Smoothness of the Beurling transform in Lipschitz domains

Martí Prats

Universitat Autònoma de Barcelona

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The Beurling Transform

The Beurling transform of a function $f \in L^p(\mathbb{C})$ is:

$$Bf(z) = c_0 \lim_{\varepsilon\to 0} \int_{|w-z|>\varepsilon} \frac{f(w)}{(z-w)^2} dm(z).$$

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Recall that $B : L^p(\mathbb{C}) \to L^p(\mathbb{C})$ is bounded for $1 < p < \infty$.
Also $B : \dot{W}^{s,p}(\mathbb{C}) \to \dot{W}^{s,p}(\mathbb{C})$ is bounded for $1 < p < \infty$ and $s > 0$.

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In particular, if $z \notin \operatorname{supp}(f)$ then Bf is analytic in an ε-neighborhood of z and

$$\partial^n Bf(z) = c_n \int_{|w-z|>\varepsilon} \frac{f(w)}{(z-w)^{n+2}} dm(z).$$

back

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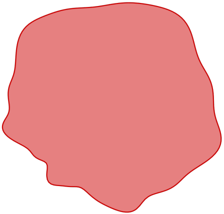
The problem we face

Let Ω be a Lipschitz domain.



When is $B : W^{s,p}(\Omega) \to W^{s,p}(\Omega)$ bounded?
We want an answer in terms of the geometry of the boundary.

Known facts, part 1

In a recent paper, Cruz, Mateu and Orobitg proved that for $0 < s \leq 1$, $1 < p < \infty$ with $sp > 2$, and $\partial\Omega$ smooth enough,

Theorem

$$B : W^{s,p}(\Omega) \to W^{s,p}(\Omega) \quad \textit{is bounded}$$

if and only if

$$B\chi_\Omega \in W^{s,p}(\Omega).$$

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One can deduce regularity of a quasiregular mapping in terms of the regularity of its Beltrami coefficient.

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Besov Spaces $B^s_{p,p}$

The geometric answer will be given in terms of Besov spaces $B^s_{p,p}$.
$B^s_{p,p}$ form a family closely related to $W^{s,p}$. They coincide for $p = 2$.
For $p < 2$, $B^s_{p,p} \subset W^{s,p}$. Otherwise $W^{s,p} \subset B^s_{p,p}$.

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Definition

For $0 < s < \infty$, $1 \leq p < \infty$, $f \in \dot{B}^s_{p,p}(\mathbb{R})$ if

$$\|f\|_{\dot{B}^s_{p,p}} = \left(\int_{\mathbb{R}} \int_{\mathbb{R}} \left| \frac{\Delta_h^{[s]+1} f(x)}{h^s} \right|^p \frac{dm(h)}{|h|} dm(x) \right)^{1/p} < \infty.$$

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Furthermore, $f \in B^s_{p,p}(\mathbb{R})$ if

$$\|f\|_{B^s_{p,p}} = \|f\|_{L^p} + \|f\|_{\dot{B}^s_{p,p}} < \infty.$$

We call them homogeneous and non-homogeneous Besov spaces respectively.

Known facts, part 2

In another recent paper, Cruz and Tolsa proved that for any $1 < p < \infty$, and Ω a Lipschitz domain,

Theorem

If the normal vector N belongs to $B_{p,p}^{1-1/p}(\partial\Omega)$, then $B(\chi_\Omega) \in W^{1,p}(\Omega)$ with

$$\|B(\chi_\Omega)\|_{\dot{W}^{1,p}(\Omega)} \leq c\|N\|_{\dot{B}_{p,p}^{1-1/p}(\partial\Omega)}.$$

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They proved also an analogous result for smoothness $0 < s < 1$. This implies

Theorem

Let $0 < s \leq 1$, $1 < p < \infty$ with $sp > 2$. If the normal vector is in the Besov space $B^{s-1/p}_{p,p}(\partial\Omega)$, then the Beurling transform is bounded in $W^{s,p}(\Omega)$.

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Tolsa proved a converse for Ω flat enough.

Main results

Main Theorem

Let Ω be smooth enough. Then we can write

$$\|\partial^n B\chi_\Omega\|^p_{L^p(\Omega)} \lesssim \|N\|^p_{B^{n-1/p}_{p,p}(\partial\Omega)} + \mathcal{H}^1(\partial\Omega)^{2-np}.$$

Main results

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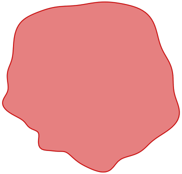
$$\|\partial^n B\chi_\Omega\|_{L^p(\Omega)}^p \lesssim \|N\|_{B_{p,p}^{n-1/p}(\partial\Omega)}^p + \mathcal{H}^1(\partial\Omega)^{2-np}.$$

Conjecture (work in progress)

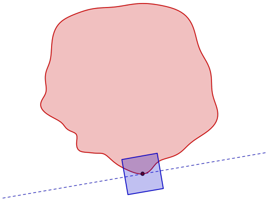
Let $2 < p < \infty$ and $1 \leq n < \infty$. Let Ω be a bounded domain smooth enough. If the exterior normal vector of Ω is in the Besov space $B_{p,p}^{n-1/p}(\partial\Omega)$, then the Beurling transform is bounded in $W^{n,p}(\Omega)$.

Reduction to local charts



- We have a domain smooth enough.

Reduction to local charts



- We have a domain smooth enough.
- In particular, at every boundary point
  we can find a cube
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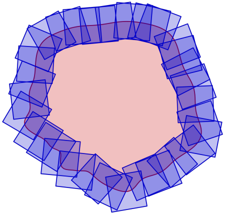

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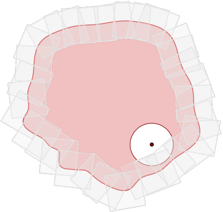
Reduction to local charts



Beurling transform

- We have a domain smooth enough.
- In particular, at every boundary point we can find a cube with fixed side-length R *parallel* to the tangent line inducing a parametrization $C^{n-1,1}$.
- We make a covering of the boundary by N of such cubes with some controlled overlapping.
- The Beurling transform of the interior points is controlled by the distance to the boundary:

$$|\partial^n B\chi_\Omega(z)| \lesssim \frac{1}{R^n}.$$

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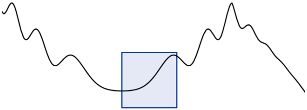
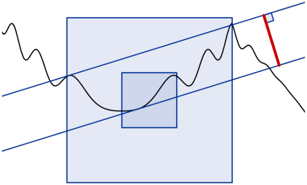
Defining some generalized betas of David-Semmes



A measure of the flatness of a set Γ:

Defining some generalized betas of David-Semmes

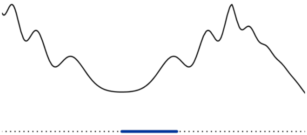


A measure of the flatness of a set Γ:

Definition (P. Jones)

$\beta_\Gamma(Q) = \inf_V \frac{w(V)}{\ell(Q)}$

Defining some generalized betas of David-Semmes



The graph of a function $y = A(x)$:
Consider $I \subset \mathbb{R}$, and define

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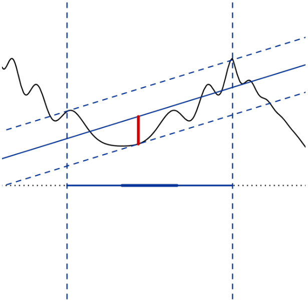

The graph of a function $y = A(x)$:
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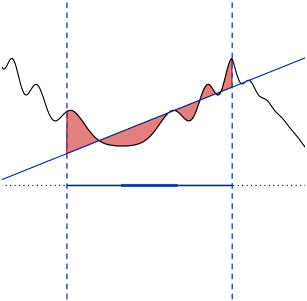
Definition

$\beta_\infty(I, A) = \inf_{P \in \mathcal{P}^1} \left\| \frac{A-P}{\ell(I)} \right\|_\infty$

Defining some generalized betas of David-Semmes



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Definition

$$\beta_p(I, A) = \inf_{P \in \mathcal{P}^1} \frac{1}{\ell(I)} \left\| \frac{A-P}{\ell(I)} \right\|_p$$

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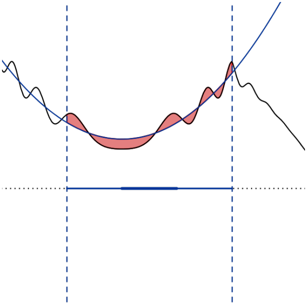

The graph of a function $y = A(x)$:
Consider $I \subset \mathbb{R}$, and define

Definition

$$\beta_{(n)}(I, A) = \inf_{P \in \mathcal{P}^n} \frac{1}{\ell(I)} \left\| \frac{A-P}{\ell(I)} \right\|_1$$

If there is no risk of confusion, we will write just $\beta_{(n)}(I)$.

Relation between $\beta_{(n)}$ and $B^n_{p,p}$

Theorem (Dorronsoro)

Let $f : \mathbb{R} \to \mathbb{R}$ be a function in the homogeneous Besov space $\dot{B}^s_{p,p}$. Then, for any $n \geq [s]$,

$$\|f\|^p_{\dot{B}^s_{p,p}} \approx \sum_{I \in \mathcal{D}} \left(\frac{\beta_{(n)}(I)}{\ell(I)^{s-1}} \right)^p \ell(I).$$

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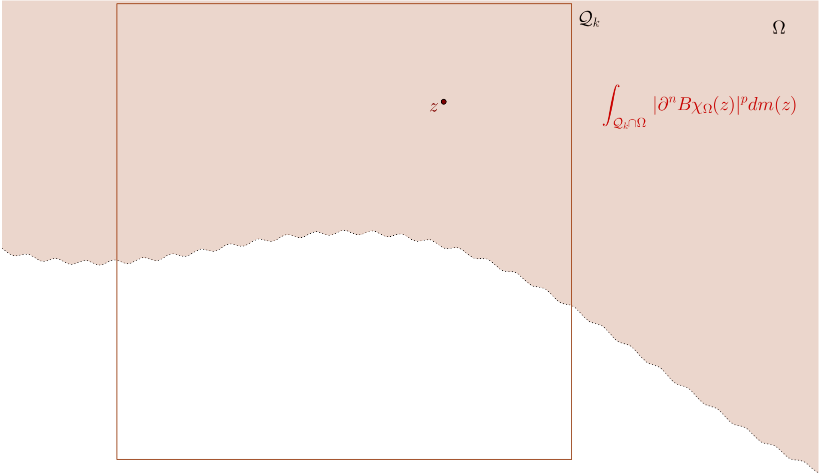
Local charts: Whitney decomposition



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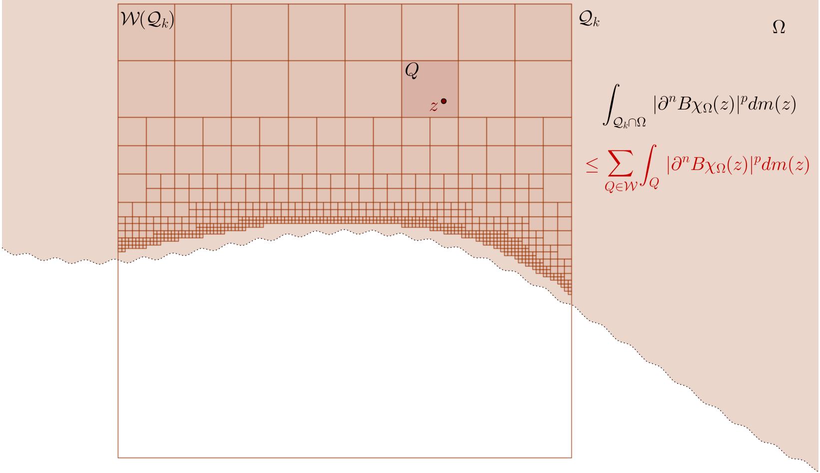

$$\int_{\mathcal{Q}_k\cap\Omega} |\partial^n B\chi_\Omega(z)|^p dm(z)$$

$$\leq \sum_{Q\in\mathcal{W}} \int_Q |\partial^n B\chi_\Omega(z)|^p dm(z)$$

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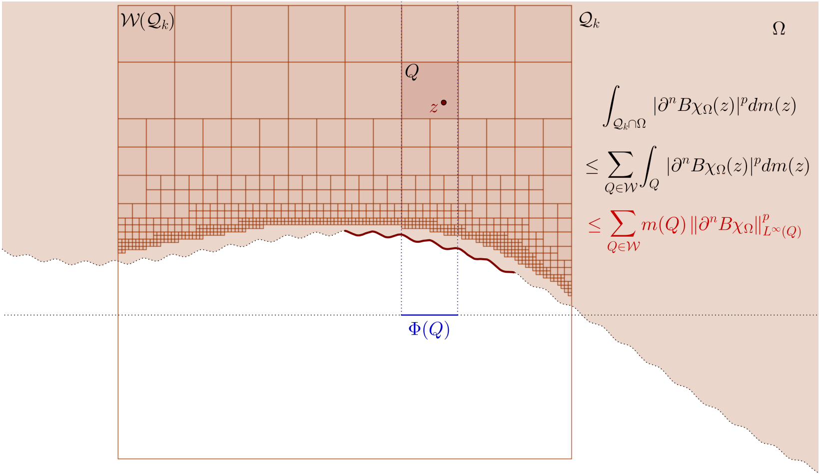

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$$\leq \sum_{Q\in\mathcal{W}} m(Q)\, \|\partial^n B\chi_\Omega\|^p_{L^\infty(Q)}$$

▸ First order derivative ▸ Second order derivative ▸ Higher order derivatives ▸▸ Skip higher order derivatives

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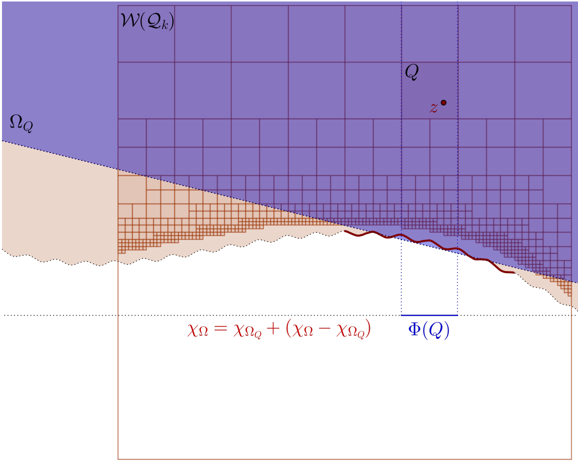
Local charts: Bounds for the first derivative



$$\int_{\mathcal{Q}_k \cap \Omega} |\partial B \chi_\Omega(z)|^p dm(z)$$
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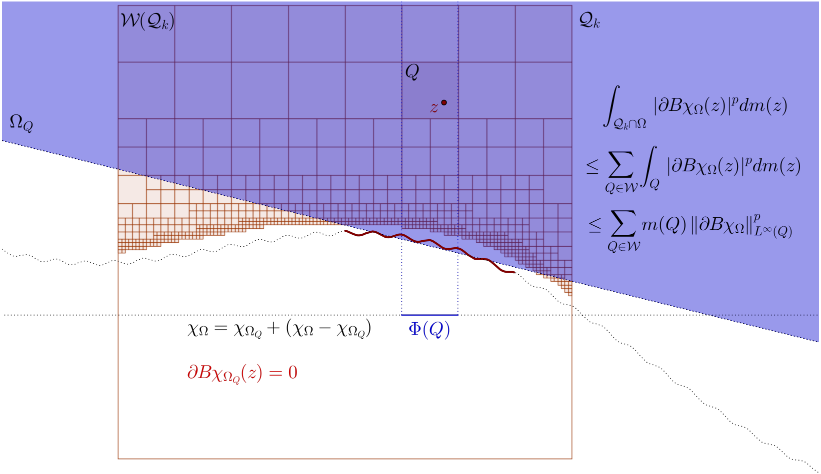

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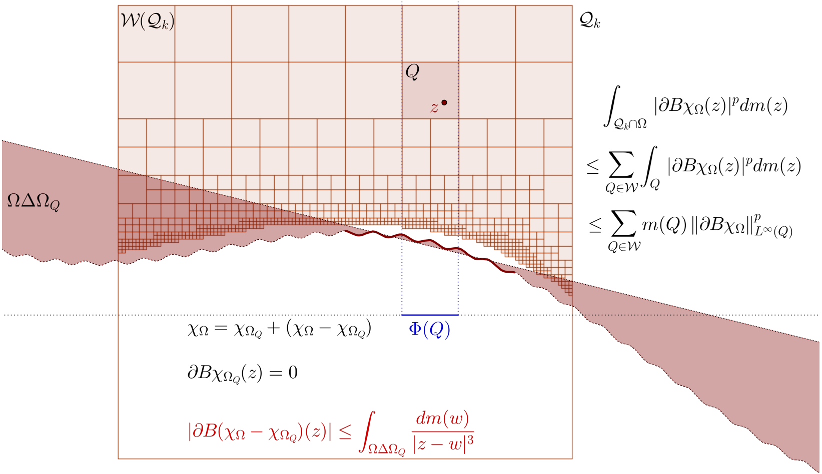

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$$\chi_\Omega = \chi_{\Omega_Q} + (\chi_\Omega - \chi_{\Omega_Q})$$

$$\partial B\chi_{\Omega_Q}(z) = 0$$

$$|\partial B(\chi_\Omega - \chi_{\Omega_Q})(z)| \leq \int_{\Omega\Delta\Omega_Q} \frac{dm(w)}{|z-w|^3}$$

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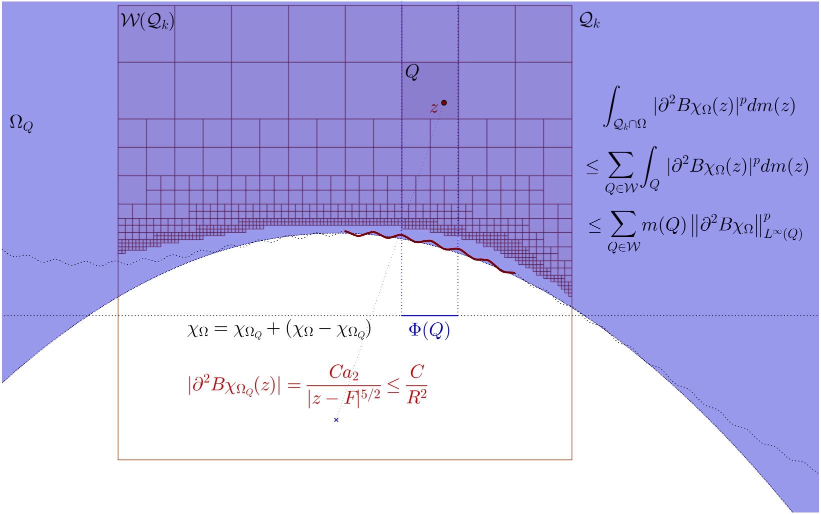
Local charts: Second order derivative



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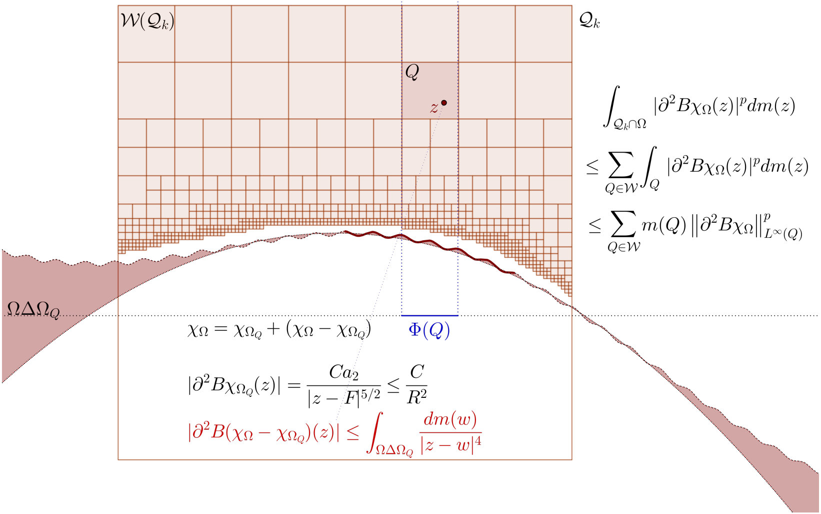
Local charts: Second order derivative



$$\int_{\mathcal{Q}_k\cap\Omega} |\partial^2 B\chi_\Omega(z)|^p dm(z)$$
$$\leq \sum_{Q\in\mathcal{W}} \int_Q |\partial^2 B\chi_\Omega(z)|^p dm(z)$$
$$\leq \sum_{Q\in\mathcal{W}} m(Q) \left\|\partial^2 B\chi_\Omega\right\|^p_{L^\infty(Q)}$$

$$\chi_\Omega = \chi_{\Omega_Q} + (\chi_\Omega - \chi_{\Omega_Q})$$

$$|\partial^2 B\chi_{\Omega_Q}(z)| = \frac{Ca_2}{|z-F|^{5/2}} \leq \frac{C}{R^2}$$

$$|\partial^2 B(\chi_\Omega - \chi_{\Omega_Q})(z)| \leq \int_{\Omega\Delta\Omega_Q} \frac{dm(w)}{|z-w|^4}$$

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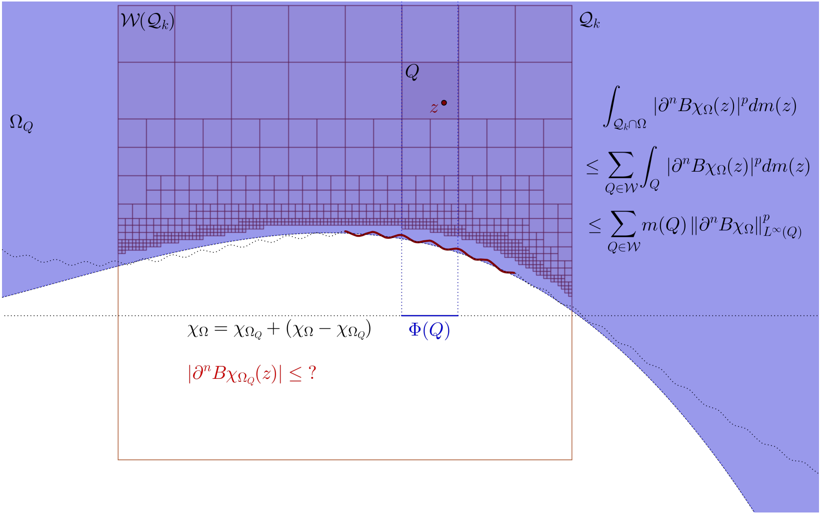
Local charts: Higher order derivatives



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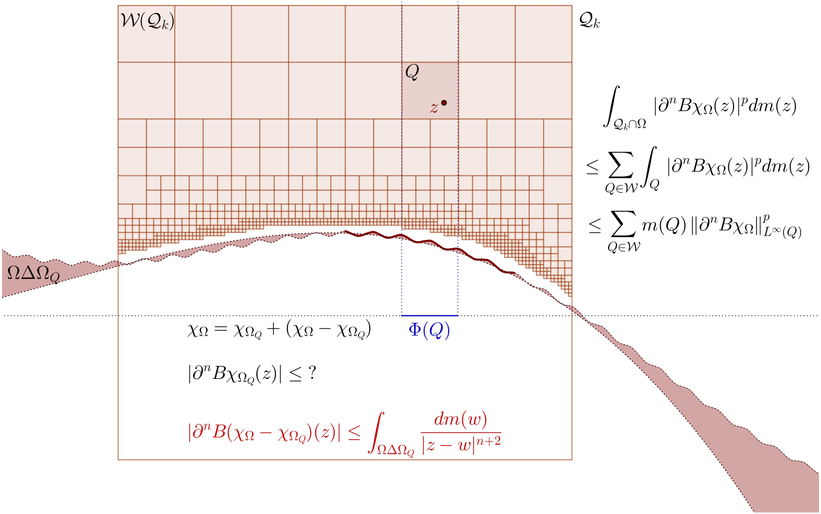
Local charts: Higher order derivatives



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$$|\partial^n B\chi_{\Omega_Q}(z)| \leq ?$$

$$|\partial^n B(\chi_\Omega - \chi_{\Omega_Q})(z)| \leq \int_{\Omega\Delta\Omega_Q} \frac{dm(w)}{|z-w|^{n+2}}$$

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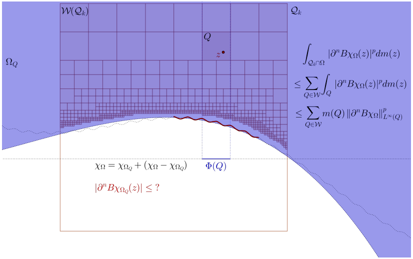
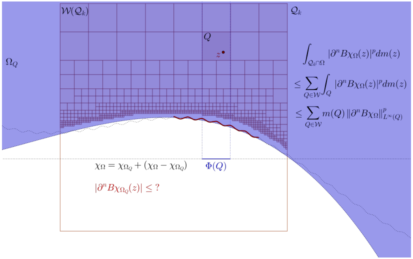
Bounding the polynomial region



We can choose R small enough (depending on the Lipschitz condition of the boundary) so that the following proposition holds:

Bounding the polynomial region



We can choose R small enough (depending on the Lipschitz condition of the boundary) so that the following proposition holds:

Proposition

If we denote by Ω_Q the region with boundary a minimizing polynomial for $\beta_{(n)}(\Phi(Q))$, we get

$$\left|\partial^n B\chi_{\Omega_Q}\right| \leq \frac{C}{R^n}.$$

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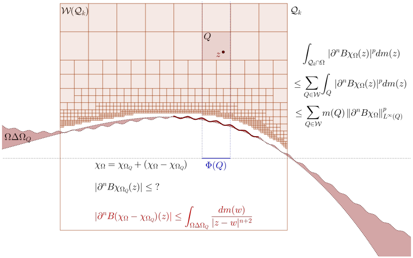
Bounding the interstitial region



Proposition

Choosing a minimizing polynomial for $\beta_{(n)}(\Phi(Q))$, we get

$$\int_{\Omega\Delta\Omega_Q} \frac{dm(w)}{|z-w|^{n+2}} \lesssim \sum_{\substack{I\in\mathcal{D} \\ \Phi(Q)\subset I\subset\Phi(\mathcal{Q}_k)}} \frac{\beta_{(n)}(I)}{\ell(I)^n} + \frac{1}{R^n}.$$

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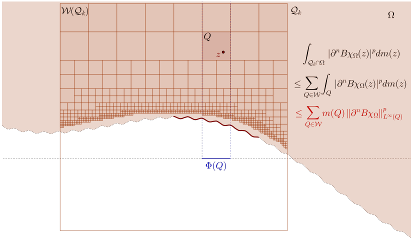
Hölder inequalities do the rest



Theorem

Let Ω be a Lipschitz domain of order n. Then, with the previous notation,

$$\|\partial^n B\chi_\Omega\|^p_{L^p(\Omega)} \lesssim \sum_{k=1}^{N} \sum_{I\in\mathcal{D}^k} \left(\frac{\beta_{(n)}(I)}{\ell(I)^{n-1/p}}\right)^p \ell(I) + \mathcal{H}^1(\partial\Omega)^{2-np}.$$

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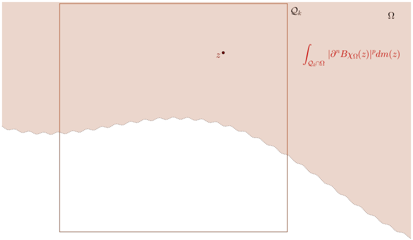

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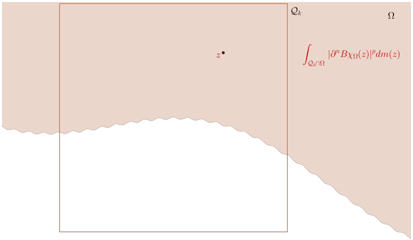
Let Ω be a Lipschitz domain of order n. Then, with the previous notation,

$$\|\partial^n B\chi_\Omega\|^p_{L^p(\Omega)} \lesssim \sum_{k=1}^{N} \|A_k\|^p_{\dot{B}^{n-1/p+1}_{p,p}} + \mathcal{H}^1(\partial\Omega)^{2-np}.$$

Hölder inequalities do the rest



Theorem

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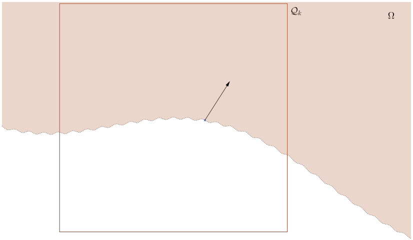
Hölder inequalities do the rest

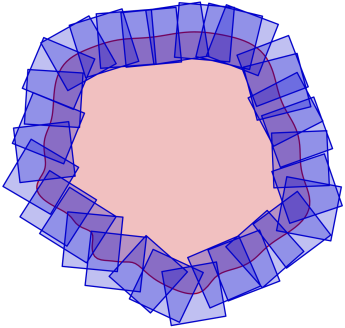


Theorem

Let Ω be a Lipschitz domain of order n. Then, with the previous notation,

$$\|\partial^n B\chi_\Omega\|_{L^p(\Omega)}^p \lesssim \sum_{k=1}^{N} \|N_{\partial\Omega\cap\mathcal{Q}_k}\|_{B^{n-1/p}_{p,p}}^p + \mathcal{H}^1(\partial\Omega)^{2-np}.$$

Hölder inequalities do the rest



Theorem

Let Ω be a Lipschitz domain of order n. Then, with the previous notation,

$$\|\partial^n B\chi_\Omega\|^p_{L^p(\Omega)} \lesssim \|N\|^p_{B^{n-1/p}_{p,p}(\partial\Omega)} + \mathcal{H}^1(\partial\Omega)^{2-np}.$$

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Conclusions

- The Besov regularity $B^{n-1/p}_{p,p}$ of the normal vector to the boundary of the domain gives us a bound of $B\chi_\Omega$ in $W^{n,p}$ (and $0 < s < 1$).

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- We think we are close to proving that if we assume $N \in B^{n-1/p}_{p,p}$, we get also the boundedness of the Beurling transform in $W^{n,p}(\Omega)$ as long as $p > 2$.
- Next steps are proving analogous results for any $s \in \mathbb{R}_+$ and giving a necessary condition for the boundedness of the Beurling transform when $p \leq 2$.

Thank you!