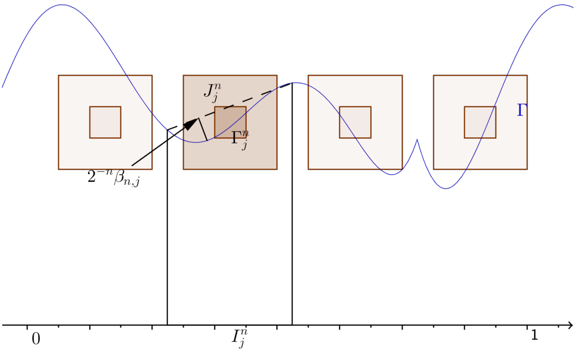

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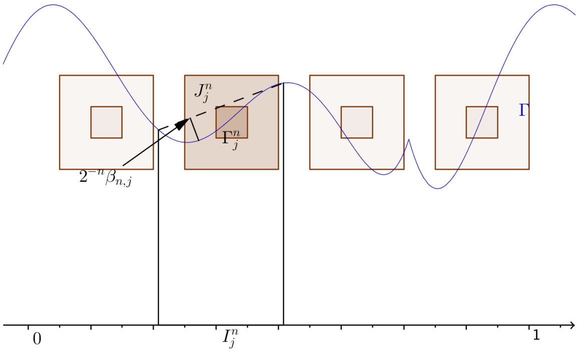

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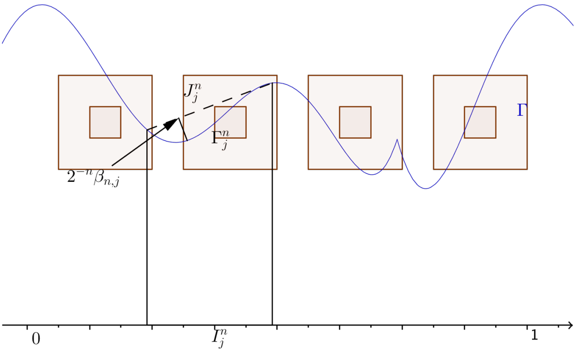

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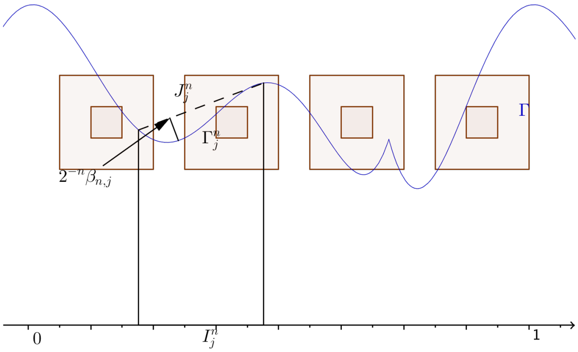

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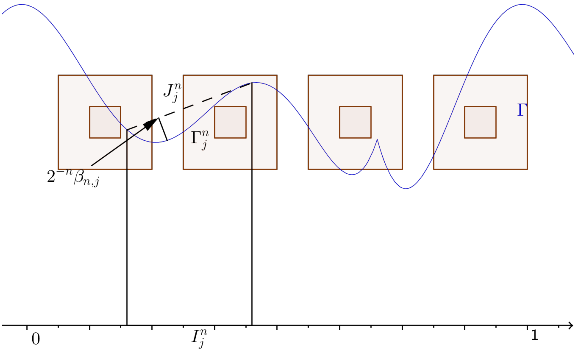

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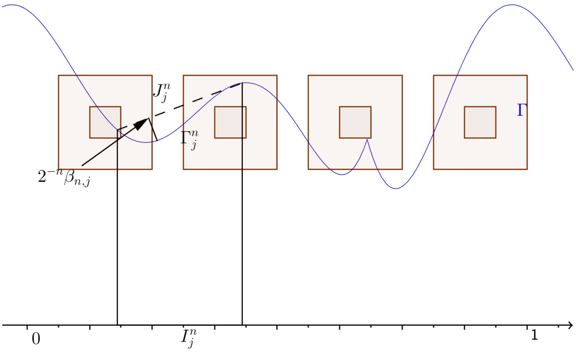

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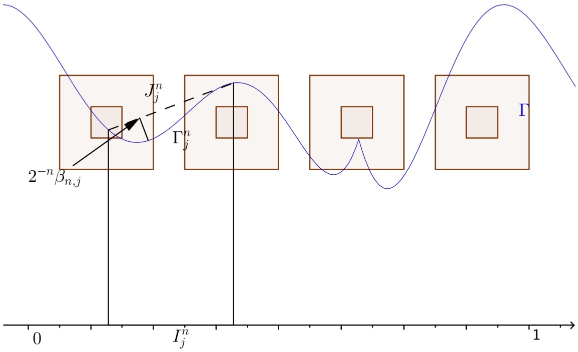

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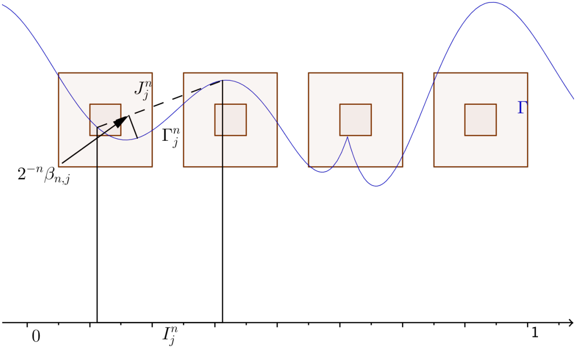

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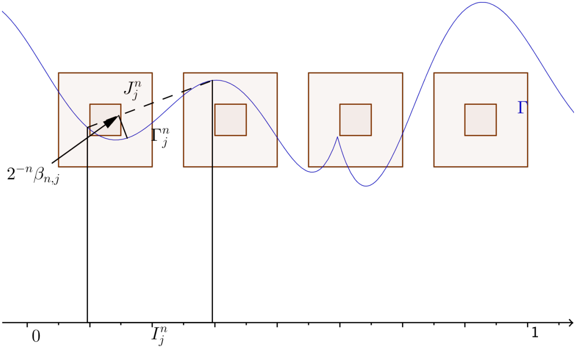

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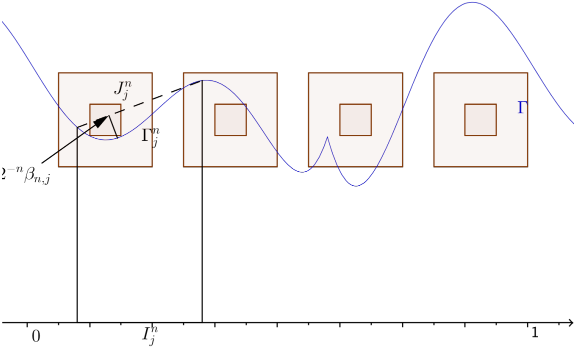

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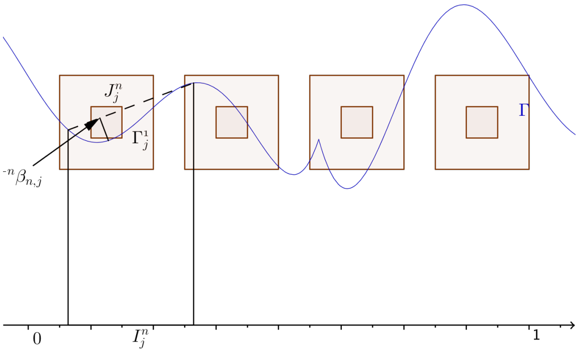

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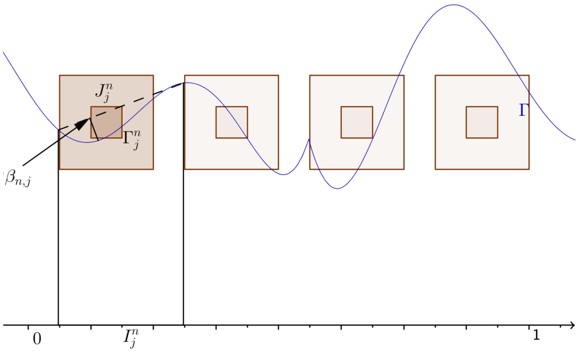

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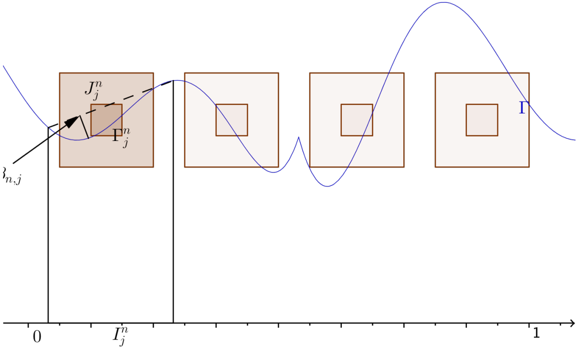

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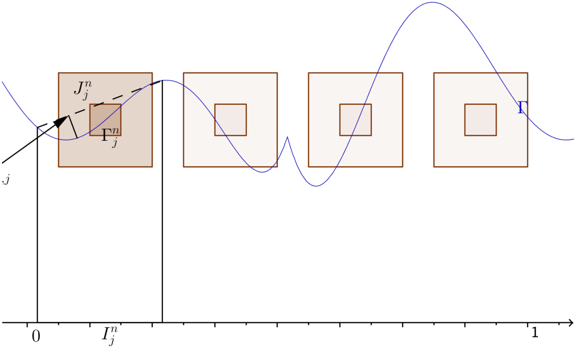

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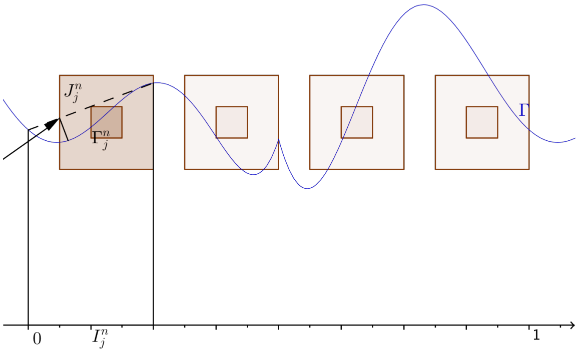

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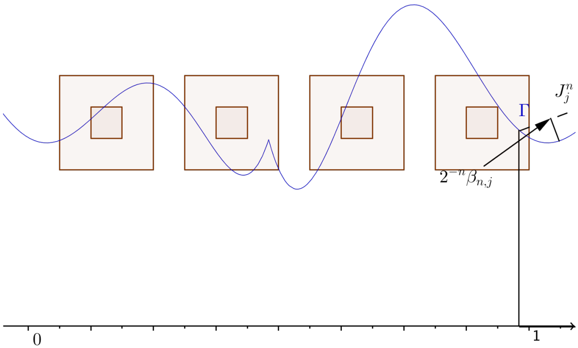

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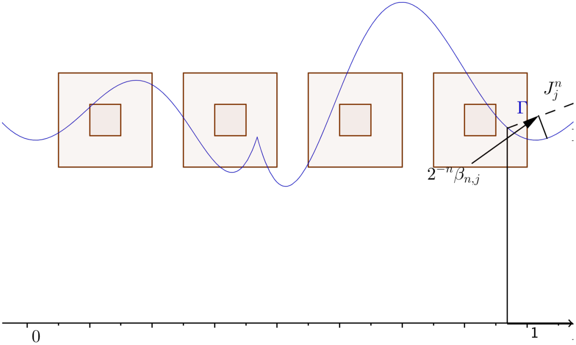

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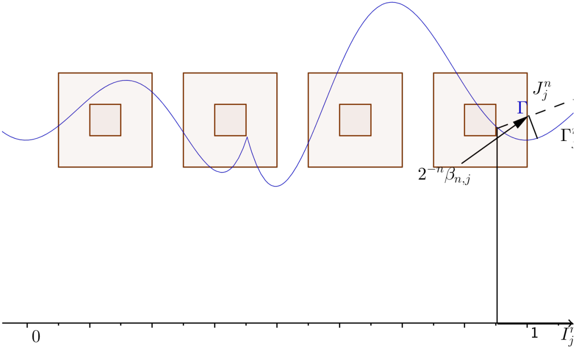

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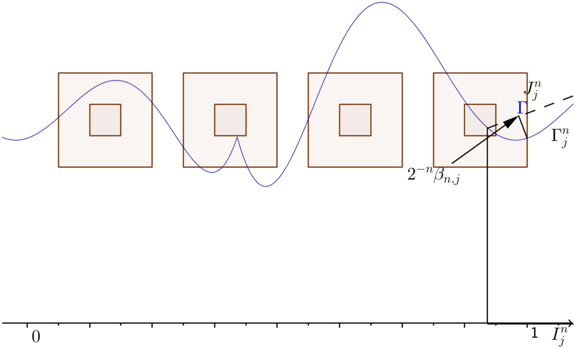

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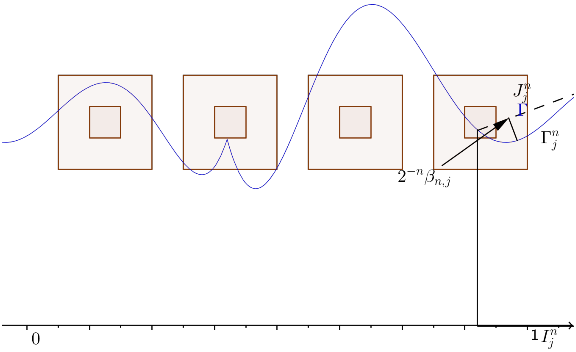

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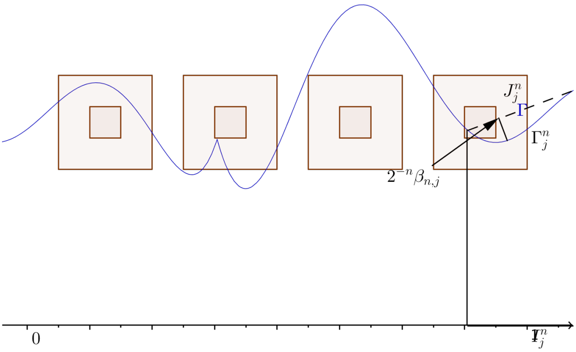

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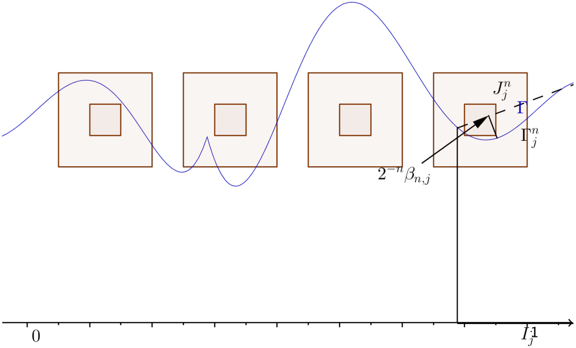

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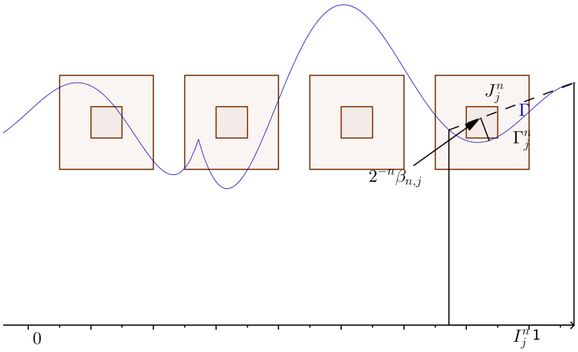

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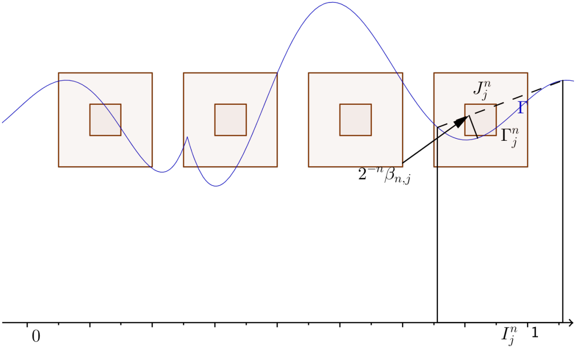

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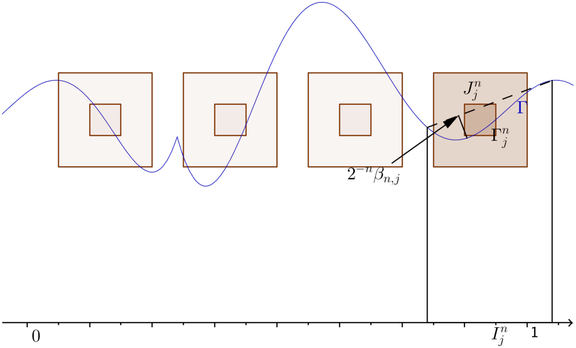

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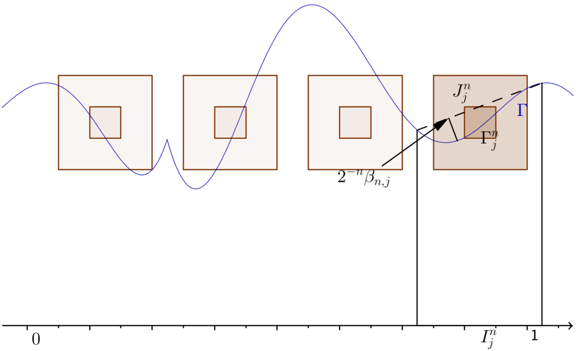

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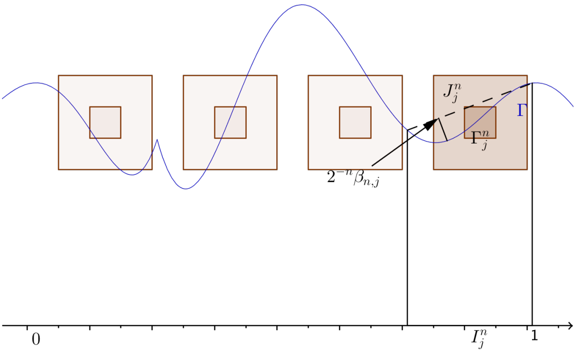

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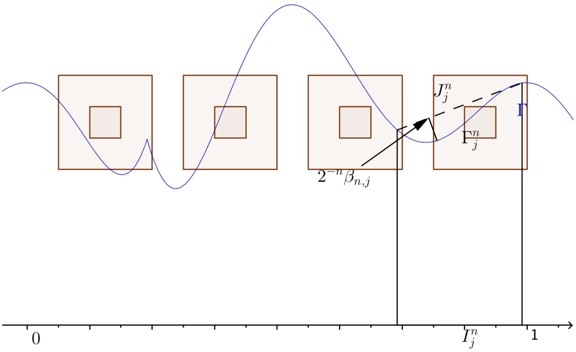

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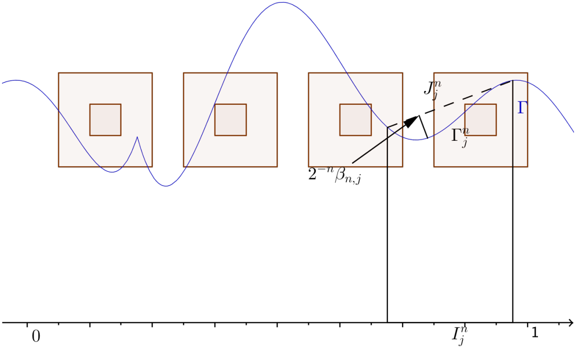

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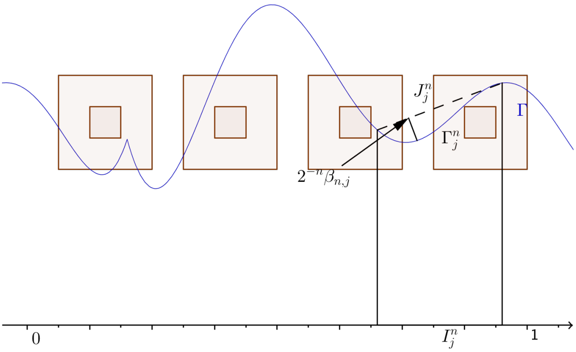

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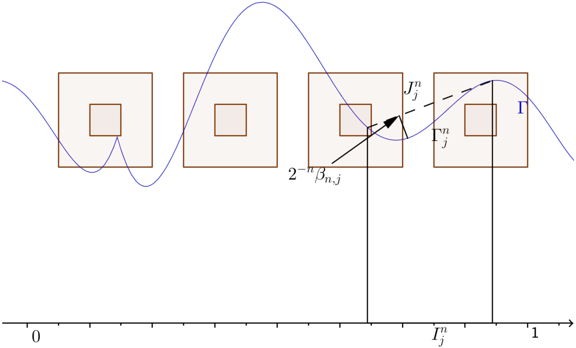

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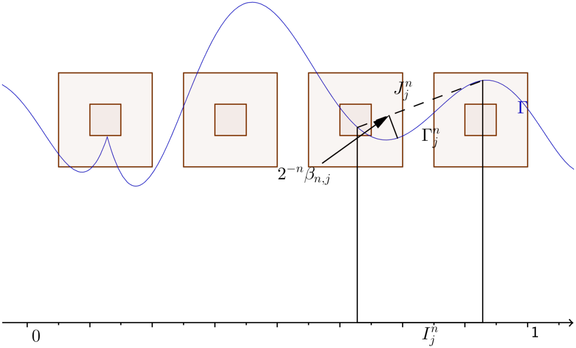

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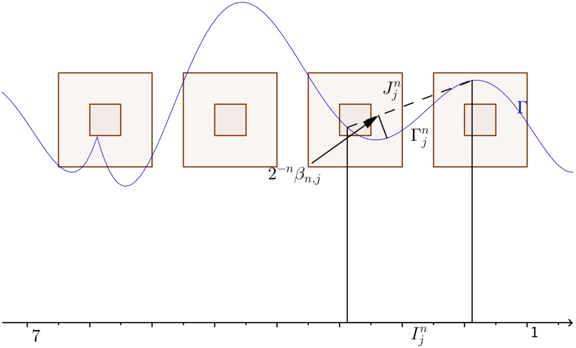

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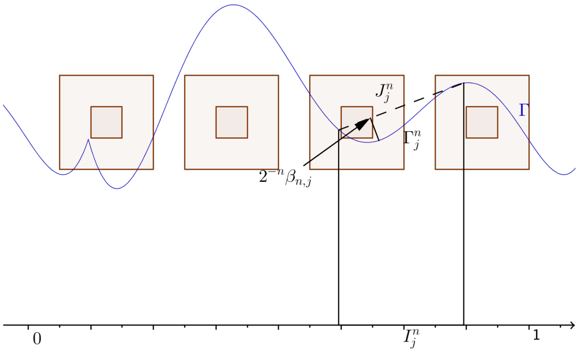

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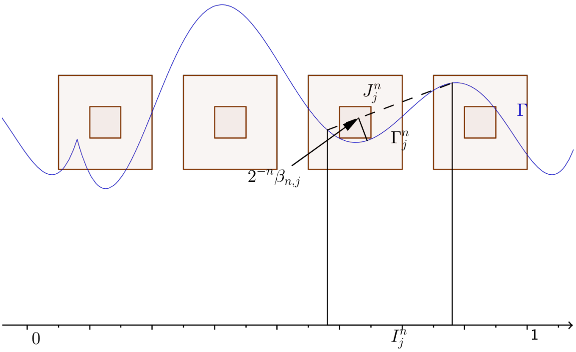

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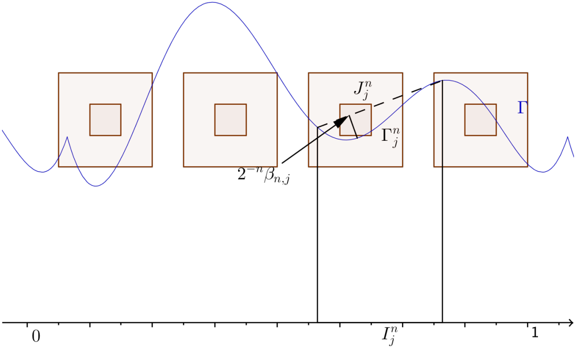

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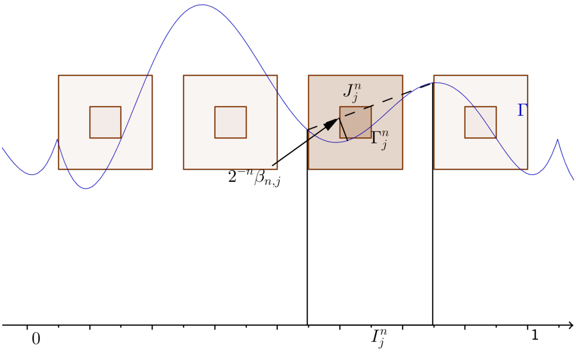

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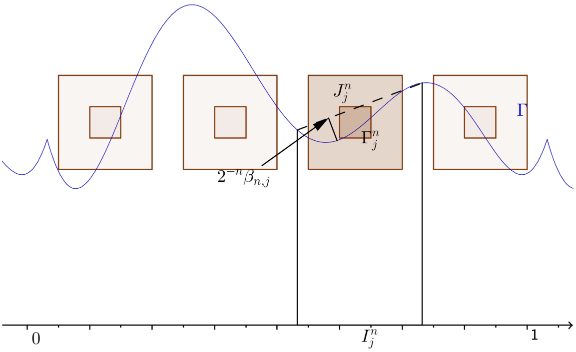

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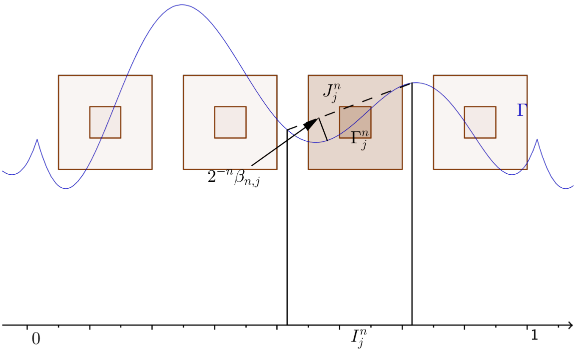

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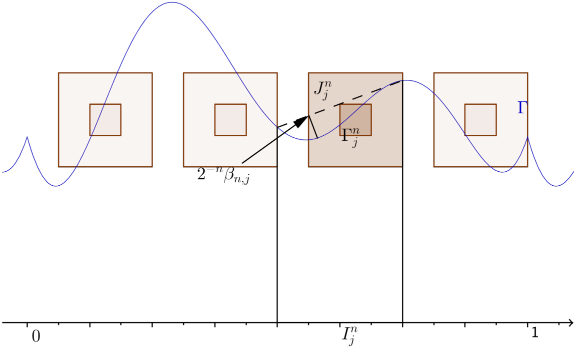
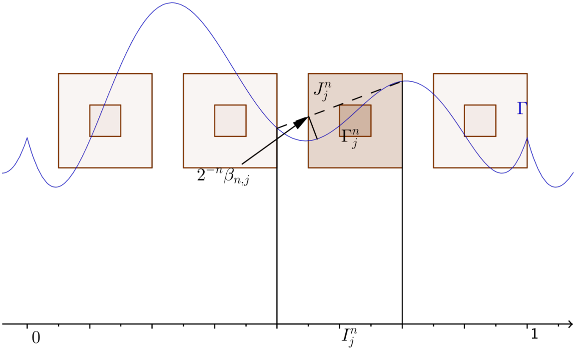


Then, for a cube Q with $\ell(Q) = 2^{-n-2}$, $3Q$ will have projection contained in the translation of an interval $I_{n,j}(t)$ with probability $1/4$ with respect to the Lebesgue measure on t.



So

$$\sum_{\ell(Q)=2^{-n-2}} \beta_\Gamma^2(Q) \lesssim \int_{-1}^{1} \sum_j \beta_{n,j}(t)^2 dt.$$

Summing with respect to n proofs the claim for Lipschitz graphs.

Second session

Back to previous steps: Main Result

Theorem

Suppose $E \subset \mathbb{C}$ is a bounded set. Then E is contained in a rectifiable curve if and only if $\beta^2(E)$ is finite. Moreover, there are constants c_1, c_2 such that

$$c_1\beta^2(E) \leq \inf_{\Gamma \supset E} \mathcal{H}^1(\Gamma) \leq c_2\beta^2(E) \tag{2}$$

where the infimum is taken over all rectifiable curves containing E.

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We have already proven left-hand side for Lipschitz graphs and right-hand side for general sets with finite β.

Purpose of the second talk

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We will do that in three steps. First we present a theorem which will allow us to make the decomposition as long as E is the boundary of a simply connected domain. The second step is a simple corollary allowing us to make such a decomposition on any connected plain set γ. Finally we will prove the relation between betas.

M-Lipschitz domains

Definition

We call an M-Lipschitz domain to a simply connected domain whose boundary can be expressed as $\{r(\theta)e^{i\theta} : 0 \leq \theta < 2\pi\}$ (i.e. it is starlike with respect to the origin), with r a Lipschitz function of coefficient M and $\frac{1}{M+1} \leq r(\theta) \leq 1$ after translation and dilation if necessary.

Decomposition theorem

Theorem

There is a constant M such that whenever Ω is a simply connected domain with $\mathcal{H}^1(\partial\Omega) < \infty$ there exists a rectifiable curve Γ such that

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- *each Ω_j is an M-Lipschitz domain,*
- *and $\sum_j \mathcal{H}^1(\partial\Omega_j) \leq M\mathcal{H}^1(\partial\Omega)$.*

Summary of the proof

Let $\varphi : \mathbb{D} \to \Omega$ be a Riemann mapping. By translating, rotating and rescaling the domain, we can assume WLOG that $\varphi(0) = 0$ and $\varphi'(0) = 1$ (i.e. $\varphi \in S$).

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On the first step we will create uniformly chord-arc domains such that we keep control on the lengths. After that we will decompose these domains into smaller domains to ensure the M-Lipschitz condition is satisfied.

Useful theorems

Theorem (Koebe's estimate, growth and distortion theorem)

Given a conformal mapping $\varphi \in S$ ($\varphi : \mathbb{D} \to \Omega$, $\varphi(0) = 0$, $\varphi(0) = 1$), we have

- $\mathrm{dist}(\varphi(z), \partial\Omega) \approx |\varphi'(z)|(1 - |z|^2)$.
- *Whitney cubes are almost invariant, with constant derivative absolute value on them.*
- $\frac{|z|}{(1+|z|)^2} \leq |\varphi(z)| \leq \frac{|z|}{(1-|z|)^2}$.
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Theorem (F. and M. Riesz Theorem)

Given a Riemann mapping φ to a Jordan domain Ω, it is bounded by a rectifiable curve if and only if $\varphi' \in H^1$, with

$$\mathcal{H}^1(\Gamma) = \|\varphi'\|_{H^1}.$$

Useful theorems

Theorem (Alexander's)

For Γ connected with finite length, call φ_i to a collection of Riemann mappings to each component of $\mathbb{C}^*$. Then

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Theorem (M. Calderon's)

Let Ω *be chord-arc domain,* $\alpha > 1$*,* $0 < p < \infty$*,* $\varphi : \Omega$ *analytic. Then*

$$\|\varphi - \varphi(z_0)\|^p_{H^p(\Omega)} \approx \|A_\alpha\varphi\|^p_{L^p(\partial\Omega)}.$$

Some tools

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A simple computation shows that $4F'(z)^2 = \varphi'(z)g'(z)^2$.

Some help from Hardy spaces

This implies that

$$\iint_{\mathbb{D}} |\varphi'(z)||g'(z)|^2 \log \frac{1}{|z|} dm(z) = 4 \iint_{\mathbb{D}} |F'(z)|^2 \log \frac{1}{|z|} dm(z).$$

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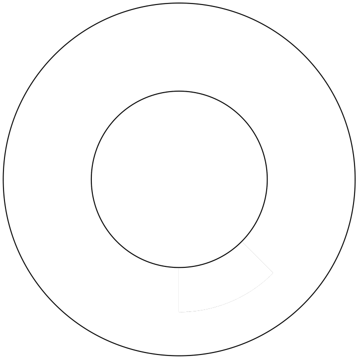
Thanks to a result due to Alexander (which somehow generalizes the F. and M. Riesz Theorem to any simply connected domain) we can see that $\varphi' \in H^1$, and $\|\varphi'\|_{H^1} \leq 2\mathcal{H}^1(\partial\Omega)$. Summing up,

$$\iint_{\mathbb{D}} |\varphi'(z)||g'(z)|^2 \log \frac{1}{|z|} dm(z) \leq 4\mathcal{H}^1(\partial\Omega).$$

The local zone



Set $\mathcal{D}_0 = \{|z| \leq 1/2\}$ and $\mathcal{U}_0 = \varphi(\mathcal{D}_0)$.

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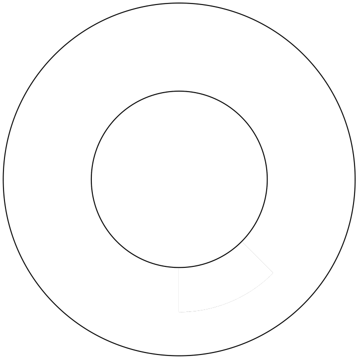
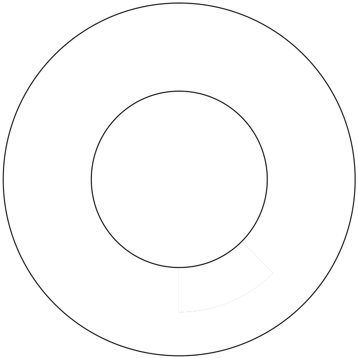


Set $\mathcal{D}_0 = \{|z| \leq 1/2\}$ and $\mathcal{U}_0 = \varphi(\mathcal{D}_0)$.By the growth theorem and the distortion theorem for univalent functions, one can see that $\mathcal{U}_0$ is an M-Lipschitz domain.

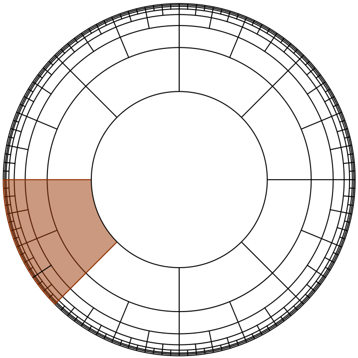
The local zone



Set $\mathcal{D}_0 = \{|z| \leq 1/2\}$ and $\mathcal{U}_0 = \varphi(\mathcal{D}_0)$.By the growth theorem and the distortion theorem for univalent functions, one can see that $\mathcal{U}_0$ is an M-Lipschitz domain.
Since $\varphi' \in H^1$ we also have

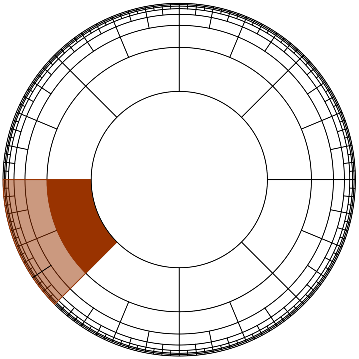
$$\mathcal{H}^1(\partial\mathcal{U}_0) \leq \mathcal{H}^1(\partial\Omega). \qquad (4)$$

Carleson boxes



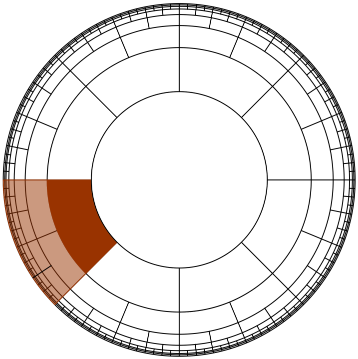
Next form the dyadic Carleson boxes

Carleson boxes



Next form the dyadic Carleson boxes and consider their top halves $T(Q) = \{z \in Q : |z| < 1 - 2^{-(n+1)}\}$. Write z_Q for the center of $T(Q)$.

Carleson boxes



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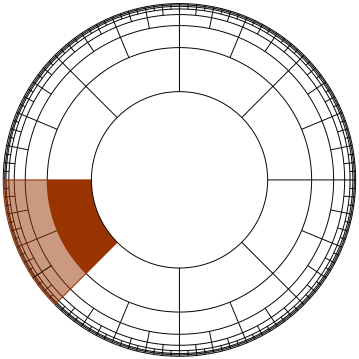

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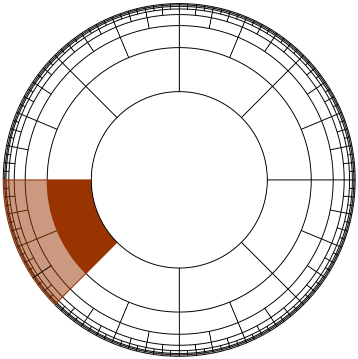

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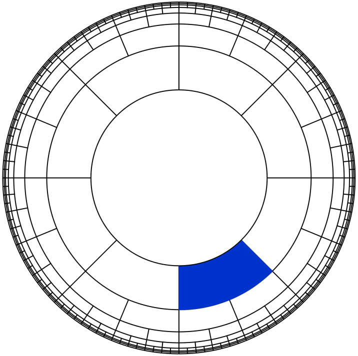
Finally we will make a subdivision of those domains to get starlike domains with uniform Lipschitz constant.

Type 0 cubes



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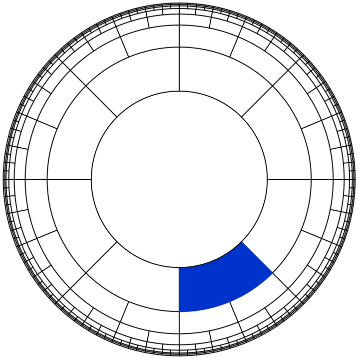

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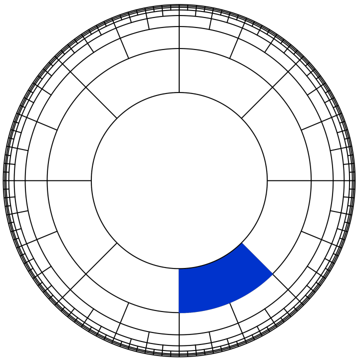
If

$$\sup_{T(Q)} |g(z) - g(z_Q)| \geq \varepsilon,$$

we say that Q is a type 0 cube and define $\mathcal{D}(Q) = T(Q)$.

Type 0 cubes



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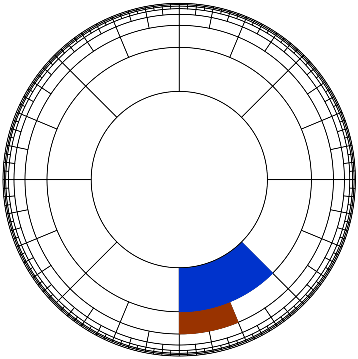
If

$$\sup_{T(Q)} |g(z) - g(z_Q)| \geq \varepsilon,$$

we say that Q is a type 0 cube and define $\mathcal{D}(Q) = T(Q)$. In that case, using the Bloch norm, we can find that $\mathcal{U}_Q = \varphi(\mathcal{D}_Q)$ is a chord-arc domain with fixed constant.

Stopping time argument: almost constant derivative



If Q is not of type 0, define $G(Q)$ to be the set of maximal boxes $Q' \subset Q$ for which

$$\sup_{T(Q')} |g(z) - g(z_Q)| \geq \varepsilon$$

and define
$\mathcal{D}(Q) = \left(Q \setminus \bigcup_{G(Q)} Q' \right)$.

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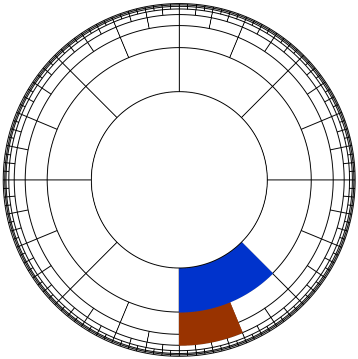

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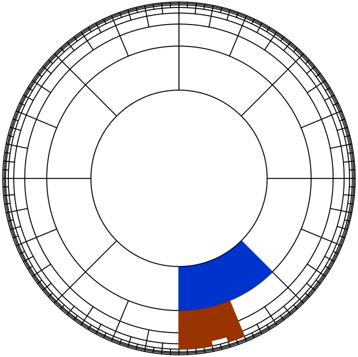

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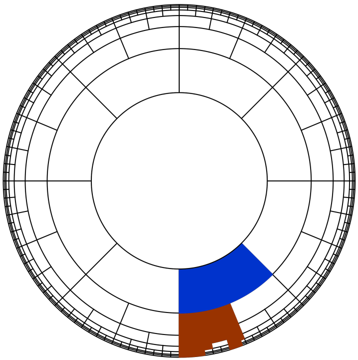

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Then, $\mathcal{D}(Q)$ is a chord-arc domain with constant 4 and

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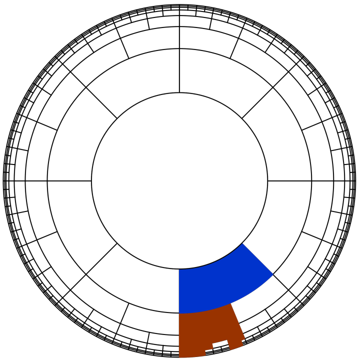

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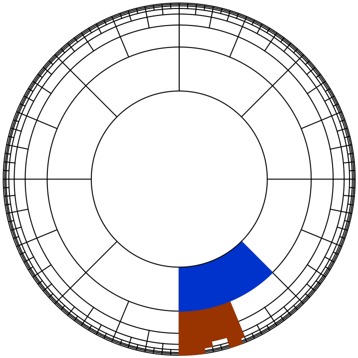
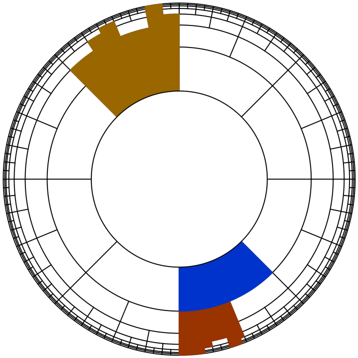
For ε small enough, $\mathcal{U}_Q = \varphi(\mathcal{D}_Q)$ will be chord-arc domains with constant 5.

Type 1 and type 2



If the domain attains the border of $\mathbb{D}$ in more than the half of the measure of $Q \cap \mathbb{D}$, then we say Q is of type 1.

Type 1 and type 2



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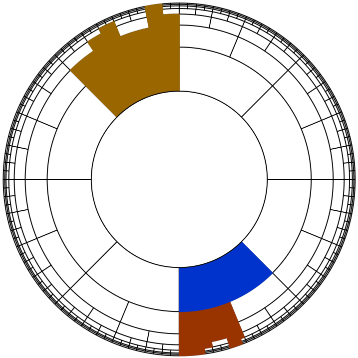

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Otherwise, we say that Q is of type 2.
Keep finding $\mathcal{D}(Q)$ for the successive remaining maximal cubes in $Q \setminus \mathcal{D}(Q)$. Then, the family $\{\mathcal{D}_j\}_{j \geq 0}$ is pairwise disjoint.

Lengths in domains of type 0

If Q is of type 0, then using the Bloch norm of g and $\sup_{T(Q')} |g(z) - g(z_Q)| \geq \varepsilon$, we see that there is a significative part of $T(Q)$ with $|g'| > C\varepsilon\ell(Q)$, so $\ell(Q)^2 \lesssim \int |g'|^2$.

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Finally, using the previous estimates on the last integral over $\mathbb{D}$,

$$\sum_{\text{type } 0} \mathcal{H}^1(\partial\mathcal{U}_j) \leq C\mathcal{H}^1(\partial\Omega).$$

Starlike domains come out

Dividing the region into a fixed number of polar rectangles, we can apply yet the previous reasoning. Furthermore, using again the Bloch estimate for g we find that the derivative is almost constant so that the image of the regions are M-Lipschitz domains.

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For Q of type 1, using F. and M. Riesz Theorem for Jordan domains we know that

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Finally, as this arcs have zero superposition in $\mathcal{H}^1$, we have using Alexander's result that

$$\sum_{\text{type 1}} \mathcal{H}^1(\partial\mathcal{U}_j) \leq C \iint_{\partial\mathbb{D}} |\varphi'(z)| \leq C\mathcal{H}^1(\partial\Omega).$$

Lengths in domains of type 2

When it comes to type 2 cubes, the reasoning is more involved. We sketch the proof.
Call $\{J_k\}$ to the top edges of the boxes in $G(Q)$. Then

$$\mathcal{H}^1(J_k) \geq \frac{\mathcal{H}^1(\partial \mathcal{D}(Q))}{12}.$$

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This allows us to prove that

$$\mathcal{H}^1(\partial\mathcal{U}_j) \lesssim \int_{\partial\mathcal{D}(Q)} |F(z) - F(z_Q)|^2.$$

Lengths in domains of type 2

By M. Calderon's Theorem,

$$\int_{\partial\mathcal{D}(Q)} |F(z) - F(z_Q)|^2 \leq C \iint_{\mathcal{D}(Q)} |F'(z)|^2 \mathcal{H}^1(B(z, 2\mathrm{dist}(z, \partial\mathcal{D}(Q))))$$

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and using that chord-arc domains are bounded by Ahlfors-regular curves,

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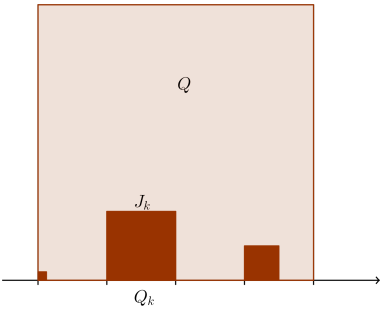
Therefore,

$$\sum_{\text{type } 2} \mathcal{H}^1(\partial\mathcal{U}_j) \leq C\mathcal{H}^1(\partial\Omega).$$

Starlike domains in type 1 or 2

It only remains to subdivide the domains $\mathcal{D}(Q)$ related to cubes of type 1 and type 2 into domains $\mathcal{D}_{Q,k}$ such that $\mathcal{U}_{Q,k} = \varphi(\mathcal{D}_{Q,k})$ are M-Lipschitz domains with lendth still bounded.

Starlike domains in type 1 or 2: visual explanation



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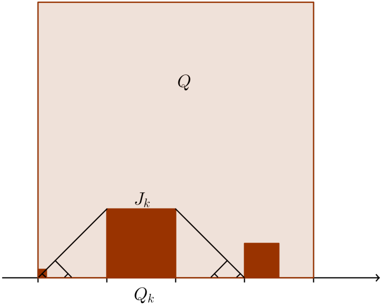

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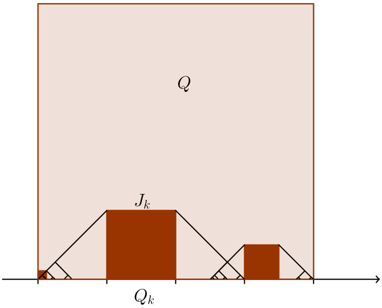

$$\mathcal{H}^1(T(Q_k)) = C\ell(Q_k)$$

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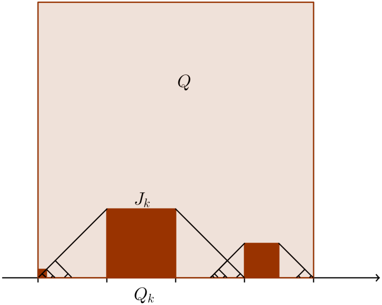

$$\mathcal{H}^1(T(Q_k)) = C\ell(Q_k)$$

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$$\sum \mathcal{H}^1(\partial\varphi(\mathcal{D}_{j,k})) \leq C|\varphi'(z_Q)|\ell(Q) \leq C\mathcal{H}^1(\mathcal{U}_j)$$

Given any connected set

Corollari

There exists a constant $M < \infty$ such that if Γ is a connected plane set with $\mathcal{H}^1(\Gamma) < \infty$, then there exists a connected plane set $\widetilde{\Gamma} \supset \Gamma$ such that $\mathcal{H}^1(\widetilde{\Gamma}) \leq M\mathcal{H}^1(\Gamma)$, the bounded components $\mathcal{D}_j$ of $\mathbb{C} \setminus \widetilde{\Gamma}$ are M-Lipschitz domains with $\Gamma \subset \bigcup \partial \mathcal{D}_j$, and the boundary of the unbounded component $\mathcal{D}_0$ of $\mathbb{C} \setminus \widetilde{\Gamma}$ is a circle at least $3\sqrt{2}\mathcal{H}^1(\Gamma)$ units from Γ.

The shortest proof

Proof.

Apply the previous result to each bounded component of the original set united to a circle big enough by a segment. □

Small domains, big domains

Now, let Γ be connected with $\mathcal{H}^1(\Gamma) < \infty$, let $\{\mathcal{D}_j\}$ be the Lipschitz domains given by the previous corollary and write $\Gamma_j = \partial\mathcal{D}_j$ and $\delta_j = \mathrm{diam}(\mathcal{D}_j)$.

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$$\mathcal{F}(Q) = \{\Gamma_j : \Gamma_j \cap 3Q \neq \emptyset, \delta_j \geq \ell(Q)\}$$

and

$$\mathcal{G}(Q) = \{\Gamma_j : \Gamma_j \cap 3Q \neq \emptyset, \delta_j < \ell(Q)\}.$$

The relation between betas

Lemma

There is a constant C *such that if* $\ell(Q) \leq \mathrm{diam}\Gamma$ *and* $\ell(Q) = \frac{1}{4}\ell(Q')$, *with* $Q \subset Q'$, *then*

$$\beta_\Gamma^2(Q) \leq C \sum_{\mathcal{F}(Q)} \beta_{\Gamma_j}^2(Q') + C_1 \frac{1}{\ell(Q)^2} \sum_{\mathcal{G}(Q)} \mathrm{Area}(\mathcal{D}_j).$$

Trivialities

WLOG, WMA that $\ell(Q) = 1$ and $\beta_\Gamma(Q) > 0$, so that $3Q \cap \Gamma_j \neq \emptyset$ for some Γ_j and $3Q' \subset \bigcup \mathcal{D}_j$.

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If $\mathcal{F}(Q) = \emptyset$ then $\sum_{\mathcal{G}(Q)} \mathrm{Area}\mathcal{D}_j \geq 9\ell(Q)^2$. Thus, we can assume there exists $\Gamma_1 \in \mathcal{F}(Q)$.

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We distinguish three cases.

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$$B \cap 3Q \subset \bigcup_{\mathcal{G}(Q)} \overline{\mathcal{D}_j}$$

and $\operatorname{Area}(B \cap 3Q) \approx d_0^2$.

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$\mathcal{F}(Q) = \{\Gamma_1\}$. Let L be a line such that

$$d = \sup_{\Gamma_1 \cap 3Q} \operatorname{dist}(z, L) \leq \beta_{\Gamma_1}(Q)\ell(3Q).$$

Let $z_0 \in \Gamma \cap 3Q$ have maximal distance $d_0 = \operatorname{dist}(z_0, \Gamma_1)$ and let $z_1 \in \Gamma_1$ have minimal distance to z_0. Call $z_2 = \frac{z_0+z_1}{2}$. Then, if $B = B\left(z_2, \frac{d_0}{2}\right)$,

$$B \cap 3Q \subset \bigcup_{\mathcal{G}(Q)} \overline{\mathcal{D}_j}$$

and $\operatorname{Area}(B \cap 3Q) \approx d_0^2$.
Hence

$$\beta_\Gamma^2(Q) \leq (d + d_0)^2 \leq 2d^2 + 2d_0^2 \lesssim \beta_{\Gamma_1}^2(Q') + \sum_{\mathcal{G}(Q)} \operatorname{Area}\mathcal{D}_j.$$

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Also because $\mathcal{D}_2$ is an M-Lipschitz domain, there exists $z_3 \in 4Q \setminus (\mathcal{D}_1 \cup \mathcal{D}_2)$ such that $\mathrm{dist}(z_3, \Gamma_j) \ge C d_1$.

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Consequently,

$$\beta^2_\Gamma(Q) \lesssim \beta^2_{\Gamma_1}(Q') + \beta_{\Gamma_2}(Q')^2 + \mathrm{d}(z_3, \Gamma_j)^2 \lesssim \sum_{\mathcal{F}(Q)} \beta^2_{\Gamma_j}(Q') + \sum_{\mathcal{G}(Q)} \mathrm{Area}(\mathcal{D}_j)$$

Case 3

$\mathcal{F}(Q)$ contains at least three distinct Γ_j.

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Then, because each $\mathcal{D}_j$ is an M-Lipschitz domain, there exist at least one $\Gamma_j \in \mathcal{F}(Q)$ such that $\beta_{\Gamma_j}(3Q') \geq C_1$, as three strips intersecting $3Q$ will always intersect one another in $3Q'$.

Proof of the theorem

To finish the proof of the main theorem, let Γ be a rectifiable curve and let $\{Gamma_j\}$ be as in the corollary. Using the lemma on Lipschitz graphs we can see that

$$\sum_{Q} \beta_{\Gamma_j}^2(Q)\ell(Q) \leq C\ell(\Gamma_j).$$

Small cubes' areas

If $\delta_j < 2^{-n}$ there are at most 25 dyadic cubes Q such that $\ell(Q) = 2^{-n}$ and $\mathcal{D}_j \in \mathcal{G}(Q)$.

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\sum_Q \frac{1}{\ell(Q)} \sum_{\mathcal{G}(Q)} \mathrm{Area}\mathcal{D}_j &= \sum_j \mathrm{Area}\mathcal{D}_j \sum_{Q:\mathcal{D}_j \in \mathcal{G}(Q)} \frac{1}{\ell(Q)} \\
&\leq 25 \sum_j \mathrm{Area}\mathcal{D}_j \sum_{m=0}^{\infty} 2^{-m}\delta^{-1} \\
&\leq 50 \sum_j \frac{\mathrm{Area}\mathcal{D}_j}{\delta_j} \\
&\leq C \sum_j \ell(\Gamma_j) \leq C\ell(\Gamma)
\end{aligned}$$

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On the other hand, in the sum

$$\sum_{Q} \sum_{\mathcal{F}(Q)} \beta^2_{\Gamma_j}(Q')\ell(Q)$$

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&\leq C\mathcal{H}^1(\Gamma) + \sum_{j} \ell(\Gamma_j) \\
&\leq C\mathcal{H}^1(\Gamma)
\end{aligned}$$

Thank you!