Regularity of stable solutions to semilinear elliptic equations on Riemannian models

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Abstract

We consider the reaction-diffusion problem $-\Delta_g u = f(u)$ in $\mathcal{B}_R$ with zero Dirichlet boundary condition, posed in a geodesic ball $\mathcal{B}_R$ with radius R of a Riemannian model (M, g). This class of Riemannian manifolds includes the classical *space forms*, *i.e.*, the Euclidean, elliptic, and hyperbolic spaces. For the class of semistable solutions we prove radial symmetry and monotonicity. Furthermore, we establish L^∞, L^p, and $W^{1,p}$ estimates which are optimal and do not depend on the nonlinearity f. As an application, under standard assumptions on the nonlinearity $\lambda f(u)$, we prove that the corresponding extremal solution u^* is bounded whenever $n \leq 9$. To establish the optimality of our regularity results we find the extremal solution for some exponential and power nonlinearities using an improved weighted Hardy inequality.

Keywords. semistable and extremal solutions, elliptic and hyperbolic spaces, *a priori* estimates, improved Hardy inequality

1 Introduction

This article is concerned with semilinear elliptic reaction-diffusion problems on Riemannian manifolds. We are interested in the class of semistable solutions, which include local minimizers, minimal solutions, extremal solutions, and also certain solutions found between a sub and a supersolution. On any geodesic ball, we show that semistable solutions are radially symmetric and decreasing. Then, we establish L^∞, L^p, and $W^{1,p}$ *a priori* estimates for solutions in this class. As an application we obtain sharp regularity results

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for extremal solutions. To show the optimality of our regularity results we find the extremal solution for some exponential and power nonlinearities. This will follow by using an improved weighted Hardy inequality for radial functions.

We point out that the regularity properties we achieve in this paper represent a geometrical extension of the ones carried out by Cabré and Capella in [5] for the Euclidean case. As in [5], our results do not depend on the specific form of the nonlinearity in the reaction term and they show that the class of semistable solutions enjoys better regularity properties than general solutions.

More specifically, let f be any locally Lipschitz positive nonlinearity and consider the following semilinear elliptic problem

$$\left\{\begin{array}{rcll} -\Delta_g u &=& f(u) & \text{in}\, \mathcal{B}_R, \\ u &>& 0 & \text{in}\, \mathcal{B}_R, \\ u &=& 0 & \text{on}\, \partial\mathcal{B}_R, \end{array}\right. \tag{1.1}$$

posed on a geodesic ball $\mathcal{B}_R$, with radius R, of a *Riemannian model* (M, g). That is, a manifold M of dimension $n \geq 2$ admitting a pole O and whose metric g is given, in spherical/polar coordinates around O, by

$$ds^2 = dr^2 + \psi(r)^2 d\Theta^2 \quad \text{for } r \in (0, R) \text{ and } \Theta \in \mathbb{S}^{n-1}, \tag{1.2}$$

where r is the geodesic distance of the point $P = (r, \Theta)$ to the pole O, ψ is a smooth positive function in $(0, R)$, and $d\Theta^2$ is the canonical metric on the unit sphere $\mathbb{S}^{n-1}$. A similar setting has been recently considered by Berchio, Ferrero, and Grillo [1] in order to study stability and qualitative properties of radial solutions to the Lane-Emden-Fowler equation, where $f(u) = |u|^{m-1}u$ with $m > 1$, on certain classes of Cartan-Hadamard manifolds with infinite volume and negative sectional curvatures.

Observe that (1.2) defines the metric only away from the origin. From [12] and [14], in order to extend in a C^2 manner the metric ds^2 to the whole $\mathbb{R}^n$ it is sufficient to impose the following conditions:

$$\psi(0) = \psi''(0) = 0 \quad \text{and} \quad \psi'(0) = 1. \tag{1.3}$$

Important consequences of the above hypotheses (1.3), as discussed in [12], are that on geodesic balls of M the Laplace-Beltrami operator $-\Delta_g$ is uniformly elliptic and its L^2 spectrum is bounded away from zero.

Our purpose is to study the regularity of semistable solutions of (1.1). We say that a classical solution $u \in C^2(\mathcal{B}_R)$ of (1.1) is *semistable* if the linearized operator at u is nonnegative definite, *i.e.*,

$$\int_{\mathcal{B}_R} |\nabla_g \xi|^2 \, dv_g \geq \int_{\mathcal{B}_R} f'(u)\xi^2 \, dv_g \quad \text{for all } \xi \in C_0^1(\mathcal{B}_R). \tag{1.4}$$

The following theorem establishes radial symmetry and monotonicity properties of semistable classical solutions $u \in C^2(\mathcal{B}_R)$. By a *radially symmetric and decreasing function* $u \in C^2(\mathcal{B}_R)$ we mean a function u such that $u = u(r)$, with $r = |x|$, and $u_r(r) = (du/dr)(r) < 0$ for all $r \in (0, R)$.

Theorem 1.1. *Let f be a locally Lipschitz positive function. Assume that $\psi \in C^2([0, R])$ is positive in $(0, R]$ and satisfies* (1.3). *If $u \in C^2(\mathcal{B}_R)$ is a semistable solution of* (1.1), *then it is radially symmetric and decreasing.*

The proof of Theorem 1.1 makes no use of moving plane arguments as usual. Instead, the radial symmetry relies on the fact that, due to the semistability, any angular derivative of u would be either a sign changing first eigenfunction of the linearized operator at u or identically zero. However, the first assertion cannot hold since the first eigenfunction of the linearized operator should be positive. The monotonicity is then a trivial consequence of the positivity of the nonlinearity f.

Our first main result establishes *a priori* estimates for semistable classical solutions of (1.1). This result is useful in order to obtain the regularity solutions, *a priori* possibly singular, that can be obtained as the limit of semistable classical solutions (see for instance the application on minimal and extremal solutions below).

Theorem 1.2. *Assume that $\psi \in C^2([0, R])$ is positive in $(0, R]$ and satisfies* (1.3). *Let f be a locally Lipschitz positive function and*

$$p_0 := \frac{2n}{n - 2\sqrt{n-1} - 4} \quad \textit{and} \quad p_1 := \frac{2n}{n - 2\sqrt{n-1} - 2}. \tag{1.5}$$

If $u \in C^2(\mathcal{B}_R)$ is a semistable solution of (1.1), *then the following assertions hold:*

(a) *If $n \le 9$ then there exists a constant $C_{n,\psi}$ depending only on n and ψ such that*

$$\|u\|_{L^\infty(\mathcal{B}_R)} \le C_{n,\psi} \|u\|_{L^1(\mathcal{B}_R)}. \tag{1.6}$$

(b) *If $n \ge 10$ then there exist constants $C_{n,\psi,p}$ and $\overline{C}_{n,\psi,p}$ depending only on n, ψ, and p such that*

$$\|u\|_{L^p(\mathcal{B}_R)} \le C_{n,\psi,p} \|u\|_{L^1(\mathcal{B}_R)} \qquad \textit{for all } p < p_0 \tag{1.7}$$

and

$$\|u\|_{W^{1,p}(\mathcal{B}_R)} \le \overline{C}_{n,\psi,p} \|u\|_{L^1(\mathcal{B}_R)} \qquad \textit{for all } p < p_1. \tag{1.8}$$

Remark 1.3. Note that the denominator of the exponent p_0 in (1.5) is positive for $n > 10$, while it vanishes for $n = 10$. This exponent has to be understood as infinity for $n = 10$.

In dimensions $n \le 9$, every solution, *a priori* possibly singular, which is limit of semistable classical solutions is bounded by Theorem 1.2 (i), and thus it is in fact a classical solution. In this sense, Theorem 1.2 may be regarded as a result on removable singularities.

Cabré and Capella [5] proved Theorem 1.2 in the Euclidean case: $\psi(r) = r$. The proof of our main theorem, as in [5], relies essentially on the following key estimate

$$\int_0^\delta u_r^2 \psi^{n-1-2\alpha}\, dr \le C_{n,\alpha,\psi} \|u\|^2_{L^1(\mathcal{B}_R)} \tag{1.9}$$

for some $\delta \in (0, R)$ and some range of explicit α (see Lemma 3.2 below). This estimate is obtained by using the radial symmetry of the solution and by choosing $\xi = |u_r|\eta$ as a new test function in the semistability condition (1.4). With this choice, we have to be careful in the computations due to the appearance of the first and second derivatives of ψ (which in the Euclidean case are identically 1 and 0, respectively). As we will see, the general assumptions (1.3) on ψ will be enough to prove (1.9).

Note that our result applies to the important case of *space forms, i.e.*, the unique complete and simply connected Riemannian manifold M of constant sectional curvature K_ψ given by

- the hyperbolic space $\mathbb{H}^n$: $\psi(r) = \sinh r$ and $K_\psi = -1$;
- the Euclidean space $\mathbb{R}^n$: $\psi(r) = r$ and $K_\psi = 0$;
- the elliptic space $\mathbb{S}^n$: $\psi(r) = \sin r$ and $K_\psi = 1$.

In Theorems 1.5 and 1.6 below we present explicit extremal solutions (which are limit of classical semistable solutions) for some exponential and power nonlinearities. These explicit solutions, as in the flat case, show the sharpness of the L^∞, L^p, and $W^{1,p}$ estimates of Theorem 1.2 in geodesic balls $\mathcal{B}_R$ of the above space forms.

As main application of Theorem 1.2, we consider the following problem

$$\left\{\begin{array}{rcll} -\Delta_g u & = & \lambda f(u) & \text{in } \Omega, \\ u & > & 0 & \text{in } \Omega, \\ u & = & 0 & \text{on } \partial\Omega, \end{array}\right. \tag{1.10}$$

where Ω is a smooth bounded domain in M, $\lambda > 0$, and f is an increasing C^1 function satisfying $f(0) > 0$ and

$$\lim_{t\to+\infty} \frac{f(t)}{t} = +\infty. \tag{1.11}$$

The study of the above nonlinear eigenvalue problem requires to extend to the general case of Riemannian models the classical results of Crandall and Rabinowitz [9] and Brezis *et al.* [2] for the Euclidean setting (see also Proposition 5.1 in [5]). More specifically, since the first eigenvalue of $-\Delta_g$ on Ω is positive (as well as the corresponding eigenfunction) and we have a comparison principle for $-\Delta_g$ (since it is uniformly elliptic), it is standard to prove that there exists a parameter value $\lambda^* \in (0, +\infty)$ such that: if $0 < \lambda < \lambda^*$ then (1.10) admits a minimal solution $u_\lambda \in C^2(\overline{\Omega})$, while for $\lambda > \lambda^*$ problem (1.10) does not admit any classical solution. Here minimal means smaller than any other supersolution of the problem. Moreover, we also have that for every $0 < \lambda < \lambda^*$ the minimal solution u_λ is semistable in the sense of (1.4). These assertions can be obtained as in Proposition 5.1 (a)-(b) of [5].

Moreover, the increasing limit of minimal solutions

$$u^* := \lim_{\lambda \uparrow \lambda^*} u_\lambda, \tag{1.12}$$

which is well defined by the pointwise increasing property of u_λ with respect to λ, becomes a *weak* solution of (1.10) for $\lambda = \lambda^*$ in the following sense: $u^* \in L^1(\mathcal{B}_R)$, $f(u^*)(R-r) \in L^1(\mathcal{B}_R)$, and

$$-\int_{\mathcal{B}_R} u \Delta_g \xi \, dv_g = \lambda \int_{\mathcal{B}_R} f(u) \xi \, dv_g \quad \text{for all } \xi \in C^1_0(\mathcal{B}_R). \tag{1.13}$$

This solution u^* is called the *extremal solution* of (1.10) for $\lambda = \lambda^*$. This statement follows as in Proposition 5.1 (c) of [5].

Applying Theorem 1.2 (a) or (b) (depending on the dimension n) to minimal solutions u_λ and letting $\lambda \uparrow \lambda^*$ it is straightforward to see that u^* enjoys the same regularity properties as the ones stated in Theorem 1.2:

Corollary 1.4. *Assume that $\psi \in C^2([0,R])$ is positive in $(0,R]$ and satisfies* (1.3)*. Let f be a C^1 positive and increasing function satisfying* (1.11)*. Let $u^* \in L^1(\mathcal{B}_R)$ be the extremal solution of* (1.10) *and p_0, p_1 the exponents defined in* (1.5)*. Then the following assertions hold:*

(i) *If $n \leq 9$ then $u^* \in L^\infty(\mathcal{B}_R)$.*

(ii) *If $n \geq 10$ then $u^* \in L^p(\mathcal{B}_R) \cap W^{1,q}(\mathcal{B}_R)$ for all $p < p_0$ and $q < p_1$.*

As second main result, we obtain the extremal solution for some exponential and power nonlinearities. More precisely, given

$$K_\psi := \begin{cases} -1 & \text{if} \quad \psi = \sinh, \\ 0 & \text{if} \quad \psi = \mathrm{Id}, \\ 1 & \text{if} \quad \psi = \sin, \end{cases} \tag{1.14}$$

we consider the following exponential and power nonlinearities:

$$f_{\mathrm{e}}(u) = \frac{e^u}{\psi(R)^2} - \frac{n-1}{n-2} K_\psi \tag{1.15}$$

and

$$f_{\mathrm{p}}(u) = (u + \psi(R)^{-\frac{2}{m-1}}) \left((u + \psi(R)^{-\frac{2}{m-1}})^{m-1} - \frac{(m-1)n - (m+1)}{(m-1)n - 2m} K_\psi \right), \tag{1.16}$$

where $m > 1$.

Note that for $\psi(r) = r$ (the Euclidean case) and $R = 1$ (the unit ball), we recover the classical nonlinearities e^u and $(1+u)^m$ studied in detail by Joseph and Lundgren [13], Crandall and Rabinowitz [9], Mignot and Puel [15], and Brezis and Vázquez [3]. For these nonlinearities the extremal parameter and the extremal solution of (1.10) are as follows:

- If $f(u) = e^u$ and $n \geq 10$ then $\lambda^* = 2(n-2)$ and $u^*(r) = \log(1/r^2)$.
- If $f(u) = (1+u)^m$ and
$$n \geq N(m) := 2 + 4\frac{m}{m-1} + 4\sqrt{\frac{m}{m-1}}, \tag{1.17}$$
then $\lambda^* = \frac{2}{m-1}\left(n - \frac{2m}{m-1}\right)$ and $u^*(r) = r^{-\frac{2}{m-1}} - 1$.

We extend this result to the hyperbolic and the elliptic spaces. In the hyperbolic space we find the extremal parameter and the extremal solution of (1.10) for both nonlinearities (the ones defined in (1.15) and (1.16)) in any geodesic ball.

Theorem 1.5. *Assume $\psi = \sinh$. Let f_{e} and f_{p} be the nonlinearities defined in* (1.15) *and* (1.16)*, respectively, and let $N(m)$ be defined in* (1.17)*. The following assertions hold:*

(i) *Let $f = f_{\mathrm{e}}$. If $n \geq 10$, then*
$$\lambda^* = 2(n-2) \quad \textit{and} \quad u^*(r) = -2\log\left(\frac{\sinh(r)}{\sinh(R)}\right).$$

(ii) *Let $f = f_{\mathrm{p}}$ with $m > 1$. If $n \geq N(m)$ then*
$$\lambda^* = \frac{2}{m-1}\left(n - \frac{2m}{m-1}\right) \quad \textit{and} \quad u^*(r) = \sinh(r)^{-\frac{2}{m-1}} - \sinh(R)^{-\frac{2}{m-1}}.$$

Instead, in the elliptic space we find the extremal parameter and the extremal solution only in sufficiently small balls.

Theorem 1.6. *Assume $\psi = \sin$. Let f_{e} and f_{p} be the nonlinearities defined in* (1.15) *and* (1.16)*, respectively, and let $N(m)$ be defined in* (1.17)*. Let*
$$R_0 := \sup\{s \in (0, \pi/2) : \frac{\sin^2 s}{(1-\cos s)^2} > n(n-2)\}. \tag{1.18}$$

The following assertions hold:

(i) *Let $f = f_{\mathrm{e}}$ and $R_{\mathrm{e}} := \arcsin\left(\sqrt{\frac{n-2}{n-1}}\right) \in (0, \pi/2)$. If $n \geq 10$ and $R < \min\{R_0, R_{\mathrm{e}}\}$, then*
$$\lambda^* = 2(n-2) \quad \textit{and} \quad u^*(r) = -2\log\left(\frac{\sin(r)}{\sin(R)}\right).$$

(ii) *Let* $f = f_{\rm p}$ *with* $m > 1$ *and* $R_{\rm p} := \arcsin\left(\sqrt{\frac{n-2}{n}}\right) \in (0, \pi/2)$. *If* $n \geq N(m)$ *and* $R < \min\{R_0, R_{\rm p}\}$ *then*

$$\lambda^* = \frac{2}{m-1}\left(n - \frac{2m}{m-1}\right) \quad \textit{and} \quad u^*(r) = \sin(r)^{-\frac{2}{m-1}} - \sin(R)^{-\frac{2}{m-1}}.$$

Remark 1.7. (i) These examples show the sharpness of our regularity results for any geodesic ball in the hyperbolic space and for geodesic balls of small enough radius in the elliptic space. For the exponential nonlinearity we obtain that the extremal solution $u^*(r) = -2\log(\psi(r)/\psi(R))$ —which is limit of semistable classical solutions— is unbounded at the origin if $n \geq 10$. This shows the optimality of Theorem 1.2 (a). Instead, for the power nonlinearity we obtain that the extremal solution $u^*(r) = \psi(r)^{-\frac{2}{m-1}} - \psi(R)^{-\frac{2}{m-1}}$ belongs exactly to the L^p and $W^{1,p}$ spaces stated in Theorem 1.2 (b). This shows the sharpness of the exponents p_0 and p_1 defined in (1.5).

(ii) In Theorem 1.6 (i) we make the assumption $R < \min\{R_0, R_{\rm e}\}$. We assume $R < R_{\rm e}$ in order to ensure that the exponential nonlinearity defined in (1.15) is positive. Instead, we assume $R < R_0$ in order to have a Hardy-type inequality (see Proposition 1.8 below). The assumptions on R in Theorem 1.6 (ii) are set exactly for the same reasons.

To prove Theorems 1.5 and 1.6 we proceed as in [3]. That is, we use the uniqueness of semistable solutions in the energy class $H_0^1(\mathcal{B}_R)$ (see Proposition 4.1 below) and the following improved Hardy inequality.

Proposition 1.8 (Improved weighted Hardy inequality)**.** *Assume* $n \geq 3$. *Let* ψ *either* $\sinh$ *or* $\sin$, *and* K_ψ *and* R_0 *be defined in* (1.14) *and* (1.18), *respectively. The following inequality holds:*

$$\int_0^R \psi^{n-1}\xi_r^2\,dr \geq \frac{(n-2)^2}{4}\int_0^R \psi^{n-1}\frac{\xi^2}{\psi^2}\,dr + H_{n,\psi}\int_0^R \psi^{n-1}\xi^2\,dr \tag{1.19}$$

for all radial $\xi \in C_0^1(\mathcal{B}_R)$, *where*

$$H_{n,\psi} = \frac{1}{4}\left(\left(\sup_{(0,R)}(\phi/\psi)\right)^{-2} - n(n-2)K_\psi\right) \tag{1.20}$$

and $\phi(r) := \int_0^r \psi(s)\,ds$ *for all* $r \in (0, R)$.

If in addition $R < R_0$ *when* $\psi = \sin$, *then* $H_{n,\psi} > 0$. *In particular,*

$$\int_0^R \psi^{n-1}\xi_r^2\,dr \geq \frac{(n-2)^2}{4}\int_0^R \psi^{n-1}\frac{\xi^2}{\psi^2}\,dr \quad \textit{for all radial } \xi \in C_0^1(\mathcal{B}_R). \tag{1.21}$$

Note inequality (1.19) is really an improved Hardy inequality only if $H_{n,\psi} > 0$. This holds for any geodesic ball in the hyperbolic space. Unfortunately, in the elliptic case we only have been able to prove it for geodesic balls of radius $R < R_0$. It would be interesting to obtain an improvement of the constant $H_{n,\psi}$ defined in (1.20) to have an (1.19) in large balls (with positive $H_{n,\psi}$).

Finally, let us to mention that the bibliography studying the regularity of extremal solutions in a general domain $\Omega \subset \mathbb{R}^n$ with the standard Euclidean metric is extensive. However, only partial answers are known for general nonlinearities f. We refer the reader to [4, 6, 10, 16, 17, 18, 19] and references therein.

Notation 1.9. We always assume that the radius R of the geodesic ball $\mathcal{B}_R$ is fixed. Therefore, all the universal constants appearing in this work, included the ones in the estimates of Theorem 1.2, may depend on R. Moreover, as usual we denote by C or M the universal constants appearing in some inequalities in this paper. The value of these constants may vary even in the same line.

The paper is organized as follows. In Section 2 we prove the radial symmetry and the monotonicity property of semistable solutions established in Theorem 1.1. Section 3 deals with the regularity of semistable and extremal solutions. We prove our L^∞, L^p, and $W^{1,p}$ estimates of Theorem 1.2 and Corollary 1.4. Finally, in Section 4 we find the extremal parameter and the extremal solution for the exponential and power nonlinearities considered in Theorems 1.5 and 1.6, establishing the sharpness of Theorem 1.2.

2 Radial symmetry of semistable solutions

This section will be devoted to the proof of Theorem 1.1. The radial symmetry of positive solutions to uniformly elliptic problems on radially symmetric domains has been subject of an extensive study, essentially started by the celebrated work of Gidas, Ni, and Nirenberg [11]. Most of these symmetry results are based on the moving plane method as well as on the use of the Maximum Principle and its generalizations. Here, we will follow a more direct approach which uses the semistability of our solutions and was applied in [7] and [8] to obtain symmetry results for semistable solutions to reaction-diffusion equations involving the p-Laplacian.

Proof of Theorem 1.1. Let $u \in C^2(\mathcal{B}_R)$ be a classical semistable solution of (1.1). Note that the semistability condition (1.4) is equivalent to the nonnegativity of the first eigenvalue of the linearized operator $-\Delta_g - f'(u)$ in $\mathcal{B}_R$, *i.e.*,

$$\lambda_1(-\Delta_g - f'(u), \mathcal{B}_R) = \inf_{\xi \in H_0^1(\mathcal{B}_R)\setminus\{0\}} \frac{\int_{\mathcal{B}_R} \{|\nabla_g \xi|^2 - f'(u)\xi^2\}\, dv_g}{\int_{\mathcal{B}_R} \xi^2\, dv_g} \geq 0. \tag{2.1}$$

Let $u_\theta = \frac{\partial u}{\partial \theta}$ be any angular derivative of u. On the one hand, by the fact that $u \in C^2(\mathcal{B}_R)$, we clearly have

$$\int_{\mathcal{B}_R} |\nabla_g u_\theta|^2 \, dv_g < \infty.$$

Moreover, the regularity up the boundary of u and the fact that $u = 0$ on $\partial\mathcal{B}_R$ trivially give that $u_\theta = 0$ on $\partial\mathcal{B}_R$. Hence, $u_\theta \in H_0^1(\mathcal{B}_R)$.

On the other hand, noting that in the spherical coordinates given by (1.2) the Riemannian Laplacian of $u = u(r, \theta_1, .., \theta_{n-1})$ is given by

$$\Delta_g u = \frac{1}{\psi(r)^{n-1}}(\psi(r)^{n-1}u_r)_r + \frac{1}{\psi(r)^2}\Delta_{\mathbb{S}^{n-1}} u,$$

where $\Delta_{\mathbb{S}^{n-1}}$ is the Riemannian Laplacian on the unit sphere $\mathbb{S}^{n-1}$, and by the radial symmetry of the weight ψ, we can differentiate problem (1.1) to see that u_θ (weakly) satisfies

$$\begin{cases} -\Delta_g u_\theta &= f'(u)u_\theta & \text{in } \mathcal{B}_R, \\ u_\theta &= 0 & \text{on } \partial\mathcal{B}_R. \end{cases}$$

Therefore, multiplying the above equation on $\mathcal{B}_R$ and integrating by parts we have

$$\int_{\mathcal{B}_R} |\nabla_g u_\theta|^2 - f'(u)u_\theta^2 \, dv_g = 0,$$

and hence, from (2.1) (taking $\xi = u_\theta$ if necessary) it follows necessarily that either $|u_\theta|$ is a first positive eigenfunction of the linearized operator at u in $\mathcal{B}_R$ or $u_\theta \equiv 0$. But by the periodicity of u with respect to θ we see that u_θ necessarily changes sign unless it is constant (equal to zero). Thus $u_\theta \equiv 0$ for any $\theta \in S^{n-1}$, which means that u is radial.

Finally, if we pass to radial coordinates we see that $u = u(r)$ satisfies

$$-\Big(\psi(r)^{n-1}u_r\Big)_r = \psi(r)^{n-1}f(u) \qquad \text{in } (0, R).$$

Integrating the previous equation from 0 to any $s \in (0, R)$ with respect to r, recalling that $f(u)$ is positive, ψ is also positive in $(0, R]$, and $u_r(0) = 0$, we have

$$\psi(s)^{n-1}u_r(s) = \int_0^s \Big(\psi(r)^{n-1}u_r(r)\Big)_r \, dr = -\int_0^s \psi(r)^{n-1}f(u(r)) \, dr < 0.$$

Thus $u_r(s) < 0$ for all $s \in (0, R)$, *i.e.*, u is decreasing. This concludes the proof. $\square$

3 Regularity of radial semistable solutions

Let us begin by rewriting problem (1.1), for radial solutions $u \in C^2(\mathcal{B}_R)$, as

$$\begin{cases} -\Big(\psi(r)^{n-1}u_r\Big)_r &= \psi(r)^{n-1}f(u) & \text{in } (0, R), \\ u &> 0 & \text{in } (0, R), \\ u_r(0) = u(R) &= 0, \end{cases} \tag{3.1}$$

and considering the quadratic form associated to the second variation of the energy functional, evaluated at u, written in radial form:

$$Q_u(\xi) := \int_0^R \psi(r)^{n-1}\{\xi_r^2 - f'(u)\xi^2\}\, dr$$

for every Lipschitz function ξ such that $\xi(R) = 0$.

We want to see that the results by Cabré and Capella in [5] for the Euclidean case carry over to the general Riemannian model setting. We start by proving the following lemma.

Lemma 3.1. *Let f be a locally Lipschitz positive function. Assume that $\psi \in C^2([0, R])$ is positive in $(0, R]$ and satisfies* (1.3)*. If $u \in C^2(\mathcal{B}_R)$ is a semistable classical solution of* (1.1)*, then*

$$(n-1)\int_0^R \psi^{n-1}u_r^2(\psi')^2\eta^2\, dr \le \int_0^R \psi^{n-1}u_r^2\{(\psi\eta)_r^2 + (n-1)\psi\psi''\eta^2\}\, dr \tag{3.2}$$

for every Lipschitz function η such that $\eta(R) = 0$.

Proof. Differentiating equation (3.1) it is easy to see that

$$-(\psi^{n-1}u_{rr})_r = \psi^{n-1}\left(f'(u) + (n-1)\left(\frac{\psi'}{\psi}\right)'\right)u_r \quad \text{in } (0, R). \tag{3.3}$$

Thanks to equation (3.3), we are able prove that for any $\eta \in H^1 \cap L^\infty(0, R)$ with support in $(0, R)$ there holds

$$Q_u(u_r\psi\eta) = \int_0^R \psi^{n-1}u_r^2\left\{(\psi\eta)_r^2 - (n-1)\Big((\psi')^2 - \psi\psi''\Big)\eta^2\right\}\, dr \ge 0.$$

In fact, integrating by parts and using (3.3) we are able to compute

$$\begin{aligned}
&Q_u(u_r\psi\eta) = \int_0^R \psi^{n-1}\left\{u_{rr}^2\psi^2\eta^2 + u_r^2(\psi\eta)_r^2 + (\psi^2\eta^2)_r u_r u_{rr} - f'(u)u_r^2\psi^2\eta^2\right\}\, dr \\
&= \int_0^R \left\{u_{rr}^2\psi^2\eta^2 + u_r^2(\psi\eta)_r^2 - f'(u)u_r^2\psi^2\eta^2\right\}\psi^{n-1} - \psi^2\eta^2(u_r u_{rr}\psi^{n-1})_r\, dr \\
&= \int_0^R \left\{u_r^2(\psi\eta)_r^2 - f'(u)u_r^2\psi^2\eta^2\right\}\psi^{n-1} - \psi^2\eta^2 u_r(u_{rr}\psi^{n-1})_r\, dr \\
&= \int_0^R \psi^{n-1}u_r^2\left\{(\psi\eta)_r^2 + (n-1)\left(\frac{\psi'}{\psi}\right)'(\psi\eta)^2\right\}\, dr \\
&= \int_0^R \psi^{n-1}u_r^2\left\{(\psi\eta)_r^2 - (n-1)\Big((\psi')^2 - \psi\psi''\Big)\eta^2\right\}\, dr.
\end{aligned}$$

Since for $n \ge 2$ we have that the singleton $\{0\}$ is of zero capacity, the fact that $u \in H_0^1(\mathcal{B}_R)$ gives that the equation above also holds for η not necessarily vanishing around 0 with $|\nabla(\psi\eta)| \in L^\infty$. ∎

Now, we are able to prove the key estimate (1.9) used in our main regularity result.

Lemma 3.2. *Let f be a locally Lipschitz positive function. Assume that $\psi \in C^2([0,R])$ is positive in $(0,R]$ and satisfies* (1.3). *Let $\delta = \delta(\psi) \in (0, R/2)$ be such that $\psi' > 0$ in $[0,\delta]$. If $u \in C^2(\mathcal{B}_R)$ is a semistable classical solution of* (1.1)*, then there exists a positive constant $C_{n,\alpha,\psi}$ depending only on n, α, and ψ such that*

$$\int_0^\delta u_r^2 \psi^{n-1-2\alpha}\, dr \le C_{n,\alpha,\psi}\, \|u\|^2_{L^1(\mathcal{B}_R)}$$

for every

$$1 \le \alpha < 1 + \sqrt{n-1}. \tag{3.4}$$

Proof. Let $\varepsilon \in (0,\delta)$ and define

$$\eta_\varepsilon(r) := \begin{cases} \psi(\varepsilon)^{-\alpha} - \psi(\delta)^{-\alpha} & \text{if } 0 \le r \le \varepsilon, \\ \psi(r)^{-\alpha} - \psi(\delta)^{-\alpha} & \text{if } \varepsilon \le r \le \delta, \\ 0 & \text{if } \delta \le r \le R. \end{cases}$$

Observe that both η_ε and $(\psi\eta_\varepsilon)_r$ are bounded. By (3.2) with $\eta = \eta_\varepsilon$ we obtain

$$\begin{aligned} &(n-1)\int_0^\varepsilon \psi^{n-1}u_r^2(\psi')^2\eta_\varepsilon^2\, dr + (n-1)\int_\varepsilon^\delta \psi^{n-1}u_r^2(\psi')^2\eta_\varepsilon^2\, dr \\ &\le \int_0^\varepsilon \psi^{n-1}u_r^2(\psi')^2\eta_\varepsilon^2\, dr + \int_\varepsilon^\delta \psi^{n-1}u_r^2(\psi')^2\Big\{(1-\alpha)\psi^{-\alpha} - \psi(\delta)^{-\alpha}\Big\}^2 dr \\ &\quad +(n-1)\int_0^\delta \psi^{n-1}u_r^2\psi|\psi''|\eta_\varepsilon^2\, dr. \end{aligned}$$

Using that $n \ge 2$ and $\eta_\varepsilon^2 \le \psi^{-2\alpha} + \psi(\delta)^{-2\alpha}$, we have

$$\begin{aligned} (n-1)\int_\varepsilon^\delta \psi^{n-1}u_r^2(\psi')^2\eta_\varepsilon^2\, dr &\le \int_\varepsilon^\delta \psi^{n-1}u_r^2(\psi')^2\Big\{(1-\alpha)\psi^{-\alpha} - \psi(\delta)^{-\alpha}\Big\}^2 dr \\ &\quad +(n-1)\int_0^\delta \psi^{n-1}u_r^2\psi|\psi''|(\psi^{-2\alpha} + \psi(\delta)^{-2\alpha})\, dr. \end{aligned}$$

Now, expanding and rearranging the terms in the integrals, and using that ψ is increasing in $(0,\delta)$, we get

$$\begin{aligned} &(n-1-(1-\alpha)^2)\int_\varepsilon^\delta \psi^{n-1}u_r^2(\psi')^2\psi^{-2\alpha}\, dr \\ &\le \int_0^\delta \psi^{n-1}u_r^2(\psi')^2\psi^{-\alpha}\psi(\delta)^{-\alpha}\Big\{2(\alpha+n-2) + \frac{\psi^\alpha}{\psi(\delta)^\alpha}\Big\} dr \\ &\quad +(n-1)\int_0^\delta \psi^{n-1}u_r^2\psi|\psi''|\psi^{-2\alpha}\Big\{1 + \frac{\psi^{2\alpha}}{\psi(\delta)^{2\alpha}}\Big\} dr \\ &\le M_{n,\alpha,\psi}\int_0^\delta \psi^{n-1}u_r^2\psi^{-\alpha}\{(\psi')^2 + \psi^{1-\alpha}\}\, dr, \end{aligned}$$

where $M_{n,\alpha,\psi}$ is a positive constant depending only on n, α, and ψ.

Using that $\inf_{(0,\delta)} \psi'$ and $\sup_{(0,\delta)} \psi'$ are positive, (3.4), and letting ε go to zero we get

$$\int_0^\delta \psi^{n-1} u_r^2 \psi^{-2\alpha}\, dr \le \frac{M_{n,\alpha,\psi}}{n-1-(1-\alpha)^2} \int_0^\delta \psi^{n-1} u_r^2 \psi^{-\alpha} \left\{1+\psi^{1-\alpha}\right\} dr. \tag{3.5}$$

Now, the fact that there exists a positive constant $C_{n,\alpha,\psi}$ depending only on n, α, and ψ such that

$$\frac{M_{n,\alpha,\psi}}{n-1-(1-\alpha)^2}\, t^{-\alpha}(1+t^{1-\alpha}) \le \frac{1}{2} t^{-2\alpha} + C_{n,\alpha,\psi} t^{n-1} \quad \text{for all } t>0$$

and (3.5) give

$$\frac{1}{2}\int_0^\delta \psi^{n-1} u_r^2 \psi^{-2\alpha}\, dr \le C_{n,\alpha,\psi} \int_0^\delta \psi^{2n-2} u_r^2\, dr. \tag{3.6}$$

Moreover, since u is positive and radially decreasing (remember that $\delta \in (0,R/2)$ only depends on ψ), we have

$$u(\delta) \le C_{n,\psi} \int_0^\delta u(r)\psi^{n-1}\, dr \le C_{n,\psi} \|u\|_{L^1(\mathcal{B}_R)} \tag{3.7}$$

and

$$-u_r(\rho) = -\frac{u(2\delta)-u(\delta)}{\delta} \le \frac{u(\delta)}{\delta} \quad \text{for some } \rho \in (\delta, 2\delta). \tag{3.8}$$

Therefore, integrating the equation (3.1) from $s \in (0,\delta)$ to ρ and noting that f is positive, we obtain

$$\begin{aligned} -u_r(s)\psi(s)^{n-1} &= -u_r(\rho)\psi(\rho)^{n-1} - \int_s^\rho f(u)\psi^{n-1}\, dr \le \frac{u(\delta)}{\delta}\psi(\rho)^{n-1} \\ &\le C_{n,\psi}\|u\|_{L^1(\mathcal{B}_R)}. \end{aligned}$$

Squaring this inequality and integrating for s between 0 and δ we get

$$\int_0^\delta u_r^2 \psi^{2n-2}\, dr \le C_{n,\psi} \|u\|^2_{L^1(\mathcal{B}_R)}.$$

We conclude the proof going back to (3.6). □

Thanks to Lemma 3.2 we are now ready to give the proof of Theorem 1.2.

Proof of Theorem 1.2. Let $\delta \in (0,R/2)$ as in Lemma 3.2. Using Schwarz inequality and (3.7) we obtain

$$\begin{aligned} |u(t)| &= \left| u(\delta) + \int_t^\delta -u_r \psi^{(n-1-2\alpha)/2} \psi^{(2\alpha-n+1)/2} dr \right| \\ &\le C_{n,\psi}\|u\|_{L^1(\mathcal{B}_R)} + \left(\int_0^\delta u_r^2 \psi^{n-1-2\alpha} dr\right)^{\frac{1}{2}} \left(\int_t^\delta \psi^{2\alpha-n+1} dr\right)^{\frac{1}{2}} \end{aligned}$$

for all $t \in (0,\delta)$. Therefore, from Lemma 3.2 we deduce

$$|u(t)| \leq C_{n,\alpha,\psi} \left\{ 1 + \left(\int_t^\delta \psi^{2\alpha-n+1}\, dr \right)^{\frac{1}{2}} \right\} \|u\|_{L^1(\mathcal{B}_R)} \tag{3.9}$$

for all $t \in (0,\delta)$ and every $\alpha \in [1, 1+\sqrt{n-1})$.

(a) L^∞ *estimate* (1.6): Assume $n \leq 9$. On the one hand, since u is radially decreasing and thanks to (3.7), we have that

$$u(t) \leq u(\delta) \leq C_{n,\psi} \|u\|_{L^1(\mathcal{B}_R)} \quad \text{for all } \delta \leq t < R. \tag{3.10}$$

On the other hand, since $\psi \in C^2([0,R])$ is positive in $(0,R]$, $\psi(0)=0$, and $\psi'(0)=1$ by assumption, we note that the integral in (3.9) is finite for $t=0$ if $2\alpha - n + 1 > -1$, *i.e.*,

$$\int_0^\delta \psi^{2\alpha-n+1} dr \leq C_\psi < +\infty \qquad \text{if} \qquad \alpha > \frac{n-2}{2}.$$

Therefore,

$$|u(t)| \leq C_{n,\alpha,\psi} \|u\|_{L^1(\mathcal{B}_R)} \quad \text{for all } 0 < t < \delta, \tag{3.11}$$

whenever

$$\max\left\{ \frac{n-2}{2}, 1 \right\} < \alpha < 1 + \sqrt{n-1}.$$

Finally, since $2 \leq n < 10$, we can choose α (depending only on n) in the previous range to obtain (3.11) with a constant $C_{n,\psi}$ depending only on n and ψ. The desired L^∞ estimate (1.6) follows from this fact and (3.10).

(b) Assume $n \geq 10$.

L^p *estimate* (1.7): On the one hand, the fact that u is decreasing and (3.7) give that

$$\left(\int_\delta^R |u|^p \psi^{n-1}\, dt \right)^{\frac{1}{p}} \leq u(\delta) \left(\int_\delta^R \psi^{n-1}\, dt \right)^{\frac{1}{p}} \leq C_{n,\psi,p} \|u\|_{L^1(\mathcal{B}_R)}. \tag{3.12}$$

On the other hand, let $s \in (0,\delta)$. By (3.9) it follows that

$$\int_s^\delta |u|^p \psi^{n-1}\, dt \leq C^p_{n,\alpha,\psi} \|u\|^p_{L^1(\mathcal{B}_R)} \int_s^\delta \left(1 + \left(\int_t^\delta \psi^{2\alpha-n+1}\, dr \right)^{\frac{1}{2}} \right)^p \psi^{n-1}\, dt$$

for every $p \geq 1$. Notice that, again by (1.3), we have:

$$\int_0^\delta \left(1 + \left(\int_t^\delta \psi^{2\alpha-n+1}\, dr \right)^{\frac{1}{2}} \right)^p \psi^{n-1}\, dt \leq C_{n,\alpha,\psi} < +\infty$$

whenever

$$\frac{2\alpha - n + 2}{2} p + n - 1 > -1, \quad \textit{i.e.}, \quad p < \frac{2n}{n - 2\alpha - 2}. \tag{3.13}$$

Therefore, for any

$$p < p_0 = \frac{2n}{n - 2\sqrt{n-1} - 4}$$

we can choose $\alpha = \alpha(n,p) \in [1, 1+\sqrt{n-1})$ such that condition (3.13) holds, obtaining

$$\left(\int_0^\delta |u|^p \psi^{n-1}\, dt \right)^{\frac{1}{p}} \leq C_{n,\psi,p} \|u\|_{L^1(\mathcal{B}_R)}.$$

Taking into account (3.12) and applying Minkowski inequality, we reach the desired L^p estimate (1.7).

$W^{1,p}$ *estimate* (1.8): Recall that every radial function u in $H^1(\mathcal{B}_R)$ also belongs (as a function of $r = |x|$) to the Sobolev space $H^1(\delta, R)$ in one dimension. Thus, by the Sobolev embedding in one dimension and (3.7), we have

$$\begin{aligned} \left(\int_\delta^R |u_r|^p \psi^{n-1}\, dr \right)^{\frac{1}{p}} &\leq C_{n,\psi,p} \left(\int_\delta^R |u_r|^p\, dr \right)^{\frac{1}{p}} \leq C_{n,\psi,p} \|u\|_{L^\infty(\delta,R)} \\ &= C_{n,\psi,p} u(\delta) \leq C_{n,\psi,p} \|u\|_{L^1(\mathcal{B}_R)}. \end{aligned} \tag{3.14}$$

Observe that by equation (3.1), and since f is positive, we have

$$u_{rr} = -(n-1)\frac{\psi'}{\psi} u_r - f(u) \leq -(n-1)\frac{\psi'}{\psi} u_r \quad \text{in } (0, R).$$

Let $\rho \in (\delta, 2\delta)$ such that (3.8) holds (as in the proof of Lemma 3.2). Integrating the previous inequality with respect to r from $t \in (0, \delta)$ to ρ, using (3.8) and (3.7), as well as Schwarz inequality, we have

$$\begin{aligned} \frac{-u_r(t)}{n-1} &\leq \frac{-u_r(\rho)}{n-1} + \int_t^\rho \frac{|\psi'|}{\psi}(-u_r)\, dr \\ &\leq \frac{u(\delta)}{(n-1)\delta} + \int_t^{2\delta} \frac{|\psi'|}{\psi} \psi^{-\frac{n-1}{2}+\alpha}(-u_r)\psi^{\frac{n-1}{2}-\alpha}\, dr \\ &\leq C_{n,\psi}\|u\|_{L^1(\mathcal{B}_R)} + \left(\int_t^{2\delta} \left(\frac{\psi'}{\psi}\right)^2 \psi^{-n+1+2\alpha}\, dr \right)^{\frac{1}{2}} \left(\int_t^{2\delta} u_r^2 \psi^{n-1-2\alpha}\, dr \right)^{\frac{1}{2}}. \end{aligned}$$

Note that at this point we can use Lemma 3.2 with δ replaced by 2δ (taking our original δ smaller if necessary). Using this fact we have

$$-u_r(t) \leq C_{n,\alpha,\psi}\|u\|_{L^1(\mathcal{B}_R)} \Big(1 + \Big(\int_t^{2\delta} \left(\frac{\psi'}{\psi}\right)^2 \psi^{-n+1+2\alpha}\, dr \Big)^{\frac{1}{2}} \Big)$$

for every $\alpha \in [1, 1+\sqrt{n-1})$. Therefore, for this range of α and given $s \in (0,\delta)$, we get

$$\int_s^\delta |u_r|^p \psi^{n-1}\, dt \leq C_{n,\alpha,\psi}^p \|u\|_{L^1(\mathcal{B}_R)}^p \int_s^\delta \Big(1 + \Big(\int_t^{2\delta} (\psi')^2 \psi^{-n-1+2\alpha}\, dr \Big)^{\frac{1}{2}} \Big)^p \psi^{n-1}\, dt.$$

Finally, note that

$$\int_0^\delta \Big(1+\Big(\int_t^{2\delta}(\psi')^2\psi^{-n-1+2\alpha}\,dr\Big)^{\frac{1}{2}}\Big)^p\psi^{n-1}\,dt \le C_{n,\psi,p} < +\infty$$

whenever

$$\frac{2\alpha-n}{2}p+n-1>-1, \quad \textit{i.e.}, \quad p<\frac{2n}{n-2\alpha} \tag{3.15}$$

(note that $n-2\alpha>0$ since $n\ge 10$ and $\alpha\in[1,1+\sqrt{n-1})$). Therefore, for any

$$p<p_1=\frac{2n}{n-2\sqrt{n-1}-2}$$

we can choose $\alpha=\alpha(n,p)\in[1,1+\sqrt{n-1})$ such that (3.15) holds, obtaining

$$\int_s^\delta |u_r|^p\psi^{n-1}\,dt \le C_{n,\psi,p}\|u\|^p_{L^1(\mathcal{B}_R)}.$$

We conclude the proof using the previous estimate, (3.14), and Minkowski inequality, proving our $W^{1,p}$ estimate (1.8). □

Finally, we prove Corollary 1.4 as an immediate consequence of Theorem 1.2.

Proof of Corollary 1.4. Since the extremal solution is a weak solution of the extremal problem (1.10) for $\lambda=\lambda^*$, and hence $u^*\in L^1(\mathcal{B}_R)$, the result follows by applying Theorem 1.2 to minimal solutions $u_\lambda\in C^2(\mathcal{B}_R)$ for $\lambda\in(0,\lambda^*)$ and letting $\lambda\uparrow\lambda^*$. □

4 Singular extremal solutions for exponential and power nonlinearities in space forms

In this section we find the extremal parameter λ^* and the extremal solution u^* of problem (1.10) for the exponential and power nonlinearities considered in Theorems 1.5 and 1.6.

This will be achieved through the use of the Improved Hardy inequality established in Proposition 1.8 as well as the following uniqueness result, due to Brezis and Vázquez [3] for the Euclidean case (see also Proposition 3.2.1 in [10]). Its proof carries over easily to our setting thanks to the fact that, as commented in the Introduction, the structural hypothesis on the weight ψ stated in (1.3) ensures that $\lambda_1(-\Delta_g;\mathcal{B}_R)>0$.

Proposition 4.1 ([3, 10]). *Let $\lambda_1(-\Delta_g;\mathcal{B}_R)>0$ denote the principal eigenvalue of the Dirichlet Laplace Beltrami operator $-\Delta_g$ in $\mathcal{B}_R$. Assume $f\in C^1(\mathbb{R})$ is convex.*

Let u_1, $u_2\in H^1_0(\mathcal{B}_R)$ be two stable weak solutions of (1.1). *Then, either $u_1=u_2$ a.e. or $f(u)=\lambda_1 u$ on the essential ranges of u_1 and u_2. In the latter case, u_1 and u_2 belong to the eigenspace associated to λ_1. In particular, they are collinear.*

Let us prove the improved Hardy-type inequality on Riemannian models following the argument of Theorem 4.1 in [3].

Proof of Proposition 1.8. Let $\xi \in C_0^1(\mathcal{B}_R)$ be a radial function and let $\varphi := \xi\psi^{\frac{n}{2}-1}$. We claim that the following Poincaré inequality holds:

$$\int_0^R \varphi_r^2\psi\,dr \geq \frac{1}{4}(\sup_{(0,R)}(\phi/\psi))^{-2}\int_0^R \varphi^2\psi\,dr. \tag{4.1}$$

Indeed, using integration by parts (note that $\varphi(0) = \varphi(R) = 0$) and Schwarz inequality we have

$$\begin{aligned}\int_0^R \varphi^2\psi\,dr &= \int_0^R \varphi^2\phi_r\,dr = -2\int_0^R \varphi\varphi_r\phi\psi^{1/2}\psi^{-1/2}\,dr \\ &\leq 2\left(\int_0^R \varphi_r^2\psi\,dr\right)^{1/2}\left(\int_0^R \varphi^2\frac{\phi^2}{\psi^2}\psi\,dr\right)^{1/2} \\ &\leq 2\sup_{(0,R)}(\phi/\psi)\left(\int_0^R \varphi_r^2\psi\,dr\right)^{1/2}\left(\int_0^R \varphi^2\psi\,dr\right)^{1/2}.\end{aligned}$$

The claim follows immediately from the previous inequality (note that $\sup_{(0,R)}(\phi/\psi) \in (0,+\infty)$ either for $\psi(r) = \sin r$, r, or $\sinh r$).

Now, using $(\psi')^2 - 1 = -K_\psi\psi^2$, $\psi''/\psi = -K_\psi$, and an integration by parts, we obtain

$$\begin{aligned}\int_0^R \left(\xi_r^2 - \frac{(n-2)^2}{4}\frac{\xi^2}{\psi^2}\right)\psi^{n-1}\,dr &= \int_0^R \varphi_r^2\psi - \frac{n-2}{2}(\varphi^2)_r\psi' - \frac{(n-2)^2}{4}K_\psi\varphi^2\psi\,dr \\ &= \int_0^R \varphi_r^2\psi + \frac{n-2}{2}\left(\frac{\psi''}{\psi} - \frac{(n-2)}{2}K_\psi\right)\varphi^2\psi\,dr \\ &= \int_0^R \varphi_r^2\psi - \frac{n(n-2)}{4}K_\psi\varphi^2\psi\,dr.\end{aligned}$$

We obtain (1.19) using Poincaré inequality (4.1).

Note that the constant $H_{n,\psi}$ defined in (1.20), for the hyperbolic and elliptic spaces, is given by

$$H_{n,\sinh} = \frac{1}{4}\left(\frac{\sinh^2 R}{(\cosh R - 1)^2} + n(n-2)\right) \tag{4.2}$$

and

$$H_{n,\sin} = \frac{1}{4}\left(\frac{\sin^2 R}{(\cos R - 1)^2} - n(n-2)\right), \tag{4.3}$$

respectively. This constant is clearly positive for all R in the hyperbolic space. Instead, in the elliptic space it is positive for all $R < R_0$ (by definition of R_0). Therefore inequality (1.21) is an immediate consequence of (1.19). $\square$

We are now able to prove Theorems 1.5 and 1.6 establishing the extremal parameter and the extremal solution of (1.10) for the exponential and the power nonlinearities defined in (1.15) and (1.16) in the hyperbolic and elliptic spaces.

4.1 Proof of Theorem 1.5 (i) and Theorem 1.6 (i): Exponential nonlinearity

Consider problem (1.10) with the exponential nonlinearity

$$f(u) = \frac{e^u}{\psi(R)^2} - \frac{n-1}{n-2}K_\psi. \tag{4.4}$$

It is clear that f is a positive increasing nonlinearity satisfying (1.11) in the hyperbolic space (since $K_\psi = -1$). Instead in the elliptic space these assumptions hold if and only if

$$R < R_{\mathrm{e}} = \sup\{s \in (0, \pi/2) : \sin^2 s < \frac{n-2}{n-1}\} = \arcsin\left(\sqrt{\frac{n-2}{n-1}}\right).$$

In these cases, as we said in the introduction, the minimal solution $u_\lambda \in C^2(\mathcal{B}_R)$ of (1.10) exists for $\lambda \in (0, \lambda^*)$ and its increasing limit u^* is a (weak) solution of the extremal problem (1.10) for $\lambda = \lambda^*$.

A simple computation shows that problem (1.10) admits the explicit singular solution

$$u^\#(r) = -2\log\left(\frac{\psi(r)}{\psi(R)}\right) \qquad \text{with } \lambda = \lambda^\# = 2(n-2).$$

Note that $u^\# \in H_0^1(\mathcal{B}_R)$ if $n \geq 3$.

We claim that $\lambda^\# = \lambda^*$ and $u^\# = u^*$ whenever $n \geq 10$ for any geodesic ball if $\psi = \sinh$ and for balls with radius $R < \min\{R_0, R_{\mathrm{e}}\}$ if $\psi = \sin$. Indeed, by Proposition 4.1 and since $u^\# \in H_0^1(\mathcal{B}_R)$ is singular at the origin, we only have to prove that $u^\#$ is semistable. That is,

$$\int_0^R \psi^{n-1}\xi_r^2\,dr \geq 2(n-2)\int_0^R \psi^{n-1}\frac{\xi^2}{\psi^2}\,dr \tag{4.5}$$

for every radial $\xi \in C_0^1(\mathcal{B}_R)$ (note that $\lambda^\# f'(u^\#) = 2(n-2)/\psi^2$). However, this inequality clearly holds by (1.21):

$$\int_0^R \psi^{n-1}\xi_r^2\,dr \geq \frac{(n-2)^2}{4}\int_0^R \psi^{n-1}\frac{\xi^2}{\psi^2}\,dr \quad \text{for all radial } \xi \in C_0^1(\mathcal{B}_R),$$

since $(n-2)^2/4 \geq 2(n-2)$ whenever $n \geq 10$. This proves Theorem 1.5 (i) and Theorem 1.6 (i).

4.2 Proof of Theorem 1.5 (ii) and Theorem 1.6 (ii): Power nonlinearity

Consider now

$$f(u) = (u + \psi(R)^{-\frac{2}{m-1}})\left((u + \psi(R)^{-\frac{2}{m-1}})^{m-1} - \frac{(m-1)n-(m+1)}{(m-1)n-2m}K_\psi\right) \tag{4.6}$$

with $m > (n+2)/(n-2)$ (*i.e.*, $n > 2(m+1)/(m-1) = 2 + 4/(m-1)$). Note that in part (ii) of Theorems 1.5 and 1.6 we assume $n \geq N(m)$, where $N(m)$ is defined in (1.17). In particular, one has $m > (n+2)/(n-2)$.

In the hyperbolic (and Euclidean) space it is clear that f is a positive increasing nonlinearity satisfying (1.11). In the elliptic space these assumptions hold whenever

$$f(0) = \sin^{-\frac{2}{n-1}} R \left(\sin^{-2} R - \frac{(m-1)n - (m+1)}{(m-1)n - 2m} \right) > 0,$$

or equivalently,

$$\sin^2 R < \frac{(m-1)n - 2m}{(m-1)n - (m+1)} =: h(m,n).$$

However, since the function h defined in the right hand side of the above inequality is increasing in m in $((n+2)/(n-2), +\infty)$, we have that $f(0) > 0$ (independently of m) if

$$R < R_{\mathrm{p}} = \sup \left\{ R \in (0, \pi/2) : \sin^2 R \leq \frac{n-2}{n} = h\left(\frac{n+2}{n-2}, n\right) \right\}.$$

Note that R_{p} coincides with the number defined in Theorem 1.6 (ii).

As a consequence, f is a positive increasing nonlinearity satisfying (1.11) in all the space forms (whenever $R < R_{\mathrm{p}}$ in the elliptic one). Therefore, the minimal solution u_λ of (1.10) exists for $\lambda \in (0, \lambda^*)$ and its increasing limit u^* is a (weak) solution of the extremal problem (1.10) for $\lambda = \lambda^*$.

In order to find the extremal solution and the extremal parameter, let us note that

$$u^\#(r) = \psi(r)^{-\frac{2}{m-1}} - \psi(R)^{-\frac{2}{m-1}}, \quad \text{with } \lambda = \lambda^\# = \frac{2}{m-1}\left(n - \frac{2m}{m-1}\right),$$

is a weak solution of (1.10). Note that, since $m > (n+2)/(n-2)$, we have $\lambda^\# > 0$ and $u^\# \in H_0^1(\mathcal{B}_R)$.

We proceed as for the exponential nonlinearity, *i.e.*, we want to prove that $u^\#$ is a semistable solution of (1.10) for $\lambda = \lambda^\#$. First, note that

$$f'(u^\#) = \frac{m}{\psi^2} - \frac{(m-1)n - (m+1)}{(m-1)n - 2m} K_\psi.$$

Therefore, semistability condition for $u^\#$ turns out to be

$$\int_0^R \psi^{n-1} \xi_r^2 \, dr \geq \lambda^\# m \int_0^R \psi^{n-1} \left(\frac{1}{\psi^2} - \frac{1}{m} \frac{(m-1)n - (m+1)}{(m-1)n - 2m} K_\psi \right) \xi^2 \, dr \qquad (4.7)$$

for every radial $\xi \in C_0^1(\mathcal{B}_R)$. By Proposition 1.8 we have

$$\int_0^R \psi^{n-1} \xi_r^2 \, dr \geq \frac{(n-2)^2}{4} \int_0^R \psi^{n-1} \frac{\xi^2}{\psi^2} \, dr + H_{n,\psi} \int_0^R \psi^{n-1} \xi^2 \, dr$$

for every radial $\xi \in C_0^1(\mathcal{B}_R)$, where $H_{n,\psi}$ is the constant defined in (1.20). Therefore, semistability condition (4.7) follows from the previous improved Hardy inequality if the following two conditions hold:

$$\frac{(n-2)^2}{4} \geq \lambda^\# m = \frac{2m}{m-1}\left(n - \frac{2m}{m-1}\right) \tag{4.8}$$

and

$$\begin{aligned} H_{n,\psi} &\geq -\frac{2m}{m-1}\left(n-\frac{2m}{m-1}\right)\frac{1}{m}\frac{(m-1)n-(m+1)}{(m-1)n-2m}K_\psi \\ &\geq -\frac{2}{(m-1)^2}\Big((m-1)n-(m+1)\Big)K_\psi. \end{aligned} \tag{4.9}$$

Note that condition (4.8) is equivalent to

$$n \geq N(m) = 2 + \frac{4m}{m-1} + 4\sqrt{\frac{m}{m-1}}. \tag{4.10}$$

In order to deal with condition (4.9) we consider the hyperbolic and the elliptic cases separately.

HYPERBOLIC CASE: Assume $\psi(r) = \sinh r$ and $K_\psi = -1$. We have (remember (4.2)) that condition (4.9) is nothing but

$$H_{n,\sinh} = \frac{1}{4}\left(\frac{\sinh^2 R}{(1-\cosh R)^2} + n(n-2)\right) \geq \frac{2}{(m-1)^2}\Big((m-1)n-(m+1)\Big).$$

It is clear that this inequality holds if

$$\frac{n(n-2)}{4} \geq \frac{2}{(m-1)^2}\Big((m-1)n-(m+1)\Big),$$

or equivalently,

$$n(n-2)(m-1)^2 \geq 8(m-1)(n-1) - 16$$

which is true whenever $m > \frac{n+2}{n-2}$. This shows that (4.9) holds independently of R and therefore $u^\#$ is a semistable solution of (1.10) for $\lambda = \lambda^\#$.

ELLIPTIC CASE: Assume $\psi(r) = \sin r$ and $K_\psi = 1$. In this case condition (4.9) is

$$H_{n,\sin} = \frac{1}{4}\left(\frac{\sin^2 R}{(1-\cos R)^2} - n(n-2)\right) \geq -\frac{2}{(m-1)^2}\Big((m-1)n-(m+1)\Big) \tag{4.11}$$

(rememeber (4.3)). This condition clearly holds since we are assuming $R < R_0$, and hence, $H_{n,\sin} > 0$. Therefore, in the elliptic case $u^\#$ is also a semistable solution.

We have thus obtained that $u^\#$ is a semistable solution of (1.10) for $\lambda = \lambda^\#$ when (4.10) holds for any geodesic ball in the hyperbolic space and for geodesic balls of radius $R < \min\{R_0, R_{\mathrm{p}}\}$ in the elliptic one. Moreover, since it is singular at the origin, we obtain that $\lambda^{\#} = \lambda^*$ and $u^{\#} = u^*$ by Proposition 4.1. This proves Theorem 1.5 (ii) and Theorem 1.6 (ii) .

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