L^p-ESTIMATES FOR THE VARIATION FOR SINGULAR INTEGRALS ON UNIFORMLY RECTIFIABLE SETS

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Abstract. The L^p $(1 < p < \infty)$ and weak-L^1 estimates for the variation for Calderón-Zygmund operators with smooth odd kernel on uniformly rectifiable measures are proven. The L^2 boundedness and the corona decomposition method are two key ingredients of the proof.

1. Introduction

This article is devoted to obtain L^p $(1 < p < \infty)$ and weak-L^1 estimates for the variation for Calderón-Zygmund operators with smooth odd kernel with respect to uniformly rectifiable measures. As a matter of fact, we prove that if the L^2 estimate holds then the L^p and weak-L^1 estimates follow; the results in [17] deal with the L^2 case.

Regarding the Calderón-Zygmund operators, given $1 \leq n < d$ integers, in this article we consider kernels $K : \mathbb{R}^d \setminus \{0\} \to \mathbb{R}$ such that $K(-x) = -K(x)$ for all $x \neq 0$ (K is odd) and

$$|K(x)| \leq \frac{C}{|x|^n}, \quad |\partial_{x_i} K(x)| \leq \frac{C}{|x|^{n+1}} \quad \text{and} \quad |\partial_{x_i}\partial_{x_j} K(x)| \leq \frac{C}{|x|^{n+2}} \tag{1}$$

for all $x = (x_1, \ldots, x_d) \in \mathbb{R}^d \setminus \{0\}$ and all $1 \leq i, j \leq d$, where and $C > 0$ is some constant. The growth estimate on the second derivatives required in (1) comes from the fact that it is also assumed in [17, Theorem 1.3 and Corollary 4.2], which are used in this article (see Theorem 3.2). We should mention that this growth estimate is usually required in what concerns to L^2 boundedness of singular integral operators and uniformly rectifiable measures, see for example [5, 6, 16, 17, 20]. However, in Theorem 1.4 below we consider more general kernels.

Given a Radon measure μ in $\mathbb{R}^d$, $f \in L^1(\mu)$ and $x \in \mathbb{R}^d$, we set

$$T^\mu_\epsilon f(x) \equiv T_\epsilon(f\mu)(x) := \int_{|x-y|>\epsilon} K(x-y) f(y)\, d\mu(y), \tag{2}$$

and we denote $T^\mu_* f(x) = \sup_{\epsilon>0} |T^\mu_\epsilon f(x)|$, $\mathcal{T} = \{T_\epsilon\}_{\epsilon>0}$ and $\mathcal{T}^\mu = \{T^\mu_\epsilon\}_{\epsilon>0}$. Given $\rho > 2$ and $f \in L^1_{loc}(\mu)$, the ρ-variation operator acting on $\mathcal{T}^\mu f = \{T^\mu_\epsilon f\}_{\epsilon>0}$ is defined as

$$(\mathcal{V}_\rho \circ \mathcal{T}^\mu) f(x) := \sup_{\{\epsilon_m\}} \left(\sum_{m\in\mathbb{Z}} |T^\mu_{\epsilon_m} f(x) - T^\mu_{\epsilon_{m+1}} f(x)|^\rho \right)^{1/\rho} \tag{3}$$

where the pointwise supremum is taken over all the non-increasing sequences of positive numbers $\{\epsilon_m\}_{m\in\mathbb{Z}}$.

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Concerning the notion of uniform rectifiability, recall that a Radon measure μ in $\mathbb{R}^d$ is called n-rectifiable if there exists a countable family of n-dimensional C^1 submanifolds $\{M_i\}_{i\in\mathbb{N}}$ in $\mathbb{R}^d$ such that $\mu(E\setminus\bigcup_{i\in\mathbb{N}}M_i)=0$ and $\mu\ll\mathcal{H}^n$, where $\mathcal{H}^n$ stands for the n-dimensional Hausdorff measure. Moreover, μ is said to be n-dimensional Ahlfors-David regular, or simply n-AD regular, if there exists some constant $C>0$ such that

$$C^{-1}r^n\le\mu(B(x,r))\le Cr^n$$

for all $x\in\operatorname{supp}\mu$ and $0<r\le\operatorname{diam}(\operatorname{supp}\mu)$. Note that if $\operatorname{diam}(\operatorname{supp}\mu)<+\infty$ then $\mu(\mathbb{R}^d)<\infty$ and so the condition $\mu(B(x,r))\le Cr^n$ in the definition of AD regularity actually holds for all $r>0$. Finally, one says that μ is uniformly n-rectifiable if it is n-AD regular and there exist $\theta,M>0$ so that, for each $x\in\operatorname{supp}\mu$ and $0<r\le\operatorname{diam}(\operatorname{supp}\mu)$, there is a Lipschitz mapping g from the n-dimensional ball $B^n(0,r)\subset\mathbb{R}^n$ into $\mathbb{R}^d$ such that $\operatorname{Lip}(g)\le M$ and

$$\mu\big(B(x,r)\cap g(B^n(0,r))\big)\ge\theta r^n,$$

where $\operatorname{Lip}(g)$ stands for the Lipschitz constant of g. In particular, uniform rectifiability implies rectifiability. A set $E\subset\mathbb{R}^d$ is called n-rectifiable (or uniformly n-rectifiable) if $\mathcal{H}^n|_E$ is n-rectifiable (or uniformly n-rectifiable, respectively).

We are ready now to state our main result. In the statement $M(\mathbb{R}^d)$ stands for the Banach space of finite real Radon measures in $\mathbb{R}^d$ equipped with the total variation norm.

Theorem 1.1. *Let μ be a uniformly n-rectifiable measure in $\mathbb{R}^d$. Let K be an odd kernel satisfying* (1) *and, for $\rho>2$, consider the associated variation operator defined in* (3). *Then*

$$\mathcal{V}_\rho\circ\mathcal{T}^\mu:L^p(\mu)\to L^p(\mu)\quad(1<p<\infty)\quad\text{and}\quad\mathcal{V}_\rho\circ\mathcal{T}:M(\mathbb{R}^d)\to L^{1,\infty}(\mu)$$

are bounded operators. In particular, $\mathcal{V}_\rho\circ\mathcal{T}^\mu:L^1(\mu)\to L^{1,\infty}(\mu)$ is bounded.

The variation operator has been studied in different contexts during the last years, being probability, ergodic theory, and harmonic analysis three areas where variational inequalities turned out to be a powerful tool to prove new results or to enhace already known ones (see for example [1, 8, 9, 10, 11, 13, 18], and the references therein). Inspired by the results on variational inequalities for Calderón-Zygmund operators in $\mathbb{R}^n$ like [2, 3], in [16] we began our study of such type of inequalities when one replaces the underlying space $\mathbb{R}^n$ and its associated Lebesgue measure by some reasonable measure in $\mathbb{R}^d$, being the Hausdorff measure on a Lipschitz graph a first natural candidate. In this regard, Theorem 1.1 should be considered as a natural generalisation of variational inequalities for Calderón-Zygmund operators in $\mathbb{R}^n$ from a geometric measure-theoretic point of view.

A big motivation to prove Theorem 1.1 is its connection to the so called David-Semmes problem regarding the Riesz transform and rectifiability. Given a Radon measure μ in $\mathbb{R}^d$, one defines the n-dimensional Riesz transform of a function $f\in L^1(\mu)$ by $R^\mu f(x)=\lim_{\epsilon\searrow0}R^\mu_\epsilon f(x)$ (whenever the limit exists), where

$$R^\mu_\epsilon f(x)=\int_{|x-y|>\epsilon}\frac{x-y}{|x-y|^{n+1}}\,f(y)\,d\mu(y),\qquad x\in\mathbb{R}^d.$$

Note that the kernel of the Riesz transform is the vector $(x^1,\dots,x^d)/|x|^{n+1}$ (so, in this case, the kernel K in (1) is vectorial). We also use the notation $\mathcal{R}^\mu f(x):=\{R^\mu_\epsilon f(x)\}_{\epsilon>0}$ and, as usual, we define the maximal operator $R^\mu_* f(x)=\sup_{\epsilon>0}|R^\mu_\epsilon f(x)|$.

G. David and S. Semmes asked more than twenty years ago the following question, which is still open (see, for example, [19, Chapter 7]):

Question 1.2. *Is it true that an n-dimensional AD regular measure μ is uniformly n-rectifiable if and only if R^μ_* is bounded in $L^2(\mu)$?*

By [5], the "only if" implication of this question above is already known to hold. Also in [5], G. David and S. Semmes gave a positive answer to the other implication if one replaces the L^2 boundedness of R^μ_* by the L^2 boundedness of T^μ_* for a wide class of odd kernels K. In the case $n = 1$ the "if" implication was proved in [14] using the notion of curvature of measures. Later on, the same implication was answered affirmatively for $n = d - 1$ in the work [12] by combining quasiorthogonality arguments with some variational estimates which use the maximum principle derived from the fact that the Riesz kernel is (a multiple) of the gradient of the fundamental solution of the Laplacian in $\mathbb{R}^d$ when $n = d - 1$. Question 1.2 is still open for the general case $1 < n < d - 1$. However, thanks to Theorem 1.1 and [17, Theorem 2.3] we get the following corollary, which characterizes uniform rectifiability in terms of variational inequalities for the Riesz transform and more general Calderón-Zygmund operators.

Corollary 1.3. *Let μ be an n-dimensional AD regular Radon measure in $\mathbb{R}^d$. Then, the following are equivalent:*

- (a) *μ is uniformly n-rectifiable,*
- (b) *for any odd kernel K as in* (1) *and any $\rho > 2$, $\mathcal{V}_\rho \circ \mathcal{T}^\mu$ is bounded in $L^p(\mu)$ for all $1 < p < \infty$, and from $L^1(\mu)$ into $L^{1,\infty}(\mu)$,*
- (c) *for some $\rho > 0$, $\mathcal{V}_\rho \circ \mathcal{R}^\mu$ is bounded in $L^2(\mu)$.*

Comparing Corollary 1.3 to Question 1.2, note that the corollary asserts that if we replace the $L^2(\mu)$ boundedness of R^μ_* by the stronger assumption that $\mathcal{V}_\rho \circ \mathcal{R}^\mu$ is bounded in $L^2(\mu)$, then μ must be uniformly rectifiable. On the other hand, the corollary claims that the variation for singular integral operators with any odd kernel satisfying (1), in particular for the n-dimensional Riesz transforms, is bounded in $L^p(\mu)$ for all $1 < p < \infty$ and it is of weak-type $(1, 1)$, which is a stronger conclusion than the one derived from an affirmative answer to Question 1.2.

The proof of $(c) \implies (a)$ in Corollary 1.3 is not as hard as the converse implications. Essentially, a combination of the arguments in [20] with the fact that, in a sense, $\mathcal{V}_\rho \circ \mathcal{R}^\mu$ controls R^μ_* does the job (see [17]). Theorem 1.1 is used to prove that $(a) \implies (b)$ in Corollary 1.3, the corresponding result in [17] was only proved for $p = 2$. Theorem 1.1 allows us to get it in full generality, completing the whole picture on variation for singular integrals and uniform rectifiability. As far as we know, neither the L^p estimates with $1 < p < \infty$ nor the weak-L^1 estimate for $\mathcal{V}_\rho \circ \mathcal{T}^\mu$ on uniform rectifiable measures μ were known, except for the case $p = 2$ treated in [17] and the case where $1 < p < \infty$ but suppμ is a Lipschitz graph with slope strictly smaller than 1, solved in [15]. Let us stress that from the latter result one can not easily deduce the L^p estimates on uniformly rectifiable measures (as in the standard situation in Calderón-Zygmund theory), basically because the good-λ method does not work properly for $\mathcal{V}_\rho \circ \mathcal{T}$. To avoid this obstacle, our method relies on the corona decomposition technique combined with some ideas from the Lipschitz case in [15] and from [2] and [13] to deal with variational inequalities, as well as the L^2 result from [17].

Finally we wish to remark that the same techniques used to prove Theorem 1.1 yield the following result, which applies to more general Calderón-Zymund operators. See Section 5 for the proof.

Theorem 1.4. *For $1 \le n < d$, let μ be a uniformly n-rectifiable measure in $\mathbb{R}^d$. Let $K : \mathbb{R}^d \times \mathbb{R}^d \setminus \{(x, y) : x = y\} \to \mathbb{R}$ be a kernel such that*

$$|K(x, y)| \le \frac{C}{|x - y|^n} \qquad \text{for all } x \ne y \in \mathbb{R}^d,$$

and

$$|K(x,y)-K(x',y)|+|K(y,x)-K(y,x')|\leq \frac{C\,|x-x'|}{|x-y|^{n+1}}$$

for all $x,x',y\in\mathbb{R}^d$ with $|x-x'|\leq\frac{1}{2}|x-y|$. For $\epsilon>0$, denote

$$T_\epsilon^\mu f(x)\equiv T_\epsilon(f\mu)(x):=\int_{|x-y|>\epsilon}K(x,y)f(y)\,d\mu(y).$$

Let $\mathcal{T}^\mu f=\{T_\epsilon^\mu f\}_{\epsilon>0}$ and let $(\mathcal{V}_\rho\circ\mathcal{T}^\mu)$ be defined as in (3)*. If $\mathcal{V}_\rho\circ\mathcal{T}^\mu$ is bounded in $L^2(\mu)$, then it is also bounded in $L^p(\mu)$ for $1<p<\infty$ and from $L^1(\mu)$ to $L^{1,\infty}(\mu)$. Also, $\mathcal{V}_\rho\circ\mathcal{T}$ is bounded from $M(\mathbb{R}^d)$ to $L^{1,\infty}(\mu)$.*

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2. Preliminaries and auxiliary results

2.1. **Notation and terminology.** As usual, in the paper the letter 'C' (or 'c') stands for some constant which may change its value at different occurrences, and which quite often only depends on n and d. Given two families of constants $A(t)$ and $B(t)$, where t stands for all the explicit or implicit parameters involving $A(t)$ and $B(t)$, the notation $A(t)\lesssim B(t)$ ($A(t)\gtrsim B(t)$) means that there is some fixed constant C such that $A(t)\leq CB(t)$ ($A(t)\geq CB(t)$) for all t, with C as above. Also, $A(t)\approx B(t)$ is equivalent to $A(t)\lesssim B(t)\lesssim A(t)$.

Throughout all the paper we assume that $1\leq n<d$ are integers and that μ is an n-dimensional AD-regular measure in $\mathbb{R}^d$. Given a bounded Borel set $A\subset\mathbb{R}^d$ and $f\in L^1_{loc}(\mu)$, we write the mean of f on A with respect to μ as follows:

$$m_A f:=\frac{1}{\mu(A)}\int_A f\,d\mu.$$

We consider the centered maximal Hardy-Littlewood operator:

$$\mathcal{M}f(x)=\sup_{r>0}m_{B(x,r)}|f|.$$

This is known to be bounded in $L^p(\mu)$, for $1<p\leq\infty$, and from $M(\mathbb{R}^d)$ to $L^{1,\infty}(\mu)$. For $1\leq q<\infty$, we also set

$$\mathcal{M}_q f:=\mathcal{M}(|f|^q)^{1/q}.$$

This is bounded in $L^p(\mu)$, for $q<p\leq\infty$, and from $L^q(\mu)$ to $L^{q,\infty}(\mu)$.

Given $0\leq a<b$, consider the closed annulus

$$A(x,a,b):=\overline{B(x,b)}\setminus B(x,a).$$

Given $k\in\mathbb{Z}$, set

$$I_k:=[2^{-k-1},2^{-k}).$$

One defines the *short* and *long variation* operators $\mathcal{V}_\rho^{\mathcal{S}}\circ\mathcal{T}^\mu$ and $\mathcal{V}_\rho^{\mathcal{L}}\circ\mathcal{T}^\mu$, respectively, by

$$(\mathcal{V}_\rho^{\mathcal{S}}\circ\mathcal{T}^\mu)f(x):=\sup_{\{\epsilon_m\}}\Bigg(\sum_{k\in\mathbb{Z}}\sum_{\epsilon_m,\epsilon_{m+1}\in I_k}|T_{\epsilon_m}^\mu f(x)-T_{\epsilon_{m+1}}^\mu f(x)|^\rho\Bigg)^{1/\rho},$$

$$(\mathcal{V}_\rho^{\mathcal{L}}\circ\mathcal{T}^\mu)f(x):=\sup_{\{\epsilon_m\}}\Bigg(\sum_{\substack{m\in\mathbb{Z}:\,\epsilon_m\in I_j,\,\epsilon_{m+1}\in I_k\\ \text{for some } j<k}}|T_{\epsilon_m}^\mu f(x)-T_{\epsilon_{m+1}}^\mu f(x)|^\rho\Bigg)^{1/\rho},$$

where, in both cases, the pointwise supremum is taken over all the non-increasing sequences of positive numbers $\{\epsilon_m\}_{m\in\mathbb{Z}}$. Given a finite Borel measure ν in $\mathbb{R}^d$, one defines $(\mathcal{V}_\rho^{\mathcal{S}}\circ\mathcal{T})\nu(x)$ and $(\mathcal{V}_\rho^{\mathcal{L}}\circ\mathcal{T})\nu(x)$ similarly. For convenience of notation, given $0<\epsilon\le\delta$ we set

$$T_{\delta,\epsilon}:=T_\delta-T_\epsilon \quad \text{and } T^\nu_{\delta,\epsilon} \text{ analogously.} \tag{4}$$

Let $\varphi_{\mathbb{R}}:[0,+\infty)\to[0,+\infty)$ be a non-decreasing $\mathcal{C}^2$ function with $\chi_{[4,\infty)}\le\varphi_{\mathbb{R}}\le\chi_{[1/4,\infty)}$ and set $\varphi_\epsilon(x)=\varphi_{\mathbb{R}}\big(|x|^2/\epsilon^2\big)$. We define

$$T_{\varphi_\epsilon}\nu(x):=\int\varphi_\epsilon(x-y)K(x-y)\,d\nu(y)\quad\text{for } x\in\mathbb{R}^d \tag{5}$$

(with $K(x-y)$ replaced by $K(x,y)$ if K is as in Theorem 1.4). Finally, write $\mathcal{T}_\varphi:=\{T_{\varphi_\epsilon}\}_{\epsilon>0}$. Compare the operator in (5) to

$$T_\epsilon\nu(x)=\int\chi_\epsilon(x-y)K(x-y)\,d\nu(y),$$

where $\chi_\epsilon(\cdot):=\chi_{(1,\infty)}(|\cdot|/\epsilon)$, and the family $\mathcal{T}_\varphi$ to $\mathcal{T}$.

2.2. **Dyadic lattices.** For the study of the uniformly rectifiable measures we will use the "dyadic cubes" built by G. David in [4, Appendix 1] (see also [6, Chapter 3 of Part I]). These dyadic cubes are not true cubes, but they play this role with respect to a given n-dimensional AD regular Radon measure μ, in a sense.

Let us explain which are the precise results and properties of this lattice of dyadic cubes. Given an n-dimensional AD regular Radon measure μ in $\mathbb{R}^d$ (for simplicity, here we may assume that $\operatorname{diam}(\operatorname{supp}\mu)=\infty$), for each $j\in\mathbb{Z}$ there exists a family $\mathcal{D}_j^\mu$ of Borel subsets of $\operatorname{supp}\mu$ (the dyadic cubes of the j-th generation) such that:

(*a*) each $\mathcal{D}_j^\mu$ is a partition of $\operatorname{supp}\mu$, i.e. $\operatorname{supp}\mu=\bigcup_{Q\in\mathcal{D}_j^\mu}Q$ and $Q\cap Q'=\varnothing$ whenever $Q,Q'\in\mathcal{D}_j^\mu$ and $Q\neq Q'$;

(*b*) if $Q\in\mathcal{D}_j^\mu$ and $Q'\in\mathcal{D}_k^\mu$ with $k\le j$, then either $Q\subset Q'$ or $Q\cap Q'=\varnothing$;

(*c*) for all $j\in\mathbb{Z}$ and $Q\in\mathcal{D}_j^\mu$, we have $2^{-j}\lesssim\operatorname{diam}(Q)\le2^{-j}$ and $\mu(Q)\approx2^{-jn}$;

(*d*) there exists $C>0$ such that, for all $j\in\mathbb{Z}$, $Q\in\mathcal{D}_j^\mu$, and $0<\tau<1$,

$$\begin{aligned}&\mu\big(\{x\in Q:\operatorname{dist}(x,\operatorname{supp}\mu\setminus Q)\le\tau2^{-j}\}\big)\\&\qquad+\mu\big(\{x\in\operatorname{supp}\mu\setminus Q:\ \operatorname{dist}(x,Q)\le\tau2^{-j}\}\big)\le C\tau^{1/C}2^{-jn}.\end{aligned}\tag{6}$$

This property is usually called the *small boundaries condition.* From (6), it follows that there is a point $z_Q\in Q$ (the center of Q) such that $\operatorname{dist}(z_Q,\operatorname{supp}\mu\setminus Q)\gtrsim2^{-j}$ (see [6, Lemma 3.5 of Part I]).

We set $\mathcal{D}^\mu:=\bigcup_{j\in\mathbb{Z}}\mathcal{D}_j^\mu$. Given a cube $Q\in\mathcal{D}_j^\mu$, we say that its side length is 2^{-j}, and we denote it by $\ell(Q)$. Notice that $\operatorname{diam}(Q)\le\ell(Q)$. For $\lambda>1$, we also write

$$\lambda Q=\big\{x\in\operatorname{supp}\mu:\ \operatorname{dist}(x,Q)\le(\lambda-1)\,\ell(Q)\big\}.$$

We denote

$$B_Q:=B(z_Q,c_1\ell(Q)), \tag{7}$$

where $c_1\ge1$ is some big constant which will be chosen below, depending on other parameters.

Let $P(Q)$ denote the cube in $\mathcal{D}_{j-1}^\mu$ which contains Q (the *parent* of Q), and set

$$\begin{aligned}\operatorname{Ch}(Q)&:=\{Q'\in\mathcal{D}_{j+1}^\mu:\ Q'\subset Q\},\\V(Q)&:=\{Q'\in\mathcal{D}_j^\mu:\ \operatorname{dist}(Q',Q)\le C_1\ell(Q)\}\end{aligned}$$

for some constant $C_1 > 0$ big enough ($\mathrm{Ch}(Q)$ are the *children* of Q, and $V(Q)$ stands for the *vicinity* of Q). Notice that $P(Q)$ is a cube from $\mathcal{D}^\mu$ but $\mathrm{Ch}(Q)$ and $V(Q)$ are collections of cubes from $\mathcal{D}^\mu$. It is not hard to show that the number of cubes in $\mathrm{Ch}(Q)$ and $V(Q)$ is bounded by some constant depending only on n and the AD regularity constant of μ, and on C_1 in the case of the vicinitiy.

The following assumptions will be used in the sequel: c_1 in (7) is big enough so that

$$Q \cup B_{Q'} \subset B_Q \text{ for all } Q' \in \mathrm{Ch}(Q)$$

and C_1 is big enough so that

$$B_Q \cap \mathrm{supp}\mu \subset \textstyle\bigcup_{Q' \in V(Q)} Q'.$$

Finally, we write

$$I_Q := I_j = [\ell(Q)/2, \ell(Q)).$$

2.3. **The corona decomposition.** Given an n-dimensional AD regular Radon measure μ on $\mathbb{R}^d$ consider the dyadic lattice $\mathcal{D}^\mu$ introduced in Subsection 2.2. Following [6, Definitions 3.13 and 3.19 of Part I], one says that μ admits a corona decomposition if, for each $\eta > 0$ and $\theta > 0$, one can find a triple $(\mathcal{B}, \mathcal{G}, \mathrm{Trs})$, where $\mathcal{B}$ and $\mathcal{G}$ are two subsets of $\mathcal{D}^\mu$ (the "bad cubes" and the "good cubes") and Trs is a family of subsets $S \subset \mathcal{G}$ (that we will call *trees*), which satisfy the following conditions::

- (a) $\mathcal{D}^\mu = \mathcal{B} \cup \mathcal{G}$ and $\mathcal{B} \cap \mathcal{G} = \varnothing$.
- (b) $\mathcal{B}$ satisfies a Carleson packing condition, i.e., $\sum_{Q \in \mathcal{B}:\, Q \subset R} \mu(Q) \lesssim \mu(R)$ for all $R \in \mathcal{D}^\mu$.
- (c) $\mathcal{G} = \biguplus_{S \in \mathrm{Trs}} S$, i.e., any $Q \in \mathcal{G}$ belongs to only one $S \in \mathrm{Trs}$.
- (d) Each $S \in \mathrm{Trs}$ is *coherent*. This means that each $S \in \mathrm{Trs}$ has a unique maximal element Q_S which contains all other elements of S as subsets, that $Q' \in S$ as soon as $Q' \in \mathcal{D}^\mu$ satisfies $Q \subset Q' \subset Q_S$ for some $Q \in S$, and that if $Q \in S$ then either all of the children of Q lie in S or none of them do (recall that if $Q \in \mathcal{D}^\mu_j$, the *children* of Q is defined as the collection of cubes $Q' \in \mathcal{D}^\mu_{j+1}$ such that $Q' \subset Q$).
- (e) The maximal cubes Q_S, for $S \in \mathrm{Trs}$, satisfy a Carleson packing condition. That is, $\sum_{S \in \mathrm{Trs}:\, Q_S \subset R} \mu(Q_S) \lesssim \mu(R)$ for all $R \in \mathcal{D}^\mu$.
- (f) For each $S \in \mathrm{Trs}$, there exists an n-dimensional Lipschitz graph Γ_S with constant smaller than η such that $\mathrm{dist}(x, \Gamma_S) \le \theta\, \mathrm{diam}(Q)$ whenever $x \in 2Q$ and $Q \in S$ (one can replace "$x \in 2Q$" by "$x \in c_2 Q$" for any constant $c_2 \ge 2$ given in advance, by [6, Lemma 3.31 of Part I]).

It is shown in [5] (see also [6]) that if μ is uniformly rectifiable then it admits a corona decomposition for all parameters $k > 2$ and $\eta, \theta > 0$. Conversely, the existence of a corona decomposition for a single set of parameters $k > 2$ and $\eta, \theta > 0$ implies that μ is uniformly rectifiable.

We set

$$\mathrm{Top}_\mathcal{G} = \{Q_S : S \in \mathrm{Trs}\} \quad \text{and} \quad \mathrm{Top} = \mathrm{Top}_\mathcal{G} \cup \mathcal{B}.$$

If μ is uniformly rectifiable, then, by the properties (b) and (e) above, for all $R \in \mathcal{D}^\mu$ we have

$$\sum_{Q \in \mathrm{Top}:\, Q \subset R} \mu(Q) \lesssim \mu(R).$$

If $R \in S$ for some $S \in \mathrm{Trs}$, we denote by $\mathrm{Tree}(R)$ the set of cubes $Q \in S$ such that $Q \subset R$ (the *tree* of R). Otherwise, that is, if $R \in \mathcal{B}$, we set $\mathrm{Tree}(R) := \{R\}$. Finally, $\mathrm{Stp}(R)$ stands for the set of cubes $Q \in \mathcal{B} \cup (\mathcal{G} \setminus \mathrm{Tree}(R))$ such that $Q \subset R$ and $P(Q) \in \mathrm{Tree}(R)$

(the *stopping* cubes relative to R), so actually $Q \subsetneq R$. Notice that if $R \in \mathcal{B}$, then we have $\mathrm{Stp}(R) = \mathrm{Ch}(R)$.

2.4. **Auxiliary results.** The following lemma follows directly from [21, Lemma 2.14] (see also [15, Lemma 2.2] for the case of Lipschitz graphs).

Lemma 2.1 (Calderón-Zygmund decomposition)**.** *Let μ be a compactly supported uniformly n-rectifiable measure in $\mathbb{R}^d$. For every positive measure $\nu \in M(\mathbb{R}^d)$ with compact support and every $\lambda > 2^{d+1}\|\nu\|/\|\mu\|$, the following hold:*

(a) *There exists a finite or countable collection of cubes $\{Q_j\}_j$ centered at $\operatorname{supp}\nu$ which are almost disjoint, that is $\sum_j \chi_{Q_j} \leq C$ (with C depending only on d), and a function $f \in L^1(\mu)$ such that*

$$\nu(Q_j) > 2^{-d-1}\lambda\mu(2Q_j), \tag{8}$$

$$\nu(\eta Q_j) \leq 2^{-d-1}\lambda\mu(2\eta Q_j) \quad \text{for } \eta > 2, \tag{9}$$

$$\nu = f\mu \text{ in } \mathbb{R}^d \setminus \Omega \text{ with } |f| \leq \lambda \ \ \mu\text{-a.e, where } \Omega = \textstyle\bigcup_j Q_j. \tag{10}$$

(b) *For each j, let $R_j := 6Q_j$ and denote $w_j := \chi_{Q_j}\left(\sum_k \chi_{Q_k}\right)^{-1}$. Then, there exists a family of functions $\{b_j\}_j$ with $\mathrm{supp} b_j \subset R_j$ and with constant sign satisfying*

$$\int b_j \, d\mu = \int w_j \, d\nu, \tag{11}$$

$$\|b_j\|_{L^\infty(\mu)}\mu(R_j) \leq C\nu(Q_j), \text{ and} \tag{12}$$

$$\textstyle\sum_j |b_j| \leq C_0\lambda, \quad \text{where } C_0 \text{ is some absolute constant.} \tag{13}$$

Let us remark that the cubes in the preceding lemma are "true cubes", i.e. they do not belong to $\mathcal{D}^\mu$.

Notice that from (9) it follows that $4.5Q_j \cap \mathrm{supp}\mu \neq \varnothing$, which implies that

$$\mu(\eta Q_j) \approx \ell(\eta Q_j)^n \qquad \text{for } \eta > 5 \text{ such that } \ell(\eta Q_j) \lesssim \mathrm{diam}(\mathrm{supp}\mu). \tag{14}$$

Additionally, if we assume that

$$\operatorname{supp}\nu \subset \mathcal{U}_{\mathrm{diam}(\mathrm{supp}\mu)}(\mathrm{supp}\mu), \tag{15}$$

where $\mathcal{U}_t(A)$ stands for the t-neighborhood of A, then we infer that $\ell(Q_j) \leq C\mathrm{diam}(\mathrm{supp}\mu)$, for all j and for some absolute constant C. Otherwise, for C big enough we would deduce that

$$\mathrm{supp}\mu \cup \mathrm{supp}\nu \subset 2Q_j,$$

and thus $\mu(2Q_j) = \|\mu\|$ and $\nu(Q_j) \leq \|\nu\|$, so by (8)

$$\|\nu\| > 2^{-d-1}\lambda\|\mu\|,$$

but this contradicts the choice of λ. In particular, under the assumption (15), we infer that

$$\mu(R_j) \approx \ell(R_j)^n \approx \ell(Q_j)^n. \tag{16}$$

We will need the following version of the dyadic Carleson embedding theorem.

Theorem 2.2 (Dyadic Carleson embedding theorem)**.** *Let μ be a Radon measure on $\mathbb{R}^d$. Let $\mathcal{D}$ be some dyadic lattice from $\mathbb{R}^d$ and let $\{a_Q\}_{Q\in\mathcal{D}}$ be a family of non-negative numbers. Suppose that for every cube $R \in \mathcal{D}$ we have*

$$\sum_{Q\in\mathcal{D}: Q\subset R} a_Q \leq c_3\,\mu(R). \tag{17}$$

Then every family of non-negative numbers $\{\gamma_Q\}_{Q\in\mathcal{D}}$ *satisfies*

$$\sum_{Q\in\mathcal{D}} \gamma_Q\, a_Q \le c_3 \int \sup_{Q\ni x} \gamma_Q \, d\mu(x). \tag{18}$$

Also, for $p\in(1,\infty)$, *if* $f\in L^p(\mu)$,

$$\sum_{Q\in\mathcal{D}} |m_Q f|^p\, a_Q \le c\, c_3 \|f\|^p_{L^p(\mu)}, \tag{19}$$

where $m_Q f = \int_Q f\, d\mu/\mu(Q)$ *and* c *is an absolute constant.*

In the preceding theorem, the lattice $\mathcal{D}$ can be, for example, either the usual dyadic lattice of $\mathbb{R}^d$ or, in the case when μ is AD-regular, the lattice of cubes associated with μ. For the proof of this classical result, see [21, Theorem 5.8], for example.

We say that $\mathcal{C}\subset\mathcal{D}$ is a Carleson family of cubes if

$$\sum_{Q\in\mathcal{C}:Q\subset R} \mu(Q) \le c_3\, \mu(R) \qquad \text{for all } R\in\mathcal{D}.$$

By (19), it follows that for such a family $\mathcal{C}$ and any $f\in L^p(\mu)$,

$$\sum_{Q\in\mathcal{C}} |m_Q f|^p\, \mu(Q) \le c\, c_3 \|f\|^p_{L^p(\mu)}.$$

Lemma 2.3. *Let* $\nu\in M(\mathbb{R}^d)$ *be a positive measure with compact support and* $\lambda > 2^{d+1}\|\nu\|/\|\mu\|$. *Consider cubes* $\{Q_j\}_j$ *and* $\{R_j\}_j$ *as in* Lemma 2.1. *Denote*

$$\nu_b := \sum_j \left(w_j\nu - b_j\mu\right),$$

where the b_j*'s satisfy* (11), (12) *and* (13), *and* $w_j := \chi_{Q_j}\left(\sum_k \chi_{Q_k}\right)^{-1}$. *Let* $\mathcal{C}\subset\mathcal{D}^\mu$ *be a family of cubes and* $\{a_S\}_{S\in\mathcal{C}}$ *be a family of non-negative numbers such that*

$$\sum_{S\in\mathcal{C}:S\subset R} a_S \le c_3\, \mu(R). \tag{20}$$

For each $S\in\mathcal{C}$ *consider the ball* B_S *given by* (7), *so it is centered on* S, $S\subset B_S$ *and* $r(B_S)\approx\ell(S)$. *Suppose that there exists some constant* $\tilde c>0$ *such that for each* $S\in\mathcal{C}$, *the ball* $\tilde c B_S$ *contains some cube* R_j. *Then, for every* $p\in(1,\infty)$,

$$\sum_{S\in\mathcal{C}} \left(\frac{|\nu_b|(B_S)}{\ell(S)^n}\right)^p a_S \lesssim \lambda^{p-1}\,\|\nu\| \tag{21}$$

and

$$\sum_{S\in\mathcal{C}} \left(\frac{\nu(B_S)}{\ell(S)^n}\right)^p a_S \lesssim \lambda^{p-1}\,\|\nu\|, \tag{22}$$

with the implicit constants depending on p, c_3, *and* $\tilde c$.

In particular, this lemma applies to the case when $a_S=1$ for all $S\in\mathcal{C}$ and $\mathcal{C}$ is a Carleson family satisfying the additional conditions stated in the lemma.

Proof. First we will show (21). By (18) in Theorem 2.2, one gets

$$\sum_{S\in\mathcal{C}} \left(\frac{|\nu_b|(B_S)}{\ell(S)^n}\right)^p a_S \le c_3 \int \left(\sup_{S\ni x} \frac{|\nu_b|(B_S)}{\ell(S)^n}\right)^p d\mu(x). \tag{23}$$

Write
$$\widetilde{\nu}_b = \sum_j w_j \nu \quad \text{and} \quad \widetilde{g} = \sum_j b_j,$$
so that, for every $S \in \mathcal{C}$,
$$|\nu_b|(B_S) \leq \widetilde{\nu}_b(B_S) + \int_{B_S} \widetilde{g}\, d\mu.$$
Note that the measure $\widetilde{\nu}_b$ and the functions b_j, $\widetilde{g}$ are positive because ν is assumed to be a positive measure. By (23) then we have
$$\sum_{S\in\mathcal{C}} \left(\frac{|\nu_b|(B_S)}{\ell(S)^n}\right)^p a_S \lesssim \int \left(\sup_{S\ni x} \frac{\widetilde{\nu}_b(B_S)}{\ell(S)^n}\right)^p d\mu(x) + \int \left(\sup_{S\ni x} m_{B_S}\widetilde{g}\right)^p d\mu(x), \tag{24}$$
where $m_{B_S}\widetilde{g} = \int_{B_S} \widetilde{g}\, d\mu/\mu(B_S)$ and we have taken into account that $\mu(B_S) \approx \ell(S)^n$.

To deal with the last integral on the right hand side of (24) we use the non-centered maximal Hardy-Littlewood operator defined by
$$\widetilde{\mathcal{M}} f(x) = \sup_{B\ni x} \frac{1}{\mu(B)} \int_B |f|\, d\mu,$$
where the supremum is taken over all the balls which contain x and whose center lies on suppμ. Recalling that $\widetilde{\mathcal{M}}$ is bounded in $L^p(\mu)$, and using that $\|\widetilde{g}\|_{L^\infty(\mu)} \leq c\,\lambda$ (by (13)) and $\|\widetilde{g}\|_{L^1(\mu)} \leq c\,\|\nu\|$ (by (12)), we obtain
$$\int \left(\sup_{S\ni x} m_{B_S}\widetilde{g}\right)^p d\mu(x) \leq c \int (\widetilde{\mathcal{M}}\widetilde{g})^p\, d\mu \leq c \int \widetilde{g}^p\, d\mu \leq c\lambda^{p-1} \int \widetilde{g}\, d\mu \leq c\lambda^{p-1}\, \|\nu\|. \tag{25}$$

Now we turn our attention to the first integral on the right hand side of (24). We write
$$\int \left(\sup_{S\ni x} \frac{\widetilde{\nu}_b(B_S)}{\ell(S)^n}\right)^p d\mu(x) = \int_{\bigcup_j 2Q_j} \ldots + \int_{\mathbb{R}^d\setminus\bigcup_j 2Q_j} \ldots =: I_1 + I_2.$$
To estimate I_1, we claim that
$$\frac{\widetilde{\nu}_b(B_S)}{\ell(S)^n} \lesssim \lambda.$$
This follows from the fact that $\tilde{c}B_S$ contains some cube R_j, which in turn implies that, for some $\eta \geq 6$ with $\eta \approx \ell(S)/\ell(Q_j)$, B_S is contained in some cube ηQ_j with $\ell(\eta Q_j) \approx \ell(S)$, and then
$$\frac{\widetilde{\nu}_b(B_S)}{\ell(S)^n} \lesssim \frac{\nu(\eta Q_j)}{\ell(\eta Q_j)^n},$$
which together with (14) and (9) yields the claim above. Then, using also (8) and the fact the cubes $\{Q_j\}_j$ have finite overlap, we deduce that
$$I_1 \lesssim \lambda^p \sum_j \mu(2Q_j) \lesssim \lambda^p \sum_j \frac{\nu(Q_j)}{\lambda} \lesssim \lambda^{p-1}\, \|\nu\|.$$

Finally we deal with the integral I_2. Consider $x \in \mathbb{R}^d \setminus \bigcup_j 2Q_j$ and S such that $x \in S \in \mathcal{C}$ (which, in particular, tells us that $S \setminus \bigcup_j 2Q_j \neq \varnothing$). Notice that
$$\widetilde{\nu}_b(B_S) \leq \sum_{i: Q_i \cap B_S \neq \varnothing} \nu(Q_i).$$

From the conditions $Q_i \cap B_S \neq \varnothing$ and $S \setminus \bigcup_j 2Q_j \neq \varnothing$, we infer that $r(B_S) \geq \frac{1}{2}\ell(Q_i)$. So we deduce that $Q_i \subset c_4 B_S$, for some constant $c_4 \gtrsim 1$. Hence,

$$\widetilde{\nu}_b(B_S) \leq \sum_{i:Q_i \subset c_4 B_S} \nu(Q_i) \leq \sum_{i:Q_i \subset c_4 B_S} \int b_i \, d\mu,$$

where we used (11) for the last estimate. Observe now that if $Q_i \subset c_4 B_S$, then $R_i \subset c_5 B_S$, for some absolute constant $c_5 \geq c_4$. So recalling that $\widetilde{g} = \sum_j b_j$, we obtain

$$\widetilde{\nu}_b(B_S) \lesssim \int_{c_5 B_S} \widetilde{g} \, d\mu,$$

Therefore,

$$\frac{\widetilde{\nu}_b(B_S)}{\ell(S)^n} \lesssim \frac{1}{\mu(B_S)} \int_{c_5 B_S} \widetilde{g} \, d\mu \lesssim \widetilde{\mathcal{M}}\widetilde{g}(x)$$

for every $x \in S$. So arguing as in (25) we deduce that

$$I_2 \lesssim \int (\widetilde{\mathcal{M}}\widetilde{g}(x))^p \, d\mu(x) \lesssim \lambda^{p-1} \|\nu\|.$$

Together with the estimate we obtained for I_1, this yields

$$\int \left(\sup_{S \ni x} \frac{\widetilde{\nu}_b(B_S)}{\ell(S)^n} \right)^p d\mu(x) \lesssim \lambda^{p-1} \|\nu\|, \tag{26}$$

and so using (25) we get (21).

In order to show (22), recall that $\nu = \widetilde{\nu}_b + f\mu$ with f as in (10). Thus,

$$\nu(B_S) = \widetilde{\nu}_b(B_S) + \int_{B_S} f \, d\mu \lesssim \widetilde{\nu}_b(B_S) + m_{B_S} f \, \ell(S)^n,$$

and then

$$\sum_{S \in \mathcal{C}} \left(\frac{\nu(B_S)}{\ell(S)^n} \right)^p a_S \lesssim \sum_{S \in \mathcal{C}} \left(\frac{\widetilde{\nu}_b(B_S)}{\ell(S)^n} \right)^p a_S + \sum_{S \in \mathcal{C}} (m_{B_S} f)^p \, a_S. \tag{27}$$

We easily get (22) from (27), combinig (18) and (19) in Theorem 2.2 with (26) and the fact that $\|f\|_{L^p(\mu)}^p \leq \lambda^{p-1} \|\nu\|$ by (10). $\square$

Let μ be a uniformly n-rectifiable measure in $\mathbb{R}^d$. Consider the splitting $\mathcal{D}^\mu = \mathcal{B} \cup (\biguplus_{T \in \mathrm{Trs}} T)$ given by the corona decomposition of μ. For a fixed constant $A \geq 1$, we denote by ∂T the family of cubes $Q \in T$ for which either $Q = Q_T$ with Q_T as in (d) in Section 2.3 or there exists some $P \in \mathcal{D}^\mu \setminus T$ such that

$$\frac{1}{2}\,\ell(P) \leq \ell(Q) \leq 2\ell(P) \quad \text{ and } \quad \mathrm{dist}(P, Q) \leq A\,\ell(Q). \tag{28}$$

We call ∂T the boundary of T. If $T = \mathrm{Tree}(R)$, with $R \in \mathrm{Top}_{\mathcal{G}}$, we also write $\partial\mathrm{Tree}(R) := \partial T$. We set

$$\partial\mathrm{Trs} := \bigcup_{T \in \mathrm{Trs}} \partial T.$$

Notice that $\partial T \subset T$.

The following lemma has been proved in [6, (3.28) in page 60].

Lemma 2.4. *Let μ be a uniformly n-rectifiable measure in $\mathbb{R}^d$. The family $\partial\mathrm{Trs}$ is a Carleson family.*

We will also need the following auxiliary result.

Lemma 2.5 (Annuli estimates)**.** *Assume that the constants η and θ in property (f) of the corona decomposition (see* Section 2.3*) are small enough. Let $Q \in \mathcal{D}^\mu$, $x \in Q$ and $\epsilon \in [\ell(Q)/2, \ell(Q)]$. Let $k \in \mathbb{Z}$ be such that $2^{-k} \leq \ell(Q)$. Given $R \in V(Q)$ and $C > 0$, denote*

$$\Lambda_k := \left\{ P \in \operatorname{Tree}(R) \cup \operatorname{Stp}(R) : \ell(P) = 2^{-k},\ P \subset A(x, \epsilon - C2^{-k}, \epsilon + C2^{-k}) \right\}.$$

Then

$$\mu\left(\bigcup\nolimits_{P \in \Lambda_k} P\right) \lesssim 2^{-k}\ell(R)^{n-1}, \tag{29}$$

where the implicit constant in the last inequality above only depends on n, d, μ and C.

In the lemma, if $\epsilon - C2^{-k} < 0$ we set $A(x, \epsilon - C2^{-k}, \epsilon + C2^{-k}) := \overline{B(x, \epsilon + C2^{-k})}$. For the proof, see [17, Lemma 5.9]. In fact, in this reference the annuli estimates are proved only for $R \in \mathcal{G}$. However, for $R \in \mathcal{B}$, the inequality (29) is trivial. Further, in [17, Lemma 5.9] one states that the result holds only for some constant C depending on n, d, and the AD-regularity constant of μ, and with a slight difference in the definition of $V(Q)$. However, it is trivial to check that this extends to the more general version above.

3. $\mathcal{V}_\rho \circ \mathcal{T} : M(\mathbb{R}^d) \to L^{1,\infty}(\mu)$ IS A BOUNDED OPERATOR

In this section we will prove the following result.

Theorem 3.1. *Let μ be a uniformly n-rectifiable measure in $\mathbb{R}^d$. Let K be an odd kernel satisfying* (1) *and consider the operator T associated to K defined in* (2)*. Then, for $\rho > 2$,*

(i) $\mathcal{V}_\rho^{\mathcal{S}} \circ \mathcal{T} : M(\mathbb{R}^d) \to L^{1,\infty}(\mu)$ is bounded,
(ii) $\mathcal{V}_\rho^{\mathcal{L}} \circ \mathcal{T} : M(\mathbb{R}^d) \to L^{1,\infty}(\mu)$ is bounded.

In particular, $\mathcal{V}_\rho \circ \mathcal{T}$ is a bounded operator from $M(\mathbb{R}^d)$ to $L^{1,\infty}(\mu)$ for all $\rho > 2$.

Notice that by the triangle inequality we can easily split the variation operator into the short and long variations, that is, $(\mathcal{V}_\rho \circ \mathcal{T}^\mu)f \leq (\mathcal{V}_\rho^{\mathcal{S}} \circ \mathcal{T}^\mu)f + (\mathcal{V}_\rho^{\mathcal{L}} \circ \mathcal{T}^\mu)f$. Therefore, that $\mathcal{V}_\rho \circ \mathcal{T}$ is a bounded operator from $M(\mathbb{R}^d)$ to $L^{1,\infty}(\mu)$ for all $\rho > 2$ follows from (i) and (ii) above, whose proofs are given below.

We will use the next result, which is contained in [17, Theorem 1.3 and Corollary 4.2].

Theorem 3.2. *Let μ be a uniformly n-rectifiable measure in $\mathbb{R}^d$. Let K be an odd kernel satisfying* (1) *and consider the operator T associated to K defined in* (2)*. Then, for $\rho > 2$,*

(i) $\mathcal{V}_\rho \circ \mathcal{T}^\mu : L^2(\mu) \to L^2(\mu)$ is bounded,
(ii) $\mathcal{V}_\rho \circ \mathcal{T}_\varphi : M(\mathbb{R}^d) \to L^{1,\infty}(\mu)$ is bounded.

Proof of Theorem 3.1(ii). We will deal with the long variation $\mathcal{V}_\rho^{\mathcal{L}} \circ \mathcal{T}$ by comparing it with the smoothened version $\mathcal{V}_\rho \circ \mathcal{T}_\varphi$, using Theorem 3.2$(ii)$, estimating the error terms by the short variation $\mathcal{V}_\rho^{\mathcal{S}} \circ \mathcal{T}$, and applying Theorem 3.1(i). More precisely, the triangle inequality yields

$$|T_\epsilon \nu(x) - T_\delta \nu(x)| \leq |T_{\varphi_\epsilon} \nu(x) - T_{\varphi_\delta} \nu(x)| + |T_\epsilon \nu(x) - T_{\varphi_\epsilon} \nu(x)| + |T_\delta \nu(x) - T_{\varphi_\delta} \nu(x)|$$

for any $0 < \delta \le \epsilon$. Therefore,

$$\begin{aligned}\big((\mathcal{V}_\rho^{\mathcal{L}}\circ\mathcal{T})\nu(x)\big)^\rho &\lesssim \big((\mathcal{V}_\rho\circ\mathcal{T}_\varphi)\nu(x)\big)^\rho\\ &\quad+\sup_{\{\epsilon_m\}}\sum_{\substack{m\in\mathbb{Z}:\,\epsilon_m\in I_j,\,\epsilon_{m+1}\in I_k\\ \text{for some } j<k}}\Big(|T_{\epsilon_m}\nu(x)-T_{\varphi_{\epsilon_m}}\nu(x)|^\rho+|T_{\epsilon_{m+1}}\nu(x)-T_{\varphi_{\epsilon_{m+1}}}\nu(x)|^\rho\Big)\\ &\lesssim \big((\mathcal{V}_\rho\circ\mathcal{T}_\varphi)\nu(x)\big)^\rho+\sup_{\substack{\{\epsilon_m\}:\,\epsilon_m\in I_m\\ \text{for all } m\in\mathbb{Z}}}\sum_{m\in\mathbb{Z}}|T_{\epsilon_m}\nu(x)-T_{\varphi_{\epsilon_m}}\nu(x)|^\rho.\end{aligned}\tag{30}$$

Let us estimate the second term on the right hand side of (30). Since $\chi_{[4,\infty)}\le\varphi_{\mathbb{R}}\le\chi_{[1/4,\infty)}$ by definition, we have

$$\chi_{[1,\infty)}(t)-\varphi_{\mathbb{R}}(t)=\int_{1/4}^4\varphi_{\mathbb{R}}'(s)(\chi_{[1,\infty)}-\chi_{[s,\infty)})(t)\,ds$$

for all $t\ge 0$. This means that $\chi_{[1,\infty)}-\varphi_{\mathbb{R}}$ is a convex combination of the functions $\chi_{[1,\infty)}-\chi_{[s,\infty)}$ for $1/4\le s\le 4$. Then, Fubini's theorem gives

$$\begin{aligned}T_\epsilon\nu(x)-T_{\varphi_\epsilon}\nu(x)&=\int\Big(\chi_{[1,\infty)}(|x-y|^2/\epsilon^2)-\varphi_{\mathbb{R}}(|x-y|^2/\epsilon^2)\Big)K(x-y)\,d\nu(y)\\ &=\int_{1/4}^4\varphi_{\mathbb{R}}'(s)\int(\chi_{[\epsilon,\infty)}-\chi_{[\epsilon\sqrt{s},\infty)})(|x-y|)K(x-y)\,d\nu(y)\,ds\\ &=\int_{1/4}^4\varphi_{\mathbb{R}}'(s)\Big(T_\epsilon\nu(x)-T_{\epsilon\sqrt{s}}\nu(x)\Big)\,ds.\end{aligned}\tag{31}$$

It is easy to see that

$$\Big(\sum_{m\in\mathbb{Z}}|T_{\epsilon_m}\nu(x)-T_{\epsilon_m\sqrt{s}}\nu(x)|^\rho\Big)^{1/\rho}\lesssim(\mathcal{V}_\rho^{\mathcal{S}}\circ\mathcal{T})\nu(x)\tag{32}$$

for all $s\in[1/4,4]$ with uniform bounds, where $\{\epsilon_m\}_{m\in\mathbb{Z}}$ is any sequence such that $\epsilon_m\in I_m$ for all $m\in\mathbb{Z}$. Using (31), Minkowski's integral inequality and (32), we get

$$\begin{aligned}&\sup_{\substack{\{\epsilon_m\}:\,\epsilon_m\in I_m\\ \text{for all } m\in\mathbb{Z}}}\Big(\sum_{m\in\mathbb{Z}}|T_{\epsilon_m}\nu(x)-T_{\varphi_{\epsilon_m}}\nu(x)|^\rho\Big)^{1/\rho}\\ &\qquad\le\sup_{\substack{\{\epsilon_m\}:\,\epsilon_m\in I_m\\ \text{for all } m\in\mathbb{Z}}}\int_{1/4}^4\varphi_{\mathbb{R}}'(s)\Big(\sum_{m\in\mathbb{Z}}|T_{\epsilon_m}\nu(x)-T_{\epsilon_m\sqrt{s}}\nu(x)|^\rho\Big)^{1/\rho}ds\\ &\qquad\lesssim\int_{1/4}^4\varphi_{\mathbb{R}}'(s)(\mathcal{V}_\rho^{\mathcal{S}}\circ\mathcal{T})\nu(x)\,ds\lesssim(\mathcal{V}_\rho^{\mathcal{S}}\circ\mathcal{T})\nu(x).\end{aligned}\tag{33}$$

Finally, applying (33) to (30) yields

$$(\mathcal{V}_\rho^{\mathcal{L}}\circ\mathcal{T})\nu(x)\lesssim(\mathcal{V}_\rho\circ\mathcal{T}_\varphi)\nu(x)+(\mathcal{V}_\rho^{\mathcal{S}}\circ\mathcal{T})\nu(x),$$

and Theorem 3.1(ii) follows by Theorems 3.2(ii) and 3.1(i). □

Proof of Theorem 3.1(i). We have to prove that there exists some constant $C>0$ such that

$$\mu\big(\{x\in\mathbb{R}^d\,:\,(\mathcal{V}_\rho^{\mathcal{S}}\circ\mathcal{T})\nu(x)>\lambda\}\big)\le\frac{C}{\lambda}\,\|\nu\|\tag{34}$$

for all $\nu \in M(\mathbb{R}^d)$ and all $\lambda > 0$. The proof of (34) combines the Calderón-Zygmund decomposition developed in Lemma 2.1, the corona decomposition of μ described in Subsection 2.3, and other standard techniques for proving variational inequalities. We will start following the lines of the proof of [15, Theorem 1.4], until the application of the corona decomposition.

Since $\mathcal{V}_\rho^\mathcal{S} \circ \mathcal{T}$ is sublinear, we can assume without loss of generality that ν is a positive measure. Let us first check that we can also assume both μ and ν to be compactly supported. Given $\nu \in M(\mathbb{R}^d)$ and $M \in \mathbb{N}$, set

$$\nu_M := \chi_{B(0,2^M)}\nu.$$

If $\operatorname{diam}(\operatorname{supp}\mu) < +\infty$ then μ is compactly supported. In case $\operatorname{diam}(\operatorname{supp}\mu) = +\infty$ we are going to restrict μ to a set $K_N \subset \mathbb{R}^d$ such that $\mu|_{K_N}$ it is still uniformly rectifiable (with constants independent of N). For this purpose, for each $N \in \mathbb{N}$ consider the family of cubes $P_i^N \in \mathcal{D}_{-N}^\mu$, $i \in I_N$, (thus $\ell(P_i^N) = 2^N$ for all $i \in I_N$) such that $B(0, 2^N) \cap P_i^N \neq \varnothing$. We denote

$$K_N = \bigcup_{i \in I_N} P_i^N \quad \text{and} \quad \mu_N = \mu|_{K_N}.$$

It is immediate to check that $\mu|_{P_i^N}$ is uniformly rectifiable for each i, N. Since K_N is a finite union of uniformly rectifiable sets (because $\# I_N$ is uniformly bounded), μ_N is also uniformly rectifiable, with constants independent of N.

Suppose that there exists some constant $C > 0$ such that

$$\mu_N\big(\{x \in \mathbb{R}^d : (\mathcal{V}_\rho^\mathcal{S} \circ \mathcal{T})\nu_M(x) > \lambda\}\big) \leq \frac{C}{\lambda} \|\nu_M\|$$

for all $\lambda > 0$, all $\nu \in M(\mathbb{R}^d)$ and all $M, N \in \mathbb{N}$. This implies that

$$\mu\big(\{x \in B(0, 2^N) : (\mathcal{V}_\rho^\mathcal{S} \circ \mathcal{T})\nu_M(x) > \lambda\}\big) \leq \frac{C}{\lambda} \|\nu_M\| \tag{35}$$

for all $\lambda > 0$, all $\nu \in M(\mathbb{R}^d)$ and all $M, N \in \mathbb{N}$. It is not hard to show that

$$\big|(\mathcal{V}_\rho^\mathcal{S} \circ \mathcal{T})\nu(x) - (\mathcal{V}_\rho^\mathcal{S} \circ \mathcal{T})\nu_N(x)\big| \leq \frac{C'}{(2^M - 2^N)^n}\, \nu\big(\mathbb{R}^d \setminus B(0, 2^M)\big)$$

for all $x \in B(0, 2^N)$ and all $M > N > 1$. In particular, if $M \to \infty$ then $(\mathcal{V}_\rho^\mathcal{S} \circ \mathcal{T})\nu_M(x) \to (\mathcal{V}_\rho^\mathcal{S} \circ \mathcal{T})\nu(x)$ uniformly in $B(0, 2^N)$. Since (35) holds for ν_M by assumption, we deduce that it also holds for ν. Now, by letting $N \to \infty$ and using monotone convergence, (35) with ν_M replaced by ν yields (34), as desired. In conclusion, for proving the theorem, we only have to verify (34) when μ and ν have compact support. Moreover, since (34) obviously holds for $\lambda \leq 2^{d+1}\|\nu\|/\|\mu\|$, we can also restrict ourselves to the case $\lambda > 2^{d+1}\|\nu\|/\|\mu\|$.

We are going to verify that we can assume (15), which will allows us to use (16) in the sequel, when we pursue the Calderón-Zygmund decomposition of ν with respect to μ. Let $M := \operatorname{diam}(\operatorname{supp}\mu) < +\infty$ and set $\nu_c := \chi_{\mathbb{R}^d \setminus \mathcal{U}_M(\operatorname{supp}\mu)}\nu$. Then $\operatorname{dist}(\operatorname{supp}\nu_c, \operatorname{supp}\mu) \geq M$. By Chebyshev's inequality,

$$\begin{aligned} \mu\big(\{x \in \mathbb{R}^d : (\mathcal{V}_\rho^\mathcal{S} \circ \mathcal{T})\nu_c(x) > \lambda\}\big) &\leq \frac{1}{\lambda} \int (\mathcal{V}_\rho^\mathcal{S} \circ \mathcal{T})\nu_c(x)\, d\mu(x) \\ &\leq \frac{C}{\lambda} \iint |x - y|^{-n}\, d\nu_c(y)\, d\mu(x) \leq \frac{C}{M^n \lambda} \|\nu_c\|\|\mu\|. \end{aligned} \tag{36}$$

For any $x \in \operatorname{supp}\mu$, $\|\mu\| = \mu(B(x, M)) \lesssim M^n$ by the AD regularity assumption on μ. Thus (36) yields

$$\mu\big(\{x \in \mathbb{R}^d : (\mathcal{V}_\rho^\mathcal{S} \circ \mathcal{T})\nu_c(x) > \lambda\}\big) \leq \frac{C}{\lambda} \|\nu_c\| \leq \frac{C}{\lambda} \|\nu\|, \tag{37}$$

with C independent of M. Note that $\nu=\nu_c+(\nu-\nu_c)$ and $\operatorname{supp}(\nu-\nu_c)\subset\mathcal{U}_{\operatorname{diam}(\operatorname{supp}\mu)}(\operatorname{supp}\mu)$. Using that $\mathcal{V}_\rho^{\mathcal{S}}\circ\mathcal{T}$ is sublinear and (37) we see that, in order to prove the theorem, it is enough to show that

$$\mu\big(\big\{x\in\mathbb{R}^d\,:\,(\mathcal{V}_\rho^{\mathcal{S}}\circ\mathcal{T})(\nu-\nu_c)(x)>\lambda\big\}\big)\leq\frac{C}{\lambda}\,\|\nu\|,$$

that is, we can assume that ν satisfies (15). In conclusion, for proving (34), from now on we assume that both μ and ν are compactly supported and they satisfy (15).

Let $\{Q_j\}_j$ be the almost disjoint family of cubes of Lemma 2.1, and set $\Omega:=\bigcup_j Q_j$ and $R_j:=6Q_j$. Then we can write $\nu=g\mu+\nu_b$, with

$$g\mu:=\chi_{\mathbb{R}^d\setminus\Omega}\nu+\sum_j b_j\mu\quad\text{and}\quad\nu_b:=\sum_j\nu_b^j:=\sum_j\left(w_j\nu-b_j\mu\right),$$

where the b_j's satisfy (11), (12) and (13), and $w_j:=\chi_{Q_j}\left(\sum_k\chi_{Q_k}\right)^{-1}$. Since (15) holds, in the sequel we can also assume that (16) holds.

Since $\mathcal{V}_\rho^{\mathcal{S}}\circ\mathcal{T}$ is sublinear,

$$\begin{aligned}&\mu\big(\big\{x\in\mathbb{R}^d\,:\,(\mathcal{V}_\rho^{\mathcal{S}}\circ\mathcal{T})\nu(x)>\lambda\big\}\big)\\&\quad\leq\mu\big(\big\{x\in\mathbb{R}^d\,:\,(\mathcal{V}_\rho^{\mathcal{S}}\circ\mathcal{T}^\mu)g(x)>\lambda/2\big\}\big)+\mu\big(\big\{x\in\mathbb{R}^d\,:\,(\mathcal{V}_\rho^{\mathcal{S}}\circ\mathcal{T})\nu_b(x)>\lambda/2\big\}\big).\end{aligned}\tag{38}$$

We obviously have $\mathcal{V}_\rho^{\mathcal{S}}\circ\mathcal{T}^\mu\leq\mathcal{V}_\rho\circ\mathcal{T}^\mu$, so Theorem 3.2($i$) yields that $\mathcal{V}_\rho^{\mathcal{S}}\circ\mathcal{T}^\mu$ is bounded in $L^2(\mu)$. Note that $|g|\leq C\lambda$ by (10) and (13). Hence, using (12),

$$\begin{aligned}\mu\big(\big\{x\in\mathbb{R}^d:(\mathcal{V}_\rho^{\mathcal{S}}\circ\mathcal{T}^\mu)g(x)>\lambda/2\big\}\big)&\lesssim\frac{1}{\lambda^2}\int|(\mathcal{V}_\rho^{\mathcal{S}}\circ\mathcal{T}^\mu)g|^2\,d\mu\lesssim\frac{1}{\lambda^2}\int|g|^2\,d\mu\\&\lesssim\frac{1}{\lambda}\int|g|\,d\mu\leq\frac{1}{\lambda}\Big(\nu(\mathbb{R}^d\setminus\Omega)+\sum_j\int_{R_j}|b_j|\,d\mu\Big)\\&\leq\frac{1}{\lambda}\Big(\nu(\mathbb{R}^d\setminus\Omega)+\sum_j\nu(Q_j)\Big)\lesssim\frac{\|\nu\|}{\lambda}.\end{aligned}\tag{39}$$

Set $\widehat{\Omega}:=\bigcup_j 2Q_j$. By (8), we have $\mu(\widehat{\Omega})\leq\sum_j\mu(2Q_j)\lesssim\lambda^{-1}\sum_j\nu(Q_j)\lesssim\lambda^{-1}\|\nu\|$. We are going to prove that

$$\mu\big(\big\{x\in\mathbb{R}^d\setminus\widehat{\Omega}\,:\,(\mathcal{V}_\rho^{\mathcal{S}}\circ\mathcal{T})\nu_b(x)>\lambda/2\big\}\big)\lesssim\frac{\|\nu\|}{\lambda}.\tag{40}$$

Then (34) follows directly from (38), (39), (40) and the estimate $\mu(\widehat{\Omega})\lesssim\lambda^{-1}\|\nu\|$ abovementioned, finishing the proof of Theorem 3.1(i).

To prove (40), given $x\in\mathbb{R}^d\setminus\widehat{\Omega}$ we first write

$$(\mathcal{V}_\rho^{\mathcal{S}}\circ\mathcal{T})\nu_b(x)\leq(\mathcal{V}_\rho^{\mathcal{S}}\circ\mathcal{T})\Big(\sum_j\chi_{2R_j}(x)\nu_b^j\Big)(x)+(\mathcal{V}_\rho^{\mathcal{S}}\circ\mathcal{T})\Big(\sum_j\chi_{\mathbb{R}^d\setminus 2R_j}(x)\nu_b^j\Big)(x).\tag{41}$$

Notice that $\chi_{2R_j}(x)$ and $\chi_{\mathbb{R}^d\setminus 2R_j}(x)$ are evaluated at the fixed point x on the right hand side.

The first term on the right hand side of (41) is easily handled using the $L^2(\mu)$ boundedness of $\mathcal{V}_\rho^{\mathcal{S}}\circ\mathcal{T}^\mu$ and standard estimates. More precisely, since $\mathcal{V}_\rho^{\mathcal{S}}\circ\mathcal{T}$ is sublinear,

$$\begin{aligned}&(\mathcal{V}_\rho^{\mathcal{S}}\circ\mathcal{T})\Big(\sum_j\chi_{2R_j}(x)\nu_b^j\Big)(x)\\&\qquad\leq\sum_j\chi_{2R_j}(x)(\mathcal{V}_\rho^{\mathcal{S}}\circ\mathcal{T}^\mu)b_j(x)+\sum_j\chi_{2R_j}(x)(\mathcal{V}_\rho^{\mathcal{S}}\circ\mathcal{T}^\nu)w_j(x)\end{aligned}\tag{42}$$

because $\nu_b^j = w_j\nu - b_j\mu$. On one hand, using Theorem 3.2(i), that $\mu(2R_j) \lesssim \mu(R_j)$ (by (16)) and (12), we get

$$\begin{aligned}\int_{2R_j} (\mathcal{V}_\rho^{\mathcal{S}} \circ \mathcal{T}^\mu) b_j \, d\mu &\leq \left(\int_{2R_j} |(\mathcal{V}_\rho^{\mathcal{S}} \circ \mathcal{T}^\mu) b_j|^2 \, d\mu \right)^{1/2} \mu(2R_j)^{1/2} \\ &\lesssim \|b_j\|_{L^2(\mu)} \mu(2R_j)^{1/2} \lesssim \|b_j\|_{L^\infty(\mu)} \mu(R_j) \lesssim \nu(Q_j).\end{aligned} \tag{43}$$

On the other hand, if $x \in 2R_j \setminus 2Q_j$ then $\operatorname{dist}(x, Q_j) \approx \ell(Q_j)$. Therefore, given $k \in \mathbb{Z}$,

$$B(x, 2^{-k}) \cap Q_j = \varnothing \iff \operatorname{dist}(x, Q_j) \geq 2^{-k} \iff \ell(Q_j) \gtrsim 2^{-k}. \tag{44}$$

Since the ℓ^ρ-norm is not bigger than the ℓ^1-norm for $\rho \geq 1$, and since $\operatorname{supp} w_j \subset Q_j$ and $|w_j| \leq 1$, from (44) and (4) we get

$$\begin{aligned}(\mathcal{V}_\rho^{\mathcal{S}} \circ \mathcal{T}^\nu) w_j(x) &\leq \sup_{\{\epsilon_m\}} \sum_{k \in \mathbb{Z}} \sum_{\epsilon_m, \epsilon_{m+1} \in I_k} |T^\nu_{\epsilon_m, \epsilon_{m+1}} w_j(x)| \\ &\lesssim \nu(Q_j) \sum_{k \in \mathbb{Z}:\, B(x, 2^{-k}) \cap Q_j \neq \varnothing} 2^{kn} \lesssim \nu(Q_j) \ell(Q_j)^{-n},\end{aligned}$$

and therefore, using again that $\mu(2R_j) \lesssim \mu(R_j) \approx \ell(R_j)^n \approx \ell(Q_j)^n$ by (16), we obtain

$$\int_{2R_j \setminus 2Q_j} (\mathcal{V}_\rho^{\mathcal{S}} \circ \mathcal{T}^\nu) w_j \, d\mu \lesssim \nu(Q_j) \ell(Q_j)^{-n} \mu(2R_j) \lesssim \nu(Q_j). \tag{45}$$

Finally, applying (43) and (45) to (42), we conclude that

$$\begin{aligned}&\int_{\mathbb{R}^d \setminus \widehat{\Omega}} (\mathcal{V}_\rho^{\mathcal{S}} \circ \mathcal{T}) \Big(\sum_j \chi_{2R_j}(x) \nu_b^j \Big)(x) \, d\mu(x) \\ &\quad \leq \sum_j \int_{2R_j} (\mathcal{V}_\rho^{\mathcal{S}} \circ \mathcal{T}^\mu) b_j \, d\mu + \sum_j \int_{2R_j \setminus 2Q_j} (\mathcal{V}_\rho^{\mathcal{S}} \circ \mathcal{T}^\nu) w_j \, d\mu \lesssim \sum_j \nu(Q_j) \lesssim \|\nu\|.\end{aligned} \tag{46}$$

Thanks to (41), (46) and Chebyshev's inequality, to prove (40) it is enough to verify that

$$\mu\left(\left\{ x \in \mathbb{R}^d \setminus \widehat{\Omega} \,:\, (\mathcal{V}_\rho^{\mathcal{S}} \circ \mathcal{T}) \Big(\sum_j \chi_{\mathbb{R}^d \setminus 2R_j}(x) \nu_b^j \Big)(x) > \lambda/4 \right\} \right) \lesssim \frac{\|\nu\|}{\lambda}. \tag{47}$$

Our task now is to prove (47). Given $x \in \operatorname{supp}\mu$, let $\{\epsilon_m\}_{m \in \mathbb{Z}}$ be a non-increasing sequence of positive numbers (which depends on x, i.e. $\epsilon_m \equiv \epsilon_m(x)$) such that

$$(\mathcal{V}_\rho^{\mathcal{S}} \circ \mathcal{T}) \Big(\sum_j \chi_{\mathbb{R}^d \setminus 2R_j}(x) \nu_b^j \Big)(x) \leq 2 \Big(\sum_{k \in \mathbb{Z}} \sum_{\epsilon_m, \epsilon_{m+1} \in I_k} \Big| \sum_j \chi_{\mathbb{R}^d \setminus 2R_j}(x) T_{\epsilon_m, \epsilon_{m+1}} \nu_b^j(x) \Big|^\rho \Big)^{1/\rho}. \tag{48}$$

Typically, the problem of the existence of such a sequence can be avoided by defining an auxiliary operator $\mathcal{V}_{\rho,I}^{\mathcal{S}} \circ \mathcal{T}$ along the same lines of $\mathcal{V}_\rho^{\mathcal{S}} \circ \mathcal{T}$ and requiring the supremum to be taken over a finite set of indices I (thus the supremum is a maximum in this case). One then proves the desired estimate for $\mathcal{V}_{\rho,I}^{\mathcal{S}} \circ \mathcal{T}$ with bounds independent of I and deduces the general result by taking the supremum over all finite sets I and using monotone convergence. For the sake of shortness, we omit the details.

Define the *interior* and *boundary sum*, respectively, by

$$S_i(x) := \bigg(\sum_{k\in\mathbb{Z}}\sum_{\epsilon_m,\epsilon_{m+1}\in I_k}\bigg|\sum_{j:\,R_j\subset A(x,\epsilon_{m+1},\epsilon_m)}\chi_{\mathbb{R}^d\setminus 2R_j}(x)T_{\epsilon_m,\epsilon_{m+1}}\nu_b^j(x)\bigg|^\rho\bigg)^{1/\rho},$$

$$S_b(x) := \bigg(\sum_{k\in\mathbb{Z}}\sum_{\epsilon_m,\epsilon_{m+1}\in I_k}\bigg|\sum_{j:\,R_j\cap\partial A(x,\epsilon_{m+1},\epsilon_m)\neq\varnothing}\chi_{\mathbb{R}^d\setminus 2R_j}(x)T_{\epsilon_m,\epsilon_{m+1}}\nu_b^j(x)\bigg|^\rho\bigg)^{1/\rho}.$$

If $R_j\cap A(x,\epsilon_{m+1},\epsilon_m)=\varnothing$ then $T_{\epsilon_m,\epsilon_{m+1}}\nu_b^j(x)=0$, thus

$$(\mathcal{V}_\rho^{\mathcal{S}}\circ\mathcal{T})\Big(\sum_j\chi_{\mathbb{R}^d\setminus 2R_j}(x)\nu_b^j\Big)(x)\leq 2(S_i+S_b)$$

by (48) and the triangle inequality, and so

$$\begin{aligned}(49)\qquad &\mu\bigg(\Big\{x\in\mathbb{R}^d\setminus\widehat{\Omega}\,:\,(\mathcal{V}_\rho^{\mathcal{S}}\circ\mathcal{T})\Big(\sum_j\chi_{\mathbb{R}^d\setminus 2R_j}(x)\nu_b^j\Big)(x)>\lambda/4\Big\}\bigg)\\ &\qquad\leq\mu\big(\{x\in\mathbb{R}^d\setminus\widehat{\Omega}\,:\,S_i(x)>\lambda/16\}\big)+\mu\big(\{x\in\mathbb{R}^d\setminus\widehat{\Omega}\,:\,S_b(x)>\lambda/16\}\big).\end{aligned}$$

To estimate $\mu\big(\{x\in\mathbb{R}^d\setminus\widehat{\Omega}\,:\,S_i(x)>\lambda/16\}\big)$ we use the fact that the ℓ^ρ-norm is not bigger than the ℓ^1-norm for $\rho\geq 1$, and that $\operatorname{supp}(\nu_b^j)\subset R_j$:

$$\begin{aligned}(50)\qquad S_i(x)&\leq\sum_{m\in\mathbb{Z}}\bigg|\sum_{j:\,R_j\subset A(x,\epsilon_{m+1},\epsilon_m)}\chi_{\mathbb{R}^d\setminus 2R_j}(x)T_{\epsilon_m,\epsilon_{m+1}}\nu_b^j(x)\bigg|\\ &\leq\sum_j\chi_{\mathbb{R}^d\setminus 2R_j}(x)\sum_{m\in\mathbb{Z}:\,A(x,\epsilon_{m+1},\epsilon_m)\supset R_j}|T_{\epsilon_m,\epsilon_{m+1}}\nu_b^j(x)|\leq\sum_j\chi_{\mathbb{R}^d\setminus 2R_j}(x)|T\nu_b^j(x)|,\end{aligned}$$

Recall that $\nu_b^j(R_j)=0$ and $\|\nu_b^j\|\lesssim\nu(Q_j)$ by (12). Thus, if z_j denotes the center of R_j, we have

$$\begin{aligned}(51)\qquad\int_{\mathbb{R}^d\setminus 2R_j}|T\nu_b^j|\,d\mu&\leq\int_{\mathbb{R}^d\setminus 2R_j}\int_{R_j}|K(x-y)-K(x-z_j)|\,d|\nu_b^j|(y)\,d\mu(x)\\ &\lesssim\int_{\mathbb{R}^d\setminus 2R_j}\int_{R_j}\frac{|y-z_j|}{|x-z_j|^{n+1}}\,d|\nu_b^j|(y)\,d\mu(x)\\ &\lesssim\|\nu_b^j\|\int_{\mathbb{R}^d\setminus 2R_j}\frac{\ell(R_j)}{|x-z_j|^{n+1}}\,d\mu(x)\lesssim\|\nu_b^j\|\lesssim\nu(Q_j).\end{aligned}$$

Finally, from Chebyshev's inequality, (50) and (51) we conclude that

$$(52)\qquad\mu\big(\{x\in\mathbb{R}^d\setminus\widehat{\Omega}\,:\,S_i(x)>\lambda/16\}\big)\leq\frac{16}{\lambda}\sum_j\int_{\mathbb{R}^d\setminus 2R_j}|T\nu_b^j|\,d\mu\lesssim\frac{1}{\lambda}\sum_j\nu(Q_j)\lesssim\frac{\|\nu\|}{\lambda}.$$

By (49), (52) and Chebyshev's inequality once again we see that, in order to prove (47), it is enough to show that

$$(53)\qquad\int_{\mathbb{R}^d\setminus\widehat{\Omega}}S_b^2\,d\mu\lesssim\lambda\|\nu\|.$$

The proof of this estimate is much more involved than the previous ones and requires the use of the corona decomposition of μ, that is, we need to introduce the splitting $\mathcal{D}^\mu=$

$\mathcal{B} \cup (\biguplus_{S\in\mathrm{Trs}} S)$. We denote

$$T_{j,m}(x) := \chi_{\mathbb{R}^d\setminus 2R_j}(x) T_{\epsilon_m,\epsilon_{m+1}} \nu_b^j(x).$$

Recall that for $P \in \mathcal{D}_k$ we write $I_P = [2^{-k-1}, 2^{-k})$. Since $\rho > 2$, the ℓ^ρ-norm is not bigger than the ℓ^2-norm, and we get

$$\begin{aligned}\int_{\mathbb{R}^d\setminus\widehat{\Omega}} S_b^2\, d\mu \le & \sum_{P\in\mathcal{B}} \int_{P\setminus\widehat{\Omega}} \sum_{\epsilon_m,\epsilon_{m+1}\in I_P} \Big| \sum_{j:\, R_j\cap\partial A(x,\epsilon_{m+1},\epsilon_m)\neq\varnothing} T_{j,m}(x)\Big|^2 d\mu(x) \\ & + \sum_{S\in\mathrm{Trs}} \sum_{P\in S} \int_{P\setminus\widehat{\Omega}} \sum_{\epsilon_m,\epsilon_{m+1}\in I_P} \Big| \sum_{j:\, R_j\cap\partial A(x,\epsilon_{m+1},\epsilon_m)\neq\varnothing} T_{j,m}(x)\Big|^2 d\mu(x).\end{aligned} \tag{54}$$

Observe that

$$|T_{j,m}(x)| \lesssim \ell(P)^{-n} \chi_{\mathbb{R}^d\setminus 2R_j}(x) |\nu_b^j|(A(x,\epsilon_{m+1},\epsilon_m)) \tag{55}$$

for all $\epsilon_m, \epsilon_{m+1} \in I_P$. If in addition $x \in P \setminus 2R_j$ and $R_j \cap \partial A(x,\epsilon_{m+1},\epsilon_m) \neq \varnothing$, taking into account that $\epsilon_m \approx \epsilon_{m+1} \approx \ell(P) \gtrsim \mathrm{dist}(x, R_j) \gtrsim \ell(R_j)$, we deduce that

$$R_j \subset B_P, \tag{56}$$

assuming the constant c_1 in (7) big enough.

Concerning the first term on the right hand side of (54), from (55) and using that $\|\nu_b^j\| \lesssim \nu(Q_j)$, that the Q_j's have bounded overlap and that $Q_j \subset B_P$ for all j such that $R_j \subset B_P$, we get

$$\begin{aligned}&\sum_{P\in\mathcal{B}} \int_{P\setminus\widehat{\Omega}} \sum_{\epsilon_m,\epsilon_{m+1}\in I_P} \Big| \sum_{j:\, R_j\cap\partial A(x,\epsilon_{m+1},\epsilon_m)\neq\varnothing} T_{j,m}(x)\Big|^2 d\mu(x) \\ &\qquad \lesssim \sum_{P\in\mathcal{B}} \int_P \Big(\sum_{\epsilon_m,\epsilon_{m+1}\in I_P} \sum_{j:\, R_j\subset B_P} \ell(P)^{-n} |\nu_b^j|(A(x,\epsilon_{m+1},\epsilon_m)) \Big)^2 d\mu(x) \\ &\qquad \lesssim \sum_{P\in\mathcal{B}} \int_P \Big(\sum_{j:\, R_j\subset B_P} \frac{\|\nu_b^j\|}{\ell(P)^n} \Big)^2 d\mu(x) \lesssim \sum_{P\in\mathcal{B}:\exists R_j\subset B_P} \Big(\frac{\nu(B_P)}{\ell(P)^n}\Big)^2 \ell(P)^n \lesssim \lambda \|\nu\|,\end{aligned} \tag{57}$$

where we also used Lemma 2.3 in the last inequality, because $\mathcal{B}$ is a Carleson family.

From now on, all our efforts are devoted to estimate the second term on the right hand side of (54).

Claim 3.3. *Assume c_1 in (7) is big enough, and let also $\alpha > 0$ be big enough depending on n, d, and on the AD regualrity constants of μ. Given $Q \in \mathrm{Top}_{\mathcal{G}}$, $P \in \mathrm{Tree}(Q)$ and $R_j \subset B_P$, at least one of the following holds:*

(*i*) *There exists $R \in \mathrm{Tree}(Q)$ such that $R \subset \alpha B_P$, $R_j \subset B_R$ and $\ell(R_j) \in I_R$.*
(*ii*) *There exists $R \in \partial\mathrm{Tree}(Q)$ such that $R \subset \alpha B_P$ and $R_j \subset B_R$.*

We postpone the proof of the preceding statement till the end of the proof of the theorem. Thanks to this claim, given $Q \in \mathrm{Top}_{\mathcal{G}}$ and $P \in \mathrm{Tree}(Q)$ we can split

$$\{j : R_j \subset B_P\} \subset J_1 \cup J_2,$$

where

$$\begin{aligned} J_1 &:= \{j : R_j \subset B_P,\ \exists\, R \in \mathrm{Tree}(Q) \text{ such that } R \subset \alpha B_P,\ R_j \subset B_R,\ \ell(R_j) \in I_R\}, \\ J_2 &:= \{j : R_j \subset B_P,\ \exists\, R \in \partial\mathrm{Tree}(Q) \text{ such that } R \subset \alpha B_P,\ R_j \subset B_R\}.\end{aligned}$$

Recall that if $x \in P \setminus 2R_j$, $\epsilon_m, \epsilon_{m+1} \in I_P$ and $R_j \cap \partial A(x, \epsilon_{m+1}, \epsilon_m) \neq \varnothing$ then $R_j \subset B_P$ (see (56)). Thus, we can decompose the second term on the right hand side of (54) using J_1 and J_2 as follows

$$\begin{aligned}
&\sum_{S\in\mathrm{Trs}}\sum_{P\in S}\int_{P\setminus\widehat{\Omega}}\sum_{\epsilon_m,\epsilon_{m+1}\in I_P}\Bigg|\sum_{j:\,R_j\cap\partial A(x,\epsilon_{m+1},\epsilon_m)\neq\varnothing}T_{j,m}(x)\Bigg|^2 d\mu(x)\\
&\quad\lesssim\sum_{Q\in\mathrm{Top}_\mathcal{G}}\sum_{P\in\mathrm{Tree}(Q)}\int_{P\setminus\widehat{\Omega}}\sum_{\epsilon_m,\epsilon_{m+1}\in I_P}\Bigg|\sum_{j\in J_1:\,R_j\cap\partial A(x,\epsilon_{m+1},\epsilon_m)\neq\varnothing}T_{j,m}(x)\Bigg|^2 d\mu(x)\\
&\qquad+\sum_{Q\in\mathrm{Top}_\mathcal{G}}\sum_{P\in\mathrm{Tree}(Q)}\int_{P\setminus\widehat{\Omega}}\sum_{\epsilon_m,\epsilon_{m+1}\in I_P}\Bigg|\sum_{j\in J_2:\,R_j\cap\partial A(x,\epsilon_{m+1},\epsilon_m)\neq\varnothing}T_{j,m}(x)\Bigg|^2 d\mu(x).
\end{aligned}\tag{58}$$

Despite that the arguments to estimate both terms on the right hand side of (58) are similar, we will deal with them separately, due to its different nature with respect to the structure of the corona decomposition.

Claim 3.4. *Let Q, P, x, ϵ_m and ϵ_{m+1} be as on the right hand side of* (58). *We have*

$$\begin{aligned}
&\Bigg|\sum_{j\in J_1:\,R_j\cap\partial A(x,\epsilon_{m+1},\epsilon_m)\neq\varnothing}|\nu_b^j|(A(x,\epsilon_{m+1},\epsilon_m))\Bigg|^2\\
&\qquad\lesssim\lambda\ell(P)^n\sum_{k:\,2^{-k}\lesssim\ell(P)}\left(\frac{2^{-k}}{\ell(P)}\right)^{1/2}\sum_{j\in J_1:\,\ell(R_j)\in I_k}|\nu_b^j|(A(x,\epsilon_{m+1},\epsilon_m)).
\end{aligned}\tag{59}$$

Given $j\in J_2$, denote by $R(j)\in\partial\mathrm{Tree}(Q)$ some cube such that $R(j)\subset\alpha B_P$ and $R_j\subset B_{R(j)}$, where $\alpha>0$ is as in Claim 3.3. *We have*

$$\begin{aligned}
&\Bigg|\sum_{j\in J_2:\,R_j\cap\partial A(x,\epsilon_{m+1},\epsilon_m)\neq\varnothing}|\nu_b^j|(A(x,\epsilon_{m+1},\epsilon_m))\Bigg|^2\\
&\qquad\lesssim\lambda^{1/2}\ell(P)^{n/2}\nu(B_P)^{1/2}\sum_{\substack{R\in\partial\mathrm{Tree}(Q):\\R\subset\alpha B_P}}\sum_{\substack{j\in J_2:\\R(j)=R}}\left(\frac{\ell(R)}{\ell(P)}\right)^{1/4}|\nu_b^j|(B_R\cap A(x,\epsilon_{m+1},\epsilon_m)).
\end{aligned}\tag{60}$$

Again we postpone the proof of the preceding claim till the end of the proof of the theorem.

For the case $j \in J_1$ in (58), using (55), (59) and (56) we get

$$\begin{aligned}
&\sum_{Q\in \mathrm{Top}_{\mathcal G}} \sum_{P\in \mathrm{Tree}(Q)} \int_{P\setminus\widehat\Omega} \sum_{\epsilon_m,\epsilon_{m+1}\in I_P} \Bigg| \sum_{j\in J_1:\, R_j\cap \partial A(x,\epsilon_{m+1},\epsilon_m)\neq\varnothing} T_{j,m}(x)\Bigg|^2 d\mu(x)\\
&\quad\lesssim \lambda \sum_{Q\in \mathrm{Top}_{\mathcal G}} \sum_{P\in \mathrm{Tree}(Q)} \ell(P)^{-n}\\
&\qquad\quad \times \int_{P\setminus\widehat\Omega} \sum_{\epsilon_m,\epsilon_{m+1}\in I_P} \sum_{k:\, 2^{-k}\lesssim \ell(P)} \left(\frac{2^{-k}}{\ell(P)}\right)^{1/2} \sum_{j\in J_1:\, \ell(R_j)\in I_k} |\nu_b^j|(A(x,\epsilon_{m+1},\epsilon_m))\, d\mu(x)\\
&\quad\lesssim \lambda \sum_{Q\in \mathrm{Top}_{\mathcal G}} \sum_{P\in \mathrm{Tree}(Q)} \sum_{k:\, 2^{-k}\lesssim \ell(P)} \left(\frac{2^{-k}}{\ell(P)}\right)^{1/2} \sum_{j\in J_1:\, \ell(R_j)\in I_k} \|\nu_b^j\|\\
&\quad\lesssim \lambda \sum_j \nu(Q_j) \sum_{k:\, \ell(R_j)\in I_k} \sum_{\substack{P\in \mathcal D^\mu:\, R_j\subset B_P\\ 2^{-k}\lesssim \ell(P)}} \left(\frac{2^{-k}}{\ell(P)}\right)^{1/2} \lesssim \lambda \sum_j \nu(Q_j) \lesssim \lambda \|\nu\|.
\end{aligned}\tag{61}$$

In the third inequality we used that $j \in J_1$ implies that $R_j \subset B_P$.

Concerning the case $j \in J_2$ in (58), by (55) and (60) we see that

$$\begin{aligned}
&\sum_{Q\in \mathrm{Top}_{\mathcal G}} \sum_{P\in \mathrm{Tree}(Q)} \int_{P\setminus\widehat\Omega} \sum_{\epsilon_m,\epsilon_{m+1}\in I_P} \Bigg| \sum_{j\in J_2:\, R_j\cap \partial A(x,\epsilon_{m+1},\epsilon_m)\neq\varnothing} T_{j,m}(x)\Bigg|^2 d\mu(x)\\
&\quad\lesssim \lambda^{1/2} \sum_{Q\in \mathrm{Top}_{\mathcal G}} \sum_{P\in \mathrm{Tree}(Q)} \ell(P)^{-n} \left(\frac{\nu(B_P)}{\ell(P)^n}\right)^{1/2}\\
&\qquad\quad \times \int_{P\setminus\widehat\Omega} \sum_{\epsilon_m,\epsilon_{m+1}\in I_P} \sum_{\substack{R\in \partial\mathrm{Tree}(Q):\\ R\subset \alpha B_P}} \sum_{\substack{j\in J_2:\\ R(j)=R}} \left(\frac{\ell(R)}{\ell(P)}\right)^{1/4} |\nu_b^j|(B_R\cap A(x,\epsilon_{m+1},\epsilon_m))\, d\mu(x)\\
&\quad\lesssim \lambda^{1/2} \sum_{Q\in \mathrm{Top}_{\mathcal G}} \sum_{P\in \mathrm{Tree}(Q)} \left(\frac{\nu(B_P)}{\ell(P)^n}\right)^{1/2} \sum_{\substack{R\in \partial\mathrm{Tree}(Q):\\ R\subset \alpha B_P}} \sum_{\substack{j\in J_2:\\ R(j)=R}} \left(\frac{\ell(R)}{\ell(P)}\right)^{1/4} \|\nu_b^j\|\\
&\quad\lesssim \lambda^{1/2} \sum_{Q\in \mathrm{Top}_{\mathcal G}} \sum_{P\in \mathrm{Tree}(Q)} \sum_{\substack{R\in \partial\mathrm{Tree}(Q):\, R\subset \alpha B_P\\ \exists R_j\subset B_R}} \left(\frac{\ell(R)}{\ell(P)}\right)^{1/4} \left(\frac{\nu(B_P)}{\ell(P)^n}\right)^{1/2} \left(\frac{\nu(B_R)}{\ell(R)^n}\right) \ell(R)^n,
\end{aligned}$$

where we also used in the last inequality above that $\|\nu_b^j\| \lesssim \nu(Q_j)$ and that the Q_j's have bounded overlap. Since $a^{1/2}b \lesssim a^{3/2} + b^{3/2}$ for all $a, b \geq 0$, we obtain

$$\sum_{Q\in\mathrm{Top}_{\mathcal{G}}} \sum_{P\in\mathrm{Tree}(Q)} \int_{P\setminus\widehat{\Omega}} \sum_{\epsilon_m,\epsilon_{m+1}\in I_P} \Bigg| \sum_{j\in J_2:\, R_j\cap\partial A(x,\epsilon_{m+1},\epsilon_m)\neq\varnothing} T_{j,m}(x)\Bigg|^2 d\mu(x)$$
$$\lesssim \lambda^{1/2} \sum_{Q\in\mathrm{Top}_{\mathcal{G}}} \sum_{P\in\mathrm{Tree}(Q)} \sum_{\substack{R\in\partial\mathrm{Tree}(Q):\, R\subset\alpha B_P\\ \exists R_j\subset B_R}} \left(\left(\frac{\nu(B_P)}{\ell(P)^n}\right)^{3/2} + \left(\frac{\nu(B_R)}{\ell(R)^n}\right)^{3/2}\right)\left(\frac{\ell(R)}{\ell(P)}\right)^{1/4} \ell(R)^n$$
$$\lesssim \lambda^{1/2} \sum_{Q\in\mathrm{Top}_{\mathcal{G}}} \sum_{\substack{P\in\mathrm{Tree}(Q)\\ \exists R_j\subset cB_P}} \left(\frac{\nu(B_P)}{\ell(P)^n}\right)^{3/2} a_P + \lambda^{1/2} \sum_{Q\in\mathrm{Top}_{\mathcal{G}}} \sum_{\substack{R\in\partial\mathrm{Tree}(Q)\\ \exists R_j\subset B_R}} \left(\frac{\nu(B_R)}{\ell(R)^n}\right)^{3/2} \ell(R)^n,$$

where we have set $a_P := \sum_{R\in\partial\mathrm{Tree}(Q):\, R\subset\alpha B_P} (\ell(R)/\ell(P))^{1/4}\ell(R)^n$ whenever $P \in \mathrm{Tree}(Q)$ for some $Q \in \mathrm{Top}_{\mathcal{G}}$ (otherwise, we set $a_P = 0$). Since $\partial\mathrm{Trs}$ is a Carleson family, we see that the a_P's satisfy a Carleson packing condition because, for a given $T \in \mathcal{D}^\mu$,

$$\sum_{P\subset T} a_P \leq \sum_{P\subset T} \sum_{Q\in\mathrm{Top}_{\mathcal{G}}:\, P\in\mathrm{Tree}(Q)} \sum_{R\in\partial\mathrm{Tree}(Q):\, R\subset\alpha B_P} \left(\frac{\ell(R)}{\ell(P)}\right)^{1/4} \ell(R)^n$$
$$\leq \sum_{P\subset T} \sum_{R\in\partial\mathrm{Trs}:\, R\subset\alpha B_P\subset\alpha B_T} \left(\frac{\ell(R)}{\ell(P)}\right)^{1/4} \ell(R)^n$$
$$\leq \sum_{R\in\partial\mathrm{Trs}:\, R\subset\alpha B_T} \ell(R)^n \sum_{P\subset T:\, R\subset\alpha B_P} \left(\frac{\ell(R)}{\ell(P)}\right)^{1/4} \lesssim \sum_{R\in\partial\mathrm{Trs}:\, R\subset\alpha B_T} \ell(R)^n \lesssim \ell(T)^n.$$

Therefore,

$$\begin{aligned} &\sum_{Q\in\mathrm{Top}_{\mathcal{G}}} \sum_{P\in\mathrm{Tree}(Q)} \int_{P\setminus\widehat{\Omega}} \sum_{\epsilon_m,\epsilon_{m+1}\in I_P} \Bigg| \sum_{j\in J_2:\, R_j\cap\partial A(x,\epsilon_{m+1},\epsilon_m)\neq\varnothing} T_{j,m}(x)\Bigg|^2 d\mu(x) \\ &\qquad\lesssim \lambda^{1/2} \sum_{P\in\mathcal{D}^\mu} \left(\frac{\nu(B_P)}{\ell(P)^n}\right)^{3/2} \big(a_P + \ell(P)^n \chi_{\partial\mathrm{Trs}}(P)\big) \lesssim \lambda\|\nu\|, \end{aligned} \tag{62}$$

because the coefficients $a_P + \ell(P)^n\chi_{\partial\mathrm{Trs}}(P)$ satisfy a Carleson packing condition and thus we can use Lemma 2.3.

Finally, (53) follows from (54), (57), (58), (61) and (62), so Theorem 3.1(i) is proved except for the claims. □

Proof of Claim 3.3. Let $Q \in \mathrm{Top}_{\mathcal{G}}$, $P \in \mathrm{Tree}(Q)$ and $R_j \subset B_P$. For the purpose of the claim, we can assume that $\ell(Q) \geq \ell(R_j)$, otherwise we can take $R = Q$ which fulfills (ii). Without loss of generality, we can also assume that $\ell(P) \geq \ell(R_j)$ (recall that $R_j \subset B_P$, so $\ell(P) \gtrsim \ell(R_j)$). Otherwise, we replace P by a suitable ancestor from $\mathrm{Tree}(Q)$ with side length comparable to $\ell(R_j)$, which must exists thanks to the previous assumption $\ell(Q) \geq \ell(R_j)$.

Let $R \in \mathrm{Tree}(Q)$ be a cube with minimal side length such that $R_j \subset B_R$ and $\ell(R) \geq \ell(R_j)$, that is, $\ell(R) \leq \ell(S)$ for all $S \in \mathrm{Tree}(Q)$ with $R_j \subset B_S$ and $\ell(S) \geq \ell(R_j)$. In particular, notice that P may coincide with R, and in any case $\ell(R) \leq \ell(P)$. If $\ell(R_j) \in I_R$, that is $\ell(R) \geq \ell(R_j) \geq \ell(R)/2$, then R fulfills (i) if α is big enough, and we are done. On the contrary, assume that $\ell(R)/2 > \ell(R_j)$. Since $R_j \subset B_R$ and $R_j \cap \mathrm{supp}\mu \neq \varnothing$, there exists $R' \in \mathcal{D}^\mu$ such that $\ell(R') = \ell(R)$, $\mathrm{dist}(R', R) \lesssim \ell(R)$ and $R' \cap R_j \neq \varnothing$. Therefore, there exists a son R'' of R' such that $R''\cap R_j\neq\varnothing$, so $R_j\subset B_{R''}$ if c_1 is big enough. By the minimality of R, we must have $R''\notin \mathrm{Tree}(Q)$, thus $R\in\partial\mathrm{Tree}(Q)$ if $A\geq 1$ in (28) is big enough, and then (ii) is fulfilled for some α big enough. $\square$

Proof of Claim 3.4. Let us first prove (59). If $j\in J_1$ then $R_j\subset B_P$ and, in particular, $\ell(R_j)\lesssim\ell(P)$. Thus, by Cauchy-Schwarz inequality,

$$
\begin{aligned}
\Big|\sum_{j\in J_1:\, R_j\cap\partial A(x,\epsilon_{m+1},\epsilon_m)\neq\varnothing} &|\nu_b^j|(A(x,\epsilon_{m+1},\epsilon_m))\Big|^2\\
&=\Big|\sum_{k:\,2^{-k}\lesssim\ell(P)}\left(\frac{2^{-k}}{\ell(P)}\right)^{1/4}\left(\frac{\ell(P)}{2^{-k}}\right)^{1/4}\sum_{\substack{j\in J_1:\,\ell(R_j)\in I_k\\ R_j\cap\partial A(x,\epsilon_{m+1},\epsilon_m)\neq\varnothing}}|\nu_b^j|(A(x,\epsilon_{m+1},\epsilon_m))\Big|^2\\
&\lesssim\sum_{k:\,2^{-k}\lesssim\ell(P)}\left(\frac{\ell(P)}{2^{-k}}\right)^{1/2}\Big|\sum_{\substack{j\in J_1:\,\ell(R_j)\in I_k\\ R_j\cap\partial A(x,\epsilon_{m+1},\epsilon_m)\neq\varnothing}}|\nu_b^j|(A(x,\epsilon_{m+1},\epsilon_m))\Big|^2.
\end{aligned}\tag{63}
$$

Using that $|\nu_b^j|(A(x,\epsilon_{m+1},\epsilon_m))\lesssim\nu(Q_j)$ and that the Q_j's have bounded overlap, from the definition of J_1 we see that

$$
\sum_{\substack{j\in J_1:\,\ell(R_j)\in I_k\\ R_j\cap\partial A(x,\epsilon_{m+1},\epsilon_m)\neq\varnothing}}|\nu_b^j|(A(x,\epsilon_{m+1},\epsilon_m))\lesssim\sum_{\substack{R\in\mathrm{Tree}(Q):\,\ell(R)\in I_k,\\ B_R\cap\partial A(x,\epsilon_{m+1},\epsilon_m)\neq\varnothing,\\ R\subset\alpha B_P,\,\exists R_j\subset B_R}}\nu(B_R).\tag{64}
$$

If $6Q_j=R_j\subset B_R$ then $\nu(6Q_j)\leq\nu(B_R)\lesssim\lambda\mu(B_R)\lesssim\lambda\mu(R)$ by (9). From (64) we infer

$$
\sum_{\substack{j\in J_1:\,\ell(R_j)\in I_k\\ R_j\cap\partial A(x,\epsilon_{m+1},\epsilon_m)\neq\varnothing}}|\nu_b^j|(A(x,\epsilon_{m+1},\epsilon_m))\lesssim\lambda\sum_{\substack{R\in\mathrm{Tree}(Q):\,\ell(R)\in I_k,\\ B_R\cap\partial A(x,\epsilon_{m+1},\epsilon_m)\neq\varnothing,\\ R\subset\alpha B_P,\,\exists R_j\subset B_R}}\mu(R).\tag{65}
$$

We want to show that the right hand side of (65) can be estimated by $\lambda 2^{-k}\ell(P)^{n-1}$. To this end, we can suppose that $\ell(R)\leq\ell(P)$, otherwise the estimate becomes trivial because we are already assuming $2^{-k}\lesssim\ell(P)$ and $\ell(R)\in I_k$ (so in this last case there is only a finite and uniformly bounded number of terms in the sum above). Suppose now that $\ell(R)\leq\ell(P)$. Since $R\subset\alpha B_P$ then $R\subset\bigcup_{P'\in V(P)}P'$ if the constant C_1 in the definition of $V(P)$ is big enough. Thus, $R\subset P'$ for some $P'\in V(P)$. Note that $P'\in\mathrm{Tree}(Q)$ because $R\in\mathrm{Tree}(Q)$, and so we finally get $R\in\mathrm{Tree}(P')$. Then, from (65) and the estimates on annuli from Lemma 2.5 we obtain

$$
\begin{aligned}
\sum_{\substack{j\in J_1:\,\ell(R_j)\in I_k\\ R_j\cap\partial A(x,\epsilon_{m+1},\epsilon_m)\neq\varnothing}}|\nu_b^j|(A(x,\epsilon_{m+1},\epsilon_m))&\lesssim\lambda\sum_{P'\in V(P)}\sum_{\substack{R\in\mathrm{Tree}(P'):\,\ell(R)\in I_k,\\ B_R\cap\partial A(x,\epsilon_{m+1},\epsilon_m)\neq\varnothing}}\mu(R)\\
&\lesssim\lambda 2^{-k}\ell(P)^{n-1},
\end{aligned}\tag{66}
$$

as desired. Finally, (59) follows from (63) and (66).

Let us turn our attention to (60) now. Recall that, given $j\in J_2$, $R(j)\in\partial\mathrm{Tree}(Q)$ denotes some cube such that $R(j)\subset\alpha B_P$ and $R_j\subset B_{R(j)}$. Similarly to (63), by Hölder's inequality we get

$$
\begin{aligned}
&\Bigg| \sum_{\substack{j\in J_2:\\ R_j\cap\partial A(x,\epsilon_{m+1},\epsilon_m)\neq\varnothing}} |\nu_b^j|(A(x,\epsilon_{m+1},\epsilon_m))\Bigg|^{3/2}\\
&\quad\leq \Bigg| \sum_{\substack{R\in\partial\mathrm{Tree}(Q):\, R\subset\alpha B_P\\ B_R\cap\partial A(x,\epsilon_{m+1},\epsilon_m)\neq\varnothing}} \sum_{j\in J_2:\, R(j)=R} |\nu_b^j|(B_R\cap A(x,\epsilon_{m+1},\epsilon_m))\Bigg|^{3/2}\\
&\quad\lesssim \sum_{k:\,2^{-k}\lesssim\ell(P)} \left(\frac{\ell(P)}{2^{-k}}\right)^{1/4} \Bigg| \sum_{\substack{R\in\partial\mathrm{Tree}(Q):\\ R\subset\alpha B_P,\,\ell(R)=2^{-k}\\ B_R\cap\partial A(x,\epsilon_{m+1},\epsilon_m)\neq\varnothing}} \sum_{j\in J_2:\, R(j)=R} |\nu_b^j|(B_R\cap A(x,\epsilon_{m+1},\epsilon_m))\Bigg|^{3/2}.
\end{aligned}
\tag{67}
$$

For the cubes $R=R(j)$ in the last sum above, note that $R_j\subset B_R$ (see the definition of J_2). So, as we did before (65), $\nu(B_R)\lesssim\lambda\mu(B_R)\lesssim\lambda\mu(R)$ by (9). Using that $\|\nu_b^j\|\lesssim\nu(Q_j)$, that the Q_j's have bounded overlap and that $\nu(B_R)\lesssim\lambda\mu(B_R)$, we deduce that

$$
\begin{aligned}
&\sum_{\substack{R\in\partial\mathrm{Tree}(Q):\, R\subset\alpha B_P,\,\ell(R)=2^{-k}\\ B_R\cap\partial A(x,\epsilon_{m+1},\epsilon_m)\neq\varnothing}} \sum_{j\in J_2:\, R(j)=R} |\nu_b^j|(B_R\cap A(x,\epsilon_{m+1},\epsilon_m))\\
&\qquad\lesssim \sum_{\substack{R\in\partial\mathrm{Tree}(Q):\\ R\subset\alpha B_P,\,\ell(R)=2^{-k}\\ B_R\cap\partial A(x,\epsilon_{m+1},\epsilon_m)\neq\varnothing}} \sum_{j\in J_2:\, R(j)=R} \nu(Q_j) \lesssim \sum_{\substack{R\in\partial\mathrm{Tree}(Q):\\ R\subset\alpha B_P,\,\ell(R)=2^{-k}\\ B_R\cap\partial A(x,\epsilon_{m+1},\epsilon_m)\neq\varnothing}} \nu(B_R)\\
&\qquad\lesssim \lambda \sum_{\substack{R\in\partial\mathrm{Tree}(Q):\, R\subset\alpha B_P,\,\ell(R)=2^{-k}\\ B_R\cap\partial A(x,\,\epsilon_{m+1},\epsilon_m)\neq\varnothing}} \mu(R).
\end{aligned}
\tag{68}
$$

As we did in the case of J_1, now we want to show that the last term above can be estimated by $\lambda 2^{-k}\ell(P)^{n-1}$. We argue similarly to what we did before (66). If R is as in the right hand side of the last inequality in (68), since $R\subset\alpha B_P$ we have $\ell(R)\lesssim\ell(P)$, and thus we can assume $\ell(R)\leq\ell(P)$ (otherwise the estimate that we want to show becomes trivial). Since $R\subset\alpha B_P$ then $R\subset\bigcup_{P'\in V(P)}P'$ if the constant C_1 in the definition of $V(P)$ is big enough. Thus, $R\subset P'$ for some $P'\in V(P)$ and $R\in\mathrm{Tree}(P')$ (recall that $R\in\partial\mathrm{Tree}(Q)$ implies $R\in\mathrm{Tree}(Q)$). Then, from (68) and the estimates on annuli from Lemma 2.5 we obtain

$$
\begin{aligned}
&\sum_{\substack{R\in\partial\mathrm{Tree}(Q):\, R\subset\alpha B_P,\,\ell(R)=2^{-k}\\ B_R\cap\partial A(x,\epsilon_{m+1},\epsilon_m)\neq\varnothing}} \sum_{j\in J_2:\, R(j)=R} |\nu_b^j|(B_R\cap A(x,\epsilon_{m+1},\epsilon_m))\\
&\qquad\lesssim \lambda \sum_{P'\in V(P)} \sum_{\substack{R\in\mathrm{Tree}(P'):\,\ell(R)\in I_k,\\ B_R\cap\partial A(x,\epsilon_{m+1},\epsilon_m)\neq\varnothing}} \mu(R) \lesssim \lambda 2^{-k}\ell(P)^{n-1},
\end{aligned}
\tag{69}
$$

as desired.

Combining (69) with (67) we get

$$\begin{aligned}
&\Bigg| \sum_{\substack{j\in J_2:\\ R_j\cap\partial A(x,\epsilon_{m+1},\epsilon_m)\neq\varnothing}} |\nu_b^j|(A(x,\epsilon_{m+1},\epsilon_m))\Bigg|^{3/2}\\
&\lesssim \lambda^{1/2}\ell(P)^{n/2}\sum_{k:\,2^{-k}\lesssim\ell(P)}\left(\frac{2^{-k}}{\ell(P)}\right)^{1/4}\sum_{\substack{R\in\partial\mathrm{Tree}(Q):\\ R\subset\alpha B_P,\,\ell(R)=2^{-k}\\ B_R\cap\partial A(x,\epsilon_{m+1},\epsilon_m)\neq\varnothing}}\ \sum_{\substack{j\in J_2:\\ R(j)=R}}|\nu_b^j|(B_R\cap A(x,\epsilon_{m+1},\epsilon_m)) \qquad (70)\\
&\lesssim \lambda^{1/2}\ell(P)^{n/2}\sum_{R\in\partial\mathrm{Tree}(Q):\,R\subset\alpha B_P}\ \sum_{j\in J_2:\,R(j)=R}\left(\frac{\ell(R)}{\ell(P)}\right)^{1/4}|\nu_b^j|(B_R\cap A(x,\epsilon_{m+1},\epsilon_m)).
\end{aligned}$$

Finally, (60) is a consequence of (70) and the trivial estimate

$$\sum_{j\in J_2:\,R_j\cap\partial A(x,\epsilon_{m+1},\epsilon_m)\neq\varnothing}|\nu_b^j|(A(x,\epsilon_{m+1},\epsilon_m))\lesssim\nu(B_P),$$

which holds if c_1 in (7) is big enough because $\|\nu_b^j\|\lesssim\nu(Q_j)$ and the Q_j's have bounded overlap.

□

4. $\mathcal{V}_\rho\circ\mathcal{T}^\mu : L^p(\mu)\to L^p(\mu)$ IS A BOUNDED OPERATOR FOR $1<p<\infty$

Under the assumptions of Theorem 1.1, the boundedness of $\mathcal{V}_\rho\circ\mathcal{T}^\mu$ in $L^p(\mu)$ for $1<p<2$ follows by interpolation, taking into account that it is bounded in $L^2(\mu)$ and from $L^1(\mu)$ to $L^{1,\infty}(\mu)$, by Theorem 3.2 and Theorem 3.1. So it only remains to prove the boundedness in $L^p(\mu)$ for $2<p<\infty$. This task is carried out in the next theorem.

Theorem 4.1. *Let μ be a uniformly n-rectifiable measure in $\mathbb{R}^d$. Let K be an odd kernel satisfying* (1) *and consider the operator T associated to K defined in* (2). *Then $\mathcal{V}_\rho\circ\mathcal{T}^\mu$ is a bounded operator in $L^p(\mu)$ for all $\rho>2$ and all $2<p<\infty$.*

Proof. We are going to prove that if μ is a uniformly n-rectifiable measure then $\mathcal{M}^\sharp_{\mathcal{D}^\mu}\circ\mathcal{V}_\rho\circ\mathcal{T}^\mu$ is a bounded operator in $L^p(\mu)$ for all $2<p<\infty$, where $\mathcal{M}^\sharp_{\mathcal{D}^\mu}$ denotes the dyadic sharp maximal function, that is,

$$\mathcal{M}^\sharp_{\mathcal{D}^\mu}f(x)=\sup_{D\in\mathcal{D}^\mu:\,x\in D}m_D|f-m_Df|.$$

The theorem will then follow from the fact that the maximal operator defined by $\mathcal{M}_{\mathcal{D}^\mu}f(x)=\sup_{D\in\mathcal{D}^\mu:\,x\in D}m_D|f|$ can be controlled in $L^p(\mu)$ norm by $\mathcal{M}^\sharp_{\mathcal{D}^\mu}$. That is, $\|\mathcal{M}_{\mathcal{D}^\mu}f\|_{L^p(\mu)}\lesssim\|\mathcal{M}^\sharp_{\mathcal{D}^\mu}f\|_{L^p(\mu)}$ (see [7, Lemma 6.9], for example).

Fix $f\in L^p(\mu)$ and $x_0\in\operatorname{supp}\mu$. Then,

$$(\mathcal{M}^\sharp_{\mathcal{D}^\mu}\circ\mathcal{V}_\rho\circ\mathcal{T}^\mu)f(x_0)=\sup_{D\in\mathcal{D}^\mu:\,x_0\in D}m_D|(\mathcal{V}_\rho\circ\mathcal{T}^\mu)f-m_D((\mathcal{V}_\rho\circ\mathcal{T}^\mu)f)|. \qquad (71)$$

Given $D\in\mathcal{D}^\mu$ such that $x_0\in D$, we decompose $f=f_1+f_2$ with $f_1:=f\chi_{3D}$ and $f_2:=f-f_1$. Since $\mathcal{V}_\rho\circ\mathcal{T}^\mu$ is sublinear and positive, $|(\mathcal{V}_\rho\circ\mathcal{T}^\mu)f-(\mathcal{V}_\rho\circ\mathcal{T}^\mu)f_2|\le(\mathcal{V}_\rho\circ\mathcal{T}^\mu)f_1$ and so $|(\mathcal{V}_\rho\circ\mathcal{T}^\mu)f-c|\le(\mathcal{V}_\rho\circ\mathcal{T}^\mu)f_1+|(\mathcal{V}_\rho\circ\mathcal{T}^\mu)f_2-c|$ for all $c\in\mathbb{R}$. If we take $c=(\mathcal{V}_\rho\circ\mathcal{T}^\mu)f_2(z_D)$, where z_D denotes the center of D (we may assume that $c<\infty$), then

$$\begin{aligned}m_D|(\mathcal{V}_\rho\circ\mathcal{T}^\mu)f-&m_D((\mathcal{V}_\rho\circ\mathcal{T}^\mu)f)|\\ &\leq 2m_D|(\mathcal{V}_\rho\circ\mathcal{T}^\mu)f-(\mathcal{V}_\rho\circ\mathcal{T}^\mu)f_2(z_D)|\\ &\lesssim m_D(\mathcal{V}_\rho\circ\mathcal{T}^\mu)f_1+m_D|(\mathcal{V}_\rho\circ\mathcal{T}^\mu)f_2-(\mathcal{V}_\rho\circ\mathcal{T}^\mu)f_2(z_D)|\\ &=: I_1+I_2.\end{aligned}\tag{72}$$

A good estimate for I_1 can be easily derived using Cauchy-Schwarz's inequality, Theorem 3.2(i) and that μ is n-AD regular. More precisely,

$$I_1\lesssim\left(\frac{1}{\mu(D)}\int_D|(\mathcal{V}_\rho\circ\mathcal{T}^\mu)f_1|^2\,d\mu\right)^{1/2}\lesssim\left(\frac{1}{\mu(D)}\int_{3D}|f|^2\,d\mu\right)^{1/2}\lesssim\mathcal{M}_2f(x_0).\tag{73}$$

The estimate of I_2 is much more involved. Given $x\in D$, by the triangle inequality we have

$$\begin{aligned}|(\mathcal{V}_\rho\circ\mathcal{T}^\mu)&f_2(x)-(\mathcal{V}_\rho\circ\mathcal{T}^\mu)f_2(z_D)|\\ &\leq\sup_{\{\epsilon_m\}_{m\in\mathbb{Z}}}\left(\sum_{m\in\mathbb{Z}}\left|T^\mu_{\epsilon_m,\epsilon_{m+1}}f_2(x)-T^\mu_{\epsilon_m,\epsilon_{m+1}}f_2(z_D)\right|^\rho\right)^{1/\rho},\end{aligned}\tag{74}$$

where the supremum is taken over all non-increasing sequences $\{\epsilon_m\}_{m\in\mathbb{Z}}$ of positive numbers ϵ_m. In order to estimate the right hand side of (74), take one of such sequences $\{\epsilon_m\}_{m\in\mathbb{Z}}$ and note that, by the triangle inequality again,

$$\begin{aligned}\Big|T^\mu_{\epsilon_m,\epsilon_{m+1}}\,&f_2(x)-T^\mu_{\epsilon_m,\epsilon_{m+1}}f_2(z_D)\Big|\\ &\leq\int\chi_{(\epsilon_{m+1},\epsilon_m]}(|x-y|)\,|K(x-y)-K(z_D-y)|\,|f_2(y)|\,d\mu(y)\\ &\quad+\int\left|\chi_{(\epsilon_{m+1},\epsilon_m]}(|x-y|)-\chi_{(\epsilon_{m+1},\epsilon_m]}(|z_D-y|)\right|\,|K(z_D-y)||f_2(y)|\,d\mu(y)\\ &=: a_m+b_m.\end{aligned}\tag{75}$$

Since x and z_D belong to D and f_2 vanishes in $3D$, we can assume that $\epsilon_{m+1}>\ell(D)$ in the definition of a_m and b_m for all $m\in\mathbb{Z}$.

Let us first look at the sum relative to the a_m's for $m\in\mathbb{Z}$. Using that $\rho>1$, the regularity of the kernel K, that f_2 vanishes in $3D$, and that μ is n-AD regular, for each $x\in D$ we have

$$\begin{aligned}\left(\sum_{m\in\mathbb{Z}}a_m^\rho\right)^{1/\rho}&\leq\sum_{m\in\mathbb{Z}}\int_{\epsilon_{m+1}<|x-y|\leq\epsilon_m}|K(x-y)-K(z_D-y)||f_2(y)|\,d\mu(y)\\ &\lesssim\ell(D)\sum_{m\in\mathbb{Z}}\int_{\epsilon_{m+1}<|x-y|\leq\epsilon_m}\frac{|f_2(y)|}{|y-z_D|^{n+1}}\,d\mu(y)\\ &\leq\ell(D)\int_{\mathbb{R}^d\setminus 3D}\frac{|f(y)|}{|y-z_D|^{n+1}}\,d\mu(y)\lesssim\mathcal{M}f(x_0)\leq\mathcal{M}_2f(x_0),\end{aligned}\tag{76}$$

where we also used Cauchy-Schwarz's inequality in the last estimate above.

The sum relative to the b_m's for $m\in\mathbb{Z}$ requires a more delicate analysis. We split $\mathbb{Z}=J_1\cup J_2$, where

$$J_1:=\{m\in\mathbb{Z}:\epsilon_m-\epsilon_{m+1}>\ell(D)\},$$
$$J_2:=\{m\in\mathbb{Z}:\epsilon_m-\epsilon_{m+1}\leq\ell(D)\}.$$

To shorten notation, we also set

$$A_m^1(z_D) := A(z_D, \epsilon_m - \ell(D), \epsilon_m + \ell(D)) \quad \text{and} \quad A_m^2(x) := A(x, \epsilon_{m+1}, \epsilon_m).$$

Since we are assuming $\epsilon_{m+1} > \ell(D)$ for all $m \in \mathbb{Z}$, both $A_m^1(z_D)$ and $A_{m+1}^1(z_D)$ are well defined for all $m \in J_1$. Moreover, since $|x - z_D| \leq \ell(D)$ for all $x \in D$, we easily get

$$\begin{aligned} &\left|\chi_{(\epsilon_{m+1},\epsilon_m]}(|x-\cdot|) - \chi_{(\epsilon_{m+1},\epsilon_m]}(|z_D-\cdot|)\right| \leq \chi_{A_m^1(z_D)} + \chi_{A_{m+1}^1(z_D)} \quad \text{for all } m \in J_1, \\ &\left|\chi_{(\epsilon_{m+1},\epsilon_m]}(|x-\cdot|) - \chi_{(\epsilon_{m+1},\epsilon_m]}(|z_D-\cdot|)\right| \leq \chi_{A_m^2(z_D)} + \chi_{A_m^2(x)} \quad \text{for all } m \in J_2. \end{aligned} \tag{77}$$

We are going to split the sum associated with the b_m's in terms of J_1 and J_2, using in each case the corresponding estimate from (77).

Concerning the sum over J_1, since $\rho > 2$, (77) yields

$$\begin{aligned} \Big(\sum_{m\in J_1} b_m^\rho\Big)^{1/\rho} &\lesssim \bigg(\sum_{m\in J_1}\Big(\int_{A_m^1(z_D)} |K(z_D-y)||f_2(y)|\,d\mu(y)\Big)^2\bigg)^{1/2} \\ &\quad + \bigg(\sum_{m\in J_1}\Big(\int_{A_{m+1}^1(z_D)} |K(z_D-y)||f_2(y)|\,d\mu(y)\Big)^2\bigg)^{1/2} \\ &=: S_1 + S_2. \end{aligned} \tag{78}$$

The arguments for estimating S_1 and S_2 are almost the same, so we will only give the details for S_1. Since f_2 vanishes in $3D$,

$$S_1^2 = \sum_{k\in\mathbb{Z}} \sum_{\substack{m\in J_1:\\ \epsilon_m\in I_k}} \Big(\int_{A_m^1(z_D)} |K(z_D-y)||f_2(y)|\,d\mu(y)\Big)^2 \lesssim \sum_{\substack{Q\in\mathcal{D}^\mu:\\ Q\supset D}} \sum_{\substack{m\in J_1:\\ \epsilon_m\in I_Q}} \frac{\left|(|f_2|\mu)\left(A_m^1(z_D)\right)\right|^2}{\ell(Q)^{2n}}. \tag{79}$$

Our task now is to bound $\left|(|f_2|\mu)\left(A_m^1(z_D)\right)\right|^2$. This is done by splitting the annulus $A_m^1(z_D)$, whose width equals $2\ell(D)$, into disjoint cubes $P \in \mathcal{D}^\mu$ such that $\ell(P) = \ell(D)$ and grouping them properly in terms of the corona decomposition, in order to be able to apply Carleson's embedding theorem later. More precisely, for $Q \supset D$ and $\epsilon_m \in I_Q$, we have

$$A_m^1(z_D) \cap \operatorname{supp}(\mu) \subset \bigcup_{R\in V(Q)} R \subset \Bigg(\bigcup_{R\in V(Q)} \bigcup_{\substack{P\in\mathrm{Tree}(R):\\ \ell(P)=\ell(D)}} P\Bigg) \cup \Bigg(\bigcup_{R\in V(Q)} \bigcup_{\substack{P\in\mathrm{Stp}(R):\\ \ell(P)\geq\ell(D)}} P\Bigg).$$

Recall also that the number of cubes in $V(Q)$ is bounded independently of Q. Therefore,

$$\begin{aligned} \left|(|f_2|\mu)\left(A_m^1(z_D)\right)\right|^2 &\lesssim \sum_{R\in V(Q)} \Bigg|\sum_{\substack{P\in\mathrm{Tree}(R):\\ \ell(P)=\ell(D)}} (|f_2|\mu)\left(A_m^1(z_D)\cap P\right)\Bigg|^2 \\ &\quad + \sum_{R\in V(Q)} \Bigg|\sum_{\substack{P\in\mathrm{Stp}(R):\\ \ell(P)\geq\ell(D)}} (|f_2|\mu)\left(A_m^1(z_D)\cap P\right)\Bigg|^2. \end{aligned} \tag{80}$$

The first term on the right hand side of (80) can be easily estimated using Cauchy-Schwarz's inequality, that the P's such that $\ell(P)=\ell(D)$ are disjoint and Lemma 2.5. That is,

$$
\begin{aligned}
\Bigg|\sum_{\substack{P\in\mathrm{Tree}(R):\\ \ell(P)=\ell(D)}}(|f_2|\mu)\left(A^1_m(z_D)\cap P\right)\Bigg|^2 &= \left|\int\Bigg(\sum_{\substack{P\in\mathrm{Tree}(R):\\ \ell(P)=\ell(D)}}\chi_{A^1_m(z_D)\cap P}\Bigg)|f_2|\,d\mu\right|^2\\
&\leq \Bigg(\sum_{\substack{P\in\mathrm{Tree}(R):\\ \ell(P)=\ell(D)}}\mu\left(A^1_m(z_D)\cap P\right)\Bigg)\Bigg(\sum_{\substack{P\in\mathrm{Tree}(R):\\ \ell(P)=\ell(D)}}(|f_2|^2\mu)\left(A^1_m(z_D)\cap P\right)\Bigg)\\
&\lesssim \ell(D)\ell(R)^{n-1}\sum_{\substack{P\in\mathrm{Tree}(R):\\ \ell(P)=\ell(D)}}\left(|f_2|^2\mu\right)\left(A^1_m(z_D)\cap P\right).
\end{aligned}\tag{81}
$$

The second term on the right hand side of (80) is estimated similarly but, since the cubes in $\mathrm{Stp}(R)$ may have different side length, we need to introduce an auxiliary splitting of the sum in terms of the side length. This extra splitting, combined with an application of Cauchy-Schwarz inequality yields

$$
\begin{aligned}
\Bigg|\sum_{\substack{P\in\mathrm{Stp}(R):\\ \ell(P)\geq\ell(D)}}(|f_2|\mu)\left(A^1_m(z_D)\cap P\right)\Bigg|^2 &= \Bigg|\sum_{j\geq0}\frac{2^{j/4}}{2^{j/4}}\sum_{\substack{P\in\mathrm{Stp}(R):\,\ell(P)\geq\ell(D)\\ \ell(P)=2^{-j}\ell(R)}}(|f_2|\mu)\left(A^1_m(z_D)\cap P\right)\Bigg|^2\\
&\lesssim \sum_{j\geq0}2^{j/2}\Bigg|\sum_{\substack{P\in\mathrm{Stp}(R):\,\ell(P)\geq\ell(D)\\ \ell(P)=2^{-j}\ell(R)}}(|f_2|\mu)\left(A^1_m(z_D)\cap P\right)\Bigg|^2\\
&\leq \sum_{j\geq0}2^{j/2}\Bigg(\sum_{\substack{P\in\mathrm{Stp}(R):\\ \ell(P)\geq\ell(D)\\ \ell(P)=2^{-j}\ell(R)}}\mu\left(A^1_m(z_D)\cap P\right)\Bigg)\Bigg(\sum_{\substack{P\in\mathrm{Stp}(R):\\ \ell(P)\geq\ell(D)\\ \ell(P)=2^{-j}\ell(R)}}\left(|f_2|^2\mu\right)\left(A^1_m(z_D)\cap P\right)\Bigg),
\end{aligned}\tag{82}
$$

where we also used in the last inequality above that the P's which belong to $\mathrm{Stp}(R)$ are disjoint and Cauchy-Schwarz's inequality. Since the width of the annulus $A^1_m(z_D)$ equals $2\ell(D)$, if $P\in\mathrm{Stp}(R)$ is such that $\ell(P)=2^{-j}\ell(R)\geq\ell(D)$ and $A^1_m(z_D)\cap P\neq\varnothing$ then

$$P\subset A(z_D,\epsilon_m-C2^{-j}\ell(R),\epsilon_m+C2^{-j}\ell(R))$$

for some $C>0$ depending only on n, d and μ. Hence, Lemma 2.5 gives

$$\sum_{\substack{P\in\mathrm{Stp}(R):\,\ell(P)\geq\ell(D)\\ \ell(P)=2^{-j}\ell(R)}}\mu\left(A^1_m(z_D)\cap P\right)\lesssim 2^{-j}\ell(R)^n,$$

which plugged into (82) yields

$$
\begin{aligned}
\Bigg|\sum_{\substack{P\in\mathrm{Stp}(R):\\ \ell(P)\geq\ell(D)}}(|f_2|\mu)\left(A^1_m(z_D)\cap P\right)\Bigg|^2 &\lesssim \sum_{j\geq0}2^{-j/2}\ell(R)^n\sum_{\substack{P\in\mathrm{Stp}(R):\\ \ell(P)\geq\ell(D)\\ \ell(P)=2^{-j}\ell(R)}}\left(|f_2|^2\mu\right)\left(A^1_m(z_D)\cap P\right)\\
&\leq \sum_{P\in\mathrm{Stp}(R)}\left(\frac{\ell(P)}{\ell(R)}\right)^{1/2}\ell(R)^n\left(|f_2|^2\mu\right)\left(A^1_m(z_D)\cap P\right).
\end{aligned}\tag{83}
$$

Applying (81) and (83) to (80), we see that

$$\begin{aligned}\left|(|f_2|\mu)\left(A^1_m(z_D)\right)\right|^2 \lesssim & \sum_{R\in V(Q)} \sum_{\substack{P\in \mathrm{Tree}(R):\\ \ell(P)=\ell(D)}} \frac{\ell(D)}{\ell(R)}\,\ell(R)^n \left(|f_2|^2\mu\right)\left(A^1_m(z_D)\cap P\right)\\ & + \sum_{R\in V(Q)} \sum_{P\in \mathrm{Stp}(R)} \left(\frac{\ell(P)}{\ell(R)}\right)^{1/2} \ell(R)^n \left(|f_2|^2\mu\right)\left(A^1_m(z_D)\cap P\right).\end{aligned}\tag{84}$$

Now that we have estimated $\left|(|f_2|\mu)\left(A^1_m(z_D)\right)\right|^2$, we can derive a bound for S_1^2. Since $\ell(Q)=\ell(R)$ for all $R\in V(Q)$, (79) and (84) imply that

$$\begin{aligned}S_1^2 \lesssim & \sum_{\substack{Q\in\mathcal{D}^\mu:\\ Q\supset D}} \sum_{\substack{m\in J_1:\\ \epsilon_m\in I_Q}} \sum_{R\in V(Q)} \sum_{\substack{P\in \mathrm{Tree}(R):\\ \ell(P)=\ell(D)}} \frac{\ell(D)}{\ell(R)}\,\ell(R)^{-n} \left(|f_2|^2\mu\right)\left(A^1_m(z_D)\cap P\right)\\ & + \sum_{\substack{Q\in\mathcal{D}^\mu:\\ Q\supset D}} \sum_{\substack{m\in J_1:\\ \epsilon_m\in I_Q}} \sum_{R\in V(Q)} \sum_{P\in \mathrm{Stp}(R)} \left(\frac{\ell(P)}{\ell(R)}\right)^{1/2} \ell(R)^{-n} \left(|f_2|^2\mu\right)\left(A^1_m(z_D)\cap P\right).\end{aligned}\tag{85}$$

Note that, for $m\in J_1$, each (closed) annulus $A^1_m(z_D)$ overlaps only with the two neighbors $A^1_{m-1}(z_D)$, $A^1_{m+1}(z_D)$ at the boundaries because $\{\epsilon_m\}_{m\in\mathbb{Z}}$ is a non-increasing sequence. Therefore, from (85) we deduce that

$$\begin{aligned}S_1^2 \lesssim & \sum_{\substack{Q\in\mathcal{D}^\mu:\\ Q\supset D}} \sum_{R\in V(Q)} \sum_{\substack{P\in \mathrm{Tree}(R):\\ \ell(P)=\ell(D)}} \frac{\ell(D)}{\ell(R)}\,\ell(R)^{-n} \left(|f_2|^2\mu\right)(P)\\ & + \sum_{\substack{Q\in\mathcal{D}^\mu:\\ Q\supset D}} \sum_{R\in V(Q)} \sum_{P\in \mathrm{Stp}(R)} \left(\frac{\ell(P)}{\ell(R)}\right)^{1/2} \ell(R)^{-n} \left(|f_2|^2\mu\right)(P).\end{aligned}\tag{86}$$

For the first term on the right hand side of (86), using that the P's in $\mathcal{D}^\mu$ such that $\ell(P)=\ell(D)$ are disjoint, that μ is n-AD regular and that $x_0\in D$, we have

$$\begin{aligned}\sum_{\substack{Q\in\mathcal{D}^\mu:\\ Q\supset D}} \sum_{R\in V(Q)} \sum_{\substack{P\in \mathrm{Tree}(R):\\ \ell(P)=\ell(D)}} \frac{\ell(D)}{\ell(R)}\,\ell(R)^{-n} \left(|f_2|^2\mu\right)(P) &\leq \sum_{\substack{Q\in\mathcal{D}^\mu:\\ Q\supset D}} \sum_{R\in V(Q)} \frac{\ell(D)}{\ell(R)} \frac{\left(|f_2|^2\mu\right)(R)}{\ell(R)^n}\\ &\lesssim \sum_{\substack{Q\in\mathcal{D}^\mu:\\ Q\supset D}} \sum_{R\in V(Q)} \frac{\ell(D)}{\ell(Q)}\,\mathcal{M}_2 f(x_0)^2\\ &\lesssim \mathcal{M}_2 f(x_0)^2.\end{aligned}\tag{87}$$

In order to estimate the second term on the right hand side of (86), note that $R\in V(Q)$ if and only if $Q\in V(R)$ and that if $D\subset Q$ and $R\in V(Q)$ then $D\subset 3R$, thus by changing the order of summation and using that the number of cubes in $V(R)$ is bounded independently of R and that $\mathcal{D}^\mu = \bigcup_{S\in\mathrm{Top}} \mathrm{Tree}(S)$ we see that

$$\begin{aligned}
&\sum_{\substack{Q\in\mathcal{D}^\mu:\\ Q\supset D}} \sum_{R\in V(Q)} \sum_{P\in\mathrm{Stp}(R)} \left(\frac{\ell(P)}{\ell(R)}\right)^{1/2} \ell(R)^{-n} \left(|f_2|^2\mu\right)(P) \\
&\qquad\le \sum_{\substack{R\in\mathcal{D}^\mu:\\ 3R\supset D}} \sum_{Q\in V(R)} \sum_{P\in\mathrm{Stp}(R)} \left(\frac{\ell(P)}{\ell(R)}\right)^{1/2} \ell(R)^{-n} \left(|f_2|^2\mu\right)(P) \\
&\qquad\lesssim \sum_{S\in\mathrm{Top}} \sum_{\substack{R\in\mathrm{Tree}(S):\\ 3R\supset D}} \sum_{P\in\mathrm{Stp}(R)} \left(\frac{\ell(P)}{\ell(R)}\right)^{1/2} \ell(R)^{-n} \left(|f_2|^2\mu\right)(P) \\
&\qquad\lesssim \sum_{S\in\mathrm{Top}} \sum_{P\in\mathrm{Stp}(S)} \ell(P)^{1/2} \left(|f_2|^2\mu\right)(P) \sum_{\substack{R\in\mathrm{Tree}(S):\\ 3R\supset D\cup P}} \ell(R)^{-n-1/2},
\end{aligned} \tag{88}$$

where we also used in the last inequality above that, for $S \in \mathrm{Top}$, if $P \in \mathrm{Stp}(R)$ for some $R \in \mathrm{Tree}(S)$ then $P \in \mathrm{Stp}(S)$ and $P \subset R$. Moreover, denoting

$$D(P, D) := \ell(P) + \mathrm{dist}(P, D) + \ell(D),$$

we have

$$\begin{aligned}
\sum_{\substack{R\in\mathrm{Tree}(S):\\ 3R\supset D\cup P}} \ell(R)^{-n-1/2} &\lesssim \sum_{j\in\mathbb{Z}} \sum_{\substack{R\in\mathrm{Tree}(S):\, 3R\supset D\cup P,\\ 2^j D(P,D)<\ell(R)\le 2^{j+1}D(P,D)}} (2^j D(P, D))^{-n-1/2} \\
&\lesssim D(P, D)^{-n-1/2},
\end{aligned} \tag{89}$$

because the number of cubes $R \in \mathcal{D}^\mu$ such that $3R \supset D \cup P$ and $2^j D(P, D) < \ell(R) \le 2^{j+1}D(P, D)$ is bounded independently of $j \in \mathbb{Z}$, and the statements "$3R \supset D \cup P$" and "$2^j D(P, D) < \ell(R) \le 2^{j+1}D(P, D)$" are compatible each other only if $j \ge j_0$ for some $j_0 \in \mathbb{Z}$ which only depends on d, n and μ. Plugging (89) into (88), we get

$$\begin{aligned}
&\sum_{\substack{Q\in\mathcal{D}^\mu:\\ Q\supset D}} \sum_{R\in V(Q)} \sum_{P\in\mathrm{Stp}(R)} \left(\frac{\ell(P)}{\ell(R)}\right)^{1/2} \ell(R)^{-n} \left(|f_2|^2\mu\right)(P) \\
&\qquad\lesssim \sum_{S\in\mathrm{Top}} \sum_{P\in\mathrm{Stp}(S)} \left(\frac{\ell(P)}{D(P, D)}\right)^{n+1/2} \frac{\left(|f_2|^2\mu\right)(P)}{\ell(P)^n}.
\end{aligned} \tag{90}$$

Finally, by (87), (90), and (86), we conclude that

$$S_1^2 \lesssim \mathcal{M}_2 f(x_0)^2 + \sum_{S\in\mathrm{Top}} \sum_{P\in\mathrm{Stp}(S)} \left(\frac{\ell(P)}{D(P, D)}\right)^{n+1/2} m_P\left(|f|^2\right). \tag{91}$$

As we pointed out before, the same estimate holds for S_2^2, because the only properties that we used from the annuli $A^1_m(z_D)$'s are that they have bounded overlap for $m \in J_1$, that their width is comparable to $\ell(D)$, that they are centered in some point lying in $D \subset Q$ and that they have diameter comparable to $\ell(Q)$. Of course, these properties are also shared by the annuli $A^1_{m+1}(z_D)$'s. Actually, for estimating S_2, one can argue exactly as in the case of S_1 but replacing $\{m \in J_1 : \epsilon_m \in I_Q\}$ by $\{m \in J_1 : \epsilon_{m+1} \in I_Q\}$ in the involved arguments.

Therefore, by (91), the analogous estimate for S_2, and (78), we see that

$$\Big(\sum_{m\in J_1} b_m^\rho\Big)^{1/\rho} \lesssim \mathcal{M}_2 f(x_0) + \bigg(\sum_{S\in\mathrm{Top}}\sum_{P\in\mathrm{Stp}(S)}\Big(\frac{\ell(P)}{D(P,D)}\Big)^{n+1/2} m_P\big(|f|^2\big)\bigg)^{1/2}. \tag{92}$$

We now deal with the sum relative to the b_m's for $m\in J_2$. The estimates are essentially as in the case of $m\in J_1$, but we include the sketch of the arguments for the reader's convenience. Since $\rho>2$, (77) yields

$$\begin{aligned}\Big(\sum_{m\in J_2} b_m^\rho\Big)^{1/\rho} &\lesssim \bigg(\sum_{m\in J_2}\Big(\int_{A_m^2(z_D)}|K(z_D-y)||f_2(y)|\,d\mu(y)\Big)^2\bigg)^{1/2}\\ &\quad+\bigg(\sum_{m\in J_2}\Big(\int_{A_m^2(x)}|K(z_D-y)||f_2(y)|\,d\mu(y)\Big)^2\bigg)^{1/2}\\ &=: S_3+S_4.\end{aligned} \tag{93}$$

The arguments to estimate S_3 and S_4 are almost the same, so we will only give the details for S_3. Since f_2 vanishes in $3D$,

$$S_3^2=\sum_{k\in\mathbb{Z}}\sum_{\substack{m\in J_2:\\ \epsilon_m\in I_k}}\Big(\int_{A_m^2(z_D)}|K(z_D-y)||f_2(y)|\,d\mu(y)\Big)^2\lesssim\sum_{\substack{Q\in\mathcal{D}^\mu:\\ Q\supset D}}\sum_{\substack{m\in J_2:\\ \epsilon_m\in I_Q}}\frac{\big|(|f_2|\mu)\big(A_m^2(z_D)\big)\big|^2}{\ell(Q)^{2n}}. \tag{94}$$

Once again, our task now is to estimate $\big|(|f_2|\mu)\big(A_m^2(z_D)\big)\big|^2$. As before, this is done by splitting the annulus $A_m^2(z_D)$, whose width is $\epsilon_m-\epsilon_{m+1}$, in disjoint cubes $P\in\mathcal{D}^\mu$ such that $\epsilon_m-\epsilon_{m+1}\in I_P$ and grouping them properly in terms of the corona decomposition. Arguing as in (80), we now have

$$\begin{aligned}\big|(|f_2|\mu)\big(A_m^2(z_D)\big)\big|^2 &\lesssim \sum_{R\in V(Q)}\bigg|\sum_{\substack{P\in\mathrm{Tree}(R):\\ \epsilon_m-\epsilon_{m+1}\in I_P}}(|f_2|\mu)\big(A_m^2(z_D)\cap P\big)\bigg|^2\\ &\quad+\sum_{R\in V(Q)}\bigg|\sum_{\substack{P\in\mathrm{Stp}(R):\\ \ell(P)\geq\epsilon_m-\epsilon_{m+1}}}(|f_2|\mu)\big(A_m^2(z_D)\cap P\big)\bigg|^2.\end{aligned} \tag{95}$$

The first term on the right hand side of (95) can be easily estimated using Cauchy-Schwarz's inequality, that the P's in $\mathrm{Tree}(R)$ such that $\epsilon_m-\epsilon_{m+1}\in I_P$ are disjoint and Lemma 2.5. Similarly to what we did in (81), we now obtain

$$\begin{aligned}&\bigg|\sum_{\substack{P\in\mathrm{Tree}(R):\\ \epsilon_m-\epsilon_{m+1}\in I_P}}(|f_2|\mu)\big(A_m^2(z_D)\cap P\big)\bigg|^2\\ &\qquad\lesssim(\epsilon_m-\epsilon_{m+1})\ell(R)^{n-1}\sum_{\substack{P\in\mathrm{Tree}(R):\\ \epsilon_m-\epsilon_{m+1}\in I_P}}\big(|f_2|^2\mu\big)\big(A_m^2(z_D)\cap P\big)\\ &\qquad\leq\ell(D)\ell(R)^{n-1}\big(|f_2|^2\mu\big)\big(A_m^2(z_D)\cap R\big),\end{aligned} \tag{96}$$

where we also used in the last inequality above that $\epsilon_m-\epsilon_{m+1}\leq\ell(D)$, because we are assuming $m\in J_2$. As before, the second term on the right hand side of (95) is estimated similarly to (96) but introducing an auxiliary splitting of the sum in terms of the side length of the cubes. By applying the Cauchy-Schwarz inequality, we can proceed exactly as in (82) and (83), but replacing $\ell(D)$ by $\epsilon_m - \epsilon_{m+1}$, and then we deduce that

$$\begin{aligned}\Bigg|\sum_{\substack{P\in \mathrm{Stp}(R):\\ \ell(P)\geq \epsilon_m-\epsilon_{m+1}}} (|f_2|\mu)\left(A^1_m(z_D)\cap P\right)\Bigg|^2 \\ \lesssim \sum_{P\in\mathrm{Stp}(R)}\left(\frac{\ell(P)}{\ell(R)}\right)^{1/2}\ell(R)^n\left(|f_2|^2\mu\right)\left(A^2_m(z_D)\cap P\right).\end{aligned} \tag{97}$$

Combining (94) and (95) with (96) and (97), and using that $\ell(R)=\ell(Q)$ for all $R\in V(Q)$ and that, for $m\in\mathbb{Z}$, the closed annuli $A^2_m(z_D)$'s overlap only with the neighboring annuli because $\{\epsilon_m\}_{m\in\mathbb{Z}}$ is a non-increasing sequence, we conclude that

$$\begin{aligned}S_3^2 &\lesssim \sum_{\substack{Q\in\mathcal{D}^\mu:\\ Q\supset D}}\sum_{R\in V(Q)}\frac{\ell(D)}{\ell(R)}\,\ell(R)^{-n}\left(|f_2|^2\mu\right)(R)\\ &\quad+\sum_{\substack{Q\in\mathcal{D}^\mu:\\ Q\supset D}}\sum_{R\in V(Q)}\sum_{P\in\mathrm{Stp}(R)}\left(\frac{\ell(P)}{\ell(R)}\right)^{1/2}\ell(R)^{-n}\left(|f_2|^2\mu\right)(P).\end{aligned} \tag{98}$$

Plugging (87) and (90) into (98) finally yields

$$S_3^2\lesssim \mathcal{M}_2f(x_0)^2+\sum_{S\in\mathrm{Top}}\sum_{P\in\mathrm{Stp}(S)}\left(\frac{\ell(P)}{D(P,D)}\right)^{n+1/2} m_P\left(|f|^2\right). \tag{99}$$

Similarly to what we said below (91), the same estimate that we have for S_3 also holds for S_4. Therefore, applying (99) (and the same estimate for S_4) to (93), we see that

$$\Bigg(\sum_{m\in J_2} b_m^\rho\Bigg)^{1/\rho}\lesssim \mathcal{M}_2f(x_0)+\Bigg(\sum_{S\in\mathrm{Top}}\sum_{P\in\mathrm{Stp}(S)}\left(\frac{\ell(P)}{D(P,D)}\right)^{n+1/2} m_P\left(|f|^2\right)\Bigg)^{1/2}. \tag{100}$$

To complete the proof of the theorem it only remains to put all the estimates together and to use standard arguments. From (76), (92) and (100), we see that

$$\Bigg(\sum_{m\in\mathbb{Z}}(a_m+b_m)^\rho\Bigg)^{1/\rho}\lesssim \mathcal{M}_2f(x_0)+\Bigg(\sum_{S\in\mathrm{Top}}\sum_{P\in\mathrm{Stp}(S)}\left(\frac{\ell(P)}{D(P,D)}\right)^{n+1/2} m_P\left(|f|^2\right)\Bigg)^{1/2},$$

which, by (74) and (75), implies that

$$\begin{aligned}I_2&=\frac{1}{\mu(D)}\int_D\left|(\mathcal{V}_\rho\circ\mathcal{T}^\mu)f_2-(\mathcal{V}_\rho\circ\mathcal{T}^\mu)f_2(z_D)\right|d\mu\\ &\lesssim \mathcal{M}_2f(x_0)+\Bigg(\sum_{S\in\mathrm{Top}}\sum_{P\in\mathrm{Stp}(S)}\left(\frac{\ell(P)}{D(P,D)}\right)^{n+1/2} m_P\left(|f|^2\right)\Bigg)^{1/2}.\end{aligned} \tag{101}$$

Finally, combining (71) and (72) with (73) and (101), and using that $\bigcup_{S\in\mathrm{Top}}\mathrm{Stp}(S)\subset\mathrm{Top}$, we conclude that

(102)
$$\begin{aligned}(\mathcal{M}^{\sharp}_{\mathcal{D}^\mu}\circ\mathcal{V}_\rho\circ\mathcal{T}^\mu)f(x_0)&\\ &\lesssim \mathcal{M}_2 f(x_0)+\sup_{D\in\mathcal{D}^\mu:\,x_0\in D}\Bigg(\sum_{S\in\mathrm{Top}}\sum_{P\in\mathrm{Stp}(S)}\left(\frac{\ell(P)}{D(P,D)}\right)^{n+1/2} m_P\left(|f|^2\right)\Bigg)^{1/2}\\ &\lesssim \mathcal{M}_2 f(x_0)+\Bigg(\sum_{P\in\mathrm{Top}}\left(\frac{\ell(P)}{D(P,x_0)}\right)^{n+1/2} m_P\left(|f|^2\right)\Bigg)^{1/2}\\ &=: \mathcal{M}_2 f(x_0)+\mathcal{E}_{1/2}f(x_0),\end{aligned}$$

for all $x_0\in\mathrm{supp}(\mu)$, where we denoted

$$D(P,x_0):=\ell(P)+\mathrm{dist}(P,x_0). \tag{103}$$

In Lemma 4.2 below we prove that $\mathcal{E}_{1/2}$ is a bounded operator in $L^p(\mu)$ for all $2<p<\infty$. Assuming this for the moment, by (102) and the $L^p(\mu)$-boundedness of $\mathcal{M}_2$, we see that $\mathcal{M}^{\sharp}_{\mathcal{D}^\mu}\circ\mathcal{V}_\rho\circ\mathcal{T}^\mu$ is also bounded in $L^p(\mu)$ for all $2<p<\infty$. Then we obtain

$$\|(\mathcal{V}_\rho\circ\mathcal{T}^\mu)f\|_{L^p(\mu)}\le\|(\mathcal{M}_{\mathcal{D}^\mu}\circ\mathcal{V}_\rho\circ\mathcal{T}^\mu)f\|_{L^p(\mu)}\lesssim\|(\mathcal{M}^{\sharp}_{\mathcal{D}^\mu}\circ\mathcal{V}_\rho\circ\mathcal{T}^\mu)f\|_{L^p(\mu)}\lesssim\|f\|_{L^p(\mu)}$$

for all $2<p<\infty$, and the theorem is proved. □

Lemma 4.2. *Given $\delta>0$, set*

$$\mathcal{E}_\delta f(x):=\Bigg(\sum_{P\in\mathrm{Top}}\left(\frac{\ell(P)}{D(P,x)}\right)^{n+\delta} m_P\left(|f|^2\right)\Bigg)^{1/2}$$

for $f\in L^p(\mu)$ and $x\in\mathbb{R}^d$, where $D(P,x)$ is defined in (103)*. Then $\mathcal{E}_\delta$ is a bounded operator in $L^p(\mu)$ for all $2<p<\infty$.*

Proof. The proof follows by duality and Carleson's embedding theorem. Since $2<p<\infty$, if q is such that $2/p+1/q=1$ then $1<q<\infty$, thus

$$\|\mathcal{E}_\delta f\|_{L^p(\mu)}=\|(\mathcal{E}_\delta f)^2\|^{1/2}_{L^{p/2}(\mu)}=\sup_{\|g\|_{L^q(\mu)}\le1}\left|\int(\mathcal{E}_\delta f)^2 g\,d\mu\right|^{1/2}. \tag{104}$$

Note that

$$\left|\int(\mathcal{E}_\delta f)^2 g\,d\mu\right|\le\sum_{P\in\mathrm{Top}} m_P\left(|f|^2\right)\int\left(\frac{\ell(P)}{D(P,x)}\right)^{n+\delta}|g(x)|\,d\mu(x). \tag{105}$$

Integrating over dyadic annuli and using that μ is n-AD regular, it is easy to check that

$$\frac{1}{\mu(P)}\int\left(\frac{\ell(P)}{D(P,x)}\right)^{n+\delta}|g(x)|\,d\mu(x)\lesssim\mathcal{M}g(y)\quad\text{for all } y\in P \tag{106}$$

(here it is crucial that $\delta > 0$). Thus, by (105), (106), Hölder's inequality and Carleson's embedding Theorem 2.2 (recall that $p/2$ and q belong to $(1,\infty)$),

$$\begin{aligned}\left|\int (\mathcal{E}_\delta f)^2 g\, d\mu\right| &\lesssim \sum_{P\in\mathrm{Top}} m_P\left(|f|^2\right) m_P(\mathcal{M}g)\mu(P)\\
&\leq \Big(\sum_{P\in\mathrm{Top}} \left(m_P\left(|f|^2\right)\right)^{p/2}\mu(P)\Big)^{2/p}\Big(\sum_{P\in\mathrm{Top}} (m_P(\mathcal{M}g))^q\mu(P)\Big)^{1/q}\\
&\lesssim \||f|^2\|_{L^{p/2}(\mu)}\|\mathcal{M}g\|_{L^q(\mu)} \lesssim \|f\|^2_{L^p(\mu)}\|g\|_{L^q(\mu)}.\end{aligned} \tag{107}$$

From (104) and (107) we conclude that $\|\mathcal{E}_\delta f\|_{L^p(\mu)} \lesssim \|f\|_{L^p(\mu)}$, as wished. □

5. The proof of Theorem 1.4

The arguments are very similar to the ones for the proof of Theorem 1.1 and so we will only sketch the main ideas.

When K is an odd kernel satisfying (1), one of the main ingredients of the proof of the boundedness of $\mathcal{V}_\rho\circ\mathcal{T}$ from $M(\mathbb{R}^d)$ to $L^{1,\infty}(\mu)$ in Section 3 and of $\mathcal{V}_\rho\circ\mathcal{T}^\mu$ in $L^p(\mu)$ for $2<p<\infty$ in Section 4 is Theorem 3.2, which ensures the boundedness of $\mathcal{V}_\rho\circ\mathcal{T}^\mu$ in $L^2(\mu)\to L^2(\mu)$ and of $\mathcal{V}_\rho\circ\mathcal{T}_\varphi$ from $M(\mathbb{R}^d)$ to $L^{1,\infty}(\mu)$. The reader can easily check that exactly the same arguments contained in Sections 3 and 4 show that if $K(\cdot,\cdot)$ is a Calderón-Zygmund kernel as in Theorem 1.4 and T is the associated operator, and moreover the following assumptions hold:

(*i*) $\mathcal{V}_\rho\circ\mathcal{T}^\mu : L^2(\mu)\to L^2(\mu)$ is bounded,
(*ii*) $\mathcal{V}_\rho\circ\mathcal{T}_\varphi : M(\mathbb{R}^d)\to L^{1,\infty}(\mu)$ is bounded,

then $\mathcal{V}_\rho\circ\mathcal{T} : M(\mathbb{R}^d)\to L^{1,\infty}(\mu)$ and $\mathcal{V}_\rho\circ\mathcal{T}^\mu : L^p(\mu)\to L^p(\mu)$, $2<p<\infty$, are also bounded. That is, the same conclusions of Theorems 3.1 and 4.1 hold.

Thus, by interpolation, to conclude the proof of Theorem 1.4 it just remains to check that the conditions (*i*) and (*ii*) above hold. This is obvious in the case of condition (*i*) because this is indeed one of the main assumptions of Theorem 1.4. Concerning (*ii*), note first that the boundedness of $\mathcal{V}_\rho\circ\mathcal{T}^\mu$ in $L^2(\mu)$ implies that $\mathcal{V}_\rho\circ\mathcal{T}^\mu_\varphi$ is also bounded in $L^2(\mu)$. This is an immediate consequence of the pointwise estimate

$$\mathcal{V}_\rho\circ\mathcal{T}^\mu_\varphi(f)(x) \lesssim \mathcal{V}_\rho\circ\mathcal{T}^\mu(f)(x),$$

which can be obtained by writing

$$T_{\varphi_\epsilon}(f\mu)(x) := \int \varphi_\epsilon(x-y)K(x,y)\, f(y)\, d\mu(y)$$

in terms of a convex combination of functions of the form

$$T_\delta(f\mu)(x) := \int_{|x-y|>\delta} K(x,y)\, f(y)\, d\mu(y),$$

for $\delta>0$ belonging to some interval depending on ϵ and then applying Minkowski's integral inequality. The arguments are quite similar to the ones in (31)-(33) and we omit them.

Then, basically the same arguments for the proof of Theorem 2.5 in [17] show that the boundedness of $\mathcal{V}_\rho\circ\mathcal{T}^\mu_\varphi$ in $L^2(\mu)$ implies that $\mathcal{V}_\rho\circ\mathcal{T}_\varphi$ is bounded from $M(\mathbb{R}^d)$ to $L^{1,\infty}(\mu)$. This is shown in [17] for the case when K is an odd kernel satisfying (1) and μ is the Hausdorff measure $\mathcal{H}^n$ on a Lipschitz graph. However, the same proof with very minor changes works in the more general situation when $K(\cdot,\cdot)$ is a kernel such as in Theorem 1.4 and μ is just and n-dimensional AD-regular measure.

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